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The standard realizations for the K-theory of varieties

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Abstract

The Grothendieck ring of varieties admits well-known realization maps to, say, mixed Hodge structures or compactly supported ℓ-adic cohomology. These are often called motivic measures. Zakharevich and Campbell have developed a spectral refinement of the Grothendieck ring of varieties. We develop a realization map from their refinement to the K-theory of mixed motives. Composing the latter with the standard realizations of motives, this yields spectral refinements of all the usual motivic measures.

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1. Motivation

Let k be a perfect field. We write $K_0(\mathcal{V}_k)$ for the Grothendieck ring of varieties. There are the standard motivic realizations, also known as motivic measures, such as the one to ℓ-adic representations

$$K_0(\mathcal{V}_k) \longrightarrow K_0(\mathrm{Rep}_{\mathrm{Gal}(k^{\mathrm{sep}}/k)}(\mathbb{Q}_\ell))$$

for $\ell \in k^\times$, or to mixed Hodge structures when $k \subset \mathbb{C}$, for example in the format of Hodge–Deligne polynomials

$$\begin{aligned} K_0(\mathcal{V}_k) &\longrightarrow \mathbb{Z}[u,v] \\ X &\longmapsto \sum_{p,q\in\mathbb{Z}} (-1)^{p+q} h_c^{p,q}(X)\cdot u^p v^q, \end{aligned}$$

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where $h_c^{p,q}(X)$ denotes the Hodge numbers of compactly supported cohomology, i.e., $\dim H_c^{p,q}(X)$. The purpose of this paper is to extend such realizations beyond K_0. For the Betti and étale realization this problem was already solved by Campbell, Wolfson and Zakharevich in [7]. In this paper we develop an axiomatic approach: *Every invariant of smooth varieties which satisfies h-hyperdescent and $\mathbb{A}^1$-invariance gives rise to a canonical realization.*

This transports a paradigm from the philosophy of mixed motives to the setting of the spectral Grothendieck ring of varieties. There are two pathways to extend the classical Grothendieck ring of varieties to an E_∞-ring spectrum $K(\mathcal{V}_k)$ such that

$$\pi_0 K(\mathcal{V}_k) = K_0(\mathcal{V}_k),$$

where $K_0(\mathcal{V}_k)$ on the right refers to the classical definition. The first approach to set up such a spectral refinement was pioneered by Zakharevich [29, 30]. A little later, Campbell [4] introduced an alternative idea for such a construction. It was shown in [8] that both strategies give rise to the same concept, the same spectrum. Still, either model has their individual benefits depending on what one wants to do with it. Our realizations will be set up as maps of spectra mapping out of Campbell's model.

We construct the realizations in two steps: (1) First we construct a realization to geometric mixed motives in the sense of Voevodsky. (2) Then we use that invariants satisfying h-hyperdescent and $\mathbb{A}^1$-invariance pin down realization functors for such motives. Composing both maps produces the desired realizations.

Let A be a commutative unital ring, serving as our ring of coefficients. We write $\mathcal{DM}^{\mathrm{eff}}_{\mathrm{gm},t}(k; A)$ for the A-linear DG category of geometric mixed motives in the topology $t \in \{\acute{e}t, Nis\}$. We extract a Waldhausen category from such a DG category such that the weak equivalences are quasi-isomorphisms in the DG sense.

Suppose $\mathcal{V}_k$ denotes the SW-category of k-varieties (see §1.1 for details).

Theorem 1.1. *Suppose k is a perfect field. If k has positive characteristic $p > 0$, we assume that $\frac{1}{p} \in A$. We construct a weakly W-exact functor $F = (F_!, F^!, F^w)$ from $\mathcal{V}_k$ to the Waldhausen DG category $\mathcal{DM}^{\mathrm{eff}}_{\mathrm{gm},t}(k; A)$. In particular, it induces a map of spectra*

$$K(F)\colon K(\mathcal{V}_k) \longrightarrow K(\mathcal{DM}^{\mathrm{eff}}_{\mathrm{gm},t}(k; A)).$$

Moreover, for any variety $X \in \mathcal{V}_k$ we have

$$F(X) = M^c(X)$$

in the homotopy category $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm},t}(k; A)$, i.e., $F(X)$ represents the motive with compact support attached to X.

See Theorem 2.14. The above motivic realization leads to the following very general mechanism to produce realizations: For every DG enhanced *realization functor* (or DG quasi-functor)

$$\mathcal{DM}^{\mathrm{eff}}_{\mathrm{gm},t}(k; A) \to \mathcal{D}$$

to a DG category $\mathcal{D}$, we obtain a realization map

$$K(\mathcal{V}_k) \to K(\mathcal{DM}^{\mathrm{eff}}_{\mathrm{gm},t}(k; A)) \to K(\mathcal{D}).$$

The first K denotes the K-theory of varieties, while the other two instances of K refer to the usual K-theory of a Waldhausen/DG category. As this and other constructions show, it is quite sensible not to enforce a notation which stresses the distinct underlying definitions. Both flavours of K-theory interact seamlessly. The DG category $\mathcal{DM}^{\text{eff}}_{\text{gm},t}(k;A)$ possesses a universal property, to be described below, which simplifies the construction of realization functors. Take A as in the previous theorem. Suppose $\mathcal{D}$ is a triangulated cocomplete and compactly generated A-linear DG category. Suppose

$$\mathcal{D}_{\text{finite}} \subseteq \mathcal{D}$$

is a suitable subcategory (we will list precise conditions in the main body of the paper). Following Vologodsky, we write $\mathcal{T}^{h,\Delta}(A[Sm_k],-)$ for DG quasi-functors on smooth k-varieties (with finite correspondences with A-coefficients as morphisms) which satisfy $\mathbb{A}^1$-invariance and h-hyperdescent.

Theorem 1.2. *Suppose k is a perfect field. If k has positive characteristic $p > 0$, we assume that $\frac{1}{p} \in A$. Further, suppose k has finite cohomological dimension with respect to A-coefficients in the sense of Equation 2.2. Suppose we are given a DG quasi-functor*

$$\phi \in \mathcal{T}^{h,\Delta}(A[Sm_k], \mathcal{D}_{\text{finite}}).$$

Then there is a map of spectra

$$\mathrm{R}^{\phi} \colon K(\mathcal{V}_k) \longrightarrow K(\mathcal{D}_{\text{finite}})$$

such that the following hold:

1. *Suppose X is a smooth proper k-variety. In K_0 we get*

   $$\mathrm{R}^{\phi}([X]) = [\phi(X)]$$

   and if $f\colon X \xrightarrow{\sim} X$ is any automorphism, we get in K_1 that

   $$\mathrm{R}^{\phi}([f\colon X \xrightarrow{\sim} X]) = \left[\phi(f)\colon \phi(X) \xrightarrow{\sim} \phi(X)\right].$$

2. *Suppose X is a smooth k-variety, $\overline{X}$ a smooth compactification with a smooth closed subvariety $Z \subseteq \overline{X}$ such that $X = \overline{X} \setminus Z$. Then in K_0,*

   $$\mathrm{R}^{\phi}([X]) = [\phi(\overline{X})] - [\phi(Z)]$$

   and if one can extend an automorphism f on X such that

   $$\begin{array}{ccc} Z & \hookrightarrow & \overline{X} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \hookrightarrow & \overline{X} \end{array}$$

   commutes, then $\mathrm{R}^{\phi}([f\colon X \xrightarrow{\sim} X]) = \mathrm{R}^{\phi}([f\colon \overline{X} \xrightarrow{\sim} \overline{X}]) - \mathrm{R}^{\phi}([f\mid_Z\colon Z \xrightarrow{\sim} Z])$.

See Theorem 3.2. Realizations like, for example, the

- Betti realization to finitely generated A-modules for A some Noetherian ring,
- mixed Hodge realization,
- ℓ-adic étale realizations (for various ℓ),
- or more broadly any mixed Weil cohomology theory in the sense of Cisinski and Déglise,

may be used as input for the above theorem, producing corresponding realizations for $K(\mathcal{V}_k)$.

This is not the first construction of realizations for $K(\mathcal{V}_k)$. The Betti and ℓ-adic realization were constructed by Campbell, Wolfson and Zakharevich in [7] and we crucially use their device of weakly W-exact functors. That we add a mixed Hodge realization solves *Problem 7.4* loc. cit. and confirms the expectation mentioned loc. cit. that other Weil cohomology theories admit realizations through the Cisinski–Déglise mechanism.

The idea to extend realization functors uniquely to motives by demanding them to exist on smooth varieties and have suitable descent and $\mathbb{A}^1$-invariance properties can be found in work of Vologodsky [27] and Robalo [20], but these ideas have a long series of predecessors constructing various realizations in the setting of mixed motives, among others by Huber [12, 13] or Ivorra [14, 15]. For the purposes of this paper, we follow the framework of Vologodsky. In his work, Vologodsky at times assumes $\mathrm{char}(k) = 0$, but a careful inspection of his proof reveals that he only does so in order to use resolution of singularities, and this in turn is only used to be able to use the technology of the *cdh*-topology. Facing this issue, one could do one of two things: We could have written this paper under the standing assumption that the base field k permits resolution of singularities[1] and hope that it will be proven also in positive characteristic one day in a far and distant future. We chose not to do this. Instead, we favoured to follow the ideas of Kelly [16], which establish all of the tools that Vologodsky's work relies on in positive characteristic, under the additional assumption that we work with coefficients in which p is invertible. This amounts to replacing the *cdh*-topology in Vologodsky by the ℓdh-topology, and re-adjusting some references to Voevodsky's foundations to Kelly's work.

Regarding mixed motives, we will follow Voevodsky's theory, most of what we need is explained in the book [18], with three noteworthy exceptions:

- Not all necessary theorems are actually proven in the book. However, in these cases we just refer to the original book of Friedlander–Suslin–Voevodsky, in particular the article [10]. This is unproblematic.
- In characteristic $p > 0$ we would like to use results which, in Voevodsky's theory, are only available under the assumption that resolution of singularities will be established also in positive characteristic. Should this ever occur, the better. However, since at present this is not available, we instead use Kelly's work based on alterations [16], which explains that all the results in [26] for $\mathcal{DM}^{\mathrm{eff}}_{\mathrm{gm},Nis}(k, A)$ hold when k is a perfect field with characteristic p, but at the price of $p \in A$ being invertible.
- The original literature, such as [10], as well as the book [18], only develop mixed motives on the level of triangulated categories. However, we crucially need a DG enhancement, detailed in the work of Beilinson and Vologodsky in [1]. Again, in the case of positive characteristic, use the ℓdh-topology therein instead of the *cdh*-topology. In characteristic zero, [1] can be used verbatim.

For the future, considering realizations as in the work of Röndigs [21] offers an interesting perspective. We did not pursue this in the present text.

1.1 Running conventions. The word *scheme* will always refer to a separated scheme over a field. We stress this because both sources [23] and [18] also use such assumptions tacitly (they both announce early on that all their schemes will satisfy conditions which are implied by the

[1] so, basically, assuming that a suitable form of resolution of singularities is possible for all varieties over k as an axiom

above). Separatedness is required to have a well-behaved notion of pullbacks of relative cycles.

By a *k-variety* we mean a reduced, separated scheme of finite type over k. This is in line with the usage in [4, Def. 2.1], specialized to a base field. Morphisms between k-varieties will always be assumed to be k-morphisms.

As summarized in [7, Def. 2.9], an *SW-category* is a (1-)category $\mathcal{C}$ equipped with three distinguished subcategories:

1. Cofibrations, denoted $\mathbf{co}(\mathcal{C})$, and whose morphisms are indicated by the arrow style $\hookrightarrow$.
2. Complement maps, denoted $\mathbf{comp}(\mathcal{C})$, and whose morphisms are indicated by the arrow style $\xrightarrow{\circ}$.
3. Weak equivalences, denoted $\mathbf{w}(\mathcal{C})$, and whose morphisms are indicated by the arrow style $\xrightarrow{\simeq}$.

Further, $\mathcal{C}$ comes with a distinguished collection of *subtraction sequences* $\{Z \to X \leftarrow U\}$. They are required to satisfy axioms spelled out in Definitions 3.7, 3.13, and 3.24 of [4] and mimic those of Waldhausen, replacing cofiber sequences with subtraction sequences.

For complexes we use the terms *bounded below/above* with reference to homological indexing, the same convention as in [18].

2. Realization to Mixed Motives

2.1 Introduction. We set up a map

$$K(\mathcal{V}_k) \longrightarrow K(\mathrm{C}^{\mathrm{gm}}),$$

where C^{gm} is the Waldhausen category of geometric effective mixed motives (see Eq. (2.8) for a precise definition). The key difficulty is to attach to each smooth variety X in $\mathcal{V}_k$ a complex which is strictly functorial in both closed immersions as well as open immersions. To this end, we use the $z_{equi}(X,0)$-cycle sheaves from the theory of motives. They are a concrete representative of the compactly supported motive $M^c(X)$.

2.2 Preliminary remarks on DG categories. For a DG category $\mathcal{A}$ we will denote by $\mathrm{mod}(\mathcal{A}^{\mathrm{op}})$ the DG category of *right* DG $\mathcal{A}$-modules. The full subcategory of compact objects (= perfect DG modules) will be denoted by $\mathrm{perf}(\mathcal{A}^{\mathrm{op}})$. We first recall [24, Definition 2.3.2]:

Definition 2.1. *A DG functor between DG categories $f : T \to T'$ is*

1. quasi-fully faithful *if for any two objects $x, y \in T$, the morphism of complexes $f_{x,y} : T(x,y) \to T'(f(x), f(y))$ is a quasi-isomorphism of complexes,*
2. quasi-essentially surjective *if the induced functor on homotopy categories is essentially surjective,*
3. *a* quasi-equivalence *if it is both quasi-fully faithful and quasi-essentially surjective.*

Definition 2.2. *A* DG quasi-functor *between two DG categories $\mathcal{A} \to \mathcal{B}$ is a diagram of DG functors*

$$\mathcal{A} \xleftarrow{\simeq} \mathcal{C} \to \mathcal{B},$$

where $\mathcal{C} \to \mathcal{A}$ is a quasi-equivalence.

There is a Yoneda-style quasi-fully faithful functor

$$\mathcal{A} \to \mathrm{perf}(\mathcal{A}^{\mathrm{op}}). \tag{2.1}$$

A DG category is called *triangulated* if the functor (2.1) is a quasi-equivalence, i.e., the induced functor between homotopy categories

$$Ho(\mathcal{A}) \to Ho(\mathrm{perf}(\mathcal{A}^{\mathrm{op}}))$$

is an equivalence.

Remark 2.3. By abuse of language we will denote both $X \in \mathcal{A}$ and the associated DG module in $\mathrm{perf}(\mathcal{A}^{\mathrm{op}})$ by X. Similarly, we will at times gloss over the difference between a triangulated DG category $\mathcal{A}$ and $\mathrm{perf}(\mathcal{A}^{\mathrm{op}})$.

The content of the following proposition is discussed at the beginning of [24, Section 3].

Proposition 2.4. *Let $\mathcal{A}$ be a DG category. There is a cofibrantly generated model structure on* $\mathrm{mod}(\mathcal{A}^{\mathrm{op}})$ *such that a morphism*

$$F \to G$$

of right DG modules is a weak equivalence (resp. a fibration) if and only if for every $X \in \mathcal{A}$ the induced morphism of complexes

$$F(X) \to G(X)$$

is a weak equivalence (respectively a fibration).[2]

We therefore have the structure of a cofibrantly generated model category on

$$\mathrm{mod}(\mathcal{A})$$

such that every object is fibrant. This structure will play an important yet transient role in Subsection 2.5.

Following [24, Subsection 5.2(a)] we define the Waldhausen 1-category associated to a DG category $\mathcal{A}$ to be

$$\mathrm{perf}_{\mathrm{co}}(\mathcal{A}^{\mathrm{op}}),$$

viewed as a 1-category, i.e., the full subcategory of cofibrant and compact right DG modules, where the morphisms are cycles in degree 0. The aforementioned model category structure endows $\mathcal{A}$ with the requisite class of cofibrations and weak equivalences. The algebraic K-theory space $K(\mathcal{A})$ is defined to be the Waldhausen K-theory of $\mathrm{perf}_{\mathrm{co}}(\mathcal{A}^{\mathrm{op}})$. The embedding (2.1) factors through $\mathrm{perf}_{\mathrm{co}}(\mathcal{A}^{\mathrm{op}})$.

Remark 2.5. Note that oftentimes $\mathcal{A}$ already comes equipped with a cofibrantly generated model (and therefore Waldhausen) structure. However, it is not always straightforward to promote DG functors out of $\mathcal{A}$ to maps of spectra in this way. Still, these DG functors will immediately produce exact functors out of $\mathrm{perf}_{\mathrm{co}}(\mathcal{A}^{\mathrm{op}})$.

Remark 2.6. Using the work of Blumberg and Mandell [2], one can check that the K-theory spectrum given by said Waldhausen structure on $\mathrm{perf}_{\mathrm{co}}(\mathcal{A}^{\mathrm{op}})$ is the same as the one associated to $\mathcal{A}$ as a pointed ∞-category if it admits finite colimits as in the work of Lurie [17, Remark 1.2.2.5]. In brief, $\mathrm{perf}_{\mathrm{co}}(\mathcal{A}^{\mathrm{op}})$ has functorial factorizations of weak cofibrations (see §2.5 for further discussion) with the Yoneda image of every morphism in $\mathcal{A}$ being a weak cofibration, and Yoneda $\mathcal{A} \to \mathrm{perf}_{\mathrm{co}}(\mathcal{A}^{\mathrm{op}})$ reflects weak equivalences and induces an equivalence on homotopy categories. One can then use [2, Theorem 1.3] to conclude that the Yoneda functor will induce an equivalence of spectra.

[2]This is known as the *projective model structure* on $\mathrm{mod}(\mathcal{A}^{\mathrm{op}})$, as in the conventions of [25].

Critical Convention: Below, we shall use calligraphic letters, as in $\mathcal{C}$, for a DG category and roman letters, as in C, to denote classical (e.g., triangulated) 1-categories. In particular, if $\mathcal{C}$ is a triangulated DG category, we may (and will) henceforth simply write

$$\mathrm{C} := Ho(\mathcal{C})$$

for its homotopy category. This is compatible with the notation in [27].

2.3 Recollections. Let A be a commutative unital ring. It will serve as our ring of coefficients, e.g., it could be $A := \mathbb{Z}$.

We fix a perfect field k such that there exists some N such that for all $r > N$ we have

$$H^r(\mathrm{Gal}(k^{\mathrm{sep}}/k), M) = 0 \tag{2.2}$$

for all A-modules M. This is the same condition as imposed in §2.2 of [27], and serves to ensure that the étale realization lands in bounded complexes.

Example 2.7. *This condition is usually harmless.*

1. *If k is separably closed, this condition is automatically satisfied, so $k = \mathbb{C}$ is fine.*
2. *If k has finite (strict) cohomological dimension (in the classical sense of Galois cohomology), the condition is satisfied, because $H^r(\mathrm{Gal}(k^{\mathrm{sep}}/k), M)$ is a torsion module for $r > 0$ and if all its p-primary torsion parts vanish, the entire module vanishes.*
3. *In particular, the condition is met if k is a finite field.*
4. *For $A = \mathbb{Z}$ and $k = \mathbb{R}$ the condition is* not *satisfied because, in positive degrees, the cohomology is periodic of period 2.*

Let Sm_k be the category of smooth separated k-varieties and k-morphisms (this is the notation of Beilinson and Vologodsky [1], [27]; the book [18] uses the marginally different notation Sm/k).

Next, $A_{tr}[Sm_k]$ denotes the category of finite correspondences over k with A-coefficients. This category has the same objects as Sm_k, and in addition to genuine k-morphisms, we consider all finite correspondences with coefficients in the ring A (this is the notation of Beilinson and Vologodsky; the book [18] has $A = \mathbb{Z}$ instead and denotes the same category by Cor_k).

2.4 DG category of mixed motives. Let $t \in \{\acute{e}t, Nis\}$ be either the étale or the Nisnevich topology. Next, one sets up a triangulated DG category of effective mixed motives $\mathcal{DM}_t^{\mathrm{eff}}(k; A)$ over the base field k and with coefficients in A. For $t = \acute{e}t$ this is described in [27, §2.2, p. 378, last paragraph], and for $t = Nis$ in [27, Remark 2.6] and there is a DG quasi-functor (see Definition 2.2)

$$\mathcal{DM}_{Nis}^{\mathrm{eff}}(k; A) \longrightarrow \mathcal{DM}_{\acute{e}t}^{\mathrm{eff}}(k; A), \tag{2.3}$$

also described in loc. cit. which corresponds to enforcing étale descent as opposed to the weaker Nisnevich descent. Following the aforementioned convention, the homotopy categories are denoted by $\mathrm{DM}_t^{\mathrm{eff}}(k; A)$ and we get an induced triangulated functor

$$\mathrm{DM}_{Nis}^{\mathrm{eff}}(k; A) \longrightarrow \mathrm{DM}_{\acute{e}t}^{\mathrm{eff}}(k; A) \tag{2.4}$$

of triangulated categories. If A is a $\mathbb{Q}$-algebra, Equation (2.3) (and therefore Equation (2.4)) are equivalences by Proposition 3.3.2 of [26].

Example 2.8. *As Chow groups do not satisfy étale descent, Chow groups are representable in $\mathcal{DM}^{\mathrm{eff}}_{Nis}(k;A)$, but not in $\mathcal{DM}^{\mathrm{eff}}_{\acute{e}t}(k;A)$. See 2.1.4 of [26] for further discussion.*

Next, one defines the DG category of *geometric mixed motives*

$$\mathcal{DM}^{\mathrm{eff}}_{\mathrm{gm},t}(k;A) := \mathcal{DM}^{\mathrm{eff}}_{t}(k;A)^{\mathrm{perf}} \tag{2.5}$$

as the full DG subcategory of $\mathcal{DM}^{\mathrm{eff}}_{t}(k;A)$ such that the objects are compact in the homotopy category $\mathrm{DM}^{\mathrm{eff}}_{t}(k;A) = Ho(\mathcal{DM}^{\mathrm{eff}}_{t}(k;A))$. As a shorthand for later use, we define

$$\begin{aligned} \mathcal{C} &:= \mathrm{perf}_{\mathrm{co}}(\mathcal{DM}^{\mathrm{eff}}_{Nis}(k;A)^{\mathrm{op}}), \\ \mathcal{C}^{\mathrm{gm}} &:= \mathrm{perf}_{\mathrm{co}}(\mathcal{DM}^{\mathrm{eff}}_{\mathrm{gm},Nis}(k;A)^{\mathrm{op}}). \end{aligned} \tag{2.6}$$

Since the functor (2.1) induces an equivalence of homotopy categories for triangulated DG categories, the categories above are DG enhancements for the triangulated categories

$$\mathrm{C} = \mathrm{DM}^{\mathrm{eff}}_{Nis}(k;A), \tag{2.7}$$

$$\mathrm{C}^{\mathrm{gm}} = \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm},Nis}(k;A). \tag{2.8}$$

These are the triangulated categories of (effective, resp. geometric) Nisnevich mixed motives, and at least the latter is precisely the same as in Voevodsky's original formalism. In the case of the former, there is a subtle but essentially irrelevant difference since Beilinson–Vologodsky allow arbitrary unbounded complexes, whereas Voevodsky imposes a boundedness condition.

2.5 Functorial factorizations in $\mathcal{C}^{\mathrm{gm}}$.

Recall from Subsection 2.2 that we may regard $\mathcal{C}^{\mathrm{gm}}$ as a Waldhausen category such that its weak equivalences are the quasi-isomorphisms of the DG structure and its cofibrations are the cofibrations of Proposition 2.4.

Remark 2.9. We record the following, essentially obvious, facts.

1. Any square in $\mathcal{C}$

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

   such that the induced triangle $A \longrightarrow B \oplus C \longrightarrow D \longrightarrow A[1]$ is distinguished in the triangulated category C is weakly equivalent by zig-zags to a genuinely cartesian square of complexes in $\mathcal{C}$.
2. If $Ho_{(\mathrm{Waldhausen})}(\mathcal{C})$ denotes the DG localization of $\mathcal{C}$ by its subcategory of weak equivalences given by its Waldhausen structure, then the 1-category $Ho_{(\mathrm{Waldhausen})}(\mathcal{C})$ is equivalent to C (as triangulated categories).

Lemma 2.10. *$\mathcal{C}^{\mathrm{gm}}$ has functorial factorization of weak cofibrations (FFWC).*

Proof. First, we consider $\mathcal{C}$. We use that the Waldhausen structure comes from a cofibrantly generated model structure (Prop. 2.4). In particular, by the small object argument, we have functorial factorizations in this model category.

Its functorial factorizations into cofibrations followed by acyclic fibrations can be written as a functor

$$\varphi\colon \mathrm{Fun}([1],\mathcal{C}) \longrightarrow \mathrm{Fun}^{c,w}([2],\mathcal{C}),$$

using the notation of [7, Appendix A] (the superscript c, w just means that the first arrow is a cofibration and the second a weak equivalence). For FFWC we only need functorial factorizations for weak cofibrations, i.e., those which are through a zig-zag equivalent to a genuine cofibration, but φ solves the problem even for arbitrary maps. Hence,

$$\text{mod }(\mathcal{A}^{\text{op}})$$

has FFWC where $\mathcal{A}$ denotes a DG category.

The mere restriction of φ to functors with values in $\mathcal{C}^{\text{gm}}$ solves the problem and gives

$$\varphi' \colon \text{Fun}([1], \mathcal{C}^{\text{gm}}) \longrightarrow \text{Fun}^{c,w}([2], \mathcal{C}^{\text{gm}}).$$

For any factorization in the model category of right DG modules made by the above φ, but with $A, B \in \mathcal{C}^{\text{gm}}$,

$$A \overset{i}{\hookrightarrow} T \overset{p}{\longrightarrow} B$$

we get $T \in \mathcal{C}^{\text{gm}}$ because A is cofibrant and $A \hookrightarrow T$ is a cofibration, and thus

$$0 \hookrightarrow A \hookrightarrow T$$

is also a cofibration. This shows that T is cofibrant. Perfectness of T follows from the fact that p is a weak equivalence (i.e., quasi-isomorphism), and the fact that B is perfect by assumption. □

2.6 Motivic realization functor. For any scheme T of finite type over k, there is the sheaf of *equidimensional cycles* $z_{equi}(T, 0) \in Sh_{Nis}(Cor_k)$, [18, Definition 16.1]. There are also sheaves $z_{equi}(T, r)$ for $r \in \mathbb{Z}$, but we only need those for $r = 0$ and chose to keep the second parameter in order to remain fully compatible to the literature.

Remark 2.11. The notation $Sh_{Nis}(Cor_k)$ is in line with the cited book [18]. As we shall later see (in the proof of Theorem 2.14) that these objects define geometric motives, there is no problem to regard them as objects in the DG category $\mathcal{C}^{\text{gm}} = \mathcal{DM}^{\text{eff}}_{\text{gm},Nis}(k; A)$ of the Beilinson–Vologodsky framework. Before having proven this, one may also consider them as objects in the DG category of presheaves $PSh^{A}_{tr}(Sm_k)$ considered in [27, §2.2].

We recall the construction. Suppose $S \in Sm_k$ and $Z \subseteq S \times_k T$ is an irreducible closed subscheme. We introduce a condition:

(♯) We say that Z satisfies (♯) if Z is dominant over some irreducible component of S, and moreover for each $s \in S$ the scheme-theoretic fiber $Z_s = Z \times_S \kappa(s)$ is a finite $\kappa(s)$-scheme.

Property (♯) is equivalent to demanding that the composed morphism

$$\begin{array}{ccc} Z & \hookrightarrow & S \times_k T \\ & \searrow & \downarrow \\ & & S \end{array}$$

is equidimensional of relative dimension zero in the sense of [23, Def. 2.1.2].

For any $S \in Sm_k$ define

$$z_{equi}(T, 0)(S) = A\left[Z \subseteq S \times_k T \mid Z \text{ satisfies } (\sharp)\right].$$

The notation refers to the free A-module having the $[Z]$ as an A-basis.

Constructed in 3.5.9 of [23], there is a Nisnevich sheaf given by $S \mapsto z_{equi}(T, 0)(S)$ with transfers, so we have $z_{equi}(T, 0) \in Sh_{Nis}(Cor_k)$. We need the following maps:

1. Suppose $i : T' \hookrightarrow T$ is a closed immersion. Then there is an induced map
$$i_* : z_{equi}(T',0)(S) \longrightarrow z_{equi}(T,0)(S)$$
for every S by identifying subschemes of T' with subschemes of T, inducing a corresponding morphism of sheaves. Moreover, if i_1, i_2 are two composable closed immersions, we have
$$i_{2*} \circ i_{1*} = (i_2 \circ i_1)_*.$$
2. Suppose $j : T'' \hookrightarrow T$ is an open immersion. Then there is an induced map
$$j^* : z_{equi}(T,0)(S) \longrightarrow z_{equi}(T'',0)(S)$$
for every S, induced by intersecting subschemes of T with T'', and thus a corresponding morphism of sheaves, and for composable open immersions, we have
$$j_1^* \circ j_2^* = (j_2 \circ j_1)^*.$$

All these properties are discussed briefly in [18, Lec. 16]. For (1), the details can be found in [23, Corollary 3.6.3] for z_{equi} and noting that closed immersions are of course proper. For (2), details are found in the paragraph before [23, Prop. 3.6.5], using that open immersions are of course flat.

Remark 2.12. It is perhaps worth to stress that while the Nisnevich sheaves $z_{equi}(T,-)$ are sheaves on smooth schemes Sm_k, the scheme T can be *any* k-variety in the sense of §1.1, no matter how singular. This is important because smooth varieties do not form an SW-category [4, Remark 3.17]. Note that if the base scheme S is non-regular, the definition of $z_{equi}(T,-)(S)$ is more restrictive. In this case, heuristically speaking, one must consider the subgroup of relative cycles with well-defined pullbacks for all $s \in S$.

Now we define a weakly W-exact functor in the sense of [7, Def. 2.17]. Suppose $\mathcal{V}_k$ denotes the SW-category of k-varieties (recall the convention on what this entails, see §1.1).

This is properly defined in [4, Corollary 3.16, $\mathbf{Var}_{/k}$], and we quickly recall that the cofibrations $\mathbf{co}(\mathcal{V}_k)$ are the closed immersions of k-varieties, the complement maps $\mathbf{comp}(\mathcal{V}_k)$ are the open immersions, and for the weak equivalences $\mathbf{w}(\mathcal{V}_k)$ we choose k-isomorphisms of k-varieties. Lastly, subtraction sequences are those of the shape
$$Z \hookrightarrow X \overset{\circ}{\longleftarrow} Y,$$
where $Y = X \setminus Z$. As varieties by the running conventions are required to be reduced, note that Z only appears with its reduced structure and no nil-thickenings can appear within $\mathcal{V}_k$.

Now [4, Corollary 3.16] shows that $\mathcal{V}_k$ is a subtractive category (i.e., a category endowed with a collection of subtraction sequences) and, along with the weak equivalences, this equips it with the structure of an SW-category.

A weakly W-exact functor
$$F : \mathcal{V}_k \longrightarrow \mathcal{C}^{\mathrm{gm}}$$
is given by a triple $(F_!, F^!, F^w)$ of functors which agree on objects, but differ on morphisms. The definition is given in [7, Def. 2.17].

Remark 2.13. Just as the formalization of mixed motives using the category Cor_k encodes the existence of a covariant functoriality alongside a contravariant functoriality, the axiomatization of weakly W-exact functors just uses the three functors $F_!, F^!, F^w$ to differentiate between various types of co- and contravariant transfers.

We recall that for any presheaf $\mathcal{F}$ of abelian groups, there is a simplicial presheaf

$$(sC)_n(\mathcal{F})(X) = \mathcal{F}(X \times \Delta^n).$$

A precise definition is given in [18, Def. 2.14] or with more details [10, §4]. To $sC_\bullet$ one also attaches a complex of presheaves, concentrated in non-positive degrees, and calls it C_*. The construction is functorial in morphisms of presheaves. We note that if $0 \to \mathcal{F}' \to \mathcal{F} \twoheadrightarrow \mathcal{F}'' \to 0$ is an exact sequence of presheaves, then

$$0 \to C_*\mathcal{F}' \to C_*\mathcal{F} \twoheadrightarrow C_*\mathcal{F}'' \to 0$$

is an exact sequence of complexes. Moreover, if the input $\mathcal{F}$ is a presheaf with transfers, so is $C_*\mathcal{F}$.

The main idea behind C_* is that it is a universal construction enforcing $\mathbb{A}^1$-homotopy invariance on a sheaf, [26, Proposition 3.2.3].

We define

$$\begin{aligned} F_? \colon \mathcal{V}_k &\longrightarrow \mathcal{C}^{\mathrm{gm}} \\ T &\longmapsto C_* z_{equi}(T, 0) \end{aligned} \tag{2.9}$$

on objects, and $F_! \colon \mathbf{co}(\mathcal{V}_k) \to \mathcal{C}^{\mathrm{gm}}$ is just sending closed immersions i to i_* of z_{equi}, and $F^! \colon \mathbf{comp}(\mathcal{V}_k) \to \mathcal{C}^{\mathrm{gm}}$ sends open immersions j to j^* of z_{equi} (as i_*, j^* are morphisms of sheaves, they define morphisms in $\mathcal{C}$). Finally, $F^w \colon \mathbf{w}(\mathcal{V}_k) \to \mathbf{w}(\mathcal{C}^{\mathrm{gm}})$ sends a k-isomorphism $f \colon X \to X$ to its pushforward f_* on z_{equi}. As f^{-1} exists, $(f^{-1})_*$ is a strict inverse $(f^{-1})_* f_* = f_*(f^{-1})_* = \mathrm{id}_{z_{equi}}$, and in particular this is an isomorphism of sheaves, and therefore (trivially) a weak equivalence in $\mathcal{C}$. This completes defining $(F_!, F^!, F^w)$ and settles axioms (1)-(4) of a weakly W-exact functor.

Theorem 2.14. *Suppose k is a perfect field. Let A be a commutative unital ring. If k has positive characteristic $p > 0$, we assume that $\frac{1}{p} \in A$. Then $F = (F_!, F^!, F^w)$ is a weakly W-exact functor from $\mathcal{V}_k$ to $\mathcal{C}^{\mathrm{gm}}$. In particular, it induces a map of spectra*

$$K(F) \colon K(\mathcal{V}_k) \longrightarrow K(\mathcal{C}^{\mathrm{gm}}). \tag{2.10}$$

Moreover, for any variety $X \in \mathcal{V}_k$ we have

$$F(X) = M^c(X) \tag{2.11}$$

in the homotopy category $\mathrm{C}^{\mathrm{gm}} = \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm},Nis}(k; A)$, i.e., $F(X)$ represents the motive with compact support attached to X.

Proof. For the sake of legibility, we give the proof for characteristic zero first, and only comment on the necessary changes if the characteristic is $p > 0$. First of all, regarding Equation 2.9 we need to check that $C_* z_{equi}(T, 0)$ lies in $\mathcal{C}^{\mathrm{gm}}$ at all. In view of the definition of geometric motives in Equation 2.5, we need to show that its image in the homotopy category lies in $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm},Nis}$, but this is the subject of [18, Corollary 16.17], and furthermore $C_* z_{equi}(T, 0)$ represents $M^c(T)$ in the notation of the theory of mixed motives, [18, Def. 16.13]. It remains to verify axioms (5)-(7) of a weakly W-exact functor. Given the diagram below on the left,

$$\begin{array}{ccc} X & \xhookrightarrow{\;i\;} & Z \\ {\scriptstyle j}\big\downarrow\circ & & \circ\big\downarrow{\scriptstyle j'} \\ Y & \xhookrightarrow[\;i'\;]{} & W \end{array} \qquad\qquad \begin{array}{ccc} F(X) & \xrightarrow{\;i_*\;} & F(Z) \\ {\scriptstyle j^*}\big\uparrow & & \big\uparrow{\scriptstyle j'^*} \\ F(Y) & \xrightarrow[\;i'_*\;]{} & F(W) \end{array}$$

and assuming this is a cartesian diagram in $\mathcal{V}_k$, axiom (5) demands that the strict functorialities result in the diagram above on the right. This follows in the case of z_{equi} by direct computation. Next, given a subtraction sequence

$$\begin{array}{ccc} Z & \overset{i}{\hookrightarrow} & X \\ & & {\scriptstyle j}\uparrow{\scriptstyle \circ} \\ & & X - Z \end{array}$$

in $\mathcal{V}_k$, axiom (6) is equivalent to demanding that

$$C_* z_{equi}(Z) \xrightarrow{i_*} C_* z_{equi}(X) \xrightarrow{j^*} C_* z_{equi}(X - Z) \longrightarrow C_* z_{equi}(Z)[1]$$

is a distinguished triangle in $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm},Nis}$, because if it is, the square stated in axiom (6) will be weakly cocartesian. This is settled by the first part of [10, Theorem 5.11] if the field k admits a resolution of singularities. For positive characteristic, Kelly's foundations from [16] can be used as a replacement when p is invertible in the coefficient ring A. Axiom (7) really makes two statements: The one for $F_!$ follows directly from the strict functoriality of the pushforward along closed immersions as our weak equivalences are also induced by a pushforward. The other part, for $F^!$, is a special case of Axiom (5). Having settled that F is weakly W-exact, [7, Proposition 2.19] produces a map of K-theory spectra. □

Variant 2.15. One could also define a variant

$$\begin{aligned} \tilde{F}_? \colon \mathcal{V}_k &\longrightarrow \mathcal{C}^{\mathrm{gm}} \\ T &\longmapsto z_{equi}(T, 0) \end{aligned}$$

without using C_*. As z_{equi} is, regarded as a complex, concentrated in the single degree zero, we may employ [26, Proposition 3.2.3], which tells us that we have a natural equivalence in T:

$$z_{equi}(T, 0) \xrightarrow{\simeq} C_*(z_{equi}(T, 0)) \tag{2.12}$$

in $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm},Nis}$. A variation of the above proof goes through, but one has to plug in the equivalence of Equation 2.12 and exploit its functoriality in the verification of the axioms (6) and (7).

3. Realization maps

3.1 Generalities on realizations for motives. We shall employ Vologodsky's technique to produce realizations of mixed motives, as developed in his paper [27]. The idea is that any functor defined on smooth k-varieties and satisfying good descent properties automatically extends *uniquely* to a functor on étale mixed motives.

We recall the idea in some more detail. If $\mathcal{C}_1, \mathcal{C}_2$ are DG categories, write $\mathcal{T}(\mathcal{C}_1, \mathcal{C}_2)$ for the category of DG quasi-functors

$$H \colon \mathcal{C}_1 \longrightarrow \mathcal{C}_2.$$

Any such functor induces a functor of homotopy categories

$$Ho(H) \colon Ho(\mathcal{C}_1) \longrightarrow Ho(\mathcal{C}_2)$$

(which can alternatively be written as $\mathrm{C}_1 \longrightarrow \mathrm{C}_2$, following our running conventions on notation) and if $\mathcal{C}_1, \mathcal{C}_2$ are triangulated, this is a triangulated functor of triangulated categories (see [24, Ex. 5.1.4]). We write

$$\mathcal{T}^c(\mathcal{C}_1, \mathcal{C}_2) \subseteq \mathcal{T}(\mathcal{C}_1, \mathcal{C}_2)$$

for the full subcategory of DG quasi-functors H such that $Ho(H)$ commutes with arbitrary direct sums.

If $\mathcal{C}$ is a DG category, $\underrightarrow{\mathcal{C}}$ denotes the DG ind-completion. This is a triangulated cocomplete DG category such that $\mathcal{C} \subseteq (\underrightarrow{\mathcal{C}})^{\mathrm{perf}}$ and defining a DG quasi-functor on $\underrightarrow{\mathcal{C}_1}$ commuting with all direct sums is equivalent to providing a DG quasi-functor only on $\mathcal{C}_1$ itself:

$$\mathcal{T}^c(\underrightarrow{\mathcal{C}_1}, \mathcal{C}_2) \xrightarrow{\sim} \mathcal{T}(\mathcal{C}_1, \mathcal{C}_2).$$

Now take $\mathcal{C}_1 := A[Sm_k]$, the category whose objects are smooth k-varieties (denoted by $A[X]$ for X the smooth k-variety) and morphisms are

$$\mathrm{Hom}_{A[Sm_k]}(A[\sqcup_i X_i], A[Y]) := \coprod_i A[\mathrm{Hom}_k(X_i, Y)]$$

for each X_i connected. See [27, §2.4].

Vologodsky now introduces the full subcategory $\mathcal{T}^{h,\Delta}(-)$,

$$\mathcal{T}^{h,\Delta}(A[Sm_k], \mathcal{C}_2) \subseteq \mathcal{T}(A[Sm_k], \mathcal{C}_2)$$

consisting of those DG quasi-functors H which

(h) satisfy hyperdescent with respect to the h-topology (i.e., if $U_\bullet \to X$ is an h-hypercovering, then H sends

$$A[U_\bullet] \longrightarrow A[X]$$

to a quasi-isomorphism),

(Δ) satisfy $\mathbb{A}^1$-invariance (i.e., H sends

$$A[X \times \mathbb{A}^1_k] \longrightarrow A[X]$$

to a quasi-isomorphism).

Theorem 3.1 (Vologodsky, [27, Theorem 2])**.** *Suppose $\mathcal{C}$ is a triangulated cocomplete and compactly generated A-linear DG category. There is an equivalence of categories*

$$\Phi\colon \mathcal{T}^c(\mathcal{DM}^{\mathrm{eff}}_{\acute{e}t}(k; A), \mathcal{C}) \xrightarrow{\sim} \mathcal{T}^{h,\Delta}(A[Sm_k], \mathcal{C}),$$

such that if X is a smooth proper variety and $M(X)$ its motive, then we get

$$\Phi^{-1}(H)(M(X)) = H(X)$$

for any $H \in \mathcal{T}^{h,\Delta}(A[Sm_k], \mathcal{C})$.

We just state this as the existence of Φ, but really the equivalence is constructed in a straightforward fashion; for details, see [27].

In other words: As soon as we exhibit a DG quasi-functor H defined on $A[Sm_k]$ which satisfies (h) and (Δ), this uniquely determines an extension $\Phi^{-1}(H)$ to all étale effective mixed motives.

Vologodsky then constructs all the standard realizations of mixed motives by first setting them up on $A[Sm_k]$ (the details are a little more involved, but this is the essential point).

We can now deduce the corresponding realization in the setting of this paper.

3.2 The realization theorem. Suppose k is a perfect field. As before, let A be a commutative unital ring. If k has positive characteristic $p > 0$, we assume $\frac{1}{p} \in A$.

Suppose $\mathcal{D}$ is a triangulated cocomplete and compactly generated A-linear DG category. Let

$$\mathcal{D}_{\text{finite}} \subseteq \mathcal{D}$$

be some full DG subcategory. Usually, $\mathcal{D}_{\text{finite}} := \mathcal{D}^{\text{perf}}$ will be the choice that we are interested in; the full DG subcategory of objects whose image in the homotopy category D is compact. As discussed earlier, we can define the algebraic K-theory $K(\mathcal{D}_{\text{finite}})$ as the Waldhausen K-theory of the Waldhausen category $\text{perf}_{\text{co}}(\mathcal{D}^{\text{op}}_{\text{finite}})$, see Remark 2.6. We remind the reader of Remark 2.3, according to which we do not notationally distinguish between an object in $\mathcal{D}_{\text{finite}}$ and the induced DG module (given by the Yoneda embedding).

Theorem 3.2. *Suppose k is a perfect field. If k has positive characteristic $p > 0$, we assume that $\frac{1}{p} \in A$. Further, suppose k has finite cohomological dimension with respect to A-coefficients in the sense of Equation 2.2. Suppose we are given a DG quasi-functor*

$$\phi \in \mathcal{T}^{h,\Delta}(A[Sm_k], \mathcal{D}_{\text{finite}}).$$

Then there is a map of spectra

$$\mathrm{R}^{\phi} \colon K(\mathcal{V}_k) \longrightarrow K(\mathcal{D}_{\text{finite}})$$

such that the following hold:

1. *Suppose X is a smooth proper k-variety. In K_0 we get*

$$\mathrm{R}^{\phi}([X]) = [\phi(X)]$$

and if $f \colon X \xrightarrow{\sim} X$ is any automorphism, we get in K_1 that

$$\mathrm{R}^{\phi}([f \colon X \xrightarrow{\sim} X]) = \left[\phi(f) \colon \phi(X) \xrightarrow{\sim} \phi(X)\right].$$

2. *Suppose X is a smooth k-variety, $\overline{X}$ a smooth compactification with a smooth closed subvariety $Z \subseteq \overline{X}$ such that $X = \overline{X} \setminus Z$. Then in K_0,*

$$\mathrm{R}^{\phi}([X]) = [\phi(\overline{X})] - [\phi(Z)]$$

and if one can extend an automorphism f such that

$$\begin{CD} Z @>>> \overline{X} \\ @VfVV @VVfV \\ Z @>>> \overline{X} \end{CD} \tag{3.1}$$

commutes, then $\mathrm{R}^{\phi}([f \colon X \xrightarrow{\sim} X]) = \mathrm{R}^{\phi}([f \colon \overline{X} \xrightarrow{\sim} \overline{X}]) - \mathrm{R}^{\phi}([f\,|_Z \colon Z \xrightarrow{\sim} Z])$.

Proof. By Theorem 2.14 we obtain a map of spectra

$$K(F) \colon K(\mathcal{V}_k) \longrightarrow K(\mathcal{C}^{\text{gm}}).$$

The DG quasi-functor $\phi \in \mathcal{T}^{h,\Delta}(A[Sm_k], \mathcal{D}_{\text{finite}})$ defines (tautologically) an element of the category $\mathcal{T}^{h,\Delta}(A[Sm_k], \mathcal{D})$ by $\mathcal{D}_{\text{finite}} \subseteq \mathcal{D}$. From Vologodsky's Theorem (Theorem 3.1) we therefore obtain a DG quasi-functor

$$\tilde{\phi} \in \mathcal{T}^{c}(\mathcal{DM}^{\text{eff}}_{\acute{e}t}(k; A), \mathcal{D}),$$

i.e.,

$$\tilde{\phi}\colon \mathcal{DM}^{\mathrm{eff}}_{\acute{e}t}(k;A) \longrightarrow \mathcal{D}.$$

Note that the values of $\tilde{\phi}$ need *not* lie in $\mathcal{D}_{\mathrm{finite}}$ because of the colimit procedures carried out in the construction of $\tilde{\phi}$. However, we have

$$\tilde{\phi}(M(X)) = \phi(X) \in \mathcal{D}_{\mathrm{finite}}$$

if X is a smooth k-variety and $M(X)$ its motive, still by Theorem 3.1. As the triangulated category $\mathrm{DM}^{\mathrm{eff}}_{\acute{e}t}(k;A)$ of geometric motives is compactly generated and the compact objects are generated by the $M(X)$ ([27, Corollary 2.4 and below]), we obtain a DG quasi-functor to $\mathcal{D}_{\mathrm{finite}}$,

$$\tilde{\phi}\mid_{\mathcal{DM}^{\mathrm{eff}}_{\mathrm{gm},\acute{e}t}(k;A)}\colon\ \mathcal{DM}^{\mathrm{eff}}_{\mathrm{gm},\acute{e}t}(k;A) \longrightarrow \mathcal{D}_{\mathrm{finite}}.$$

Now form the composition of DG quasi-functors

$$\mathcal{C}^{\mathrm{gm}} = \mathcal{DM}^{\mathrm{eff}}_{\mathrm{gm},Nis}(k;A) \overset{(\#)}{\longrightarrow} \mathcal{DM}^{\mathrm{eff}}_{\mathrm{gm},\acute{e}t}(k;A) \overset{\tilde{\phi}|_{(\ldots)}}{\longrightarrow} \mathcal{D}_{\mathrm{finite}},$$

where $(\#)$ is the change-of-topology quasi-functor from Equation (2.3). In the case that X is a smooth *proper* k-variety, we first observe that $M^c(X) = M(X)$. Now we can prove (1). On the level of K_0, we have

$$K(F)_0\colon K_0(\mathcal{V}_k) \longrightarrow K_0(\mathcal{C}^{\mathrm{gm}}) \longrightarrow K_0(\mathcal{D}_{\mathrm{finite}})$$

sending

$$[X] \mapsto [M^c(X)] = [M(X)] \mapsto [\tilde{\phi}(X)]$$

and on K_1 we can basically use the same argument. We now prove (2). Using [18, Theorem 16.15] we have a triangle

$$M^c(Z) \overset{i_*}{\longrightarrow} M^c(\overline{X}) \overset{j^*}{\longrightarrow} M^c(X) \longrightarrow M^c(Z)[1]$$

and thus a canonical quasi-isomorphism between $M^c(X)$ and the cone (on the DG category level) below on the left,

$$\mathrm{cone}\left(M^c(Z) \overset{i_*}{\longrightarrow} M^c(\overline{X})\right) \overset{\sim}{\longrightarrow}_{qis} \mathrm{cone}\left(M(Z) \overset{i_*}{\longrightarrow} M(\overline{X})\right), \tag{3.2}$$

and the quasi-isomorphism in the middle holds since Z and $\overline{X}$ are both smooth and proper. Hence, we can reduce our claim about K_0 to (1) and the cone of complexes is easy to see to correspond to a difference in K_0. For the claim about f, note that Equation 3.1 ensures that f also acts on the cones in Equation 3.2, and then also reduces to (1). □

4. Example: The Betti realization

We illustrate our constructions with the concrete example of the Betti realization. Suppose $k \subseteq \mathbb{C}$ is any subfield of the complex numbers. We do *not* demand that k has finite cohomological dimension in the sense of Equation 2.2 in this section. If X is a k-variety, we write $X_{\mathbb{C}}$ for the complex manifold attached to the smooth $\mathbb{C}$-variety $X \times_k \operatorname{Spec}\mathbb{C}$.

Suppose A is a commutative Noetherian unital ring which will serve as the coefficient ring for the Betti realization. We write $C^{\mathrm{sing}}(X_{\mathbb{C}}, A)$ for the singular chain complex with coefficients in A.

Remark 4.1. One can drop the Noetherian assumption if one works with coherent A-modules instead of finitely generated ones below. We prefer to stick to the Noetherian assumption for the sake of simplicity.

Theorem 4.2. *There is a map of spectra*

$$\mathrm{R}_A^{Betti} \colon K(\mathcal{V}_k) \longrightarrow K(\mathsf{Mod}_{fg}(A))$$

such that the following hold:

1. *Suppose X is a smooth proper k-variety. In K_0 we get*
$$\mathrm{R}_A^{Betti}([X]) = \sum_i (-1)^i [H_i^{Betti}(X_{\mathbb{C}}, A)], \tag{4.1}$$
the cohomology of the complex manifold $X_{\mathbb{C}}$. If $\varphi \colon X \xrightarrow{\sim} X$ is any automorphism, we get
$$\mathrm{R}_A^{Betti}([\varphi \colon X \xrightarrow{\sim} X]) = \sum_i (-1)^i [\varphi_* H_i^{Betti}(X_{\mathbb{C}}, A)]$$
in K_1.
2. *Suppose X is a smooth k-variety, $\overline{X}$ a smooth compactification with a smooth closed subvariety $Z \subseteq \overline{X}$ such that $X = \overline{X} \setminus Z$. Then in K_0,*
$$\mathrm{R}_A^{Betti}([X]) = \mathrm{R}_A^{Betti}([\overline{X}] - [Z])$$
and if
$$\begin{array}{ccc} Z & \hookrightarrow & \overline{X} \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi} \\ Z & \hookrightarrow & \overline{X} \end{array} \tag{4.2}$$
commutes, then $\mathrm{R}_A^{Betti}([\varphi \colon X \xrightarrow{\sim} X]) = \mathrm{R}_A^{Betti}([\varphi \colon \overline{X} \xrightarrow{\sim} \overline{X}] - [\varphi \colon Z \xrightarrow{\sim} Z])$.

Remark 4.3. If A is regular, this yields a map

$$\mathrm{R}_A^{Betti} \colon K(\mathcal{V}_k) \longrightarrow K(A)$$

with the same properties, except that each H_i^{Betti} is tacitly replaced by a finite projective resolution.

Proof. Condition 2.2 is void in this setting because we can first base-change

$$\mathcal{V}_k \longrightarrow \mathcal{V}_{\mathbb{C}}$$

and work in the latter situation, where the condition is tautologically satisfied (Example 2.7). So without loss of generality, $k = \mathbb{C}$. We use Theorem 3.2 for the Betti realization, as in [27, §2.7]. A smooth variety X gets sent to the singular chain complex with coefficients in A,

$$C^{\mathrm{sing}}(X_{\mathbb{C}}, A).$$

For the motive $M(X)$ of a smooth variety the complex $C^{\mathrm{sing}}(X_{\mathbb{C}}, A)$ has finite homological support (namely concentrated in degrees $[0, 2\dim X]$) and each cohomology group is finitely generated. Hence, we may take bounded complexes of finitely generated A-modules

$$\mathcal{D}_{\mathrm{finite}} := C^b(\mathsf{Mod}_{fg}(A)) \qquad \text{and} \qquad \mathcal{D} := C(\mathsf{Mod}(A))$$

inside all complexes and all modules (all with the standard DG structure). If we use Theorem 3.2 now, we obtain almost our claim, except for the map being

$$K(\mathcal{V}_k) \longrightarrow K(C^b(\mathsf{Mod}_{fg}(A)))$$

Note that the Waldhausen structure on $C^b(\mathsf{Mod}_{fg}(A))$ is such that the weak equivalences are the quasi-isomorphisms, so within K-theory language one would perhaps stress this by writing $K(qC^b(\mathsf{Mod}_{fg}(A)))$. By the Gillet–Waldhausen Theorem ([28, Chapter V, Theorem 2.2]) there is an equivalence

$$K(\mathsf{Mod}_{fg}(A)) \longrightarrow K(qC^b(\mathsf{Mod}_{fg}(A))),$$

where the left side is the ordinary K-theory of an abelian category and the right side is the aforementioned Waldhausen K-theory with respect to weak equivalences. The map is induced from sending an object $M \in \mathsf{Mod}_{fg}(A)$ to the complex concentrated in degree zero. The inverse map, on the level of K_0 (and analogously K_1), then is

$$[C^\bullet] \mapsto \sum_i (-1)^i [C^i]$$

for any bounded complex $C^\bullet$. This yields the statement of our theorem, in particular the signs in Equation 4.1. $\square$

5. Example: The Hodge realization

Let $k = \mathbb{C}$ and $\mathbb{Z} \subseteq A \subseteq \mathbb{Q}$. Let MHS^A_{eff} be the category of effective polarizable A-Hodge structures (as in [27, §2.8]). Vologodsky characterizes effectiveness by

$$F^1 = 0.$$

This property is evidently extension-closed, and thus MHS^A_{eff} is endowed with the structure of an exact category.

Remark 5.1. For a pure Hodge structure M of weight w with

$$M \otimes_A \mathbb{C} = \bigoplus_{p+q=w} M^{p,q}$$

effectiveness is equivalent to

$$M^{p,q} = 0$$

for $p > 0$ or $q > 0$. This is because we follow Vologodsky's convention which models the properties of effective pure Hodge structures on Hodge structures on singular *homology* of a smooth projective variety.

Theorem 5.2. *There is a map of spectra*

$$\mathrm{R}^{Hodge}_A \colon K(\mathcal{V}_k) \longrightarrow K(MHS^A_{\text{eff}})$$

such that the following hold:

1. *Suppose X is a smooth proper k-variety. In K_0 we get*

$$\mathrm{R}_A^{Hodge}([X]) = \sum_i (-1)^i [H_i(X_{\mathbb{C}}, A)],$$

the cohomology of the complex manifold $X_{\mathbb{C}}$. If $\varphi\colon X \xrightarrow{\sim} X$ is any automorphism, we get

$$\mathrm{R}_A^{Hodge}([\varphi\colon X \xrightarrow{\sim} X]) = \sum_i (-1)^i [\varphi_* H_i(X_{\mathbb{C}}, A)]$$

in K_1.

2. *Suppose X is a smooth k-variety, $\overline{X}$ a smooth compactification with a smooth closed subvariety $Z \subseteq \overline{X}$ such that $X = \overline{X} \setminus Z$. Then in K_0,*

$$\mathrm{R}_A^{Hodge}([X]) = \mathrm{R}_A^{Hodge}([\overline{X}] - [Z])$$

and if

$$\begin{array}{ccc} Z & \hookrightarrow & \overline{X} \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi} \\ Z & \hookrightarrow & \overline{X} \end{array}$$

commutes, then $\mathrm{R}_A^{Hodge}([\varphi\colon X \xrightarrow{\sim} X]) = \mathrm{R}_A^{Hodge}([\varphi\colon \overline{X} \xrightarrow{\sim} \overline{X}] - [\varphi\colon Z \xrightarrow{\sim} Z])$.

Proof. We proceed as for the Betti realization, but with

$$\mathcal{D}_{\text{finite}} := C^b(MHS^A_{\text{eff}}) \qquad \text{and} \qquad \mathcal{D} := C(\mathsf{Lex}(MHS^A_{\text{eff}})),$$

where

$$MHS^A_{\text{eff}} \hookrightarrow \mathsf{Lex}(MHS^A_{\text{eff}})$$

is the Quillen embedding, realizing the exact category as an extension-closed full subcategory of a Grothendieck abelian category. Again, by Gillet–Waldhausen (as in the plain Betti situation) reduce from bounded complexes up to quasi-isomorphism to $K(MHS^A_{\text{eff}})$. □

6. Other realizations

By the work of Cisinski and Déglise any mixed Weil cohomology (as well as what they call a *stable cohomology theory* loc. cit.) satisfies h-descent (and therefore h-hyperdescent) along with $\mathbb{A}^1$-invariance, see [6, Corollary 17.2.6]. This should cover most interesting realizations whose coefficients are a $\mathbb{Q}$-algebra. For example (besides the aforementioned Betti realization if one takes $\mathbb{Q}$-coefficients) de Rham cohomology or syntomic cohomology over p-adic fields ([9], or use [19, Theorem A.1]). We leave carefully choosing categories of values $\mathcal{D}_{\text{finite}}$ to the reader, as this varies from case to case, and the optimal details may depend on what concrete applications the reader will have in mind.

Another realization of interest can be obtained from [22]. In loc. cit., Sosnilo constructs an exact functor of stable ∞-categories $\mathcal{C} \to \mathrm{Com}^b(\underline{Hw})$, where $\mathcal{C}$ is a stable ∞-category endowed with a bounded weight structure in the sense of Bondarko's [3]. The notation $\underline{Hw}$ refers to the heart of the weight structure. The DG category $\mathcal{DM}^{\text{eff}}_{\text{gm},t}(k; A)$ possesses such a weight structure, whenever the characteristic of k is invertible in A or zero. In this case, the heart is given by the additive category of effective Chow motives. Therefore, we obtain a realization map

$$K(\mathcal{V}_k) \to K(\mathcal{C}^{\text{gm}}) \to K(\mathrm{Com}^b(\mathrm{Chow}^{\text{eff}})).$$

On the level of K_0 this recovers a well-known construction of Gillet–Soulé [11]. Details will appear elsewhere. In [7, Problem 7.5], the authors speculate about the existence of such a *Gillet–Soulé realization* and pose several questions about its properties.

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