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Configuration spaces form a Segal semi-dendroidal space

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Abstract

The purpose of this short note is to illustrate the utility of (semi-) dendroidal objects in describing certain 'up-to-homotopy' operads. Specifically, we exhibit a semi-dendroidal space satisfying the Segal condition, whose evaluation at a k-corolla is the space of ordered configurations of k points in the n-dimensional unit ball.

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By forgetting radii, the kth space of the little n-disks operad is homotopy equivalent to the configuration space of k ordered points in the unit n-disk. The latter collection of spaces (as k varies) does not admit the structure of an operad, but in light of this homotopy equivalence should admit the structure of an up-to-homotopy operad. Corollary 5 gives one way to make this precise using *semi-dendroidal spaces* satisfying a *Segal condition*.

The reader should recall the dendroidal category Ω from [4, 5]; the former reference provides a good overview of basic dendroidal theory, and we will often adopt the same notation in this work. We let Ω_{inj} be the wide subcategory generated by isomorphisms and face maps. A space-valued presheaf on Ω_{inj} (resp. Ω) will be called a semi-dendroidal (resp. dendroidal) space. A planar structure on a (rooted) tree is an assignment, for each tree T and each vertex $v \in T$, a bijection $b_v : \{1, \dots, k_v\} \to in(v)$. For notational reasons it will be convenient to assume that any given tree comes equipped with a planar structure, though we will not require maps to preserve this extra structure (so we are actually working with the equivalent category which was called Ω' in Example 2.8 of [1]). Given any colored (symmetric) operad $\mathscr{O}$ in topological spaces, there is an associated dendroidal space (the *dendroidal nerve*) $N_d(\mathscr{O})$ with $N_d(\mathscr{O})_T = Oper(\Omega(T), \mathscr{O})$. A point in $N_d(\mathscr{O})_T$ may be identified with a pair (f_0, f_1), where $f_0 : E(T) \to col(\mathscr{O})$ is a function and f_1 assigns to each vertex v of T a point of $\mathscr{O}(f_0 b_v(1), \dots, f_0 b_v(k_v); f_0(out(v)))$.

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Let $\mathscr{O}$ be a 2-colored operad in the category of topological spaces; we write $\{1,2\}$ as the color set of $\mathscr{O}$. We ask that $\mathscr{O}$ satisfies

$$\mathscr{O}(\boldsymbol{\ell};2) = \begin{cases} * & \text{if } \boldsymbol{\ell} = 2 \\ \varnothing & \text{otherwise.} \end{cases} \tag{1}$$

Here we are using the notation $\boldsymbol{\ell} = l_1 l_2 \dots l_p$ with $l_i \in \{1,2\}$ and $|\boldsymbol{\ell}| = p \geq 0$ for (ordered) lists in the set $\{1,2\}$. Given two such lists $\boldsymbol{\ell} = l_1 l_2 \dots l_p$ and $\boldsymbol{\ell}' = l_1' l_2' \dots l_q'$, write $\boldsymbol{\ell} \circ_i \boldsymbol{\ell}' = l_1 \dots l_{i-1} l_1' \dots l_q' l_{i+1} \dots l_p$. There is a natural partial order on the set of lists of a fixed length, given by entrywise comparison: $l_1 \dots l_p \leq l_1' \dots l_p'$ if and only if $l_i \leq l_i'$ for all i. We will write $\mathbf{m}^p$ (resp. $\mathbf{M}^p$) for the list of length p with every entry 1 (resp. 2). The operad $\mathscr{O}$ should come equipped with a collection of weak homotopy equivalences $\mathfrak{s}_{\boldsymbol{\ell},\boldsymbol{\ell}'} : \mathscr{O}(\boldsymbol{\ell};1) \to \mathscr{O}(\boldsymbol{\ell}';1)$ whenever $\boldsymbol{\ell} \leq \boldsymbol{\ell}'$. These maps should respect the partial order, that is

$$\mathfrak{s}_{\boldsymbol{\ell}',\boldsymbol{\ell}''} \circ \mathfrak{s}_{\boldsymbol{\ell},\boldsymbol{\ell}'} = \mathfrak{s}_{\boldsymbol{\ell},\boldsymbol{\ell}''} \qquad \& \qquad \mathfrak{s}_{\boldsymbol{\ell},\boldsymbol{\ell}} = \mathrm{id}_{\mathscr{O}(\boldsymbol{\ell};1)} .$$

Further, these should be compatible with the operad structure, in the sense that

$$(\mathfrak{s}_{\boldsymbol{\ell},\boldsymbol{\ell}'} x) \circ_i (\mathfrak{s}_{\boldsymbol{\ell}'',\boldsymbol{\ell}'''} y) = \mathfrak{s}_{\boldsymbol{\ell}\circ_i\boldsymbol{\ell}'',\boldsymbol{\ell}'\circ_i\boldsymbol{\ell}'''}(x \circ_i y) \tag{2}$$

whenever both sides are defined (that is, whenever $\boldsymbol{\ell} \leq \boldsymbol{\ell}'$, $\boldsymbol{\ell}'' \leq \boldsymbol{\ell}'''$, and $l_i = l_i' = 1$) and $\sigma^* \circ \mathfrak{s}_{\boldsymbol{\ell},\boldsymbol{\ell}'} = \mathfrak{s}_{\boldsymbol{\ell}\cdot\sigma,\boldsymbol{\ell}'\cdot\sigma} \circ \sigma^*$ for $\sigma \in \Sigma_p$. Such a 2-colored operad $\mathscr{O}$ along with the data $\{\mathfrak{s}_{\boldsymbol{\ell},\boldsymbol{\ell}'}\}$ shall be called an *operad with shifts*.

Example 1. Our main example is the *operad of points and little disks*. The spaces $\mathscr{O}(\boldsymbol{\ell};2)$ are determined by (1), while a point of $\mathscr{O}(\boldsymbol{\ell};1)$ consists of $|\boldsymbol{\ell}|$ pieces of data:

- If $l_i = 1$, an affine embedding of the open unit disk $\mathbb{D} \subset \mathbb{R}^n$ into $\mathbb{D}$; we will write $a_i : \mathbb{D} \to \mathbb{D}$ for the embedding and D_i for its image.
- If $l_i = 2$, a point $d_i \in \mathbb{D}$.

Writing $\overline{D}_i$ for the closure of D_i, these data should be a configuration in the sense that, unless $i = j$, we have $D_i \cap D_j = \varnothing$, $d_i \neq d_j$, and $d_i \notin \overline{D}_j$. In order to topologize $\mathscr{O}(\boldsymbol{\ell};1)$, notice that each affine map $a_i(t) = r_i t + c_i$ may be identified with a point $(r_i, c_i) \in \mathbb{R}_{>0} \times \mathbb{R}^n$, while $d_j \in \mathbb{D} \subset \mathbb{R}^n$. We thus regard $\mathscr{O}(\boldsymbol{\ell};1)$ as a subspace of $\mathbb{R}^{p_1(n+1)+p_2 n}$, where $p_j = |\{i \,|\, l_i = j\}|$. The operad structure is a variation on the usual one for the little n-disks operad: when $l_i = 1$, the map

$$\circ_i : \mathscr{O}(\boldsymbol{\ell};1) \times \mathscr{O}(\boldsymbol{\ell}';1) \to \mathscr{O}(\boldsymbol{\ell} \circ_i \boldsymbol{\ell}';1)$$

is given on a pair $(\mathbf{x},\mathbf{y})$ by applying the affine transformation $x_i = a_i : \mathbb{D} \to \mathbb{D}$ to all of the disks and points that constitute $\mathbf{y}$, to end up with

$$\mathbf{x} \circ_i \mathbf{y} = x_1, \dots, x_{i-1}, a_i y_1, \dots, a_i y_{|\boldsymbol{\ell}'|}, x_{i+1}, \dots, x_{|\boldsymbol{\ell}|}.$$

This is an operad with shifts: define $\mathfrak{s}_{\boldsymbol{\ell},\boldsymbol{\ell}'}(\mathbf{x}) = \mathbf{y}$ componentwise as follows. If $l_i = l_i'$ then set $y_i = x_i$. If $l_i < l_i'$, then $l_i = 1$ and $l_i' = 2$, so x_i is a disk embedding a_i, while y_i is meant to to be a point. In this case, we set $y_i = a_i(0)$, the center of the disk x_i. At the two extremes, we have that $\mathscr{O}(\mathbf{m}^k;1) = \mathscr{O}(1\overset{k}{\cdots}1;1)$ is the kth space of the usual little n-disks operad and $\mathscr{O}(\mathbf{M}^k;1) = \mathscr{O}(2\overset{k}{\cdots}2;1)$ is the configuration space of k points in $\mathbb{D}$.

To show that $\mathfrak{s}_{\boldsymbol{\ell},\boldsymbol{\ell}'}$ is a homotopy equivalence, it is enough do so when $\boldsymbol{\ell} < \boldsymbol{\ell}'$ with $l_i = l_i'$ for all i except a single i_0 where $1 = l_{i_0} < l_{i_0}' = 2$. Define a continuous $\epsilon : \mathscr{O}(\boldsymbol{\ell}';1) \to \mathbb{R}_{>0}$

by letting $\epsilon(\mathbf{y})$ be the minimum of the distances of $d_{i_0} = y_{i_0}$ to d_i ($i \neq i_0$), D_i, and $\partial\mathbb{D}$. Let $g : \mathscr{O}(\boldsymbol{\ell}'; 1) \to \mathscr{O}(\boldsymbol{\ell}; 1)$, $g(\mathbf{y}) = \mathbf{x}$ be the right inverse to $\mathfrak{s}_{\boldsymbol{\ell},\boldsymbol{\ell}'}$ which is given by $x_i = y_i$ for $i \neq i_0$, and x_{i_0} is the affine embedding $u \mapsto \frac{1}{2}\epsilon(\mathbf{y})u + d_{i_0}$. Writing $x_{i_0} = (u \mapsto r_{\mathbf{x}} u + c_{\mathbf{x}})$, define

$$H_t(\mathbf{x}) = x_1, \dots, x_{i_0-1}, \left(u \mapsto \left[\left(tr_{\mathbf{x}} + (1-t)\frac{1}{2}\epsilon(\mathfrak{s}_{\boldsymbol{\ell},\boldsymbol{\ell}'}(\mathbf{x}))\right)u + c_{\mathbf{x}}\right]\right), x_{i_0+1}, \dots, x_{|\boldsymbol{\ell}|}$$

which is a homotopy from $g \circ \mathfrak{s}_{\boldsymbol{\ell},\boldsymbol{\ell}'}$ to the identity of $\mathscr{O}(\boldsymbol{\ell}; 1)$.

Given any operad with shifts $(\mathscr{O}, \mathfrak{s}_{\boldsymbol{\ell},\boldsymbol{\ell}'})$, we will presently define an associated semi-dendroidal space X. Let $X_\eta = *$. The most concise description of the spaces X_T, $T \neq \eta$, are as subspaces of $N_d(\mathscr{O})_T = Oper(\Omega(T), \mathscr{O})$. We say that a colored operad map $\Omega(T) \to \mathscr{O}$ is in X_T if and only if the map on color sets $E(T) \to \{1, 2\}$ sends all of the leaves to 2, the root to 1, and all of the internal edges to 1. Define two subspaces X_T^L and X_T^{IR} of $N_d(\mathscr{O})_T$. The space X_T^L is the subspace consisting of those maps which send all leaves to 2, and X_T^{IR} is the subspace consisting of those maps which send the root and all internal edges to 1. If $T \neq \eta$, then $X_T = X_T^L \cap X_T^{IR}$. For use in later equations, we will write $\mathfrak{i}' = \mathfrak{i}'_T : X_T^L \to N_d(\mathscr{O})_T$, $\mathfrak{i}'' = \mathfrak{i}''_T : X_T \to X_T^L$, and $\mathfrak{i} = \mathfrak{i}' \circ \mathfrak{i}'' = \mathfrak{i}_T : X_T \to N_d(\mathscr{O})_T$ for the inclusions.

For each non-trivial tree T, there is a map $\mathfrak{s}_T : N_d(\mathscr{O})_T \to X_T^L$. Fix a planar structure on T, which allows us to identify $N_d(\mathscr{O})_T$ with the space of pairs (f_0, f_1) as in the first paragraph. Writing $\mathfrak{s}_T(f_0, f_1) = (f'_0, f'_1)$, we first set $f'_0(e) = 2$ if e is a leaf and $f'_0(e) = f_0(e)$ otherwise. If v is a vertex, write $\boldsymbol{\ell}_v = f_0 b_v 1, \dots f_0 b_v k_v$ and $\boldsymbol{\ell}'_v = f'_0 b_v 1, \dots, f'_0 b_v k_v$. We have $\boldsymbol{\ell}_v \leq \boldsymbol{\ell}'_v$ for all v, so we can define $f'_1(v) = \mathfrak{s}_{\boldsymbol{\ell}_v,\boldsymbol{\ell}'_v} f_1(v)$.

Let $\alpha : S \to T$ be a map of Ω satisfying the following property: if v is a vertex of S so that $\alpha(v)$ is an edge e of T, then e is not a leaf edge. Note that this property is not closed under composition, but every map in Ω_{inj} satisfies it. If α is such a map and $S \neq \eta$, then the composite $X_T^{IR} \hookrightarrow N_d(\mathscr{O})_T \xrightarrow{\alpha^*} N_d(\mathscr{O})_S \xrightarrow{\mathfrak{s}_S} X_S^L$ actually lands in the subspace $X_S \subseteq X_S^L$. To distinguish from the operator α^* in the dendroidal nerve, we will write $\hat{\alpha} : X_T \to X_S$ for the map defined by $\mathfrak{i}''_S \hat{\alpha} = \mathfrak{s}_S \alpha^* \mathfrak{i}_T$ (equivalently $\mathfrak{i}\hat{\alpha} = \mathfrak{i}'\mathfrak{s}\alpha^*\mathfrak{i}$). If $S = \eta$, then $\alpha : \eta \to T$ automatically satisfies the indicated property; we will write $\hat{\alpha}$ for the unique function $X_T \to X_\eta = *$.

We wish to show that the relations which hold among maps in Ω_{inj} also hold among the hat-maps. For this purpose it would be enough to show that hat anticommutes with composition, which we can show in several cases.

Lemma 2. *Let $\alpha : S \to T$ and $\beta : R \to S$ be two maps of Ω_{inj} so that α induces a bijection on leaves. Then $\hat{\beta}\hat{\alpha} = \widehat{\alpha \circ \beta}$. The same equality holds if α is an arbitrary map of Ω_{inj} and β is an isomorphism.*

Proof. Since α sends leaves to leaves, $\alpha^*\mathfrak{i} : X_T \to N_d(\mathscr{O})_T \to N_d(\mathscr{O})_S$ already lands in the subspace X_S^L, and on this subspace $\mathfrak{i}'\mathfrak{s}(x) = x$. Thus $\alpha^*\mathfrak{i} = \mathfrak{i}'\mathfrak{s}\alpha^*\mathfrak{i} = \mathfrak{i}\hat{\alpha}$. This equality implies the second in $\mathfrak{i}\hat{\beta}\hat{\alpha} = \mathfrak{i}'\mathfrak{s}\beta^*\mathfrak{i}\hat{\alpha} = \mathfrak{i}'\mathfrak{s}\beta^*\alpha^*\mathfrak{i} = \mathfrak{i}'\mathfrak{s}(\alpha \circ \beta)^*\mathfrak{i} = \mathfrak{i}\widehat{\alpha \circ \beta}$. Since $\mathfrak{i}$ is an injection, the conclusion follows.

Let us address the second statement. Write $\beta_L^* : X_S^L \to X_R^L$ for the restriction of β_*. It is immediate that $\mathfrak{i}'_R \beta_L^* = \beta^* \mathfrak{i}'_S$. Since $\sigma^* \circ \mathfrak{s}_{\boldsymbol{\ell},\boldsymbol{\ell}'} = \mathfrak{s}_{\boldsymbol{\ell}\cdot\sigma,\boldsymbol{\ell}'\cdot\sigma} \circ \sigma^*$ whenever $\sigma \in \Sigma_p$ (see page 235), we have $\mathfrak{s}_S \beta^* = \beta_L^* \mathfrak{s}_R$. As in the first paragraph, $\beta^*\mathfrak{i} = \mathfrak{i}\hat{\beta}$ because β sends leaves to leaves. Putting these three facts together, we have $\mathfrak{i}\widehat{\alpha \circ \beta} = \mathfrak{i}'\mathfrak{s}(\alpha \circ \beta)^*\mathfrak{i} = \mathfrak{i}'\mathfrak{s}\beta^*\alpha^*\mathfrak{i} = \mathfrak{i}'\beta_L^*\mathfrak{s}\alpha^*\mathfrak{i} = \beta^*\mathfrak{i}'\mathfrak{s}\alpha^*\mathfrak{i} = \beta^*\mathfrak{i}\hat{\alpha} = \mathfrak{i}\hat{\beta}\hat{\alpha}$, hence $\hat{\beta}\hat{\alpha} = \widehat{\alpha \circ \beta}$. □

Theorem 3. *The collection $\{X_T\}$ together with the operators $\hat{\alpha}$ for $\alpha \in \Omega_{\mathrm{inj}}$ constitute a semi-dendroidal space.*

Proof. It is enough to show, given a commutative diagram $\begin{tikzcd} T_0 \arrow[r, "\delta"] \arrow[d, "\partial'"] & T_1 \arrow[d, "\partial"] \\ T_2 \arrow[r, "\delta'"] & T_3 \end{tikzcd}$ where ∂', ∂ are face maps and δ, δ' are either both face maps or isomorphisms, that $\hat{\delta}\hat{\partial} = \hat{\partial}'\hat{\delta}'$. This certainly follows if $\hat{\delta}\hat{\partial} = \widehat{\partial \circ \delta}$ and $\hat{\partial}'\hat{\delta}' = \widehat{\delta' \circ \partial'}$. Several cases of these equalities have been established in Lemma 2; note also that they are both obvious if $T_0 = \eta$ since $X_\eta = *$. We will now sweep up the few remaining cases.

Consider a composition $T_0 \xrightarrow{\delta} T_1 \xrightarrow{\partial_{v_0}} T_2$ of face maps with $T_0 \neq \eta$ and ∂_{v_0} an outer face map which chops off a vertex v_0 whose incoming edges are leaves. Since $|T_2| > 1$, the output of v_0 is not the root edge, so $e_0 = out(v_0)$ is the i_0th input of some other vertex w_0.

Fix an arbitrary $(f_0, f_1) \in X_{T_2} \subseteq N_d(\mathscr{O})_{T_2}$ and write $\mathfrak{s}_{T_1}\partial^*_{v_0}(f_0, f_1) = (g_0, g_1)$, $\mathfrak{s}_{T_0}\delta^*(g_0, g_1) = (h_0, h_1)$, and $\mathfrak{s}_{T_0}(\partial_{v_0} \circ \delta)^*(f_0, f_1) = (h'_0, h'_1)$. Since (h_0, h_1) and (h'_0, h'_1) are both in X_{T_0}, we know that $h_0 = h'_0$ (both send all leaves to 2 and all other edges to 1). Showing $(h_0, h_1) = (h'_0, h'_1)$ is the same as showing $h_1 = h'_1$, and we split this task into several cases. Notice that since ∂_{v_0} is external, we can immediately compute g_1:

$$g_1(v) = \begin{cases} f_1(v) & v \neq w_0 \\ \mathfrak{s}_{\boldsymbol{\ell},\boldsymbol{\ell}'} f_1(w_0) & v = w_0 \end{cases} \tag{3}$$

where $\boldsymbol{\ell}$ and $\boldsymbol{\ell}'$ are identical except at entry i_0. It will be convenient to write $\mathfrak{s}_{\boldsymbol{\ell},\boldsymbol{\ell}'}$ as $\mathfrak{s}^{i_0}$, indicating which entry has changed. More generally, given $\boldsymbol{\ell} \leq \boldsymbol{\ell}'$, let I be the set so that $i \in I$ if and only if $l_i < l'_i$ and write $\mathfrak{s}_{\boldsymbol{\ell},\boldsymbol{\ell}'} = \mathfrak{s}^I$. We can then rewrite (3) using the planar structure as $g_1(v) = \mathfrak{s}^{b_v^{-1}(e_0)} f_1(v)$.

In all three cases below, write $(\partial_{v_0} \circ \delta)^*(f_0, f_1) = (\tilde{h}_0, \tilde{h}_1)$.

Case 1: δ is external at the root. As in the proof of Lemma 2, $(h_0, h_1) = \delta^*(g_0, g_1)$ since δ is a bijection on leaves. Further, since δ is external, h_1 is a restriction of g_1. Since ∂_{v_0} and δ are external, we have $\tilde{h}_1(v) = f_1(v)$ whenever the left hand side is defined. Calculating $h'_1(v)$, we see that it is just $f_1(v)$ unless $v = w_0$, in which case we have $h'_1(w_0) = \mathfrak{s}_{\boldsymbol{\ell},\boldsymbol{\ell}'} f_1(w_0)$. Thus $h'_1(v) = g_1(v) = h_1(v)$.

Case 2: δ is external at a leaf vertex v_1. Let e_1 be the output edge of v_1. As above, $\tilde{h}_1$ is just a restriction of f_1. We have $h'_1(v) = \mathfrak{s}^{b_v^{-1}\{e_0,e_1\}} f_1(v) = \mathfrak{s}^{b_v^{-1}(e_1)} \mathfrak{s}^{b_v^{-1}(e_0)} f_1(v) = \mathfrak{s}^{b_v^{-1}(e_1)} g_1(v) = h_1(v)$, so $h_1 = h'_1$.

Case 3: δ is internal at an edge e_1. As in the proof of Lemma 2, $(h_0, h_1) = \delta^*(g_0, g_1)$ since δ is a bijection on leaves. We will write v_1 and w_1 for the two vertices that e_1 connects, with $e_1 = out(v_1)$ and $h_{w_1}(i_1) = e_1 \in in(w_1)$. It is possible that $w_0 = v_1$ or $w_0 = w_1$. Write $V(T_0) = \{\bar{v}\} \sqcup V(T_1) \setminus \{v_1, w_1\}$ with $\delta(\bar{v}) = w_1 \circ_{e_1} v_1$. We have that $\tilde{h}_1(\bar{v}) = f_1(w_1) \circ_{e_1} f_1(v_1)$ and otherwise $\tilde{h}_1(v) = f_1(v)$. Then

$$h'_1(\bar{v}) = \mathfrak{s}^{b_{\bar{v}}^{-1}(e_0)}(f_1(w_1) \circ_{i_1} f_1(v_1)) \overset{(2)}{=} [\mathfrak{s}^{b_{w_1}^{-1}(e_0)} f_1(w_1)] \circ_{i_1} [\mathfrak{s}^{b_{v_1}^{-1}(e_0)} f_1(v_1)] = g_1(w_1) \circ_{i_1} g_1(v_1) = h_1(\bar{v})$$

and otherwise $h'_1(v) = \mathfrak{s}^{b_{\bar{v}}^{-1}(e_0)} f_1(v) = g_1(v) = h_1(v)$. Thus $h_1 = h'_1$. □

Given any (semi-)dendroidal space Z, there is the *Segal map*,

$$Z_T \to \prod_{v \in T} Z_{C_v}$$

induced by the corolla inclusions $C_v \to T$ as v ranges over all vertices of T. If Z is the dendroidal nerve of a one-colored operad, then the Segal map is an isomorphism. It is interesting to weaken this and ask that the Segal map is merely a weak equivalence – as in [2, Theorem 1.1] and [3, Section 9] we expect this notion to be closely related to one-colored topological operads.

Theorem 4. *Suppose that $\mathscr{O}$ is an operad with shifts and X is the associated semi-dendroidal space. Then X satisfies the* Segal condition*, that is, for $T \neq \eta$, the Segal map $X_T \to \prod_{v\in T} X_{C_v}$ is a weak equivalence and $X_\eta = *$.*

Proof. By definition, X_η is a point. Let T be a nontrivial tree, and let $f : E(T) \to \{1, 2\}$ be the function which takes the leaves to 2, the internal edges to 1, and the root to 1. Write $\boldsymbol{\ell}_v = fb_v(1), \dots, fb_v(k_v)$. Then $X_T = \prod_{v\in T} \mathscr{O}(\boldsymbol{\ell}_v; 1)$. Suppose that $w \in T$ and let $\alpha : C_w \to T$ be the corolla inclusion. Then $\hat{\alpha} : X_T \to X_{C_w}$ is the composite

$$X_T = \prod_{v\in T} \mathscr{O}(\ell_v; 1) \xrightarrow{\pi_w} \mathscr{O}(\boldsymbol{\ell}_w; 1) \xrightarrow{\mathfrak{s}_{\ell_w,\mathbf{M}^{|w|}}} \mathscr{O}(\mathbf{M}^{|w|}; 1) = X_{C_w}.$$

Then the Segal map $X_T = \prod_v \mathscr{O}(\boldsymbol{\ell}_v; 1) \to \prod_v \mathscr{O}(\mathbf{M}^{|v|}; 1) = \prod_v X_{C_v}$ is the product of weak homotopy equivalences, hence a weak homotopy equivalence. □

Corollary 5. *There is a semi-dendroidal space X satisfying the Segal condition so that X_{C_k} is the configuration space of k points in $\mathbb{D}$.*

Proof. Apply the previous theorem to the operad with shifts given by configurations of points and disks in the unit disk. As we mentioned above, $X_{C_k} = \mathscr{O}(\mathbf{M}^k; 1)$ is the ordered configuration space of k points in the disk. □

It is natural to ask whether the semi-dendroidal space X admits the structure of a dendroidal space, that is, whether one can define degeneracy operators which are compatible with the existing face maps. The reader may have noticed that we have already defined $\hat{\alpha} : X_T \to X_S$ for many maps of Ω which were not in Ω_{inj}, including all degeneracy maps except those that are degenerate at a leaf. Further, the proof of Lemma 2 shows that many of the expected relations among faces and degeneracies hold with these definitions.

Nevertheless, we will now show that we cannot in general extend to a dendroidal structure.

Proposition 6. *The semi-dendroidal space X from Corollary 5 does not admit the structure of a dendroidal space.*

We devote the remainder of the paper to the proof of this proposition. We will prove this by looking at its underlying semi-simplicial space (also called X) and showing that no choice of degeneracy operators gives a simplicial space. For convenience, we will write points of X_k as lists $[a_1, \dots, a_{k-1}, P]$, where $a_1, \dots, a_{k-1} \in \mathscr{O}(1; 1) \subset (0, 1] \times \mathbb{D}$ and $P \in \mathscr{O}(2; 1) = \mathbb{D}$. Each a_i is an embedding $\mathbb{D} \to \mathbb{D}$ of the form $a_i(x) = r_i x + c_i$ where $c_i \in \mathbb{D} \subset \mathbb{R}^n$ and $r_i > 0$. For $k \geq 1$, the face maps are

$$d_i[a_1, \dots, a_{k-1}, P] = \begin{cases} [a_2, \dots, a_{k-1}, P] & i = 0 \\ [a_1, \dots, a_i \circ a_{i+1}, \dots, a_{k-1}, P] & 1 \leq i \leq k-2 \\ [a_1, \dots, a_{k-2}, a_{k-1}(P)] & i = k-1 \neq 0 \\ [a_1, \dots, a_{k-2}, a_{k-1}(\mathbf{0})] & i = k. \end{cases}$$

We now attempt to construct degeneracy operators in low degrees, and eventually show they cannot be chosen to satisfy all of the simplicial identities. We only need information about three of the degeneracy maps, namely $s_i : X_i \to X_{i+1}$ for $i = 0, 1, 2$. The map $s_0 : * = X_0 \to X_1 = \mathbb{D}$ just picks out a point, which we will call A for the moment. Let us examine $s_1 : X_1 \to X_2$. Since $d_0 s_1[P] = s_0 d_0[P] = [A]$, we have $s_1[P] = [r^P x + c^P, A]$, where $r : \mathbb{D} \to (0, 1]$ and $c : \mathbb{D} \to \mathbb{D}$ are continuous maps satisfying $0 < r^P \leq dist(c^P, \partial\mathbb{D})$ for all P. Since $[P] = d_2 s_1[P]$, we have $P = r^P \mathbf{0} + c^P = c^P$, hence c is the identity map on $\mathbb{D}$ and $s_1[P] = [r^P x + P, A]$. Finally, since $[P] = d_1 s_1[P] = [r^P A + P]$, we conclude that $A = \mathbf{0}$. Thus $s_0[\,] = [\mathbf{0}]$ and $s_1[P] = [r^P x + P, \mathbf{0}]$ for some unspecified function r^P.

We now turn to $s_2 : X_2 \to X_3$. We immediately know that $s_2[a, P]$ is of the form

$$s_2[a, P] = [a, R^{a,P} x + C^{a,P}, Q^{a,P}]$$

by examining the first term of $[a, P] = d_3 s_2[a, P]$. Here, $R : X_2 \to (0, 1]$, $C : X_2 \to \mathbb{D}$, and $Q : X_2 \to \mathbb{D}$ are continuous functions. The second term of $[a, P] = d_3 s_2[a, P]$ is $R^{a,P}\mathbf{0} + C^{a,P}$, so $C^{a,P} = P$. Comparing the second entries in

$$[a, P] = d_2 s_2[a, P] = d_2[a, R^{a,P} x + P, Q^{a,P}] = [a, R^{a,P} Q^{a,P} + P],$$

we see that $Q^{a,P} = \mathbf{0}$ since $R^{a,P}$ is never zero. Thus $s_2[a, P] = [a, R^{a,P} x + P, \mathbf{0}]$ for some function $R^{a,P}$.

In giving form to s_2, we used only the identities $d_3 s_2 = \mathrm{id} = d_2 s_2$. The main trouble is with the identity $d_1 s_2 = s_1 d_1$. We can calculate (writing $a(x) = r_1 x + c_1$)

$$d_1 s_2[a, P] = d_1[a, R^{a,P} x + P, \mathbf{0}] = [r_1 R^{a,P} x + r_1 P + c_1, \mathbf{0}]$$

and

$$s_1 d_1[a, P] = s_1[a(P)] = [r^{a(P)} x + a(P), \mathbf{0}].$$

If $d_1 s_2 = s_1 d_1$, then we would have $r_1 R^{a,P} = r^{a(P)}$ for all a and P. In the next paragraph, we will show that this is not possible.

Fix $P, c_1 \in \mathbb{D}$ and let $B = 1 - |c_1|$. Write $f(t) = R^{tx+c_1,P}$ and $g(t) = r^{tP+c_1}$, which are positive real-valued functions. The function f is defined on $(0, B]$ and is bounded $|f(t)| \leq 1$. The function g is defined on the closed interval $[0, B]$ and is continuous from the right at zero. Assuming $d_1 s_2 = s_1 d_1$ in the previous paragraph implies that $tf(t) = g(t)$ for all $t \in (0, B]$. Since f is bounded, the left hand side approaches 0 as t goes to zero, while the limit of the right hand side is $g(0) = r^{c_1} > 0$. This is a contradiction, hence $d_1 s_2 \neq s_1 d_1$.

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