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Motives on pro-algebraic stacks

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Abstract

We define an ∞-category of rational motives for inverse limits of algebraic stacks, so-called pro-algebraic stacks. We show that it admits a 6-functor formalism for certain classes of morphisms. On pro-schemes, we show that this 6-functor formalism is in the sense of Liu-Zheng. This theory yields an approach to the theory of motives for non-representable algebraic stacks and non-finite type morphisms.

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1. Introduction

Motivation: Barsotti–Tate groups have been central to many studies in arithmetic geometry. Consider the stack BT of Barsotti–Tate groups over $\mathrm{Spec}(\mathbb{F}_p)$. Geometrically, BT is difficult to understand, as it is not an algebraic stack. However, we can express BT as a pro-system of algebraic stacks, $\mathrm{BT} \simeq \lim_{n\geq 0} \mathrm{BT}_n$, where BT_n denotes the algebraic stack of n-truncated Barsotti–Tate groups. Given the importance of BT, it is natural to ask about its "motive" in $\mathrm{DM}(\mathbb{F}_p, \mathbb{Q})$. Since BT is not algebraic, this question does not a priori make sense, though one can formally define

$$M(\mathrm{BT}) \coloneqq \operatorname*{colim}_{n\geq 0} M(\mathrm{BT}_n).$$

Alternatively, we could formally extend $\mathrm{DM}^!_{\mathbb{Q}}$ to arbitrary prestacks and define $M(\mathrm{BT})$ as the !-pull/push of the unit. However, there is no general reason why these two definitions should agree. This issue is often referred to as continuity. If $S = \lim_i S_i$ is a projective system of noetherian finite-dimensional schemes with affine transition maps, then

$$M(S) \cong \operatorname*{colim}_{i} M(S_i)$$

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provided that S is also noetherian of finite dimension [7, Thm. 14.3.1].

Returning to BT, both of the above definitions seem reasonable. However, the formal extension $\mathrm{DM}^!_{\mathbb{Q}}$ does not come with the six-functor formalism one would typically expect in a theory of motives. For computations, such a formalism would be very useful. This becomes clearer when considering the stack Disp of so-called displays [14]. This stack also admits a presentation $\mathrm{Disp} \simeq \lim_{n\geq 0} \mathrm{Disp}_n$ as a pro-system of algebraic stacks over $\mathrm{Spec}(\mathbb{F}_p)$, and we have a map of pro-systems of algebraic stacks $\phi\colon \mathrm{BT} \to \mathrm{Disp}$ [14]. This morphism is level-wise smooth and a universal homeomorphism. The motive of each Disp_n is in fact Tate (cf. Theorem 4.16), and hence easier to understand than the motive of each BT_n. To transfer information from Disp to BT, a definition of the motive of a pro-system of algebraic stacks should be compatible with morphisms, at least those of pro-(locally of finite type).

In this article, we will investigate such situations more closely. To be more precise, we will study the category of rational motives on pro-systems of algebraic stacks $X \simeq \lim_i X_i$ defined via $\mathrm{colim}_i\, \mathrm{DM}(X_i)$, and investigate its functorial properties.

Motives on algebraic stacks: We can define $\mathrm{DM}(F)$, the DG-category of motives for any prestack $F\colon (\mathrm{Ring}) \to \mathbb{S}$ (here $\mathbb{S}$ denotes the ∞-category of Kan-complexes) via right Kan extension along the Yoneda embedding [21]. In this way, we can define $\mathrm{DM}(X)$ for any algebraic stack locally of finite type over a field K for rational motives. Since rational motives satisfy étale descent, the ∞-category of rational motives on an algebraic stack X with smooth atlas $\tilde{X}$ can be computed via the formula

$$\mathrm{DM}(X) \coloneqq \lim(\mathrm{DM}(\tilde{X}) \rightrightarrows \mathrm{DM}(\tilde{X} \times_X \tilde{X}) \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} \dots)$$

[21]. The problem with these definitions is that the computation of the motive of a limit $\lim_i X_i$ as above is not straightforward. To circumvent this problem, we will work with an alternative definition and show that in classical situations our definition agrees with the usual one.

Motives on pro-algebraic stacks: Let us fix a filtered ∞-category $\mathcal{I}$.

Definition 1. Let K be a field and $x\colon \mathcal{I} \to \mathrm{PreStk}_K^{\mathrm{op}}$ be an inverse system of prestacks. Assume that each $X_i \coloneqq x(i)$ is representable by an algebraic stack locally of finite type over K and let us denote by $X \coloneqq \lim x$. Then we call the tuple (x, X) a *pro-$\mathcal{I}$-algebraic stack*.

We call (x, X) *tame*, if x_{ij} is smooth for all $i \to j \in \mathcal{I}$.

Note that there is an obvious notion of morphisms of pro-$\mathcal{I}$-algebraic stacks.

Definition 2. Let (x, X) be a pro-$\mathcal{I}$-algebraic stack. Then we define the ∞-category of rational motives on (x, X) to be

$$\mathrm{DM}(x, X) \coloneqq \operatorname*{colim}_{i\in\mathcal{I},x^*} \mathrm{DM}(X_i),$$

where the colimit is computed in $\mathrm{DGCat}_{\mathrm{cont}} \coloneqq \mathrm{Mod}_{\mathcal{D}(\mathbb{Q})}(\mathrm{Pr}^L_{\mathrm{st}})$.

With this construction, we will obtain a full 6-functor formalism on pro-$\mathcal{I}$-algebraic stacks, which is not provided by the right Kan extended version. The caveat is that this 6-functor formalism does not exist for all morphisms of pro-$\mathcal{I}$-algebraic stacks. We have to make restrictions depending if $\mathcal{I} \simeq \mathbb{N}$, the poset $(\mathbb{N}, \leq)$ viewed as an ∞-category, or not. We will explain this in the main text of this article.

To keep this introduction light, we will only state a summarized version of the existence of a (motivic) 6-functor formalism and refer for a precise version to the main text.

Theorem 3 (2.9)**.** *The ∞-category $\mathrm{DM}(x,X)$ is closed symmetric monoidal, compactly generated and the assignment $(x,X) \mapsto \mathrm{DM}(x,X)$ is contravariantly functorial in (x,X).*

Let $f\colon (x,X) \to (y,Y)$ be a morphisms of tame pro-$\mathbb{N}$-algebraic stacks that is levelwise locally of finite type. Then $f^* := \mathrm{DM}(f)$ is part of a 6-functor formalism if we assume that the exchange map $f_{n+1!}x_n^! \to y_n^! f_{n!}$ induced by the diagram is

$$\begin{array}{ccc} X_{n+1} & \xrightarrow{x_n} & X_n \\ {\scriptstyle f_{n+1}}\downarrow & & \downarrow{\scriptstyle f_n} \\ Y_{n+1} & \xrightarrow{y_n} & Y_n \end{array}$$

an equivalence for all $n \in \mathbb{N}$. Moreover, if $f_n^* y_{n*} \to x_{n*} f_{n+1}^*$ is an equivalence, then f^* defines a pullback formalism with respect to smooth maps.

Further, this 6-functor formalism satisfies homotopy invariance, T-stability, localization and purity in a suitable sense.

Finally, if we assume that each square above is cartesian, the same results hold for any locally of finite type morphism of pro-$\mathcal{I}$-algebraic stacks for any filtered ∞-category without any assumption on tameness and the exchange map.

The assumption on the base change map is satisfied, for example, if (x,X) and (y,Y) are tame and square above is cartesian. Surprisingly, the condition on $f_{n+1!}x_n^! \to y_n^! f_{n!}$ only appears if the square above is not cartesian, since otherwise, we can use the $!/*$-base change equivalence. More importantly, we will see that it also holds for the structure map $(\tau, \mathrm{Disp}) \to \mathrm{Spec}(\mathbb{F}_p)$, (where $\tau_n\colon \mathrm{Disp}_{n+1} \to \mathrm{Disp}_n$ denotes the truncation map on displays), for which each square is not cartesian. In particular, we can define the motive of Disp via !-pull/push of the unit. Moreover, we will see that in the special case of Disp, we have continuity

$$\mathrm{DM}(\mathrm{Disp}) \simeq \mathrm{DM}(\tau, \mathrm{Disp})$$

(see Proposition 4.15). By Theorem 3, we obtain adjoint functors $p_\sharp \dashv p^* \dashv p_*$ and $p_! \dashv p^!$ with the expected compatibilities for the structure map $p\colon \mathrm{Disp} \to \mathrm{Spec}(\mathbb{F}_p)$. These do not follow formally from the definition of DM(Disp) via Kan extension.

The definition via $\mathrm{DM}(\tau, \mathrm{Disp})$ allows us to compute $M(\mathrm{Disp})$ more explicitly.

Theorem 4 (4.16)**.** *The pullback along the truncation map $\mathrm{Disp} \to \mathrm{Disp}_1$ induces a fully faithful embedding.*

$$\mathrm{DM}(\mathrm{Disp}_1) \hookrightarrow \mathrm{DM}(\mathrm{Disp})$$

In particular, we have an equivalence of motives

$$M(\mathrm{Disp}) \simeq M(\mathrm{Disp}_1)$$

inside $\mathrm{DM}(\mathbb{F}_p)$ and moreover it is contained in the full stable cocomplete subcategory of $\mathrm{DM}(\mathbb{F}_p)$ generated by Tate motives.

As a consequence, working with pro-algebraic stacks, we see that

$$H^{*,*}(BT, \mathbb{Q}) \cong H^{*,*}(\mathrm{Disp}_1, \mathbb{Q})$$

(see Corollary 4.19).

Generalization: Since most arguments in the proof of Theorem 3 apply in a more general context, we will frequently work with compactly generated 6-functor formalisms. Although our primary interest lies in rational motives, we will formulate the necessary statements as generally as possible. This broader perspective encompasses both compactly generated pullback formalisms [8] and 6-functor formalisms in the sense of Liu–Zheng [16]. The advantage of this generality is that the key computations remain valid in a wider range of settings, such as pro-systems of ind-schemes. The étale derived category of such systems, for instance, is considered by Bouthier, Kazhdan and Varshavsky [3].

Structure of this article. We start by defining pro-$\mathcal{I}$-algebraic stacks and listing some examples. We define the ∞-category of rational motives on pro-$\mathcal{I}$-algebraic stacks.

Afterward, we prove, that on certain classes of morphisms, this definition yields a (motivic) 6-functor formalism.

Lastly, we will have a closer look at motivic cohomology for pro-$\mathcal{I}$-algebraic stacks and apply our results to the stack of displays and arbitrary Galois extensions.

1.1 Assumptions and notations.

In the following, we want to fix some categorical and algebraic notation that is used throughout this article.

Categorical notation. Throughout, we fix some inaccessible regular cardinal κ. By *small*, we will mean κ*-small*. We will freely work with $(\infty,1)$-categories in the sense of [17] and if we write ∞-category, we will always mean $(\infty,1)$-category. When we say *category*, we always mean 1-category and view it as an ∞-category via the nerve functor.

- Let K be a poset. Then we view K as an ∞-category with morphisms induced by the ordering.
- If $F\colon C\to D$ is a functor of ∞-categories that admits a right adjoint G, then we will denote the adjunction of F and G by the symbol $F \dashv G$.
- The ∞-category of small ∞-groupoids is denoted by $\mathbb{S}$.
- The ∞-category of small ∞-categories is denotes by Cat_∞.
- The ∞-category of presentable ∞-categories with continuous functors, i.e. colimit preserving functors, is denoted by Pr^L.
- We denote by $\mathrm{Pr}^{L,\otimes}$ the ∞-operad, whose underlying ∞-category is Pr^L.
- The ∞-category of presentable stable ∞-categories with continuous functors is denoted by $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}} := \mathrm{Mod}_{\mathrm{Sp}}(\mathrm{Pr}^{L,\otimes})$.
- Let $\mathcal{D}(\mathbb{Q})$ denote the category derived category of $\mathbb{Q}$-vector spaces. This is an algebra object in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$. The ∞-category $\mathrm{DGCat}_{\mathrm{cont}}$ denotes the ∞-category of $\mathcal{D}(\mathbb{Q})$-modules in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$.
- Throughout this article, we will deal with diagrams $X\in\mathrm{Fun}(K,\mathcal{C})$, where K and $\mathcal{C}$ are ∞-categories. For any $k\in K$, we will denote by X_k the value of X at k and for any morphism $\alpha\colon k\to l$ in K, we will denote the induced morphism in $\mathcal{C}$ by X_α. We will often write X_{kl} whenever the morphism $k\to l$ is clear from the context. This abusive notation is for convenience of the reader.

  If $f\colon X\to Y$ is a morphisms of diagrams in $\mathrm{Fun}(K,\mathcal{C})$, then we write $f_k\colon X_k\to Y_k$ for the morphism in $\mathcal{C}$ associated to $k\in K$.

Algebraic notation. Throughout, let S be an excellent noetherian scheme of finite dimension.

- An algebraic stack is in the sense of [22], also referred to as Artin stack.
- Let X be an S-scheme of finite type. By $\mathrm{DM}(X)$, we denote the DG-category of Beilinson-motives [7, §14]. Roughly, these are $\mathbb{A}^1$-local étale sheaves on X with values in $\mathcal{D}(\mathbb{Q})$ such that $\mathbb{P}^1$ is $\otimes$-invertible.

2. Motives on pro-algebraic stacks

In this article, we aim to extend the six-functor formalism to limits of algebraic stacks, such as the stack of Barsotti–Tate groups. This stack can be realized as a limit of its truncations. We develop a general theory for inverse systems of algebraic stacks, allowing for non-representable transition maps.

Before introducing our main constructions and definitions, we recall the theory of rational motives on algebraic stacks. Following the construction of Richarz–Scholbach, we obtain the DG-category of rational motives on prestacks as defined in [21, Def. 2.2.1]. We will later specialize this to our specific context. In what follows, we omit some details and refer the reader to *loc. cit.* for an explicit construction.

Let $\mathrm{AffSch}_S^{\mathrm{ft}}$ denote the category of affine schemes of finite type over S. Since $\mathrm{AffSch}_S^{\mathrm{ft}}$ is essentially small, we replace it with a small skeleton consisting of the relevant objects for our purposes. We write $\mathrm{AffSch}_S^{\kappa}$ for the pro-κ-completion of $\mathrm{AffSch}_S^{\mathrm{ft}}$, which is again small by [12, Lem. 6.1.2].

A prestack over S is a functor from $\mathrm{AffSch}_S^{\kappa}$ to ∞-groupoids. The ∞-category of prestacks over S is denoted by PreStk_S.

The functor $\mathrm{DM}\colon X \mapsto \mathrm{DM}(X)$ denotes the functor of Beilinson motives on finite type S-schemes. We extend DM to prestacks as follows. First, we perform a left Kan extension of DM from $\mathrm{AffSch}_S^{\mathrm{ft}}$ to $\mathrm{AffSch}_S^{\kappa}$, followed by a right Kan extension along the Yoneda embedding $\mathrm{AffSch}_S^{\kappa} \hookrightarrow \mathrm{PreStk}_S$. The transition maps in this process are given by $*$-pullbacks. This yields a functor

$$\mathrm{DM}^*\colon \mathrm{PreStk}_S^{\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}.$$

If we instead use !-pullbacks in place of $*$-pullbacks, we denote the resulting functor by $\mathrm{DM}^!$. However, when restricted to schemes locally of finite type over S, these two constructions are equivalent. This equivalence follows from the fact that, in our setting, DM satisfies h-descent on schemes [7]. Consequently, both $\mathrm{DM}^!$ and DM^* can be computed via descent. Indeed, schemes locally of finite type admit open covers by affine schemes of finite type, and !- and $*$-pullbacks agree up to shift and twist. The same reasoning shows that $\mathrm{DM}^*(X) \simeq \mathrm{DM}^!(X)$ for any algebraic space, since the diagonal of an algebraic space is representable by a scheme.

Using étale descent, we may glue the six-functor formalism of DM to the restriction of DM^* to algebraic spaces, as in [13], or equivalently via the DESCENT algorithm of Liu–Zheng [16]. Similar constructions appear in [21], where ind-algebraic stacks and ind-schemes are treated.

We may now apply the same arguments to glue the six-functor formalism to the restriction of DM^* to algebraic stacks locally of finite type over S. As before, the existence of a smooth cover implies that $\mathrm{DM}^!(X) \simeq \mathrm{DM}^*(X)$ for any such stack X.

To summarize, there is (up to equivalence) no distinction between working with $\mathrm{DM}^!$ and DM^*. Furthermore, we obtain a complete six-functor formalism in the sense of Liu–Zheng for algebraic stacks locally of finite type over S, i.e., the functor DM^* extends to a symmetric monoidal functor

$$\mathrm{DM}^*\colon \mathrm{Corr}(\mathrm{Stk}_S^{\mathrm{lft}})_{\mathrm{lft}} \to \mathrm{CAlg}(\mathrm{Pr}_{\omega}^{L}).$$

To simplify notation, we will henceforth write DM for the extension DM^*. We retain the notation $\mathrm{DM}^!$ for the !-variant.

Notation 2.1. Throughout this article, we will denote by $\mathcal{I}$ a filtered ∞-category.

Definition 2.2. Recall the definition of a pro-$\mathcal{I}$-algebraic stack of Definition 1. We simply say *pro-algebraic* if we do not want to specify $\mathcal{I}$.

- If (x, X) is a pro-$\mathcal{I}$-algebraic stack, then we denote by $X_i := x(i)$ and the transition maps $X_j \to X_i$ by x_{ij}, for $i \to j \in \mathcal{I}$.
- Let $f\colon (x, X) \to (y, Y)$ be a morphism of pro-$\mathcal{I}$-algebraic stacks. If $\mathbf{P}$ is a property of a morphism of algebraic stacks, we will say that f has property $\mathbf{P}$ if each $f_i\colon X_i \to Y_i$ has property $\mathbf{P}$.
- If X is an algebraic stack, we will abuse notation to denote the constant diagram $i \mapsto X$ (here the transition maps are given by id_X) with X itself.

We will define some important properties of pro-algebraic stacks inspired from examples of the introduction.

Definition 2.3. Let (x, X) be a pro-$\mathcal{I}$-algebraic stack. Then we say that (x, X) is

1. *classical* if x_{ij} is affine for all $i \to j \in \mathcal{I}$,
2. *strict* if $x_{ij}^*\colon \mathrm{DM}(X_i) \to \mathrm{DM}(X_j)$ is fully faithful for all $i \to j \in \mathcal{I}$.
3. *tame* if each x_{ij} is smooth[1].

Remark 2.4. The property strict will not be used heavily in this article. Nevertheless, we want to remark that the stack of displays by Lau admits a presentation as a strict pro-$\mathbb{N}$-algebraic stack. Perhaps a more straightforward example is the infinite dimensional affine space, via the projections $\mathbb{A}^{n+1} \to \mathbb{A}^n$.

As remarked in the introduction, we only get a 6-functor formalism for certain classes of morphisms. Let us define these.

Definition 2.5. If $f\colon (x, X) \to (y, Y)$ is a morphism of pro-$\mathcal{I}$-algebraic stacks, then we call f *cartesian* if each square

$$\begin{array}{ccc} X_j & \xrightarrow{x_{ij}} & X_i \\ {\scriptstyle f_j}\downarrow & & \downarrow{\scriptstyle f_i} \\ Y_j & \xrightarrow{y_{ij}} & Y_i \end{array}$$

is a pullback diagram. We call a morphism $f\colon (x, X) \to (y, Y)$ of pro-$\mathcal{I}$-algebraic stacks $*$-*adjointable* if the exchange morphism $f_i^* y_{ij*} \to x_{ij*} f_j^*$ induced by each square above is an equivalence and !-*adjointable* if $f_{j!} x_{ij}^! \to y_{ij}^! f_{i!}$ is an equivalence.

The property *cartesian* is perhaps a more natural restriction for the existence of a 6-functor formalism. Especially, when considering that homotopy invariance, T-stability and localization requires a good notion of the affine line, the projective space resp. taking complement in the category of pro-algebraic stacks. However, it turns out that the property *adjointable* is enough for just the existence of a 6-functor formalism if $\mathcal{I} \simeq \mathbb{N}$. We will make this more precise late but let us give an example of adjointable morphisms of pro-algebraic stacks.

[1] In practice the pro-algebraic stacks we consider are always tame. However, not every assertion in our construction will need this condition.

Example 2.6. (1) Let $f\colon (x,X)\to(y,Y)$ be a cartesian morphism of pro-$\mathcal{I}$-algebraic stacks, then f is adjointable.

(2) Let $f\colon (x,X)\to(y,Y)$ be a morphism of pro-$\mathcal{I}$-algebraic stacks. If y_{ij*} is fully faithful for all $i\to j\in\mathcal{I}$ (e.g. when $(y,Y)\cong S$) and (x,X) is strict, then a simple computation shows that f is $*$-adjointable.

Indeed, we use that $\mathrm{id}\to x_{ij*}x_{ij}^*$ and $y_{ij}^*y_{ij*}\to\mathrm{id}$ are invertible to conclude that the morphism

$$f_i^*y_{ij*}\to x_{ij*}x_{ij}^*f_i^*y_{ij*}\simeq x_{ij*}f_j^*y_{ij}^*y_{ij*}\to x_{ij*}f_j^*$$

is also invertible.

If (x,X) and (y,Y) in addition tame and $\mathrm{id}\to y_{ij}^*y_{ij\sharp}$ is invertible (again this holds if $(y,Y)\cong S$), then the same argument together with purity shows that f is !-adjointable.

Example 2.7. Let L/K be an algebraic extension of fields. Let $\mathcal{F}_K^L$ be the filtered category of all finite field extensions $L/E/K$. Let $\mathcal{F}_K^L\to\mathrm{Fields}$ be the associated diagram in the category of fields. Then $\mathrm{Spec}(L)\cong\lim_{E\in\mathcal{F}_K^L}\mathrm{Spec}(E)$. Now let $\mathfrak{X}$ be an algebraic stack over K. For any $E\in\mathcal{F}_K^L$, we set $\mathfrak{X}_E:=\mathfrak{X}\times_K E$. Let $x_{E'/E}\colon\mathfrak{X}_E\to\mathfrak{X}_{E'}$ be the projection for a field extension E'/E. Then $(x,\mathfrak{X}_L)$ is a classical pro-$\mathcal{F}_K^L$-algebraic stack. Naturally, we also get a classical pro-$\mathcal{F}_K^L$-algebraic stack $(\iota,\mathrm{Spec}(L))$ defined by the inclusions of field extensions in $\mathcal{F}_K^L$. The associated morphism $(x,\mathfrak{X}_L)\to(\iota,\mathrm{Spec}(L))$ is cartesian. Note that $(\iota,\mathrm{Spec}(L))$ and thus also $(x,\mathfrak{X}_L)$ is not strict as for example any Galois extension $E\hookrightarrow E$ is a $\mathrm{Gal}(E'/E)$-torsor.

The above can be generalized to the following case. We can replace $\mathrm{Spec}(K)$ by S and assume that S is connected. Let $\bar{s}$ be a geometric point of S. Then we can define F_S as the category of pointed étale covers over $(S,\bar{s})$. The construction above can be repeated in this case for any algebraic stack $\mathfrak{X}$ over S.

Now let us define the DG-category of rational motives associated to a pro-algebraic stack.

Definition 2.8. Let (x,X) be a pro-$\mathcal{I}$-algebraic stack. Then we define the ∞*-category of rational motives on* (x,X) as

$$\mathrm{DM}(x,X):=\operatorname*{colim}_{i\in\mathcal{I},x^*}\mathrm{DM}^*(X_i),$$

where the colimit is taken in $\mathrm{DGCat}_{\mathrm{cont}}$.

One might wonder why we chose DM^* and not $\mathrm{DM}^!$. Our main reason is convenience. We conjecture that we have an equivalence between both possible definitions. But by working with DM^*, monoidality of $\mathrm{DM}(x,X)$ follows by definition. Also, we will see that if $\mathcal{I}\simeq\mathbb{N}$, then both possible definitions are equivalent.

Let us now formalize Theorem 3 of the introduction. We will postpone the technical proof to Section 3.

Theorem 2.9. *Let $f\colon(x,X)\to(y,Y)$ be a morphism of pro-$\mathcal{I}$-algebraic stacks. Then there is an adjunction*

$$f^*\colon\mathrm{DM}(y,Y)\rightleftarrows\mathrm{DM}(x,X)\colon f_*,$$

where f^* is symmetric monoidal and preserves compact objects. The functor f^* extends to a functor $(x,X)\mapsto\mathrm{DM}(x,X)$. Moreover, the functor DM *satisfies the following property on pro-$\mathcal{I}$-algebraic stacks.*

(PB) If f is smooth and $*$-adjointable, then f^* admits a left adjoint $f_\sharp$ satisfying base change and projection formula.

Assume now that f locally of finite type. Further, assume one of the following holds true

($*$) *f is cartesian, or*

(†) *$\mathcal{I} \simeq \mathbb{N}$, f is !-adjointable, (x, X) and (y, Y) are tame.*

Then we can upgrade f^* to be part of a 6-functor formalism in the following sense. There exists an adjunction

$$f_!\colon \mathrm{DM}(x, X) \rightleftarrows \mathrm{DM}(y, Y)\colon f^!,$$

where $f_!$ defines a covariant functor on pro-algebraic stacks satisfying ($*$) or (†). Further, the adjunctions above satisfy the following properties.

(F1) If f is cartesian and proper, we have a canonical equivalence $f_* \simeq f_!$.

(F2) The functor $f^!$ commutes with arbitrary colimits.

(F3) (Base change) For any pullback square

$$\begin{array}{ccc} (w, W) & \xrightarrow{f'} & (z, Z) \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ (x, X) & \xrightarrow{f} & (y, Y) \end{array}$$

of pro-algebraic stacks with morphisms satisfying ($*$) resp. (†), the exchange morphisms

$$f^* g_! \to g'_! f'^*$$
$$g^! f_* \to f'_* g'^!$$

are equivalences.

(F4) (Projection formula) The exchange transformation $f_!(- \otimes f^*(-)) \to f_!(-) \otimes -$ is an equivalence.

Furthermore, the following properties hold.

(M1) (Homotopy invariance) Let us define $(q, \mathbb{A}^1_X)$ via the system $i \mapsto \mathbb{A}^1_{X_i}$. Then p^* is fully faithful, where $p\colon (q, \mathbb{A}^1_X) \to (x, X)$ denotes the projection.

(M2) (Localization) Let $i\colon (z, Z) \hookrightarrow (x, X)$ be cartesian closed immersion of pro-algebraic stacks with open complement $j\colon (u, U) \to (x, X)$. Then there exist fiber sequences

$$i_! i^! \to \mathrm{id} \to j_* j^*$$
$$j_! j^! \to \mathrm{id} \to i_* i^*.$$

Assuming that $\mathcal{I}$ admits an initial object $0 \in \mathcal{I}$, we moreover have the following properties.

(M3) (T-stability) Let $V_0 \to X_0$ be a vector bundle and let $p\colon (v, V) \to (x, X)$ be the induced morphism of pro-algebraic stacks. The zero section $s_0\colon X_0 \to V_0$ induces a section $s\colon (x, X) \to (v, V)$. Then the Thom-map $\mathrm{Th}(p, s) := \mathrm{Th}(V_0) := p_\sharp s_*$ is an equivalence of DG-categories.

(M4) (Orientation) Let $(q, \mathbb{G}_{\mathrm{m},X})$ be the pro-algebraic stack defined by $n \mapsto \mathbb{G}_{\mathrm{m},X_i}$. We naturally obtain a projection $p\colon (q, \mathbb{G}_{\mathrm{m},X}) \to (x, X)$. For $M \in \mathrm{DM}(X)$, we set $M(1) := \mathrm{cofib}(p_\sharp p^* M \to M)[-1]$. Let $V_0 \to X_0$ be a vector bundle of rank n. Then there is a functorial equivalence $\mathrm{Th}(V_0)1_X \simeq 1_X(n)[2n]$.

(M5) (Purity) If f is cartesian, we have a canonical equivalence $f^! \simeq \mathrm{Th}(\Omega_{f_0})1_X \otimes f^*$.

If we further restrict ourselves to pro-$\mathcal{I}$-schemes, i.e. inverse systems of schemes locally of finite type over S indexed over $\mathcal{I}$, then f^* extends to a 6-functor formalism in the sense of Liu-Zheng [16].

Before we continue with the proof, we want to show that $\mathrm{DM}(x, X) \simeq \mathrm{DM}(X)$ for classical pro-algebraic stacks, as one would expect. We will need the following result.

Lemma 2.10. *Let $\mathcal{E}$ be an ∞-category that admits finite limits. Let $F\colon \mathcal{E} \to \mathrm{Pr}^L_\omega$ be a functor. Denote by $F_\mathcal{I}$ the composition of $\mathrm{Fun}(\mathcal{I}, F)$ with the colimit functor $\mathrm{Fun}(\mathcal{I}, \mathrm{Pr}^L_\omega) \to \mathrm{Pr}^L_\omega$. Then we have the following.*

(1) Let $X \in \mathrm{Fun}(\mathcal{I}, \mathcal{E})$. We denote by X^{op} the opposite functor. Further, let us denote by $F_*\colon \mathcal{E}^{\mathrm{op}} \to \mathrm{Pr}^R$ the functor induced by the equivalence $\mathrm{Pr}^L \simeq (\mathrm{Pr}^R)^{\mathrm{op}}$ [17, Cor. 5.5.3.4]. Then we have

$$F_\mathcal{I}(X) \simeq \lim_{\mathrm{Pr}^R} F_* \circ X^{\mathrm{op}},$$

where the equivalence is in Cat_∞.

(2) For any $i_0, i_1 \in \mathcal{I}$ the composition

$$F(X_{i_0}) \xrightarrow{\mathrm{ins}^X_{i_0}} F_\mathcal{I}(X) \simeq \lim_{\mathrm{Pr}^R} F_* \circ X \xrightarrow{p^X_{i_1}} F(X_{i_1})$$

is canonically equivalent to

$$\operatorname*{colim}_{k\in\mathcal{I}, \alpha\colon i_0 \to k, \beta\colon i_1\to k} F_*(X_\beta) \circ F_\mathcal{I}(X_\alpha)$$

where the colimit is taken in $\mathrm{Fun}_{\mathrm{Pr}^L}(F(X_{i_0}), F(X_{i_1}))$.

Proof. The first assertion follows from [17, Cor. 5.5.3.4] (see also Remark 2.11 below). A proof for (2) is given by Gaitsgory in the setting of DG-categories (cf. [9, Lem. 1.3.6] - note that the proof is completely analogous in this slightly more general setting). □

Remark 2.11. The equivalence in Lemma 2.10 for $X \in \mathrm{Fun}(\mathcal{I}, \mathcal{E})$ is induced via the following process [9, Lem. 1.3.2]. The projections $p^X_i\colon \lim_{\mathrm{Pr}^R} F_* \circ X \to X_i$ admit left adjoints, denoted by γ^X_i. These left adjoints assemble to an equivalence

$$h_X\colon F_\mathcal{I}(X) \xrightarrow{\sim} \lim_{i\in\mathcal{I}^{\mathrm{op}}, x_*} F(X_i). \tag{1}$$

This is an equivalence in Pr^L, where we view the RHS naturally as an object in Pr^L. The inverse of h_X is equivalent to $\mathrm{colim}_{i\in\mathcal{I}} \mathrm{ins}^X_i \circ p^X_i$, where $\mathrm{ins}^X_i\colon F(X_i) \to F_\mathcal{I}(X)$ denotes the canonical map.

Let us fix the above notation, as we will need it later on.

Notation 2.12. We fix the notation as in Remark 2.11, i.e. for a pro-$\mathcal{I}$-algebraic stack (x, X), we denote by h_X the equivalence

$$h_X\colon \mathrm{DM}(x, X) \to \lim_{i\in\mathcal{I}, x_*} \mathrm{DM}(X_i),$$

we denote for any $i \in \mathcal{I}$ by

$$\mathrm{ins}^X_i\colon \mathrm{DM}(X_i) \to \mathrm{DM}(x, X) \text{ resp. } p^X_i\colon \lim_{i\in\mathcal{I}, x_*} \mathrm{DM}(X_i) \to \mathrm{DM}(X_i)$$

the canonical inclusion resp. projections and we set $\gamma^X_i \coloneqq h_X \circ \mathrm{ins}^X_i$.

Proposition 2.13. *Assume that $\mathcal{I}$ admits an initial object $0 \in \mathcal{I}$. Let (x, X) be a classical pro-$\mathcal{I}$-algebraic stack. Then the natural map*

$$\mathrm{DM}(x, X) \to \mathrm{DM}(X)$$

is an equivalence of DG-categories.

Proof. For the proof we will use two immediate facts. The canonical map $\mathrm{DM}(x, X) \to \mathrm{DM}(X)$ is by construction monoidal and preserves colimits. Moreover, to show that $\mathrm{DM}(x, X) \to \mathrm{DM}(X)$ is an equivalence, we may work with the right adjoints

$$\mathrm{DM}(X) \to \lim_{i \in \mathcal{I}^{\mathrm{op}}, x_*} \mathrm{DM}(X_i)$$

(this follows from Lemma 2.10 applied to $F = \mathrm{DM} : \mathrm{Stk}_S^{\mathrm{lft,op}} \to \mathrm{Pr}^L_\omega$). By assumption X_0 admits a smooth cover by affine finite type S-schemes $U_{0,k} = \mathrm{Spec}(A_k)$, for some set K and $k \in K$. As $X_i \to X_0$ is affine for all $i \in \mathcal{I}$, the pullback $U_{i,k} := X_i \times_{X_0} U_{i,k}$ is also a smooth cover by affines. The same holds after passage to limits, i.e. for $U_k := X \times_{X_0} U_{0,i}$ the projection $\coprod_{k \in K} U_i \to X$ is an effective epimorphism [22, 01YZ]. In particular, as DM satisfies h-descent (cf. [21, Thm. 2.2.16]), we have $\lim_\Delta \mathrm{DM}(\check{C}_\bullet(\coprod_{k \in K} U_i/X)) \simeq \mathrm{DM}(X)$. By construction, each of the U_i is equivalent to $\lim_{i \in \mathcal{I}} U_{i,k}$. As the $U_{i,k}$ are affine of finite type over S, we have

$$\mathrm{DM}(U_i) \simeq \mathrm{DM}(\lim_{i \in \mathcal{I}} U_{i,k}) \simeq \operatorname*{colim}_{i \in \mathcal{I}, x^*} \mathrm{DM}(U_{i,k}) \simeq \lim_{i \in \mathcal{I}^{\mathrm{op}}, x_*} \mathrm{DM}(U_{i,k}).$$

This yields the equivalence

$$\mathrm{DM}(\check{C}_\bullet(\coprod_{k \in K} U_i/X)) \simeq \lim_{i \in \mathcal{I}, x_*} \mathrm{DM}(\check{C}_\bullet(\coprod_{k \in K} U_{i,k} X_i)).$$

Putting all of this together, we claim that this yields

$$\begin{aligned}\mathrm{DM}(X) \simeq \lim_\Delta \mathrm{DM}(\check{C}_\bullet(\coprod_{k \in K} U_i/X)) &\simeq \lim_{i \in \mathcal{I}^{\mathrm{op}}, x_*} \lim_\Delta \mathrm{DM}(\check{C}_\bullet(\coprod_{k \in K} U_{i,k}/X_i)) \\ &\simeq \lim_{i \in \mathcal{I}^{\mathrm{op}}, x_*} \mathrm{DM}(X_i) \simeq \operatorname*{colim}_{i \in \mathcal{I}, x^*} \mathrm{DM}(X_i)\end{aligned}$$

as desired.

Indeed, the only thing to check is the second equivalence. For this it is enough to see that x_{ij}^* induces a map of simplicial objects

$$\mathrm{DM}(\check{C}_\bullet(\coprod_{k \in K} U_{j,k}/X_j)) \to \mathrm{DM}(\check{C}_\bullet(\coprod_{k \in K} U_{i,k}/X_i)).$$

But this follows immediately from smooth base change, as

$$\check{C}_\bullet(\coprod_{k \in K} U_{j,k}/X_j) \simeq \check{C}_\bullet(\coprod_{k \in K} U_{i,k}/X_i) \times_{X_i} X_j.$$ □

Remark 2.14. Let Y be a scheme and let (x, X) be a classical pro-algebraic stack such that each X_i is representable by a finite type Y-scheme. The proof of Proposition 2.13 can be repeated to obtain the following more general result on schemes.

- Let $F \colon \mathrm{Sch}_Y^{\mathrm{ft,op}} \to \mathrm{CALg}(\mathrm{Pr}^{L,\otimes})$ be a Zariski sheaf, then $\operatorname{colim} F \circ x \simeq F(X)$.

This includes for example motives with integral coefficients and the stable homotopy category of Morel-Voevodsky.

By definition if (x, X) is a strict pro-algebraic stack then the transition maps x^*_{ij} are fully faithful for all $i \to j \in \mathcal{I}$. In particular, we would expect that the pullback $\mathrm{DM}(X_0) \to \mathrm{DM}(X)$ is fully faithful. At least when we work with $\mathrm{DM}(x, X)$ this is true, as our explicit analysis will show. If (x, X) is classical, we see in particular that $\mathrm{DM}(X_0) \to \mathrm{DM}(X)$ is indeed fully faithful.

Remark 2.15 (Underling motive of strict pro-algebraic stacks)**.** Let us show how to compute the underlying motive in $\mathrm{DM}(S)$ of a strict pro-$\mathcal{I}$-algebraic stack, when $\mathcal{I}$ admits an initial object 0.

Let (x, X) be a strict pro-algebraic stack. Then by Example 2.6 (2), the projection $c_0 \colon (x, X) \to X_0$ is an adjointable map of pro-algebraic stacks. Thus, by Theorem 2.9, we obtain an adjunction

$$c_{0*} \colon \mathrm{DM}(x, X) \rightleftarrows \mathrm{DM}(X_0) \colon c_0^*$$

Moreover, we see from explicit computations in Lemma 2.10 (2) that c_0^* is fully faithful, since c_0^* corresponds to canonical map $\mathrm{DM}(X_0) \to \mathrm{DM}(x, X)$ and c_{0*} to the projection.

In particular, we will see ath if $f \colon (x, X) \to S$ is strict, tame and $f_0 \colon X_0 \to S$ denote the structure map, we have for any $M \in \mathrm{DM}(S)$ the equivalence $f_! f^! M \simeq f_{0!} f_0^! M$.

3. (Motivic) six functor formalism on pro-algebraic stacks

In this section we will prove Theorem 2.9.

Moreover, we will not restrict us to the category of algebraic stacks but rather work in the most general context, as we will only need formal properties, such as a geometric setup in the sense of Mann [20]. While this is a more abstract formalism we develop, the benefit is that we can also apply this formalism to other settings, for example ind-schemes. Such inverse limits of ind-schemes and étale cohomology were considered by Bouthier-Kazhdan-Varshavsky [3] where they were called *placid*.

3.1 Pullback formalisms.

We start by analyzing filtered colimits of pullback formalisms after Drew-Gallauer [8]. We will analyze such structures more generally in this subsection without using pro-algebraic stacks but work with general diagrams in an ∞-category $\mathcal{E}$ admitting finite limits.

Definition 3.1 ([8])**.** Let $\mathcal{E}$ be an ∞-category that admits finite limits and let us fix a class of morphisms $P \subseteq \mathrm{Fun}(\Delta^1, \mathcal{E})$ that is closed under pullback, equivalences and composition. A *compactly generated pullback formalism on $\mathcal{E}$ with respect to P* is a functor

$$\mathcal{M}^{\otimes} \colon \mathcal{E}^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^{L,\otimes}_{\omega}), \quad f \mapsto f^* \quad \text{(we denote the right adjoint of } f^* \text{ by } f_*)$$

satisfying the following conditions.

(1) If a morphism $f \colon X \to Y$ of $\mathcal{E}$ lies in P, then there exists a left adjoint $f_\sharp$ of f^*.

(2) For each pullback square

$$\begin{array}{ccc} X \times_Z Y & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{g} & Z \end{array}$$

in $\mathcal{E}$ such that f lies in P, the exchange transformation

$$f'_\sharp g'^* \to f'_\sharp g'^* f^* f_\sharp \to f'_\sharp f'^* g^* f_\sharp \to g^* f_\sharp$$

is equivalence.

(3) For each morphism $f\colon X \to Y$ in P the exchange transformation

$$f_\sharp(M \otimes_{\mathcal{M}^\otimes(X)} f^*N) \to f_\sharp M \otimes_{\mathcal{M}^\otimes(Y)} N$$

is an equivalence, for any $M \in \mathcal{M}^\otimes(X)$ and $N \in \mathcal{M}^\otimes(Y)$.

We call the triple $(\mathcal{M}^\otimes, \mathcal{E}, \mathcal{P})$ as above a *pullback formalism*.

Remark 3.2. For property (2) in Definition 3.1 we may equivalently ask for the $*$-exchange morphism $f^*g_* \to g'_*f'^*$ to be an equivalence [1, Prop. 1.1.9].

Example 3.3. Recall that in the assumptions 1.1, we fixed an excellent noetherian scheme S finite dimension. By [7, Thm. 16.1.4, Cor. 6.2.2] the ∞-category $\mathrm{DM}(X, \mathbb{Q})$ is compactly generated for any finite type S-scheme and f^* preserves compact objects. In particular, $\mathrm{DM}_{\mathbb{Q}}$ induces a pullback formalism on finite type S-schemes. By descent this can be extended to locally of finite type algebraic stacks over S [21].

We will start by extending $\mathcal{M}^\otimes$ to diagrams in $\mathcal{I}$, via taking colimits.

Remark 3.4. Let us remark that Pr^L_ω is presentable and thus so is $\mathrm{Mod}_{\mathrm{Sp}}(\mathrm{Pr}^{L,\otimes}_\omega)$ and $\mathrm{CAlg}(\mathrm{Pr}^{L,\otimes}_\omega)$ [18, Lem. 5.3.2.9, Cor. 4.2.3.7, Prop. 3.2.3.5].

Notation 3.5. Consider the functor

$$\mathrm{Fun}(\mathcal{I}, \mathcal{E}^{\mathrm{op}}) \to \mathrm{Fun}(\mathcal{I}, \mathrm{CAlg}(\mathrm{Pr}^{L,\otimes}_\omega))$$

induced by postcomposition with $\mathcal{M}^\otimes$. Composing this functor with the colimit functor, we obtain

$$\mathcal{M}^\otimes_{\mathcal{I}}\colon \mathrm{Fun}(\mathcal{I}, \mathcal{E}^{\mathrm{op}}) \to \mathrm{CAlg}(\mathrm{Pr}^{L,\otimes}_\omega).$$

If (x, X) is a pro-algebraic stack, we see that the following notation agrees Definition 2.8.

Notation 3.6. Let $\mathcal{M}^\otimes$ be a pullback formalism on locally of finite type Artin S-stacks with respect to smooth morphisms. Let (x, X) be a pro-$\mathcal{I}$-algebraic stack. Then we define

$$\mathcal{M}(x, X) \coloneqq \mathcal{M}^\otimes_{\mathcal{I}}(x).$$

Proposition 3.7. *Let $(\mathcal{M}^\otimes, \mathcal{E}, \mathcal{P})$ be a pullback formalism. Let $\mathcal{P}_{\mathcal{I}}$ denote the class of morphisms in $\mathrm{Fun}(\mathcal{I}, \mathcal{E}^{\mathrm{op}})$ such that $f \in \mathrm{Fun}(\mathcal{I}, \mathcal{E}^{\mathrm{op}})$ belongs to $\mathcal{P}_{\mathcal{I}}$ if and only if each f_i belongs to $\mathcal{P}$ and each square in $\mathcal{M}_{\mathcal{I}}(f)$ is adjointable, i.e. for any morphism $i \to j \in \mathcal{I}$ the induced morphism $f_i^* y_{ij*} \to x_{ij*} f_j^*$ is an equivalence [18, Def. 4.7.4.13]. Then $\mathcal{M}_{\mathcal{I}}$ is a pullback formalism on $\mathrm{Fun}(\mathcal{I}, \mathcal{E}^{\mathrm{op}})$ for the class $\mathcal{P}_{\mathcal{I}}$.*

Proposition 3.7 is one of the crucial points in extending the motivic six functor formalism on DM for Artin-stacks to pro-algebraic stacks. But before we can prove this proposition, we need some additional lemmas.

Lemma 3.8. *Let $F\colon \mathcal{E}^{\mathrm{op}} \to \mathrm{Pr}^L_\omega$ be a functor. Denote by $F_{\mathcal{I}}$ the composition of $\mathrm{Fun}(\mathcal{I}, F)$ with the colimit functor. Let X be in $\mathrm{Fun}(\mathcal{I}, \mathcal{E}^{\mathrm{op}})$. Then we have*

$$F_{\mathcal{I}}(X) \simeq \mathrm{Ind}(\operatorname*{colim}_{i\in\mathcal{I},x^*} F(X_i)^c)$$

and any $M \in F_{\mathcal{I}}(X)$ is compact if and only if there exists an $i \in \mathcal{I}$ and $M_i \in F(X_i)^c$ that lifts M.

Proof. The first assertion follows from [10, Cor. 7.2.7], where the colimit on the right is taken in Cat_∞.

Note that the filtered colimit of idempotent complete ∞-categories is idempotent complete [17, Cor. 4.4.5.21]. Thus, any compact object comes from an object of the colimit on the right hand side (cf. [17, Lem. 5.4.2.4] and [19, 02LG]). □

Lemma 3.9. *If $f\colon X \to Y$ is a morphism in $\mathrm{Fun}(\mathcal{I}, \mathcal{E}^{\mathrm{op}})$ such that each f_i admits a left adjoint for all $i \in \mathcal{I}$ and each square is adjointable (in the sense of Proposition 3.7), then $f^* := F_{\mathcal{I}}(f)$ admits a left adjoint.*

Proof. This lemma is stated without proof in [3, Prop. 5.1.8 (c)]. As far as we know, there is no written proof for this result in the literature, so we provide one.

By Lemma 2.10 (1), we see that $f_*\colon \lim_{\mathcal{I}} F_* \circ X \to \lim_{\mathcal{I}} F_* \circ Y$ admits a left adjoint f^*, that has to be compatible with the identification of $\lim_{\mathcal{I}} F_*$ and $F_{\mathcal{I}}$. We claim that the following diagram is commutative (up to homotopy)

$$\begin{array}{ccc} \lim_{\mathcal{I}} F_* \circ Y & \xrightarrow{p_i^Y} & F(Y_i) \\ \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f_i^*} \\ \lim_{\mathcal{I}} F_* \circ X & \xrightarrow{p_i^X} & F(X) \end{array}$$

for all $i \in \mathcal{I}$. This claim implies that f^* commutes with limits and therefore admits a left adjoint $f_\sharp$.

Let us show the claim. For this, note that we have the following diagram with commutative squares (up to homotopy)

$$\begin{array}{ccccc} F(Y_j) & \xrightarrow{\mathrm{ins}_Y^j} & F_{\mathcal{I}}(Y) & \xrightarrow{h_Y} & \lim_{\mathcal{I}} F_* \circ Y \\ \downarrow{\scriptstyle f_j^*} & & \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} \\ F(X_j) & \xrightarrow{\mathrm{ins}_X^j} & F_{\mathcal{I}}(X) & \xrightarrow{h_X} & \lim_{\mathcal{I}} F_* \circ X \end{array}$$

for any $j \in \mathcal{I}$. First, we show that $p_i^Y \circ f^*$ commutes with colimits. Indeed, it is enough to see that $p_i^Y \circ f^* \circ h_X \circ \mathrm{ins}_X^j$ commutes with colimits, then is equivalent to the induced map $F_{\mathcal{I}}(X) \to F(Y_i)$ in Pr^L, which necessarily commutes with colimits. By Lemma 2.10 (2), we have

$$p_i^X \circ f^* \circ h_Y \circ \mathrm{ins}_Y^j \simeq p_i^Y \circ h_X \circ \mathrm{ins}_X^j \circ f_j^* \simeq \operatorname*{colim}_{i\to k, j\to k} X_{ik*} \circ X_{jk}^* \circ f_j^*$$

(recall our notations in the introduction under Section 1.1). By construction each of the functors X_{ik*}, X_{jk}^*, f_j^* commutes with colimits and thus also the left hand side of the equivalence. Analogously, one can show that $f_i^* \circ p_i^Y$ commutes with colimits. Therefore, it is enough to show that

$$p_i^X \circ f^* \circ h_Y \circ \mathrm{ins}_Y^j \simeq f_i^* \circ p_i^Y \circ h_Y \circ \mathrm{ins}_Y^j.$$

The left hand side is equivalent to $\mathrm{colim}_{i\to k, j\to k} X_{ik*} \circ X_{jk}^* \circ f_j^*$ by the above and analogously, we see that the right hand side is equivalent to $\mathrm{colim}_{i\to k, j\to k} f_i^* \circ Y_{ik*} \circ Y_{jk}^*$. Since each square in $F(f)$ is right adjointable these colimits are in fact equivalent, proving the claim and therefore assertion. □

Lemma 3.10. *Let f be a morphism in $\mathrm{Fun}(\mathcal{I}, \mathcal{E}^{\mathrm{op}})$ such that f_i admits a left adjoint for all $i \in \mathcal{I}$. Let $f_\sharp$ denote the left adjoint of f given by Lemma 3.9. Then the following square*

$$\begin{array}{ccc} F(X_i) & \xrightarrow{\mathrm{ins}_X^i} & F_{\mathcal{I}}(X) \\ \big\downarrow{\scriptstyle f_{i\sharp}} & & \big\downarrow{\scriptstyle f_\sharp} \\ F(Y_i) & \xrightarrow{\mathrm{ins}_Y^i} & F_{\mathcal{I}}(Y) \end{array}$$

commutes (up to homotopy) for all $i \in \mathcal{I}$.

Proof. By construction of $f_\sharp$ the square

$$\begin{array}{ccc} F_{\mathcal{I}}(X) & \xrightarrow{h_X} & \lim_{\mathcal{I}} F_* \circ X \\ \big\downarrow{\scriptstyle f_\sharp} & & \big\downarrow{\scriptstyle f_\sharp} \\ F_{\mathcal{I}}(Y) & \xrightarrow{h_Y} & \lim_{\mathcal{I}} F_* \circ Y \end{array}$$

commutes (up to homotopy). Thus, we may check if the square in the lemma commutes after applying h_X. But the proof of Lemma 3.9 shows that the square

$$\begin{array}{ccc} \lim_{\mathcal{I}} F_* \circ Y & \xrightarrow{p_i^Y} & F(Y_i) \\ \big\downarrow{\scriptstyle f^*} & & \big\downarrow{\scriptstyle f_i^*} \\ \lim_{\mathcal{I}} F_* \circ X & \xrightarrow{p_i^X} & F(X_i) \end{array}$$

commutes (up to homotopy) for all $i \in \mathcal{I}$. So in particular, we conclude by passing to left adjoints. □

Proof of Proposition 3.7. Assertion (1) follows from Lemma 3.9.

We are left to show (2) and (3). Invoking Lemma 2.10 resp. Lemma 3.10 and the fact that each compact of $\mathcal{M}_{\mathcal{I}}(X)$ comes from some level $i \in \mathcal{I}$ (cf. Lemma 3.8). So, we can check (2) and (3) on some $k \in \mathcal{I}$ (note that we use that $\mathcal{I}$ is filtered) and this follows from the definition of $\mathcal{M}^{\otimes}$. □

3.2 Six functor formalism. In this section, we want to prove Theorem 2.9. We will first give a proof in the (†) case. Afterwards, we will give a proof in the (∗) case and complete the proof.

Before we continue, we want to show that properties (M1)-(M5) are a formal consequence of the compact generation of $\mathrm{DM}(x, X)$ (see Lemma 3.8).

Lemma 3.11. *In the setting of Theorem 2.9, assuming $f_! \dashv f^!$ exists, satisfying (F1)-(F4), the properties (M1) and (M2) hold. If $\mathcal{I}$ admits an initial object $0 \in \mathcal{I}$, then also (M3) - (M5) hold.*

Proof. Arguing as in the proof of Proposition 3.7 and using descent the properties (M1) - (M3) reduce to questions of DM on schemes, which is well known [7]. The property (M4) follows from the localization sequence.

If f is cartesian, smooth and then the relative dimension of f_i and f_0 agree for all $i \in \mathcal{I}$. From here (M5) follows similarly as before. □

3.2.1 Proof of Theorem **??** *for adjointable maps* In this subsection, we will assume that we are in the setting (†) and we simply say pro-algebraic instead of pro-$\mathbb{N}_0$-algebraic.

Lemma 3.12. *Let $\mathcal{C}$ be an ∞-category. Let* $\mathrm{Spine}[\mathbb{N}_0]$ *be the simplicial subset of* $\mathbb{N}_0$ *of finite vertices that are joined by edges (or more informally, we forget about all $n \geq 2$ simplices in $\mathbb{N}_0$). Then the restriction* $\mathrm{Fun}(\mathbb{N}_0, \mathcal{C}) \to \mathrm{Fun}(\mathrm{Spine}[\mathbb{N}_0], \mathcal{C})$ *is a trivial Kan fibration.*

In particular, any natural transformation of functors $\alpha\colon \mathrm{Spine}[\mathbb{N}_0] \times \Delta^1 \to \mathcal{C}$ *can be lifted to a natural transformation in* $\hat{\alpha}\colon \mathbb{N}_0 \times \Delta^1 \to \mathcal{C}$.

Proof. Indeed, the functor $\mathrm{Spine}[\mathbb{N}_0] \to \mathbb{N}_0$ is inner anodyne [19, 03HK] proving the claim. □

Proposition 3.13. *Let (x, X) be a tame pro-algebraic stack, then*

$$\mathrm{DM}(x, X) \simeq \operatorname*{colim}_{i \in \mathbb{N}_0, x^!} \mathrm{DM}^!(X_i).$$

Proof. By descent, we see that for each $n \in \mathbb{N}_0$ we have $\tilde{p}_i\colon \mathrm{DM}^!(X_i) \simeq \mathrm{DM}^*(X_i)$ as X_i admits a smooth covering by a scheme. We will construct equivalences

$$p_i\colon \mathrm{DM}^!(X_i) \simeq \mathrm{DM}^*(X_i)$$

out of $\tilde{p}_n$ that are compatible with the presentation as a colimit. This is easily achieved by twisting with the Thom-motives associated to the smooth maps x_{0i}. But let us be more precise.

Let us define $p_i := \tilde{p}_n \otimes \mathrm{Th}(\Omega_{x_{0n}})1_{X_n}$, where Th denotes the Thom-motive[2] associated to $\mathbb{V}(\Omega_{x_{0i}}) \to X_n$. Then p_n is still an equivalence of $\mathrm{DM}^!(X_n)$ and $\mathrm{DM}^*(X_n)$. By compatibility with the purity equivalence with composition [7, Rem. 2.4.52], the p_i are compatible with the presentation of $\mathrm{DM}(x, X)$ as a colimit by Lemma 3.12. □

Remark 3.14. To generalize Proposition 3.13 to arbitrary $\mathcal{I}$ it is enough to know that the level-wise equivalences p_i in the proof yield an equivalence of diagrams. While on the homotopy categorical level this is clear, we were not able to write a map down for higher simplices.

Proof of Theorem 2.9. The property (PB) is Proposition 3.7. By Proposition 3.13, we obtain a functor $f^!\colon \mathrm{DM}(y, Y) \to \mathrm{DM}(x, X)$. Applying Proposition 3.7 to $f^!$, we obtain a left adjoint $f_!$ of $f^!$. Note that we use that each $f_n^!$ preserves colimits, since $f_{n!}$ preserves compact objects. In particular, (F2) follows from construction. The properties (F3) and (F4) can be checked on compacts, which is clear by design and Lemma 3.8.

The rest follows from Lemma 3.11 □

Remark 3.15. Let (x, X) be a strict tame pro-algebraic stack and (y, Y) a tame pro-algebraic stack together with a morphism $f\colon (x, X) \to (y, Y)$. Assume that for $y_n\colon Y_{n+1} \to Y_n$ and $n \geq 0$ the unit $\mathrm{id} \to y_n^* y_{n\sharp}$ is an equivalence. In this setting, we can make $f_!$ more explicit.

For each $n \geq 0$, we obtain a functor $f_{n!}$. We want to construct a morphism of diagrams $f_{\bullet!}\colon \mathbb{N}_0 \times \Delta^{1,\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}$, which after taking the colimit then yields $f_!$. Since our morphisms are not cartesian, we have to take the Thom-twist of $x_n\colon X_{n+1} \to X_n$ and y_n into account. Set $\tilde{f}_{n!} := (\mathrm{Th}(\Omega_{y_{0n}})1_{Y_n})^{-1} \otimes f_{n!} \circ \mathrm{Th}(\Omega_{x_{0n}})1_{X_n} \otimes -$. Then, we obtain

$$\begin{aligned}
y_n^* \tilde{f}_{n!} &\simeq y_n^* (\mathrm{Th}(\Omega_{y_{0n}})1_{Y_n})^{-1} \otimes f_{n!} \circ \mathrm{Th}(\Omega_{x_{0n}})1_{X_n} \otimes x_{n!} x_n^! \\
&\simeq y_n^* (\mathrm{Th}(\Omega_{y_{0n}})1_{Y_n})^{-1} \otimes y_{n!} f_{n+1!} \circ \mathrm{Th}(\Omega_{x_{0n+1}})1_{X_{n+1}} \otimes x_n^* \\
&\simeq y_n^* y_{n\sharp} \circ (\mathrm{Th}(\Omega_{y_{0n+1}})1_{Y_{n+1}})^{-1} \otimes f_{n+1!} \circ \mathrm{Th}(\Omega_{x_{0n+1}})1_{X_{n+1}} \otimes x_n^* \simeq \tilde{f}_{n+1!} x_n^*,
\end{aligned}$$

[2] Let $p\colon V \to X_i$ be a vector bundle with zero section s, then $\mathrm{Th}(V) := p_\sharp s_*$.

where we use compatibility of Thom twists with composition in the sense of [7, Rem. 2.4.52] and the projection formula.

In particular, we obtain the desired diagram $f_{\bullet !}\colon \mathbb{N}_0 \times \Delta^{1,\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}$ that after taking colimits induces a functor $f_!\colon \mathrm{DM}(x,X) \to \mathrm{DM}(y,Y)$. This is a colimit preserving functor and therefore admits a right adjoint. By construction of the comparison morphism in Proposition 3.13, this agrees with the construction of $f_!$ in the proof of Theorem 2.9.

This construction also shows why (F1) fails in this setting. Even when f is proper, the resulting $f_!$ is an infinite twist of f_*. Note however, if f is cartesian, then by design the twists appearing in $\tilde{f}$ cancel each other out, so this caveat appears only in the non-cartesian setting.

3.2.2 Proof of Theorem 2.9 for cartesian maps. We now proceed to prove Theorem 2.9 in the $(*)$ case. Throughout, we use the term *pro-algebraic* in place of *pro-$\mathcal{I}$-algebraic.*

To that end, we need to extend the six-functor formalism to pro-algebraic stacks. We will freely use the frameworks developed by Gaitsgory–Rozenblyum, Liu–Zheng, and Mann (cf. [10], [16], [20]). Our arguments rely on results from the latter, and for details we refer the reader to [20, §A.5]. We will not recall the main definitions here, and instead use the language of geometric setups and correspondences as developed in *loc. cit.*

Let $(\mathcal{C}, E)$ be a *geometric setup*, meaning an ∞-category $\mathcal{C}$ admitting finite limits, together with a class of morphisms $E \subset \mathrm{Mor}(\mathcal{C})$ that is stable under pullbacks, compositions, and equivalences. Let

$$\mathcal{D}\colon \mathrm{Corr}(\mathcal{C})_E \to \mathrm{Cat}_\infty$$

be a six-functor formalism in the sense of [20, §A.5], where $\mathrm{Corr}(\mathcal{C})_E$ denotes the ∞-category of correspondences as defined in [16].

We denote by

$$\mathcal{D}_{\mathcal{C}}\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Cat}_\infty)$$

the restriction of $\mathcal{D}$ along the canonical inclusion $\mathcal{C}^{\mathrm{op}} \hookrightarrow \mathrm{Corr}(\mathcal{C})_E$. We assume that $\mathcal{D}_{\mathcal{C}}$ factors through $\mathrm{CAlg}(\mathrm{Pr}^{L,\otimes}_\omega)$. We will use notation analogous to that of Section 3.1.

We denote by $\mathcal{D}_{\mathcal{I}}$ the composition

$$\mathcal{D}_{\mathcal{I}}\colon\ \mathrm{Fun}(\mathcal{I}, \mathcal{C}^{\mathrm{op}}) \xrightarrow{\mathcal{D}_{\mathcal{C}}} \mathrm{Fun}(\mathcal{I}, \mathrm{CAlg}(\mathrm{Pr}^{L,\otimes}_\omega)) \xrightarrow{\mathrm{colim}} \mathrm{CAlg}(\mathrm{Pr}^{L,\otimes}_\omega).$$

We define $E_{\mathcal{I}}$ to be the class of those morphisms $f\colon X \to Y$ in $\mathrm{Fun}(\mathcal{I}, \mathcal{C}^{\mathrm{op}})$ such that f_i is in E for all $i \in \mathcal{I}$ and for all $i \to j \in \mathcal{I}$ the square

$$\begin{array}{ccc} X_j & \xrightarrow{X_{ij}} & X_i \\ \downarrow{\scriptstyle f_j} & & \downarrow{\scriptstyle f_i} \\ Y_j & \xrightarrow{Y_{ij}} & Y_i \end{array}$$

is cartesian.

Proposition 3.16. *Assume that $\mathcal{I}$ admits an initial object 0 and E admits a suitable decomposition J, F that satisfies the assumptions of [20, Prop. A.5.10]. Then $\mathcal{D}_{\mathcal{I}}$ can be upgraded to a 6-functor formalism*

$$\mathcal{D}_{\mathcal{I}}\colon \mathrm{Corr}(\mathrm{Fun}(\mathcal{I}, \mathcal{C}^{\mathrm{op}}))_{E_{\mathcal{I}}} \to \mathrm{CAlg}(\mathrm{Pr}^{L,\otimes}_\omega).$$

Further, if $f \in E$ is a morphism such that for all $i \in \mathcal{I}$ we have $f_i \in F$ resp. $f(i) \in J$, then $f_! \simeq f_*$ resp. $f_!$ is left adjoint to f^*.

Proof. We define $J_{\mathcal{I}}$ resp. $F_{\mathcal{I}}$ to be the class of those morphisms f in $E_{\mathcal{I}}$ such that $f(i)$ is in J resp. F for all $i \in \mathcal{I}$. Then we claim that $J_{\mathcal{I}}, F_{\mathcal{I}}$ is a suitable decomposition of E.

Indeed, since for a map f in E all squares of the associated diagram are pullback squares. Thus, all of the properties follow immediately (note that we need that $\mathcal{I}$ admits an initial object for the decomposition property).

We will use Proposition 3.7 and the criterion in [20, Proposition. A.5.10] to prove this proposition.

By Proposition 3.7 the functor $\mathcal{D}_{\mathcal{I}}$ defines a pullback formalism on $\mathrm{Fun}(\mathcal{I}, \mathcal{C}^{\mathrm{op}})$ with respect to $J_{\mathcal{I}}$.

By construction, the pullback f^* along any morphism $f \in \mathrm{Fun}(\mathcal{I}, \mathcal{C}^{\mathrm{op}})$ admits a right adjoint. We need to check that for morphisms in $F_{\mathcal{I}}$ we have base change and projection formula. But similar to the proof of Proposition 3.7, we can use that any compact object comes from some compact on the i-th level, for some $i \in \mathcal{I}$ (by *mutas mutandis* of Lemma 3.8). This argument shows that for any pullback square

$$\begin{array}{ccc} W & \xrightarrow{f'} & X \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ Y & \xrightarrow{f} & Z \end{array}$$

with $g \in J_{\mathcal{I}}$ and $f \in F_{\mathcal{I}}$ we have $g_\sharp f'_* \simeq f_* g'_\sharp$ completing the proof. □

The above proposition a priori only applies if the geometric setup has a suitable decomposition. For schemes this is usually not a problem as in practice we can decompose separated morphisms via compactifications into open immersions and proper maps. For algebraic stacks this only holds locally. Nevertheless, the existence of the !-adjunction extends to cartesian morphisms of pro-$\mathcal{I}$-algebraic stacks by the base change formula, as seen below.

Proposition 3.17. *Let $f \colon X \to Y$ be a morphism inside $E_{\mathcal{I}} \subseteq \mathrm{Fun}(\mathcal{I}^{\mathrm{op}}, \mathcal{C})^{\mathrm{op}}$. Then there exists an adjunction*

$$f_! \colon \mathcal{D}_{\mathcal{I}}(X) \rightleftarrows \mathcal{D}_{\mathcal{I}}(Y) \colon f^!$$

satisfying base change and projection formula with respect to $f^* := \mathcal{D}_{\mathcal{I}}(f)$.

Proof. First, let us note that the morphism f corresponds to a diagram $\mathcal{I}^{\mathrm{op}} \times \Delta^{1,\mathrm{op}} \to \mathcal{C}$. The idea of the proof is to take $f_! := \mathrm{colim}_{\mathcal{I}} f_{!\bullet}$. However, as $\mathcal{I}$ is arbitrary, we do not immediately obtain a a diagram $f_{!\bullet} \colon \mathcal{I} \times \Delta^{1,\mathrm{op}} \to \mathrm{DGCat}_{\mathrm{cont}}$ as in the proof of the $(*)$ case. Instead, we will make use of the fact that f is cartesian and we will construct a diagram $\mathcal{I} \times \Delta^{1,\mathrm{op}} \to \mathrm{Corr}(\mathcal{C})_E$. Taking the colimit of this diagram will then yield $f_!$. To make this process homotopy coherent, we will use the description[3] of $\mathrm{Corr}(\mathcal{C})_E$ in terms of bisimplicial sets.

Let us consider the marked simplicial set $(\mathcal{C}, E, \mathrm{ALL})$, where E is as before and ALL denotes the class of all edges in $\mathcal{C}$. By the assignment $([n_1],[n_2]) \mapsto \mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^{n_1} \times \Delta^{n_2}, \mathcal{C})$ we obtain a bisimplicial set $\delta_* \mathcal{C}$. We denote by $\mathcal{C}_E^{\mathrm{cart}}$ the bisimplicial subset such that the vertical edges are contained in E and each square is cartesian (for a rigorous construction see [6]). By a similar construction, we obtain a bisimplicial set $(\mathcal{I}^{\mathrm{op}} \times \Delta^{1,\mathrm{op}})^{\mathrm{ALL}}_{\mathrm{ALL}}$ associated to the marked simplicial set $(\mathcal{I}^{\mathrm{op}} \times \Delta^{1,\mathrm{op}}, \mathrm{ALL}, \mathrm{ALL})$. This process is functorial in marked simplicial sets with marked squares, so we obtain a map $\alpha \colon (\mathcal{I}^{\mathrm{op}} \times \Delta^{1,\mathrm{op}})^{\mathrm{ALL}}_{\mathrm{ALL}} \to \mathcal{C}_E^{\mathrm{cart}}$. We take opposites of the horizontal arrows, which we see as a functor op^2 and apply this to α. Finally, we let $K\mathrm{pt}^n$ denote the

[3]We were made aware of this alternative description by Chirantan Chowdhury.

bisimplicial subset of $\Delta^{(n,n)}$ spanned by edges (k,l) for $0 \leq k \leq l \leq n$. Then we define the simplicial set by

$$\delta^*_{2\nabla} \operatorname{op}^2 \mathcal{C}^{\mathrm{cart}}_E \colon n \mapsto \operatorname{Hom}_{\mathrm{Set}_{2\Delta}}(K\mathrm{pt}^n, \operatorname{op}^2 \mathcal{C}^{\mathrm{cart}}_E).$$

By functoriality of the construction, we obtain a diagram

$$\tilde{\alpha} \colon \delta^*_{2\nabla} \operatorname{op}^2(\mathcal{I}^{\mathrm{op}} \times \Delta^{1,\mathrm{op}})^{\mathrm{ALL}}_{\mathrm{ALL}} \to \delta^*_{2\nabla} \operatorname{op}^2 \mathcal{C}^{\mathrm{cart}}_E.$$

It is not hard to see from the construction that $\delta^*_{2\nabla} \operatorname{op}^2 \mathcal{C}^{\mathrm{cart}}_E \cong \operatorname{Corr}(\mathcal{C})_E$ as simplicial sets. The simplicial set $\delta^*_{2\nabla} \operatorname{op}^2(\mathcal{I}^{\mathrm{op}} \times \Delta^{1,\mathrm{op}})^{\mathrm{ALL}}_{\mathrm{ALL}}$ is very explicit and we can construct a map of simplicial sets $\mathcal{I} \times \Delta^{1,\mathrm{op}} \to \delta^*_{2\nabla} \operatorname{op}^2(\mathcal{I}^{\mathrm{op}} \times \Delta^{1,\mathrm{op}})^{\mathrm{ALL}}_{\mathrm{ALL}}$ in the following way.

Note that we have a map

$$\operatorname{Hom}_{\mathrm{Set}_\Delta}(\Delta^n \times \Delta^{n,\mathrm{op}}, \mathcal{I}^{\mathrm{op}} \times \Delta^{1,\mathrm{op}}) \to (\delta^*_{2\nabla} \operatorname{op}^2(\mathcal{I}^{\mathrm{op}} \times \Delta^{1,\mathrm{op}})^{\mathrm{ALL}}_{\mathrm{ALL}})_n.$$

Let $\Delta^n \to \mathcal{I} \times \Delta^{1,\mathrm{op}}$ be a map of simplicial sets. This map is uniquely determined by projections to $\mathcal{I}$ and $\Delta^{1,\mathrm{op}}$. Let us denote these maps by $p_1 \colon \Delta^n \to \mathcal{I}$ and $p_2 \colon \Delta^n \to \Delta^{1,\mathrm{op}}$. From these two maps, we get a map $\Delta^n \times \Delta^{n,\mathrm{op}} \to \mathcal{I}^{\mathrm{op}} \times \Delta^{1,\mathrm{op}}$ of simplicial sets via

$$\varphi_1 \colon \Delta^n \times \Delta^{n,\mathrm{op}} \xrightarrow{\mathrm{pr}_2} \Delta^{n,\mathrm{op}} \xrightarrow{p_1^{\mathrm{op}}} \mathcal{I}^{\mathrm{op}},$$
$$\varphi_2 \colon \Delta^n \times \Delta^{n,\mathrm{op}} \xrightarrow{\mathrm{pr}_1} \Delta^n \xrightarrow{p_2} \Delta^{1,\mathrm{op}},$$

where $\mathrm{pr_i}$ denotes the i-th projection. In particular, we thus obtain a morphism of simplicial sets

$$\phi \colon \mathcal{I} \times \Delta^{1,\mathrm{op}} \to \delta^*_{2\nabla} \operatorname{op}^2(\mathcal{I}^{\mathrm{op}} \times \Delta^{1,\mathrm{op}})^{\mathrm{ALL}}_{\mathrm{ALL}} \to \operatorname{Corr}(\mathcal{C})_E.$$

Let $\widetilde{\mathcal{D}}$ denote the composition

$$\operatorname{Fun}(\mathcal{I}, \operatorname{Corr}(\mathcal{C})_E)^{\mathrm{op}} \xrightarrow{(\mathcal{D}\circ -)^{\mathrm{op}}} \operatorname{Fun}(\mathcal{I}, \operatorname{CAlg}(\mathrm{Pr}^{L,\otimes}_\omega))^{\mathrm{op}} \xrightarrow{\mathrm{colim}} \operatorname{CAlg}(\mathrm{Pr}^{L,\otimes}_\omega).$$

By design, we have $\widetilde{\mathcal{D}}(X) \simeq \mathcal{D}_{\mathcal{I}}(X)$, where we view X as an objet of the left hand side via the map $\operatorname{Fun}(\mathcal{I}, \mathcal{C}^{\mathrm{op}})^{\mathrm{op}} \to \operatorname{Fun}(\mathcal{I}, \operatorname{Corr}(\mathcal{C})_E)^{\mathrm{op}}$. In particular, by applying $\widetilde{\mathcal{D}}$ to ϕ, which we view as an edge in $\operatorname{Fun}(\mathcal{I}, \operatorname{Corr}(\mathcal{C})_E)^{\mathrm{op}}$, we obtain a colimit preserving map that preserves compact objects

$$f_! \coloneqq \widetilde{\mathcal{D}}(\phi) \colon \mathcal{D}_{\mathcal{I}}(X) \to \mathcal{D}_{\mathcal{I}}(Y).$$

The remaining assertions follow by Lemma 3.8 similar to the proof in the (†) case. □

Remark 3.18. In the proof of Proposition 3.17, we can apply the same method to obtain a diagram $\mathcal{I} \times \Delta^1 \to \operatorname{Corr}(\mathcal{C})_E$ by working with the projection $\Delta^n \times \Delta^{n,\mathrm{op}} \to \Delta^{n,\mathrm{op}}$. The colimit along this map then yields the pullback functor f^*.

Remark 3.19. As is evident from the proof of Proposition 3.17, if $\mathcal{D}^!_{|\mathcal{C}_E}$ commutes with colimits, then so does

$$f^! \coloneqq \mathcal{D}^!_{\mathcal{I}}(f)$$

for all $f \in E_{\mathcal{I}}$. This follows from the fact that $\mathcal{D}_{\mathcal{I}}(X)$ is compactly generated for all $X \in \operatorname{Fun}(\mathcal{I}, \mathcal{C})$, by Lemma 3.8.

Proof of Theorem 2.9. This is now just a consequence of Proposition 3.16, Proposition 3.17 and Lemma 3.11. Note that if f is a proper cartesian morphism of pro-$\mathcal{I}$-algebraic stacks, then the construction of $f_!$ in Proposition 3.16 shows that $f_! \simeq f_*$, using that on compacts they agree by Lemma 3.8. □

Remark 3.20. Let $f\colon (x,X)\to(y,Y)$ be a smooth morphism pro-$\mathcal{I}$-algebraic stacks. For $M\in \mathrm{DM}(y,Y)$, we can compute the "motivic global sections with values in M" directly. To be more precise, let us note that since f_*f^* is colimit preserving, its values are determined by its restriction to compacts. As any compact in $\mathrm{DM}(y,Y)$ comes from a compact in some $\mathrm{DM}(Y_i)$ (cf. Lemma 3.8), we see that

$$f_*f^*M \simeq \operatorname*{colim}_{\mathrm{ins}_i M_i\to M}\ \operatorname*{colim}_{i\to k, j\to k} \mathrm{ins}_i^Y f_{i*}f_i^* y_{ik*}y_{ij}^* M_i,$$

where the colimit is over all $i\in\mathcal{I}$ and all compacts $M_i\in \mathrm{DM}(Y_i)^c$. A similar formula can also be given for $f_!f^!$.

In particular, if for example $(y,Y)\cong S$, and (x,X) is strict, we have

$$f_*f^*1_S \simeq f_{0*}f_0^*1_S$$

as expected (cf. Remark 2.15).

Remark 3.21 (Non-finite type base change)**.** Let $f\colon (x,X)\to(y,Y)$ be a smooth adjointable morphism pro-algebraic stacks. Assume that f is either $\mathcal{I}\simeq\mathbb{N}_0$ or that f is cartesian and $\mathcal{I}$ has an initial object 0.

Our computations have shown that we have $f_*\mathrm{ins}_0^X\simeq \mathrm{ins}_0^Y f_{0*}$, where ins_0^X denotes the natural functor $\mathrm{DM}(X_0)\to\mathrm{DM}(x,X)$ (similarly for Y). This can be seen as a form of non-finite type base change equivalence of the square

$$\begin{array}{ccc}(x,X) & \longrightarrow & X_0\\ \downarrow & & \downarrow\\ (y,Y) & \longrightarrow & Y_0,\end{array}$$

which is cartesian if f is cartesian. However, note that as longs as $(x,X)\to X_0$ and $(y,Y)\to Y_0$ are not adjointable, we cannot use the base change of Theorem 2.9.

4. Motivic cohomology of pro-algebraic stacks

In this section, we want to define and highlight some properties of motivic cohomology in our setting.

In the following, we fix a morphism $f\colon (x,X)\to(y,Y)$ of pro-$\mathcal{I}$-algebraic stacks over S. Let $h\colon (x,X)\to S$ and $g\colon (y,Y)\to S$ be the morphism induced by the structure morphisms. We further assume that either f is cartesian or that $\mathcal{I}\simeq\mathbb{N}$ and f is adjointable.

By Theorem 2.9, we have adjunctions $f^*\dashv f_*$ and $f_!\dashv f^!$ between $\mathrm{DM}(x,X)$ and $\mathrm{DM}(y,Y)$ satisfying all the properties stated in the theorem. Note that even though (x,X) and (y,Y) are not assumed to be strict, we still have adjunctions $h^*\dashv h_*$ and $g^*\dashv g_*$ by Lemma 3.9.

Definition 4.1. Let $f\colon (x,X)\to(y,Y)$ and $g\colon (y,Y)\to S$ be as above. Then we define the *relative motivic cohomology of* (x,X) *with coefficients in* $M\in\mathrm{DM}(y,Y)$ as

$$R\Gamma(X,M) \coloneqq \underline{\mathrm{Hom}}_{\mathcal{D}(\mathbb{Q})}(f_!f^!1_{(y,Y)},M)\in \mathrm{DGCat}_{\mathrm{cont}}.$$

Further, we define for $n,m\in\mathbb{Z}$ the *absolute motivic cohomology of* (y,Y) *in degree* (n,m) as the $\mathbb{Q}$-vector space

$$H^{n,m}(Y,\mathbb{Q}) \coloneqq \pi_0\,\mathrm{Hom}_{\mathrm{DM}(y,Y)}(1_{(y,Y)},1_{(y,Y)}(n)[m]).$$

Remark 4.2. Let us remark that the absolute motivic cohomology of (y, Y) can also be computed relative to S in the following sense

$$H^{n,m}(Y, \mathbb{Q}) \cong \mathrm{Hom}_{h\mathrm{DM}(S)}(1_S, g_* g^* 1_S(n)[m]).$$

Further, if (y, Y) is strict and $(y, Y) \to S$ is a smooth, then

$$H^{n,m}(Y, \mathbb{Q}) \cong \pi_m \mathrm{Hom}_{\mathrm{DM}(S)}(g_\sharp g^* 1_S, 1_S(n)).$$

Assumption 4.3. From now on let us assume that $\mathcal{I}$ admits an initial object $0 \in \mathcal{I}$

Lemma 4.4. *Under the notation above, we have*

$$g_* 1_{(y,Y)} \simeq \operatorname*{colim}_{j \in \mathcal{I}} g_{j*} 1_{Y_j}.$$

Proof. By construction, we have $g^* \simeq \mathrm{ins}_0^Y g_0^*$. Using adjunctions this shows that

$$g_* \simeq g_{0*} p_0^Y h_Y.$$

By construction, we have $\mathrm{ins}_0^Y 1_{Y_0} \simeq 1_{(y,Y)}$. Thus,

$$\begin{aligned} g_* 1_{(y,Y)} \simeq g_{0*} p_0^Y \gamma_0^Y 1_{Y_0} &\simeq \operatorname*{colim}_{j \in \mathcal{I}} g_{0*} y_{0j*} y_{0j}^* 1_{Y_0} \\ &\simeq \operatorname*{colim}_{j \in \mathcal{I}} g_{j*} 1_{Y_j}. \end{aligned}$$

□

Remark 4.5. Lemma 4.4 immediately shows that for S regular and $(y, Y) \to S$ smooth, we have

$$H^{n,m}(Y, \mathbb{Q}) \cong \operatorname*{colim}_{i \in \mathcal{I}} A^n(Y_i, 2n - m)_{\mathbb{Q}}.$$

We will see below, that if (y, Y) is classical and $Y \to S$ is locally of finite type, then this colimit presentation of the absolute motivic cohomology of Y agrees with the classical motivic cohomology of Y. In this setting, we can also look at the functor induced by the usual $*$-pushforward on Artin-stacks $\mathrm{DM}(y, Y) \simeq \mathrm{DM}(Y) \to \mathrm{DM}(S)$. This agrees by construction with $g_* \colon \mathrm{DM}(y, Y) \to \mathrm{DM}(S)$.

Proposition 4.6. *Assume (y, Y) is classical such that $g \colon Y \to S$ is locally of finite type. Then*

$$H^{n,m}(Y, \mathbb{Q}) \simeq \mathrm{Hom}_{h\mathrm{DM}(Y)}(1_Y, 1_Y(n)[m]).$$

Proof. Let $\tilde{Y}_0 \to Y_0$ be a smooth cover of Y_0 by a scheme locally of finite type over S. Let $\tilde{Y}_i$ denote the base change to Y_i and $\tilde{Y}$ its limit. Then $\tilde{Y}$ is representable by a scheme and yields a smooth cover of Y. As Y is locally of finite type, so is $\tilde{Y}$. For every $i \in \mathcal{I}$, we can compute

$$g_{i*} 1_{Y_i} \simeq \operatorname*{colim}_{\Delta} g_{\check{C}(\tilde{Y}_i/Y_i)_\bullet *} 1_{\check{C}(\tilde{Y}_i/Y_i)_\bullet}$$

by descent, where $g_{\check{C}(\tilde{Y}_i/Y_i)_\bullet} \colon \check{C}(\tilde{Y}_i/Y_i)_\bullet \to S$ denotes the projection. In particular, we can write

$$f_* 1_{(y,Y)} \simeq \operatorname*{colim}_{j \in \mathcal{I}} g_{j*} 1_{Y_j} \simeq \operatorname*{colim}_{\Delta} \operatorname*{colim}_{j \in \mathcal{I}} g_{\check{C}(\tilde{Y}_i/Y_i)_\bullet *} 1_{\check{C}(\tilde{Y}_i/Y_i)_\bullet}$$

(cf. Lemma 4.4). Therefore, by continuity of DM [7, Thm. 14.3.1] and descent, we have

$$\begin{aligned} H^{n,m}(Y, \mathbb{Q}) &\cong \operatorname*{colim}_{\Delta} \operatorname*{colim}_{i \in \mathcal{I}} \mathrm{Hom}_{h\mathrm{DM}(\check{C}(\tilde{Y}_i/Y_i)_\bullet)}(1_{\check{C}(\tilde{Y}_i/Y_i)_\bullet}, 1_{\check{C}(\tilde{Y}_i/Y_i)_\bullet}(n)[m]) \\ &\cong \operatorname*{colim}_{\Delta} \mathrm{Hom}_{h\mathrm{DM}(\check{C}(\tilde{Y}/Y)_\bullet)}(1_{\check{C}(\tilde{Y}/Y)_\bullet}, 1_{\check{C}(\tilde{Y}/Y)_\bullet}(n)[m]) \\ &\cong \mathrm{Hom}_{h\mathrm{DM}(Y)}(1_Y, 1_Y(n)[m]). \end{aligned}$$

□

Example 4.7. Let us consider the situation of Example 2.7. So let L/K be an algebraic extension of fields and consider a smooth scheme $Z \to \mathrm{Spec}(K)$. We denote by $\mathcal{F}_K^L$ the set of finite field extensions E/K contained in L. We denote by $(\iota, \mathrm{Spec}(L))$ the associated classical pro-$\mathcal{F}_K^L$-algebraic stack over K and by (z, Z_L) the induced classical pro-$\mathcal{F}_K^L$-algebraic stack over K. Using Remark 4.2, we have

$$H^{n,m}(Z_L, \mathbb{Q}) \cong \operatorname*{colim}_{E \in \mathcal{F}_K^L} A^n(Z_E, 2n-m)_{\mathbb{Q}} \cong A^n(Z_L, 2n-m)_{\mathbb{Q}},$$

where the second isomorphism follows from direct computations on the level of cycles [4, Lem. 1.4.6 (i)].

Example 4.8. Assume that S is regular. Let $Z \to S$ be a smooth integral S-scheme with function field K. Let us consider the family $(U)_{U \in \mathcal{U}_K}$ of all affine open subschemes of Z. If we denote the respective inclusions by $\iota_{UU'} \colon U' \hookrightarrow U$, then this family assembles to a classical pro-$\mathcal{U}_K$-algebraic stack $(\iota, \mathrm{Spec}(K))$. Let $T \to Z$ be a flat morphism of S-schemes. Then base change of T along the system $(\iota, \mathrm{Spec}(K))$ induces a classical pro-$\mathcal{U}_K$-algebraic stack (ι_T, T_K). Using Remark 4.2, we have

$$H^{n,m}(T_K, \mathbb{Q}) \cong \operatorname*{colim}_{U} A^n(T_U, 2n-m)_{\mathbb{Q}} \cong A^n(T_K, 2n-m)_{\mathbb{Q}},$$

where the second isomorphism follows from direct computations on the level of cycles [4, Lem. 1.4.6 (ii)].

4.1 Motives on Disp. Let us fix a prime $p > 0$ and assume from now on that $S = \mathrm{Spec}(\mathbb{F}_p)$. We will denote the Frobenius with σ. In the following section, we want to define the pro-algebraic stack of (truncated) Displays over S, after the construction of Lau (cf. [14]) and show that its underlying motive in $\mathrm{DM}(S)$ is Tate. We do not want to go into the detail of the construction of (truncated) Displays and its relation to the stack of Barsotti-Tate groups, but rather give an equivalent definition following [5].

Let us consider the functor $\mathrm{Spec}(R) \mapsto \mathrm{GL}_h(W_n(R))$ for any affine S-scheme $\mathrm{Spec}(R)$, where W_n denotes the ring of n-truncated Witt-vectors. This functor is represented by an open subscheme of $\mathbb{A}^{nh^2}$, which we denote by X_n^h. Let $G_n^{h,d}(R)$ denote the group of invertible $h \times h$ matricies

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with $A \in \mathrm{GL}_{h-d}(W_n(R))$, $B \in \mathrm{End}(W_n(R)^d, W_n(R)^{h-d})$, $C \in \mathrm{End}(W_n(R)^{h-d}, I_{n+1}(R)^d)$, where $I_{n+1}(R)$ denotes the image of the Verschiebung $W_{n+1} \to W_n$ and $D \in \mathrm{GL}_d(W_n(R))$. Let us denote by $G_n^{h,d}$ the group scheme representing this functor. Then $G_n^{h,d}$ acts on X_n^h via $M \cdot x := Mx\sigma'(M)^{-1}$, where

$$\sigma'(M) := \begin{pmatrix} \sigma A & p\sigma B \\ \sigma C & \sigma D \end{pmatrix}.$$

Definition 4.9. For $0 \le d \le h$ and $n \in \mathbb{N}$, we define the stack of n-truncated displays of dimension d and height n to be

$$\mathrm{Disp}_n^{h,d} := \left[X_n^h / G_n^{h,d}\right].$$

Further, we define the stack of n-truncated displays as

$$\mathrm{Disp}_n := \coprod_{0 \le d \le h} \mathrm{Disp}_n^{h,d}.$$

Notation 4.10. For each $n \in \mathbb{N}$, there exists a truncation map $\tau_n\colon \mathrm{Disp}_{n+1} \to \mathrm{Disp}_n$. We denote by (τ, Disp) the pro-algebraic stack induced by the τ_n.

Lemma 4.11 ([5, Lem. 2.2.2])**.** *Let $K^{h,d}_{n,m}$ denote the kernel of the projection $G^{h,d}_n \to G^{h,d}_m$ for $m < n$ and $\tilde{K}^{h,d}_n$ the kernel of the projection $G^{h,d}_n \to \mathrm{GL}_{h-d} \times \mathrm{GL}_d$. Then $K^{h,d}_{n,m}$ and $\tilde{K}^{h,d}_n$ are split unipotent.*

Remark 4.12. As seen in the proof of [5, Thm. 2.3.3] the group $G^{h,d}_1$ is even a split extension of $\mathrm{GL}_{h-d} \times \mathrm{GL}_d$ by a split unipotent group U_1. The splitting is induced by the canonical inclusion $\mathrm{GL}_{h-d} \times \mathrm{GL}_d \hookrightarrow G^{h,d}_1$.

Proposition 4.13. *The pro-algebraic stack (τ, Disp) is strict and tame.*

Proof. As DM preserves finite products, it is enough to prove that for any $0 \le d \le h$ and any $n > m$ the truncation $\tau^{h,d}_{n,m}\colon \mathrm{Disp}^{h,d}_n \to \mathrm{Disp}^{h,d}_m$ is strict. By definition $\tau_{n,m}$ factors through

$$\left[X^h_n/G^{h,d}_n\right] \xrightarrow{a} \left[X^h_m/G^{h,d}_n\right] \xrightarrow{b} \left[X^h_m/G^{h,d}_m\right].$$

The map $b^*\colon \mathrm{DM}(\left[X^h_m/G^{h,d}_m\right]) \to \mathrm{DM}(\left[X^h_m/G^{h,d}_n\right])$ is an equivalence of categories by Lemma 4.11 (cf. [21, Prop. 2.2.11]).

We now claim that a^* is fully faithful (the idea is the same as in [5, Thm 2.3.1]). By definition the map $X^h_n \to X^h_m$ is a $K^{h,0}_{n,m}$-torsor. By étale descent we may replace the morphism a by the projection $K^{h,0}_{n,m} \to S$. By Lemma 4.11 this group is split unipotent, so it admits a normal series with successive quotients given by a vector bundle. Thus, applying again étale descent and induction, we may assume that $K^{h,0}_{n,m}$ is a vector bundle over S, showing that a^* is fully faithful.

The factorization also shows that τ is tame. □

Lemma 4.14 ([15])**.** *The functors $X^{h,d}_\infty := \lim_n X^{h,d}_n$ and $G^{h,d}_\infty := \lim_n G^{h,d}_n$ are representable by affine $\mathbb{F}_p$-group schemes. Moreover, $G^{h,d}_\infty$ acts on $X^{h,d}_\infty$ and we have an equivalence of algebraic stacks $\left[X^{h,d}_\infty/G^{h,d}_\infty\right] \cong \mathrm{Disp}$.*

Proposition 4.15. *The natural morphism $\mathrm{DM}(\tau, \mathrm{Disp}) \to \mathrm{DM}(\mathrm{Disp})$ induces an equivalence of DG-categories.*

Proof. Recall that $\mathrm{Disp}^{h,d}_n \cong \left[X^{h,d}_n/G^{h,d}_n\right]$. Let us consider the notation of Lemma 4.14. The pro-algebraic group scheme $G^{h,d}_\infty$ acts on each $X^{h,d}_n$ via restriction for each $n \in \mathbb{N}$.

Note that we have for any $1 \le n \le m \le \infty$ a cartesian diagram

$$\begin{array}{ccc} X^{h,d}_m & \longrightarrow & X^{h,d}_n \\ \downarrow & & \downarrow \\ \left[X^{h,d}_m/G^{h,d}_\infty\right] & \longrightarrow & \left[X^{h,d}_n/G^{h,d}_\infty\right]. \end{array} \tag{2}$$

[23, Cor. A.13]. In particular, the induced map

$$\lim_n X^{h,d}_n \cong X^{h,d}_\infty \to \left[X^{h,d}_\infty/G^{h,d}_\infty\right]$$

is an effective epimorphism of étale stacks and the associated Čech nerve is given by base change of the Čech nerve of $X^{h,d}_1 \to \left[X^{h,d}_1/G^{h,d}_\infty\right]$.

Each transition map $\left[X^{h,d}_{n+1}/G^{h,d}_\infty\right] \to \left[X^{h,d}_{n}/G^{h,d}_\infty\right]$ is an affine bundle as explained in the proof of Proposition 4.13. We also note that $G^{h,d}_\infty$ is an affine faithfully flat group scheme.

We have

$$\mathrm{DM}([X^{h,d}_n/G^{h,d}_\infty]) \simeq \lim_n(\mathrm{DM}(X^{h,d}_n) \rightrightarrows \mathrm{DM}(X^{h,d}_n \times_{\mathbb{F}_p} G^{h,d}_\infty) \substack{\rightarrow\\\rightarrow\\\rightarrow} ...),$$

for all $1 \leq n \leq \infty$ [21, Thm. 2.2.16]. Having the cartesian diagram (2) in mind, we can now repeat the proof of Proposition 2.13 and obtain

$$\mathrm{DM}(\left[X^{h,d}_\infty/G^{h,d}_\infty\right]) \simeq \operatorname*{colim}_n \mathrm{DM}(\left[X^{h,d}_n/G^{h,d}_\infty\right]).$$

Finally, by [21, Prop. 2.2.11], we have

$$\mathrm{DM}(\left[X^{h,d}_n/G^{h,d}_\infty\right]) \simeq \mathrm{DM}(\left[X^{h,d}_n/G^{h,d}_n\right])$$

concluding the proof. □

Theorem 4.16. *The pullback along the truncation map* $\mathrm{Disp} \to \mathrm{Disp}_1$ *induces a fully faithful embedding.*

$$\mathrm{DM}(\mathrm{Disp}_1) \hookrightarrow \mathrm{DM}(\mathrm{Disp})$$

In particular, we have an equivalence of motives

$$M(\mathrm{Disp}) \simeq M(\mathrm{Disp}_1)$$

inside $\mathrm{DM}(\mathbb{F}_p)$ *and moreover it is contained in the full stable cocomplete subcategory of* $\mathrm{DM}(\mathbb{F}_p)$ *generated by Tate motives.*

Proof. By Proposition 4.13 the natural map $\mathrm{DM}(\mathrm{Disp}_1) \to \mathrm{DM}(\tau, \mathrm{Disp})$ is fully faithful. By Proposition 4.15 the natural map $\mathrm{DM}(\tau, \mathrm{Disp}) \to \mathrm{DM}(\mathrm{Disp})$ is an equivalence. This proves that $\mathrm{DM}(\mathrm{Disp}_1) \to \mathrm{DM}(\mathrm{Disp})$ is fully faithful and $M(\mathrm{Disp}) \simeq M(\mathrm{Disp}_1)$. The fact, that this motive is Tate follows[4] from [24]. □

Remark 4.17. Brokemper computes the Chow groups of $\mathrm{Disp}^{h,d}_1$ explicitly in [5, Thm. 2.3.3]. Thus, Theorem 4.16 and Remark 2.15 tell us that the motivic cohomology of Disp can be computed by the Chow groups of $\mathrm{Disp}^{h,d}_1$. To be more precise, we can compute

$$\bigoplus_{n\in\mathbb{Z}} H^{n,2n}(\mathrm{Disp}, \mathbb{Q}) \cong \bigoplus_{0\leq d\leq h} \mathbb{Q}[t_1, \dots, t_h]^{S_h \times S_{h-d}}/(c_1, \dots, c_h),$$

where c_i denotes the i-th elementary symmetric polynomial in variables $t_1, \dots, t_h$.

Remark 4.18. Let BT be the stack of Barsotti-Tate groups (short, BT-groups) over S. For any $n \in \mathbb{N}$, we denote by BT_n the stack of level-n BT-groups. For any $n \geq 0$ there exists a truncation map $\tau_n\colon \mathrm{BT}_{n+1} \to \mathrm{BT}_n$ and $\lim_{n,\tau} \mathrm{BT}_n \simeq \mathrm{BT}$. The maps τ_n are smooth (cf. [11, Thm. 4.4]). Note that every level-n BT-group admits the notion of a height and dimension, which are locally constant functions over S. In particular, we can write $\mathrm{BT}_n \simeq \coprod_{0\leq d\leq h} \mathrm{BT}^{h,d}_n$, where $\mathrm{BT}^{h,d}_n$ denotes the substack of BT generated by BT-groups of dimension d and height h.

We want to remark, that there is a morphism $\phi\colon \mathrm{BT} \to \mathrm{Disp}$ compatible with truncations (cf. [14]). Moreover, each of the maps ϕ_n is smooth and an equivalence on geometric points (cf.

[4]Note that Disp_1 is isomorphic to the stack of F-zips, considered in [24].

[14, Thm. A]). In particular, ϕ_n is a universal homeomorphism. Note however, that this is not enough to see that $\mathrm{DM}(\tau, \mathrm{Disp}) \simeq \mathrm{DM}(\tau, \mathrm{BT})$ as the morphisms ϕ_n are not representable. We, conjecture that $\phi^*\colon \mathrm{DM}(\mathrm{Disp}) \to \mathrm{DM}(\mathrm{BT})$ restricts to Tate-motives proving that the underlying motive of BT and Disp are equivalent. However, as pro-algebraic stacks the motivic cohomology groups of BT and Disp are isomorphic.

Corollary 4.19. *We have an isomorphism of motivic cohomology groups*

$$H^{*,*}(\mathrm{BT}, \mathbb{Q}) \cong H^{*,*}(\mathrm{Disp}, \mathbb{Q}) \cong H^{*,*}(\mathrm{Disp}_1, \mathbb{Q}).$$

Proof. This follows form the definition of the motivic cohomology for the pro-algebraic stack (τ, BT), Theorem 4.16 and [5, Thm. 2.5.4]. □

4.2 Action of the absolute Galois group on the motivic homology spectrum. Let us come back to the setting of Example 2.7 and fix the setting for this subsection. So as in the example, let L/K be a Galois extension of fields. Let X be a smooth K-scheme. We denote by $\mathcal{F}_K^L$ the filtered category of all finite extensions of K contained in L. The inclusion along finite extensions $E \hookrightarrow F$ of K yield pro-$\mathcal{F}_K^L$-algebraic stacks (x, X_L) and (q, L). The base change of the structure map $f\colon X \to \mathrm{Spec}(K)$ to $E \in \mathcal{F}_K^L$ yields a map of diagrams

$$f_\bullet\colon (x, X_L) \to (q, L)$$

Let $\mathcal{G}_K^L$ denote the filtered subcategory of all finite Galois extensions of K. Then $\mathcal{G}_K^L \subseteq \mathcal{F}_K^L$ is cofinal. In particular, we see that

$$\mathrm{DM}(x, X_L) \simeq \operatorname*{colim}_{E \in \mathcal{G}_K^L, x^*} \mathrm{DM}(x, X_E).$$

Thus, to understand the action of $\mathrm{Gal}(L/K)$ on $f_{L*}1_{(x,X_L)}$, we may restrict (x, X_L) and (q, L) to $\mathcal{G}_K^L$ and work with their underlying pro-$\mathcal{G}_K^L$-algebraic stacks. By abuse of notation, we will keep the notation of (x, X_L) and (q, L).

We naturally get a map $g\colon (q, \mathrm{Spec}(L)) \to \mathrm{Spec}(K)$, where we view $\mathrm{Spec}(K)$ as a constant diagram. We denote the composition $g \circ f_\bullet$ by $h_\bullet$.

Remark 4.20. Note that Proposition 2.13 shows that

$$\mathrm{DM}(x, X_L) \simeq \mathrm{DM}(X_L) \text{ and } \mathrm{DM}(q, L) \simeq \mathrm{DM}(L)$$

We will see in the next lemma that our formalism enables us to compute the motive $f_{L*}1_{(x,X_L)}$ by "pulling back" f_*1_X along the natural map $\mathrm{DM}(K) \to \mathrm{DM}(q, L)$.

Assume that L/K is a *finite* extension. Then the smooth base change immediately yields

$$f_{L*}1_{(x,X_L)} \simeq f_{L*}x^*_{L/K}1_X \simeq q^*_{L/K}f_*1_X.$$

If L/K is not finite this does not hold as there is no base change formalism that shows this result. In our formalism we have an analog of a smooth base change in this case as in Remark 3.21.

Next, we claim that $\mathrm{Gal}(L/K)$ acts on $h_*1_{(x,X_L)}$. To see this, it is enough to construct actions of $\mathrm{Gal}(E/K)$ on $h_*1_{(x,X_L)}$ for any finite Galois extension E/K that is compatible with the presentation of $\mathrm{Gal}(L/K)$ as a limit of such.

Remark 4.21. Let E/K be a finite Galois extension and $\varphi \in \mathrm{Aut}_K(E)$. And $M \in \mathrm{DM}(q, \mathrm{Spec}(L))$. Then φ induces an automorphism

$$g_*M \to g_*M$$

via the following.

Let $\mathcal{G}_E^L \subseteq \mathcal{G}_K^L$ denote the subcategory of finite Galois extensions over E. Then we can restrict q along this (filtered) subcategory and we denote the induced pro-$\mathcal{G}_E^L$-algebraic stack by $(q_{|E}, \mathrm{Spec}(L))$. The K-automorphism φ induces via base change a cartesian automorphism $\varphi\colon (q_{|E}, \mathrm{Spec}(L)) \to (q_{|E}, \mathrm{Spec}(L))$. The $*$-pushforward along φ yields a functor

$$\varphi_*\colon \mathrm{DM}(q, \mathrm{Spec}(L)) \to \mathrm{DM}(q, \mathrm{Spec}(L)).$$

By construction $g_*\varphi_* \simeq g_*$, yielding the desired endomorphism above.

This endomorphism by construction induces an action of $\mathrm{Gal}(L/K)$ on g_*M, i.e. a map $B\,\mathrm{Gal}(L/K) \to \mathrm{DM}(K)$.

Remark 4.22. Combining the above, we have

$$\operatorname*{colim}_{E \in \mathcal{G}_K^L} h_* 1_X^{\mathrm{Gal}(E/K)} \simeq \operatorname*{colim}_{E \in \mathcal{G}_K^L} \operatorname*{colim}_{F \in \mathcal{G}_E^L} q_{F/K*} f_{F*} 1_{X_F}^{\mathrm{Gal}(E/K)} \simeq f_* 1_X,$$

where we use [1, Lem. 2.1.166] in the first equivalence. In particular, for a X smooth K-scheme, we have

$$A^n(X, m)_{\mathbb{Q}} \simeq A^n(X_L, m)_{\mathbb{Q}}^{\mathrm{Gal}(L/K)}$$

by Example 4.7. This recovers the computation on the level of cycles [5, Lem. 1.3.6].

There exists a notion of continuous homotopy fixed points on so-called discrete G-spectra [2]. The object $\mathrm{colim}_{E \in \mathcal{G}_K^L} h_* 1_X^{\mathrm{Gal}(E/K)}$ can be seen as an analog of continuous homotopy fixed points under the action of the absolute Galois group. By construction the action of $\mathrm{Gal}(L/K)$ is stabilized on an open and closed normal subgroup after passage to a large enough field extension.

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