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Parallel transport in principal 2-bundles

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Abstract

A nice differential-geometric framework for (non-abelian) higher gauge theory is provided by principal 2-bundles, i.e. categorified principal bundles. Their total spaces are Lie groupoids, local trivializations are kinds of Morita equivalences, and connections are Lie-2-algebra-valued 1-forms. In this article, we construct explicitly the parallel transport of a connection on a principal 2-bundle. Parallel transport along a path is a Morita equivalence between the fibres over the end points, and parallel transport along a surface is an intertwiner between Morita equivalences. We prove that our constructions fit into the general axiomatic framework for categorified parallel transport and surface holonomy.

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1. Introduction

Many different concrete models for 2-bundles (sometimes called categorified bundles or gerbes) have been developed so far. For most of them, there exists a notion of a connection. For some of them, it is proved that there exists a corresponding parallel transport along paths and surfaces. However, to my best knowledge, in none of these models the relation between the connection and the parallel transport is concretely realized. The aim of the present paper is to fill this gap by constructing the parallel transport in one of these models: principal 2-bundles.

Let me try to clarify some of above statements. First of all, our categorified bundles live over a smooth manifold M, and their structure group is a strict Lie 2-group. Familiar models of 2-bundles with connection are (non-abelian) bundle gerbes [5, 1], G-gerbes [4], (non-abelian) differential cocycles [3], and principal 2-bundles [14, 9, 13].

In joint work with Urs Schreiber [11], based on earlier work of Baez-Schreiber [2], we have developed a model-independent, axiomatic framework for the parallel transport of connections

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in categorified bundles, called "transport 2-functors". Such a transport 2-functor is a 2-functor

$$\mathrm{tra} : \mathcal{P}_2(M) \longrightarrow \mathcal{C},$$

where $\mathcal{P}_2(M)$ is the path-2-groupoid of M and $\mathcal{C}$ is some bicategory that depends on the model. The basic idea is that the objects of $\mathcal{P}_2(M)$ are the points $x \in M$, the morphisms are all smooth paths γ in M, and the 2-morphisms are fixed-ends homotopies Σ between paths ("bigons"). Then, $\mathrm{tra}(x)$ is the "fibre over x", $\mathrm{tra}(\gamma)$ is the parallel transport along the path γ, and $\mathrm{tra}(\Sigma)$ is the parallel transport along the surface Σ. The axioms of a 2-functor describe how parallel transport behaves under gluing and cutting of paths and surfaces. The most difficult aspect of this framework is to axiomatically characterize smoothness conditions for the transport 2-functor. This has been worked out in [11]. It was proved there that – after picking particular bicategories $\mathcal{C}$ – the bicategory of transport 2-functors is equivalent to several bicategories of above-mentioned models.

In all cases discussed in [11], these equivalences are given by spans of 2-functors which are in general not canonically invertible. This means, for instance, that not even for an abelian bundle gerbe with connection one can associate in a canonical way a transport 2-functor. In particular, there is no clear answer to the question, what the parallel transport of such a bundle gerbe along a path is. This is unsatisfying, in particular regarding the applications to higher gauge theory, where the parallel transport along a path constitutes the coupling of strings to gauge fields.

In the present paper we consider the model of principal 2-bundles and provide a solution to this problem. Principal 2-bundles have been introduced by Wockel [14] and further worked out by Schommer-Pries [9]. A principal 2-bundle is the most direct categorification of an ordinary principal bundle: its total space is a Lie groupoid on which a Lie 2-group Γ acts in a certain way making it fibre-wise principal. Morphisms between principal 2-bundles – in particular, local trivializations – are not smooth functors between Lie groupoids but a generalization called *anafunctor*, a kind of directed Morita equivalence. Connections on principal 2-bundles have recently been introduced in [13]. We recall the central definitions in Section 2. The main part of this article is to construct the parallel transport of these connections:

(1) If γ is a smooth path in M starting at x and ending at y, then we construct a Γ-equivariant anafunctor $F_\gamma : \mathcal{P}_x \longrightarrow \mathcal{P}_y$ between the fibres of $\mathcal{P}$ over these points. This is the content of Section 3.
(2) Suppose the connection is fake-flat. If Σ is a smooth fixed-ends homotopy between paths γ_1 and γ_2, then we construct a Γ-equivariant transformation $\varphi_\Sigma : F_{\gamma_1} \Longrightarrow F_{\gamma_2}$ between the anafunctors associated to the two paths. This is the content of Section 4.

In principle, constructions (1) and (2) are performed in a very similar way as for ordinary principal bundles. The basic idea is to lift paths and homotopies "horizontally" to the objects of the total space Lie groupoid. There are two main differences compared to ordinary principal bundles: horizontal lifts (i) exist only locally and (ii) are not uniquely determined by an initial condition. Local existence requires to compensate differences with structure on the morphisms of the total space Lie groupoid; this makes the whole construction more complex. Non-uniqueness requires to consider all possible horizontal lifts at one time; this forces us to consider anafunctors instead of ordinary functors. All these issues are carefully discussed and resolved in Sections 3 and 4. The following is the main result of this article.

Theorem 1.1. *Our constructions (1) and (2) of the parallel transport of a principal Γ-2-bundle fit into the axiomatic framework of transport 2-functors. This means:*

(a) For every principal Γ-2-bundle $\mathcal{P}$ with fake-flat connection the assignments $x \longmapsto \mathcal{P}_x$, $\gamma \longmapsto F_\gamma$, and $\Sigma \longmapsto \varphi_\Sigma$ form a transport 2-functor

$$\mathrm{tra}_{\mathcal{P}} : \mathcal{P}_2(M) \longrightarrow \Gamma\text{-}\mathcal{T}\!or$$

with target the bicategory of Γ-torsors.

(b) The assignment $\mathcal{P} \longmapsto \mathrm{tra}_{\mathcal{P}}$ is compatible with the bicategorical structure of principal Γ-2-bundles in the sense that it extends to a 2-functor

$$2\text{-}\mathcal{B}un_\Gamma^{\nabla ff}(M) \longrightarrow \mathrm{Trans}_\Gamma(M, \Gamma\text{-}\mathcal{T}\!or)$$

between the bicategories of principal Γ-2-bundles with fake-flat connections and the bicategory of transport 2-functors.

Theorem 1.1 is proved in Section 6 as Theorems 6.11 and 6.13. We will show in a forthcoming paper that the 2-functor in (b) is actually an equivalence of bicategories. This means that the model of principal Γ-2-bundles with connections comprises all aspects of categorified parallel transport.

This article is organized in a straightforward way. In Section 2 we offer a short review about principal 2-bundles and connections. This review covers all material sufficient to understand the statements and constructions of this article. By intention, understanding all details of the proofs might require to consult [13]. Therefore, all definitions, notations, and most symbols used in this article coincide with the corresponding ones in [13]. Sections 3 and 4 contain the constructions of the parallel transport, in Section 5 we reduce these constructions to two important subclasses of principal 2-bundles, and Section 6 contains the proof of our main result. In an appendix we summarize and slightly extend results of [10] about path-ordered and surface-ordered exponentials, which provide the "local" foundations for parallel transport.

Admittedly, some constructions and proofs we perform in this article are quite laborious. However, we believe that our results – once established – provide a rather complete and convenient "calculus" for categorified parallel transport in the well-established context of Lie groupoids.

2. Principal 2-bundles

We give a very short introduction to principal 2-bundles and connections. A comprehensive treatment is given in [13]. There is a bicategory $\mathcal{L}ie\mathcal{G}rpd$ whose objects are *Lie groupoids*, whose 1-morphisms are called *anafunctors* (a.k.a. bibundles, Hilsum-Skandalis maps, Morita equivalences,...), and whose 2-morphisms are called *transformations* (bibundle maps, intertwiners,...). Ordinary (smooth) functors form a proper subset among all anafunctors. Ordinary (smooth) natural transformations correspond to all transformations between functors. The purpose of enlarging the set of 1-morphisms from functors to anafunctors is to invert certain functors (called *weak equivalences*). One effect of this enlargement is that $\mathcal{L}ie\mathcal{G}rpd$ is equivalent to the bicategory of differential stacks [7].

In this paper, a *Lie 2-group* is a Lie groupoid whose objects and morphisms are equipped with Lie group structures, so that the structure maps are Lie group homomorphisms. Lie 2-groups are in one-to-one correspondence with *crossed modules* of Lie groups. Often this version of a Lie 2-group is called "strict". A *smooth right action* of a Lie 2-group Γ on a Lie groupoid $\mathcal{X}$ is a smooth functor $R : \mathcal{X} \times \Gamma \longrightarrow \mathcal{X}$ satisfying strictly the axioms of an action. Now, there is a new bicategory, whose objects are Lie groupoids equipped with smooth right Γ-actions, whose morphisms are Γ-equivariant anafunctors, and whose 2-morphisms are Γ-equivariant transformations.

Finally, we fix the following conventions. If X is a smooth manifold, we denote by X_{dis} the Lie groupoid with objects X and only identity morphisms. A smooth functor $\phi : \mathcal{X} \longrightarrow \mathcal{Y}$ is called surjective/submersive, if it is so on the level on objects.

Definition 2.1. Let M be a smooth manifold.

(a) A *principal* Γ*-2-bundle over* M is a Lie groupoid $\mathcal{P}$, a smooth, surjective and submersive functor $\pi : \mathcal{P} \longrightarrow M_{dis}$, and a smooth right action $R : \mathcal{P} \times \Gamma \longrightarrow \mathcal{P}$ such that $\pi \circ R = \pi \circ \mathrm{pr}_1$ and the smooth functor $(\mathrm{pr}_1, R) : \mathcal{P} \times \Gamma \longrightarrow \mathcal{P} \times_M \mathcal{P}$ is a weak equivalence.

(b) A *1-morphism* between principal Γ-2-bundles is a Γ-equivariant anafunctor $J : \mathcal{P}_1 \longrightarrow \mathcal{P}_2$ such that $\pi_2 \circ J = \pi_1$.

(c) A *2-morphism* between 1-morphisms is a Γ-equivariant transformation.

Principal Γ-2-bundles over M form a bigroupoid that we denote by $2\text{-}\mathcal{B}un_\Gamma(M)$. Moreover, the assignment $M \longmapsto 2\text{-}\mathcal{B}un_\Gamma(M)$ is a stack over the site of smooth manifolds [6, Theorem 6.2.1].

Remark 2.2. We describe some notation and technical features related to our 1-morphisms, which will be used later in the paper. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be principal Γ-2-bundles over M.

(a) The anafunctor underlying a 1-morphism $J : \mathcal{P}_1 \longrightarrow \mathcal{P}_2$ consists of a *total space* J, *anchor maps* $\alpha_l : J \longrightarrow \mathcal{P}_1$ and $\alpha_r : J \longrightarrow \mathcal{P}_2$, and commuting smooth groupoid actions $\rho_l : \mathrm{Mor}(\mathcal{P}_1)_s\times_{\alpha_l} J \longrightarrow J$ and $\rho_r : J_{\alpha_r}\times_t \mathrm{Mor}(\mathcal{P}_2) \longrightarrow J$, which we will often denote by $\rho \circ j$ and $j \circ \rho$, respectively. Its Γ-equivariance consists of a smooth right action $\rho : J \times \mathrm{Mor}(\Gamma) \longrightarrow J$, usually denoted by $j \cdot \gamma$, that is compatible with the groupoid actions in the sense that

$$R(\rho_1, \gamma_1) \circ (j \cdot \gamma) \circ R(\rho_2, \gamma_2) = (\rho_1 \circ j \circ \rho_2) \cdot (\gamma_1 \circ \gamma \circ \gamma_2)$$

whenever all compositions are defined, see [13, Definition 2.4.1 (b)].

(b) If $\phi : \mathcal{P}_1 \longrightarrow \mathcal{P}_2$ is a smooth functor that preserves the fibres and strictly commutes with the Γ-actions, then it induces a 1-morphism with total space $J_\phi := \mathrm{Obj}(\mathcal{P}_1)\,_\phi\times_t \mathrm{Mor}(\mathcal{P}_2)$, anchors $\alpha_l(p, \rho) := p$ and $\alpha_r(p, \rho) := s(\rho)$, groupoid actions $\eta \circ (p, \rho) := (t(\eta), \phi(\eta) \circ \rho)$ and $(p, \rho) \circ \eta := (p, \rho \circ \eta)$, and $\mathrm{Mor}(\Gamma)$-action $(p, \rho) \cdot \gamma := (R(p, t(\gamma)), R(\rho, \gamma))$, see [13, Remarks 2.3.3 (a) & 2.4.2 (b)].

(c) A smooth natural transformation $\eta : \phi \Longrightarrow \phi'$ induces a transformation $f_\eta : J_\phi \Longrightarrow J_{\phi'}$ by $f_\eta(p, \rho) := (x, \eta(p) \circ \rho)$. If η is Γ-equivariant then f_η is also Γ-equivariant, hence a 2-morphism, see [13, Remark 2.4.2 (b)].

(d) Let $\phi : \mathcal{P}_1 \longrightarrow \mathcal{P}_2$ be a smooth, fibre-preserving, Γ-equivariant functor, and let $J : \mathcal{P}_1 \longrightarrow \mathcal{P}_2$ be a 1-morphism. For a smooth map $\tilde{f} : \mathrm{Obj}(\mathcal{P}_1) \longrightarrow J$ we consider three conditions:

(T1) $\alpha_l(\tilde{f}_\gamma(p)) = p$ and $\alpha_r(\tilde{f}_\gamma(p)) = \phi(p)$

(T2) $\alpha \circ \tilde{f}_\gamma(p) \circ \beta = \tilde{f}_\gamma(t(\alpha)) \circ \phi(\alpha) \circ \beta$

(T3) $\tilde{f}_\gamma(R(p, g)) = \tilde{f}_\gamma(p) \cdot \mathrm{id}_g$.

There is a bijection between smooth maps $\tilde{f}$ satisfying (T1), (T2) and (T3) and 2-morphisms $f : J_\phi \Longrightarrow J$. This bijection is established by the relation $\tilde{f}(p) = f(p, \phi(\mathrm{id}_p))$, see [13, Remarks 2.3.3 (c) & 2.4.2 (b)].

Next we come to connections. If $\mathcal{X}$ is a Lie groupoid and γ is a Lie 2-algebra, then there is a differential graded-commutative Lie algebra $\Omega^*(\mathcal{X}, \gamma)$ of γ*-valued differential forms on* $\mathcal{X}$ [13, Section 4]. If $\phi : \mathcal{X} \longrightarrow \mathcal{Y}$ is a smooth functor, then there is a "pullback" Lie algebra homomorphism $\phi^* : \Omega^*(\mathcal{Y},\gamma) \longrightarrow \Omega^*(\mathcal{X},\gamma)$. If γ is the Lie 2-algebra of a Lie 2-group Γ, then there is an *adjoint action* of Γ on $\Omega^*(\mathcal{X},\gamma)$. Further, Γ carries a "Maurer-Cartan"-form $\Theta \in \Omega^1(\Gamma,\gamma)$.

Definition 2.3. A *connection* on a principal Γ-2-bundle $\mathcal{P}$ is a γ-valued 1-form $\Omega \in \Omega^1(\mathcal{P},\gamma)$ such that

$$R^*\Omega = \mathrm{Ad}^{-1}_{\mathrm{pr}_\Gamma}(\mathrm{pr}_{\mathcal{P}}^*\Omega) + \mathrm{pr}_\Gamma^*\Theta$$

over $\mathcal{P} \times \Gamma$, where $\mathrm{pr}_{\mathcal{P}}$ and pr_Γ are the projections to the two factors.

Let us spell out explicitly all structure and conditions that are packed into Definition 2.3. For this purpose, we assume that the Lie 2-group Γ is given as a crossed module (G,H,t,α), where $t : H \longrightarrow G$ is the Lie group homomorphism, and $\alpha : G \times H \longrightarrow H$ is the action of G on H. We will denote by $\alpha_g \in \mathrm{Aut}(H)$ the action of a fixed $g \in G$ on H, and for $h \in H$ we denote by $\tilde{\alpha}_h : G \longrightarrow H$ the map defined by $\tilde{\alpha}_h(g) := h^{-1}\alpha(g,h)$. The correspondence between Γ and (G,H,t,α) is $\mathrm{Obj}(\Gamma) = G$ and $\mathrm{Mor}(\Gamma) = H \ltimes_\alpha G$, with $s(h,g) = g$ and $t(h,g) = t(h)g$. The associated Lie 2-algebra is the crossed module $(\mathfrak{g},\mathfrak{h},t_*,\alpha_*)$, where $\mathfrak{g}$ and $\mathfrak{h}$ are the Lie algebras of G and H, respectively, and t_* and α_* are the differentials of t and α. Throughout the whole paper we will work in exactly this setting of crossed modules. We point to a formulary for calculations collected in [13, Appendix A], which we will eventually use without further mentioning.

Now, a connection Ω on a principal Γ-2-bundle $\mathcal{P}$ consists of three components $\Omega = (\Omega^a,\Omega^b,\Omega^c)$, which are ordinary differential forms:

$$\Omega^a \in \Omega^1(\mathrm{Obj}(\mathcal{P}),\mathfrak{g}) \quad , \quad \Omega^b \in \Omega^1(\mathrm{Mor}(\mathcal{P}),\mathfrak{h}) \quad \text{and} \quad \Omega^c \in \Omega^2(\mathrm{Obj}(\mathcal{P}),\mathfrak{h}).$$

These satisfy the following conditions:

$$R^*\Omega^a = \mathrm{Ad}_g^{-1}(p^*\Omega^a) + g^*\theta \qquad \text{over } \mathrm{Obj}(\mathcal{P}) \times \mathrm{Obj}(\Gamma) \tag{1}$$

$$R^*\Omega^b = (\alpha_{g^{-1}})_*\Big(\mathrm{Ad}_h^{-1}(p^*\Omega^b) + (\tilde{\alpha}_h)_*(p^*s^*\Omega^a) + h^*\theta\Big) \qquad \text{over } \mathrm{Mor}(\mathcal{P}) \times \mathrm{Mor}(\Gamma) \tag{2}$$

$$R^*\Omega^c = (\alpha_{g^{-1}})_*(p^*\Omega^c) \qquad \text{over } \mathrm{Obj}(\mathcal{P}) \times \mathrm{Obj}(\Gamma). \tag{3}$$

Here, p, g and h denote the projections to either $\mathrm{Obj}(\mathcal{P})$ or $\mathrm{Mor}(\mathcal{P})$, G and H, respectively.

The 2-form $\mathrm{curv}(\Omega) := \mathrm{D}\Omega + \frac{1}{2}[\Omega\wedge\Omega] \in \Omega^2(\mathcal{P},\gamma)$ is called the *curvature* of Ω. The connection Ω is called *flat* if $\mathrm{curv}(\Omega) = 0$. Between general connections and flat connections are *fake-flat* connections: these satisfy the conditions (with $\Delta := t^* - s^*$)

$$\mathrm{d}\Omega^a + \frac{1}{2}[\Omega^a \wedge \Omega^a] + t_*(\Omega^c) = 0 \quad \text{and} \quad \Delta\Omega^c + \mathrm{d}\Omega^b + \frac{1}{2}[\Omega^b \wedge \Omega^b] + \alpha_*(s^*\Omega^a \wedge \Omega^b) = 0.$$

If $J : \mathcal{P}_1 \longrightarrow \mathcal{P}_2$ is a 1-morphism, then pulling back a connection Ω_2 on $\mathcal{P}_2$ to $\mathcal{P}_1$ requires the following additional structure on J, as explained in [13, Sections 4.3 & 5.2].

Definition 2.4. An Ω_2*-pullback* on a 1-morphism $J : \mathcal{P}_1 \longrightarrow \mathcal{P}_2$ is a pair $\nu = (\nu_0,\nu_1)$ of differential forms $\nu_0 \in \Omega^1(J,\mathfrak{h})$ and $\nu_1 \in \Omega^2(J,\mathfrak{h})$ which are compatible with the $\mathcal{P}_2$-action ρ_r in the sense that

$$\rho_r^*\nu_0 = \mathrm{pr}_J^*\nu_0 + \mathrm{pr}^*_{\mathrm{Mor}(\mathcal{P}_2)}\Omega_2^b \quad \text{and} \quad \rho_r^*\nu_1 = \mathrm{pr}_J^*\nu_1 + \mathrm{pr}^*_{\mathrm{Mor}(\mathcal{P}_2)}\Delta\Omega_2^c$$

over $J\;{}_{\alpha_r}\!\times_t \mathrm{Mor}(\mathcal{P}_2)$. An Ω_2-pullback is called:

(a) *connective*, if it is compatible with the Mor(Γ)-action ρ in the sense that

$$\rho^*\nu_0 = (\alpha_{g^{-1}})_* \left(\mathrm{Ad}_h^{-1}(\mathrm{pr}_J^*\nu_0) + (\tilde{\alpha}_h)_*(\mathrm{pr}_J^*\alpha_r^*\Omega_2^a) + h^*\theta\right)$$
$$\rho^*\nu_1 = (\alpha_{g^{-1}})_* \left(\mathrm{Ad}_h^{-1}(\mathrm{pr}_J^*\nu_1) + (\tilde{\alpha}_h)_*(t_*(\mathrm{pr}_J^*\alpha_r^*\Omega_2^c))\right)$$

over $J \times \mathrm{Mor}(\Gamma)$, where g and h are the projections to the factors of $\mathrm{Mor}(\Gamma) = H \ltimes G$.

(b) *fake-flat*, if $\mathrm{d}\nu_0 + \frac{1}{2}[\nu_0 \wedge \nu_0] + \alpha_*(\alpha_r^*\Omega_2^a \wedge \nu_0) + \nu_1 = 0$.

Given an Ω_2-pullback ν on $\mathcal{P}_2$, one can define a 1-form $J_\nu^*\Omega_2$ on $\mathcal{P}_1$ that depends on the choice of ν. If ν is connective, then $J_\nu^*\Omega_2$ is a connection on $\mathcal{P}_1$, and if Ω_2 and ν are fake-flat, then $J_\nu^*\Omega_2$ is fake-flat ([13, Proposition 5.2.12]). If a connection Ω_1 on $\mathcal{P}_1$ is given, then we say that ν is *connection-preserving* if $\Omega_1 = J_\nu^*\Omega_2$.

A 2-morphism $f : J \Longrightarrow J'$ between 1-morphisms $J, J' : \mathcal{P}_1 \longrightarrow \mathcal{P}_2$ equipped with Ω_2-pullbacks ν and ν, respectively, is called *connection-preserving* if $\nu = f^*\nu'$. We form two bicategories of principal Γ-2-bundles with connection:

- A bicategory $2\text{-}\mathcal{B}un_\Gamma^\nabla(M)$ consisting of principal Γ-2-bundles with connections, 1-morphisms with connective, connection-preserving pullbacks, and connection-preserving 2-morphisms.
- A bicategory $2\text{-}\mathcal{B}un_\Gamma^{\nabla_{ff}}(M)$ consisting of principal Γ-2-bundles with fake-flat connections, 1-morphisms with fake-flat, connective, connection-preserving pullbacks, and connection-preserving 2-morphisms.

There is a classification result showing that these bicategories correspond to non-abelian differential cohomology [13, Theorem 5.3.4]. Moreover, it is straightforward to see that they form presheaves of bicategories over the category of smooth manifolds, i.e., there are consistent pullback 2-functors along smooth maps.

Remark 2.5. We describe how smooth functors can be turned into 1-morphisms in the setting with connections. Suppose $\phi : \mathcal{P}_1 \longrightarrow \mathcal{P}_2$ is a fibre-preserving, Γ-equivariant smooth functor between principal Γ-2-bundles equipped with connections Ω_1 and Ω_2, respectively. Let $J_\phi = \mathrm{Obj}(\mathcal{P}_1)\,{}_\phi\!\times_t \mathrm{Mor}(\mathcal{P}_2)$ be the associated anafunctor (Remark 2.2 (b)).

(a) A "canonical" Ω_2-pullback on J_ϕ is defined by $\nu_0 := \mathrm{pr}_2^*\Omega_2^b$ and $\nu_1 := -\mathrm{pr}_2^*s^*\Omega_2^c + \mathrm{pr}_1^*\phi^*\Omega_2^c$. It is always connective, fake-flat if Ω_2 is fake-flat, and connection-preserving if $\Omega_1 = \phi^*\Omega_2$. See [13, Remark 5.2.10 (a) – (c)].

(b) The canonical Ω_2-pullback ν on J_ϕ can be shifted by a pair $\kappa = (\kappa_0, \kappa_1)$ of differential forms $\kappa_0 \in \Omega^1(\mathrm{Obj}(\mathcal{P}_1), \mathfrak{h})$ and $\kappa_1 \in \Omega^2(\mathrm{Obj}(\mathcal{P}_1), \mathfrak{h})$, and the shifted pullback is again connective provided that these forms are G-equivariant in the sense that $R^*\kappa_i = (\alpha_{\mathrm{pr}_2^{-1}})_*(\mathrm{pr}_1^*\kappa_i)$ over $\mathrm{Obj}(\mathcal{P}_1) \times G$. See [13, Remark 5.2.10 (e) – (g)].

3. Parallel transport along paths

Let $\mathcal{P}$ be a principal Γ-bundle with a connection Ω. For $x \in M$ we denote by $\mathcal{P}_x := \pi^{-1}(\{x\})$ the fibre of $\mathcal{P}$ over x, which is a Lie groupoid with smooth right Γ-action. In this section we define for each path $\gamma : [0,1] \longrightarrow M$ a Γ-equivariant anafunctor

$$F_\gamma : \mathcal{P}_{\gamma(0)} \longrightarrow \mathcal{P}_{\gamma(1)},$$

which we regard as the parallel transport along γ. For this purpose, we first introduce and study in Section 3.1 the notion of a horizontal path in the total space of $\mathcal{P}$. In Section 3.2 we give a complete definition of the anafunctor F_γ. In Sections 3.3 to 3.5 we derive several properties of F_γ with respect to path composition, 1-morphisms between principal 2-bundles, and pullback.

3.1 Horizontal paths We start with some basic terminology and notation. By a *path* in a smooth manifold X we understand a smooth map $\gamma : [a,b] \longrightarrow X$, where $a, b \in \mathbb{R}$ with $a < b$. If $x := \gamma(a)$ and $y := \gamma(b)$, we use the notation $\gamma : x \longrightarrow y$. If no interval is specified, then the unit interval $[0,1]$ is assumed. The tangent vector at $t \in [a,b]$ is denoted by $\dot\gamma(t)$ or $\partial_t\gamma(t)$. The constant path at a point $x \in X$ will be denoted by x or id_x. If $f : X \longrightarrow Y$ is a smooth map, we write $f(\gamma)$ for the path $f \circ \gamma$. Further, if $R : \mathcal{X} \times \Gamma \longrightarrow \mathcal{X}$ is a right action of a Lie 2-group Γ on a Lie groupoid $\mathcal{X}$, we will write $R(\rho, g)$ instead of $R(\rho, \mathrm{id}_g)$, for $\rho \in \mathrm{Mor}(\mathcal{X})$ and $g \in G$. For instance, if β is a path in $\mathrm{Obj}(\mathcal{X})$ and g is a path in G, then $R(\beta, g)$ stands for the path $t \longmapsto R(\beta(t), \mathrm{id}_{g(t)})$.

First we discuss horizontality for paths in the objects a principal 2-bundle $\mathcal{P}$ with connection Ω. A path $\beta : [a,b] \longrightarrow \mathrm{Obj}(\mathcal{P})$ is *horizontal*, if $\Omega^a(\dot\beta(t)) = 0$ for all $t \in [a,b]$.

Proposition 3.1. *Let $\mathcal{P}$ be a principal Γ-2-bundle with connection Ω.*

(a) Suppose $\beta : [a,b] \longrightarrow \mathrm{Obj}(\mathcal{P})$ is a path. Then, there exists a unique path $g : [a,b] \longrightarrow G$ with $g(a) = 1$ such that $\beta^{hor} := R(\beta, g)$ is horizontal.

(b) Suppose $\beta : [a,b] \longrightarrow \mathrm{Obj}(\mathcal{P})$ is a horizontal path and $g \in G$. Then, $R(\beta, g)$ is horizontal.

Proof. For (a) we claim that the following statements are equivalent:

(1) g is a solution of the differential equation $\dot g(\tau) = -\Omega^a(\dot\beta(\tau))g(\tau)$.

(2) $\beta^{hor} = R(\beta, g)$ is horizontal.

Equivalence is proved by following the calculation using Eq. (1):

$$\Omega^a(\dot\beta^{hor}) = \Omega^a(\partial_t R(\beta, g)) = R^*\Omega^a(\dot\beta, \dot g) = \mathrm{Ad}_g^{-1}(\Omega^a(\dot\beta)) + g^{-1}\dot g.$$

Now, existence and uniqueness of g follow from existence and uniqueness of solutions of linear initial value problems. (b) follows immediately from the transformation behaviour of Ω^a, see Eq. (1). □

Next we turn to paths in the morphisms of $\mathcal{P}$, and collect various statements that we will use throughout this article. A path $\rho : [a,b] \longrightarrow \mathrm{Mor}(\mathcal{P})$ is *horizontal*, if $\Omega^b(\dot\rho(t)) = 0$ for all $t \in [a,b]$.

Proposition 3.2. *Let $\mathcal{P}$ be a principal Γ-2-bundle with connection Ω.*

(a) Suppose $\rho : [a,b] \longrightarrow \mathrm{Mor}(\mathcal{P})$ is a path. Then, there exists a unique path $h : [a,b] \longrightarrow H$ with $h(a) = 1$ such $\rho^{hor} := R(\rho, (h,1))$ is horizontal.

(b) Suppose $\beta : [a,b] \longrightarrow \mathrm{Obj}(\mathcal{P})$ is a path. Then, the path id_β in $\mathrm{Mor}(\mathcal{P})$ is horizontal.

(c) Suppose $\rho : [a,b] \longrightarrow \mathrm{Mor}(\mathcal{P})$ is a path. Then, ρ is horizontal if and only if its pointwise groupoid inversion ρ^{-1} is horizontal.

(d) Suppose $\rho_1, \rho_2 : [a,b] \longrightarrow \mathrm{Mor}(\mathcal{P})$ are horizontal paths with $s(\rho_2) = t(\rho_1)$. Then, their pointwise composition $\rho_2 \circ \rho_1$ is horizontal.

(e) Suppose $\rho : [a,b] \longrightarrow \mathrm{Mor}(\mathcal{P})$ is a horizontal path and $\gamma \in \mathrm{Mor}(\Gamma)$. Then, $R(\rho, \gamma)$ is horizontal.

(f) Suppose $\rho : [a,b] \longrightarrow \mathrm{Mor}(\mathcal{P})$ is a horizontal path and $g : [a,b] \longrightarrow G$ is a path. Then, $R(\rho, g)$ is horizontal.

(g) Suppose $\rho : [a,b] \longrightarrow \mathrm{Mor}(\mathcal{P})$ is horizontal, and of the paths $s(\rho)$ and $t(\rho)$ in $\mathrm{Obj}(\mathcal{P})$ one is horizontal. Then, the other is horizontal, too.

(h) Suppose $\beta_1, \beta_2 : [a,b] \longrightarrow \mathrm{Obj}(\mathcal{P})$ are horizontal, and $\pi \circ \beta_1 = \pi \circ \beta_2$. Then, there exists a horizontal path $\rho : [a,b] \longrightarrow \mathrm{Mor}(\mathcal{P})$ and $g \in G$ such that $\beta_1 = R(s(\rho), g^{-1})$ and $\beta_2 = t(\rho)$.

Moreover, if there is $\rho_0 \in \mathrm{Mor}(\mathcal{P})$ such that $s(\rho_0) = \beta_1(a)$ and $t(\rho_0) = \beta_2(a)$, then one can choose ρ and g such that $\rho(a) = \rho_0$ and $g = 1$.

(i) *Suppose $\rho, \rho' : [a, b] \longrightarrow \mathrm{Mor}(\mathcal{P})$ are horizontal paths such that $s(\rho) = s(\rho')$ is horizontal and $t(\rho) = t(\rho')$. Then, there exists a unique $h \in H$ with $t(h) = 1$ and $\rho' = R(\rho, (h, 1))$.*

Proof. For (a) we claim that the following statements are equivalent:

(1) h is a solution of the differential equation $\dot{h}(\tau) = -\Omega^b(\dot{\rho}(\tau))h(\tau) - (\alpha_{h(\tau)})_*(\Omega^a(s_*(\dot{\rho}(\tau))))$.

(2) ρ^{hor} is horizontal.

Equivalence is proved by the following equation obtained using Eq. (2),

$$\Omega^b(\dot{\rho}^{hor}) = \mathrm{Ad}_h^{-1}(\Omega^b(\dot{\rho})) + (\tilde{\alpha}_h)_*(\Omega^a(s(\dot{\rho}))) + h^{-1}\dot{h}.$$

Now, existence and uniqueness follow like in the proof of Proposition 3.1 (a). For (b) we have $\Omega^b(\partial_t \mathrm{id}_\beta) = \mathrm{id}^*\Omega^b(\dot{\beta}) = 0$ since $\mathrm{id}^*\Omega^b = 0$. For (c) we use $inv^*\Omega^b = -\Omega^b$, and for (d) $c^*\Omega^b = \mathrm{pr}_1^*\Omega^b + \mathrm{pr}_2^*\Omega^b$. (e) is trivial. For (f) we check

$$\Omega^b(\partial_t R(\rho, (1, g))) = R^*\Omega^b(\dot{\rho}, (0, \dot{g})) = (\alpha_g)_*(\Omega^b(\dot{\rho})) = 0.$$

For (g) we use that $t^*\Omega^a - s^*\Omega^a = t_*(\Omega^b)$. Since ρ is horizontal, we have

$$0 = t_*(\Omega^b(\dot{\rho})) = \Omega^a(\partial_t s(\rho)) - \Omega^a(\partial_t t(\rho)).$$

For (h) we note that (β_1, β_2) is a path in $\mathrm{Obj}(\mathcal{P}) \times_M \mathrm{Obj}(\mathcal{P})$. By [13, Lemma 3.1.6] there exists a transition span $\rho : [a, b] \longrightarrow \mathrm{Mor}(\mathcal{P})$ along (β_1, β_2) with transition function $g : [a, b] \longrightarrow G$, i.e. $\beta_1 = R(s(\rho), g^{-1})$ and $\beta_2 = t(\rho)$. If there is $\rho_0 \in \mathrm{Mor}(\mathcal{P})$ such that $s(\rho_0) = \beta_1(a)$ and $t(\rho_0) = \beta_2(a)$, then by [13, Lemma 3.1.4] there exists $h_0 \in H$ such that $R(\rho(a), (h_0, g(a)^{-1})) = \rho_0$ and $t(h_0) = g(a)$. Then we use $\tilde{\rho} := R(\rho, (h_0, t(h_0)^{-1}))$ and $\tilde{g} := gt(h_0)^{-1}$, satisfying $R(s(\tilde{\rho}), \tilde{g}^{-1}) = R(s(\rho), g^{-1}) = \beta_1$ and $t(\tilde{\rho}) = t(\rho) = \beta_2$, as well as $\tilde{\rho}(a) = \rho_0$ and $\tilde{g}(a) = 1$. By (a) there exists $h : [a, b] \longrightarrow H$ with $h(a) = 1$ such that $R(\rho, (h, 1))$ is horizontal. Then, by (f) also $\rho' := R(\rho, (h, t(h)^{-1}))$ is horizontal, and $t(\rho') = t(\rho) = \beta_2$. We set $g' := gt(h)^{-1}$. Then, $R(s(\rho'), g'^{-1}) = R(s(\rho), t(h)^{-1}g'^{-1}) = R(s(\rho), g^{-1}) = \beta_1$. Now, by (g), it follows that $s(\rho')$ is horizontal. A short calculation shows that $0 = \Omega^a(\dot{\beta}_1) = -\dot{g}'g'^{-1}$; hence g' is constant. For (i) we obtain by [13, Lemma 3.1.4] a smooth map $h : [a, b] \longrightarrow H$ with $t(h) = 1$ and $\rho' = R(\rho, (h, 1))$. Again, a short calculation shows $0 = \Omega^b(\dot{\rho}') = h^{-1}\dot{h}$; hence h is constant. ☐

Finally, we consider a 1-morphism $J : \mathcal{P}_1 \longrightarrow \mathcal{P}_2$ in $2\text{-}\mathcal{B}un^{\nabla}_{\Gamma}(M)$ between principal Γ-2-bundles over M, connections Ω_1 and Ω_2 on $\mathcal{P}_1$ and $\mathcal{P}_2$, respectively, and a connective, connection-preserving Ω_2-pullback $\nu = (\nu_0, \nu_1)$ on J. A path $\lambda : [a, b] \longrightarrow J$ is *horizontal*, if $\nu_0(\dot{\lambda}(t)) = 0$ for all $t \in [a, b]$.

Remark 3.3. Suppose $\phi : \mathcal{P}_1 \longrightarrow \mathcal{P}_2$ is a smooth functor, $J_\phi = \mathrm{Obj}(\mathcal{P}_1)\, {}_\phi\!\times_t \mathrm{Mor}(\mathcal{P}_2)$ is the associated anafunctor (Remark 2.2 (b)), and ν is the canonical Ω_2-pullback on J_ϕ (Remark 2.5). Then, a path $\lambda = (\gamma, \rho)$ in J_ϕ is horizontal if and only if ρ is horizontal in $\mathrm{Mor}(\mathcal{P}_2)$. If $\kappa = (\kappa_0, \kappa_1)$ shifts the canonical Ω_2-pullback, then λ is horizontal if and only if $\Omega_2^b(\dot{\rho}) + \kappa_0(\dot{\gamma}) = 0$.

Proposition 3.4. *Let $J : \mathcal{P}_1 \longrightarrow \mathcal{P}_2$ be a 1-morphism in $2\text{-}\mathcal{B}un^{\nabla}_{\Gamma}(M)$.*

(a) *Suppose $\lambda : [a, b] \longrightarrow J$ is horizontal, and $\gamma : [a, b] \longrightarrow G$ is a path. Then, $\lambda \cdot \mathrm{id}_\gamma$ is horizontal.*

(b) *Suppose $\lambda : [a, b] \longrightarrow J$ is horizontal, and of the paths $\alpha_l(\lambda)$ and $\alpha_r(\lambda)$ one is horizontal. Then, the other is horizontal, too.*

(c) *Suppose* $\lambda : [a, b] \longrightarrow J$ *and* $\rho : [a, b] \longrightarrow \mathrm{Mor}(\mathcal{P}_2)$ *are paths with* $\alpha_r(\lambda) = t(\rho)$*. If of the three paths* λ*,* ρ*, and* $\lambda \circ \rho$ *two are horizontal, then the third is horizontal, too.*

(d) *Suppose* $\lambda : [a, b] \longrightarrow J$ *and* $\rho : [a, b] \longrightarrow \mathrm{Mor}(\mathcal{P}_1)$ *are paths with* $s(\rho) = \alpha_l(\lambda)$*. If of the three paths* λ*,* ρ*, and* $\rho \circ \lambda$ *two are horizontal, then the third is horizontal, too.*

(e) *Suppose* $\lambda : [a, b] \longrightarrow J$ *is a path. Then, there exists a unique path* $h : [a, b] \longrightarrow H$ *with* $h(a) = 1$ *such that* $\lambda_1(t) := \lambda(t) \cdot (h(t), 1)$ *and* $\lambda_2(t) := \lambda(t) \cdot (h(t), t(h)^{-1})$ *are horizontal.*

Proof. (a) follows since ν is connective. (b) is exactly as Proposition 3.2 (g), using that ν is connection-preserving, which implies $t_*(\nu_0) = \alpha_l^*\Omega_1^a - \alpha_r^*\Omega_2^a$. For (c) we check that $\nu_0(\partial_t\rho_r(\lambda, \rho)) = \nu_0(\dot{\lambda}) + \Omega_2^b(\dot{\rho})$, and (d) is analogous. For (e) we claim that the following three statements are equivalent:

(1) λ_2 is horizontal.

(2) λ_1 is horizontal.

(3) h solves the differential equation $\dot{h}(t) = -\nu_0(\dot{\lambda}(t))h(t) - (\alpha_{h(t)})_*(\alpha_r^*\Omega_2^a(\dot{\lambda}(t)))$.

Equivalence between (1) and (2) is (a). Equivalence between (2) and (3) is proved by the following calculation, using connectivity:

$$\nu_0(\dot{\lambda}_1) = \rho^*\nu_0(\dot{\lambda}, (\dot{h}, 0)) = \mathrm{Ad}_h^{-1}(\nu_0(\dot{\lambda})) + (\tilde{\alpha}_h)_*(\alpha_r^*\Omega_2^a(\dot{\lambda})) + h^{-1}\dot{h}. \qquad \square$$

3.2 Definition of parallel transport along paths Let $\mathcal{P}$ be a principal Γ-2-bundle over M with connection Ω. Let $\gamma : [0, 1] \longrightarrow M$ be a path in M. In this section we define the anafunctor $F_\gamma : \mathcal{P}_{\gamma(0)} \longrightarrow \mathcal{P}_{\gamma(1)}$. We proceed in the following four steps: Step 1 is to define a set $F_\gamma(t)$ with respect to a fixed subdivision t of $[0, 1]$. Step 2 is to define anchors and actions for $F_\gamma(t)$ with the required algebraic properties. Step 3 is to get rid of the subdivision via a direct limit construction, resulting in a set F_γ. Step 4 is to equip F_γ with the structure of a smooth manifold, and to show that anchors and actions are smooth.

3.2.1 Step 1: Total space with respect to a fixed subdivision We consider for $0 < n \in \mathbb{N}$ the set $T_n := \{(t_i)_{i=0}^n \mid 0 = t_0 < t_1 < ... < t_n = 1\}$ of possible n-fold subdivisions of the interval $[0, 1]$, and define for $t \in T_n$ the set

$$F_\gamma(t) := \{(\{\rho_i\}_{i=0}^n, \{\gamma_i\}_{i=1}^n) \mid \rho_i \in \mathrm{Mor}(\mathcal{P}), \gamma_i : [t_{i-1}, t_i] \longrightarrow \mathrm{Obj}(\mathcal{P}) \text{ horizontal paths}, \\ \pi \circ \gamma_i = \gamma|_{[t_{i-1}, t_i]},\ t(\rho_i) = \gamma_{i+1}(t_i) \text{ and } s(\rho_i) = \gamma_i(t_i)\}/\sim \quad (4)$$

where $\sim$ is an equivalence relation defined below. In words, $F_\gamma(t)$ consists of locally defined horizontal lifts γ_i of the pieces $\gamma|_{[t_{i-1}, t_i]}$, together with morphisms ρ_i between the endpoint of each lift to the initial point of the next one. We think about the elements of $F_\gamma(t)$ as "formal" compositions of paths in $\mathrm{Obj}(\mathcal{P})$ and and morphisms of $\mathcal{P}$, and we will use the notation $\xi = \rho_n * \gamma_n * ... * \rho_2 * \gamma_1 * \rho_0$ for a representative ξ of an element in $F_\gamma(t)$.

The equivalence relation in Eq. (4) is generated by relations $\{\sim_j\}_{1 \leq j \leq n}$ defined as follows: we define a relation

$$\rho_n * \gamma_n * ... * \gamma_1 * \rho_0 \quad \sim_j \quad \rho'_n * \gamma'_n * ... * \gamma'_1 * \rho'_0 \quad (5)$$

if there exist a horizontal path $\tilde{\rho} : [t_{j-1}, t_j] \longrightarrow \mathrm{Mor}(\mathcal{P})$ such that

$$\gamma_j = s(\tilde{\rho}) \quad \text{and} \quad \gamma'_j = t(\tilde{\rho}), \quad (6)$$

and $\gamma_i' = \gamma_i$ for all $1 \leq i \leq n$, $i \neq j$, as well as

$$\rho_{j-1}' = \tilde\rho(t_{j-1}) \circ \rho_{j-1} \quad \text{and} \quad \rho_j' = \rho_j \circ \tilde\rho(t_j)^{-1} \tag{7}$$

and $\rho_i' = \rho_i$ for all $0 \leq i \leq n$, $i \neq j, j-1$. We will use the terminology that the relation Eq. (5) is *via* $\tilde\rho$. It is straightforward to check using Proposition 3.2 (g) that given one representative $\xi = \rho_n * \gamma_n * ... * \gamma_1 * \rho_0$ and a horizontal path $\tilde\rho$ with $\gamma_j = s(\tilde\rho)$, then one can turn Eqs. (6) and (7) into definitions, producing another element $\rho_n' * \gamma_n' * ... * \gamma_1' * \rho_0'$, related to ξ via $\tilde\rho$.

Lemma 3.5. *(a) For each $1 \leq j \leq n$, $\sim_j$ is an equivalence relation.*
(b) For $1 \leq i < j \leq n$ and $\xi^1 \sim_i \xi' \sim_j \xi^2$, there exists ξ'' such that $\xi^1 \sim_j \xi'' \sim_i \xi^2$.

Proof. To (a): For reflexivity put $\tilde\rho(t) := \mathrm{id}_{\gamma(t)}$, which is horizontal by Proposition 3.2 (b). For symmetry assume that $\xi \sim_j \xi'$ via $\tilde\rho$. Then, $\tilde\rho' := \tilde\rho^{-1}$ is horizontal by Proposition 3.2 (c), and we have $\xi' \sim_j \xi$ via $\tilde\rho'$. Transitivity goes analogously using Proposition 3.2 (d). To (b): We let $\xi^1 \sim_i \xi'$ be via $\tilde\rho^1$ and $\xi' \sim_j \xi^2$ be via $\tilde\rho^2$. We define $\tilde\rho^3 := \tilde\rho^2$. We define ξ'' such that $\xi^1 \sim_j \xi''$ via $\tilde\rho^3$. Now one can check that $\xi'' \sim_i \xi^2$ via $\tilde\rho^1$. ☐

3.2.2 Step 2: Anchors and actions Next we define a left $\mathcal{P}_x$-action and a right $\mathcal{P}_y$-action on the set $F_\gamma(t)$; their anchors are

$$\alpha_l : F_\gamma(t) \longrightarrow \mathcal{P}_x : \rho_n * \gamma_n * ... * \gamma_1 * \rho_0 \longmapsto s(\rho_0)$$
$$\alpha_r : F_\gamma(t) \longrightarrow \mathcal{P}_y : \rho_n * \gamma_n * ... * \gamma_1 * \rho_0 \longmapsto t(\rho_n).$$

These maps are obviously well-defined under the equivalence relation.

Lemma 3.6. *The map $\mathrm{Mor}(\mathcal{P}_x)\,{}_s\!\times_{\alpha_l} F_\gamma(t) \longrightarrow F_\gamma(t)$ defined by*

$$\rho \circ (\rho_n * \gamma_n * ... * \gamma_1 * \rho_0) := \rho_n * \gamma_n * ... * \gamma_1 * (\rho_0 \circ \rho^{-1})$$

is a well-defined left action of $\mathcal{P}_x$ on $F_\gamma(t)$ with anchor α_l, and keeps α_r invariant.

Proof. For the well-definedness, only $\sim_1$ has to be checked, which is done via Eq. (7). The other statements are obvious. ☐

Lemma 3.7. *The map $F_\gamma(t)\,{}_{\alpha_r}\!\times_t \mathrm{Mor}(\mathcal{P}_y) \longrightarrow F_\gamma(t)$ defined by*

$$(\rho_n * \gamma_n * ... * \gamma_1 * \rho_0) \circ \rho := \rho^{-1} \circ \rho_n * \gamma_n * ... * \gamma_1 * \rho_0$$

is a well-defined right action of $\mathcal{P}_y$ on $F_\gamma(t)$ with anchor α_r, it keeps α_l invariant, and it commutes with the left action of Lemma 3.6. Moreover, if $\xi, \xi' \in F_\gamma$ with $\alpha_l(\xi) = \alpha_l(\xi')$, then there exists a unique $\rho \in \mathrm{Mor}(\mathcal{P}_y)$ such that $\xi \circ \rho = \xi'$.

Proof. Well-definedness and the properties of an action are straightforward to check. More difficult is to prove existence and uniqueness of ρ; this is exactly the point where our equivalence relation becomes relevant. For existence, suppose

$$\xi = \rho_n * \gamma_n * ... * \gamma_1 * \rho_0 \quad \text{and} \quad \xi' = \rho_n' * \gamma_n' * ... * \gamma_1' * \rho_0'$$

satisfy $\alpha_l(\xi) = \alpha_l(\xi')$. This implies that $\rho_0' \circ \rho_0^{-1} \in \mathrm{Mor}(\mathcal{P})$ satisfies $s(\rho_0' \circ \rho_0^{-1}) = t(\rho_0) = \gamma_1(t_0)$ and $t(\rho_0' \circ \rho_0^{-1}) = t(\rho_0') = \gamma_1'(t_0)$. By Proposition 3.2 (h) there exists a horizontal path $\tilde\rho : [t_0, t_1] \longrightarrow \mathrm{Mor}(\mathcal{P})$ with $\tilde\rho(t_0) = \rho_0' \circ \rho_0^{-1}$, $s(\tilde\rho) = \gamma_1$ and $t(\tilde\rho) = \gamma_1'$. Via $\tilde\rho$ we obtain a relation

$$\rho_n * \gamma_n * ... * \gamma_1 * \rho_0 \sim_1 \rho_n^{(1)} * \gamma_n^{(1)} * ... * \rho_1^{(1)} * \gamma_1^{(1)} * \rho_0^{(1)},$$

with $\gamma_1^{(1)} = t(\tilde{\rho}) = \gamma_1'$ and $\rho_0^{(1)} = \tilde{\rho}(t_0) \circ \rho_0 = \rho_0'$. Now we are in the situation that $s(\rho_1^{(1)}) = s(\rho_1')$, and can use $\sim_2$ in the same manner as $\sim_1$ before. After n steps, we arrive at a relation

$$\rho_n * \gamma_n * ... * \gamma_1 * \rho_0 \sim \rho_n^{(n)} * \gamma_n' * ... * \gamma_1' * \rho_0'.$$

Now we define $\rho := \rho_n^{(n)} \circ \rho_n'^{-1}$; this definition yields $\xi \circ \rho = \xi'$.

Next we show that ρ is unique, i.e. we prove that a relation $\xi \circ \rho \sim \xi$ implies $\rho = \mathrm{id}$. Putting $\xi = \rho_n * \gamma_n * ... * \gamma_1 * \rho_0$, the assumption is

$$\rho_n * \gamma_n * ... * \gamma_1 * \rho_0 \sim (\rho^{-1} \circ \rho_n) * \gamma_n * ... * \gamma_1 * \rho_0,$$

where $\sim$ is a finite chain composed of the relations $\sim_1$, ..., $\sim_n$. By Lemma 3.5 we can assume that this chain is ordered with descending i and each i appears at most once. If the chain $\sim$ is empty, we must have $\rho_n = \rho^{-1} \circ \rho_n$; this proves $\rho = \mathrm{id}$. If it is non-empty, we proceed by induction over the minimum j of occurring relations $\sim_j$. We write the chain $\sim$ as

$$\rho_n * \gamma_n * ... * \gamma_1 * \rho_0 \sim' \rho_n' * \gamma_n' * ... * \rho_1' * \gamma_1' * \rho_0' \sim_j (\rho^{-1} \circ \rho_n) * \gamma_n * ... * \gamma_1 * \rho_0,$$

with $\sim'$ a chain composed only of the relations $\sim_{j+1}, ..., \sim_n$. Since $\sim'$ does not affect the parts before ρ_j, we have $\gamma_i' = \gamma_i$ for $1 \leq i \leq\ j$ and $\rho_i' = \rho_i$ for $0 \leq i < j$. We claim that $\sim_j$ implies coincidence of the remaining parts: (A) $\gamma_i' = \gamma_i$ for $j < i \leq n$, (B) $\rho_j' = \rho_j$ if $j < n$, (C) $\rho_i' = \rho_i$ for all $j < i < n$, and (D) $\rho_n' = (\rho^{-1} \circ \rho_n)$. Given the claim, we obtain

$$\rho_n * \gamma_n * ... * \gamma_1 * \rho_0 \sim' (\rho^{-1} \circ \rho_n) * \gamma_n * ... * \gamma_1 * \rho_0,$$

and have hence shifted the induction parameter from j to $j + 1$. At the end the minimum is shifted to $n + 1$, meaning that the chain of relations becomes empty.

In order to prove the claim, we assume that $\sim_j$ is via $\tilde{\rho}$. Since $\gamma_j' = \gamma_j$ and $\rho_{j-1}' = \rho_{j-1}$, we have $t(\tilde{\rho}) = s(\tilde{\rho})$ and $\tilde{\rho}(t_0) = \mathrm{id}$. This shows (A) and (C). If $n = j$, then we have by Eq. (7) $\rho_j' = \rho^{-1} \circ \rho_j \circ \tilde{\rho}(t_j)^{-1}$. If $j < n$, then we have by Eq. (7) $\rho_j' = \rho_j \circ \tilde{\rho}(t_j)^{-1}$ and $\rho_n' = \rho^{-1} \circ \rho_n$. We show that $\tilde{\rho}(t_j)^{-1} = \mathrm{id}$, which proves the remaining claims (B) and (C). Indeed, we observe that $\tilde{\rho}$ and id_{γ_j} satisfy the assumptions of Proposition 3.2 (i), and since $\tilde{\rho}(t_{j-1}) = \mathrm{id}_{\gamma_j(t_{j-1})}$, we have $\tilde{\rho} = \mathrm{id}_{\gamma_j}$. □

Next we define the $\mathrm{Mor}(\Gamma)$-action on $F_\gamma(t)$, which at the end constitutes the Γ-equivariance of the anafunctor F_γ.

Lemma 3.8. *The map $F_\gamma(t) \times \mathrm{Mor}(\Gamma) \longrightarrow F_\gamma(t)$ defined by*

$$(\rho_n * \gamma_n * ... * \gamma_1 * \rho_0) \cdot (h, g) := R(\rho_n, g) * R(\gamma_n, g) * ... * R(\gamma_1, g) * R(\rho_0, (h^{-1}, t(h)g))$$

is a well-defined action and compatible with the left $\mathcal{P}_x$-action and the right $\mathcal{P}_y$-action in the sense of Remark 2.2 (a).

Proof. The axioms of an action are straightforward to check on the level of representatives. In order to check well-definedness, we can then write $(h, g) = (h, 1) \cdot (1, g)$ and check separately. For well-definedness with respect to elements $(1, g) \in \mathrm{Mor}(\Gamma)$, it is straightforward to see that if

$$\rho_n * ... * \gamma_1 * \rho_0 \sim_j \rho_n' * ... * \gamma_1' * \rho_0'$$

via $\tilde{\rho}$, then

$$(\rho_n * ... * \gamma_1 * \rho_0) \cdot (1, g) \sim_j (\rho_n' * ... * \gamma_1' * \rho_0') \cdot (1, g)$$

via $R(\tilde{\rho}, g)$, which is horizontal by Proposition 3.2 (f). For well-definedness with respect to elements of the form $(h, 1)$, it suffices to consider $\sim_1$, which is easy. The compatibility with the anchors and the left $\mathcal{P}_x$-action hold on the level of representatives and are straightforward to check. Compatibility with the right $\mathcal{P}_y$-action, however, only holds on the level of equivalence classes: we claim that

$$(\xi \circ \rho) \cdot ((h, g) \circ (h_r, g_r)) = R(\rho^{-1}, g_r) * R(\xi, g_r) * R(\mathrm{id}_{s(\xi)}, (h_r^{-1}h^{-1}, t(h)g)) \tag{8}$$

and

$$(\xi \cdot (h, g)) \circ R(\rho, (h_r, g_r)) = R(\rho^{-1}, (h_r^{-1}, g)) * R(\xi, g) * R(\mathrm{id}_{s(\xi)}, (h^{-1}, t(h)g)) \tag{9}$$

are equivalent. This relation will be proved by induction over the length of ξ. We start with the case $n = 0$, i.e. $\xi = \rho_0$. Then, Eqs. (8) and (9) are, respectively,

$$\rho_0' := R(\rho^{-1}, g_r) \circ R(\rho_0, g_r) \circ R(\mathrm{id}_{s(\rho_0)}, (h_r^{-1}h^{-1}, t(h)g))$$
$$\rho_0'' := R(\rho^{-1}, (h_r^{-1}, g)) \circ R(\rho_0, g) \circ R(\mathrm{id}_{s(\rho_0)}, (h^{-1}, t(h)g)).$$

Both expressions are in fact equal; in particular, $\rho_0' \sim \rho_0''$. Now let $n > 1$ and $\xi = \rho_n * ... * \rho_0$. Then, Eq. (8) is an element ξ' consisting of

$$\rho_n' := R(\rho^{-1}, g_r) \circ R(\rho_n, g_r) \qquad \rho_i' := R(\rho_i, g_r) \text{ for } 1 \le i < n$$
$$\rho_0' := R(\rho_0, g_r) \circ R(\mathrm{id}_{s(\xi)}, (h_r^{-1}h^{-1}, t(h)g)) \qquad \gamma_i' := R(\gamma_i, g_r) \text{ for } 1 \le i \le n.$$

We use $\sim_1$ with $\tilde{\rho} := R(\mathrm{id}_{\gamma_1}, (h_r, g_r))$; this is horizontal by Propositions 3.2 (e) and 3.2 (f), and satisfies $s(\tilde{\rho}) = R(\gamma_1, g_r) = \gamma_1'$. Then, we obtain an equivalent representative ξ'' with $\xi' \sim_1 \xi''$, which consists of the components $\gamma_1'' := t(\tilde{\rho})$,

$$\begin{aligned}
\rho_1'' &:= \rho_1' \circ \tilde{\rho}(t_1)^{-1} = R(\rho_1, g_r) \circ R(\mathrm{id}_{\gamma_1(t_1)}, (h_r^{-1}, g)),\\
\rho_0'' &:= \tilde{\rho}(0) \circ \rho_0'\\
&= R(\mathrm{id}_{\gamma_1(0)}, (h_r, g_r)) \circ R(\rho_0, g_r) \circ R(\mathrm{id}_{s(\xi)}, (h_r^{-1}h^{-1}, t(h)g))\\
&= R(\rho_0, (h^{-1}, t(h)g))\\
&= R(\rho_0, g) \circ R(\mathrm{id}_{s(\xi)}, (h^{-1}, t(h)g)),
\end{aligned}$$

as well as $\gamma_i'' := \gamma_i'$ and $\rho_i'' := \rho_i'$ for all other indices i. We define $\xi^- := \rho_n * ... * \rho_1$, $g_r^- := g_r$, $\rho^- := \rho$, $h_r^- := h_r$ and $h^- := 1$, and $g^- := g$. With this new notation we can rewrite ξ'' as

$$\begin{aligned}
\xi'' &= \rho_n'' * \gamma_n'' * ... * \gamma_2'' * \rho_1'' * \gamma_1'' * \rho_0''\\
&= R(\rho^{-1}, g_r) * R(\rho_n * ... * \rho_1, g_r) * R(\mathrm{id}_{\gamma_1(t_1)}, (h_r^{-1}, g))\\
&\quad * R(\gamma_1, g) * R(\rho_0, g) \circ R(\mathrm{id}_{s(\xi)}, (h^{-1}, t(h)g))\\
&= R((\rho^-)^{-1}, g_r^-) * R(\xi^-, g_r^-) * R(\mathrm{id}_{s(\xi^-)}, ((h_r^-)^{-1}(h^-)^{-1}, t(h^-)g^-))\\
&\quad * R(\gamma_1, g) * R(\rho_0, g) \circ R(\mathrm{id}_{s(\xi)}, (h^{-1}, t(h)g)).
\end{aligned}$$

Now the first line of the last result is precisely in the form where we can apply the induction hypothesis; thus, we have

$$\begin{aligned}
\xi'' &\sim R((\rho^-)^{-1}, ((h_r^-)^{-1}, g^-)) * R(\xi^-, g^-) * R(\mathrm{id}_{s(\xi^-)}, ((h^-)^{-1}, t(h^-)g^-))\\
&\quad * R(\gamma_1, g) * R(\rho_0, g) \circ R(\mathrm{id}_{s(\xi)}, (h^{-1}, t(h)g))\\
&\sim R(\rho^{-1}, (h_r^{-1}, g)) * R(\rho_n * ... * \rho_1, g)\\
&\quad * R(\mathrm{id}_{\gamma_1(1)}, (1, g)) * R(\gamma_1, g) * R(\rho_0, g) \circ R(\mathrm{id}_{s(\xi)}, (h^{-1}, t(h)g))\\
&\sim R(\rho^{-1}, (h_r^{-1}, g)) * R(\xi, g) * R(\mathrm{id}_{s(\xi)}, (h^{-1}, t(h)g)).
\end{aligned}$$

This is Eq. (9). □

3.2.3 Step 3: Direct limit So far we have worked relative to the fixed subdivision $t \in T_n$ of the interval; the next step is to get rid of this parameter. The set $T := \bigsqcup_{n\in\mathbb{N}} T_n$ is a directed set, where $t \leq t'$ if $\{t_i\} \subseteq \{t'_i\}$ as subsets of $\mathbb{R}$. For $t \leq t'$ we have a map

$$f_{t,t'} : F_\gamma(t) \longrightarrow F_\gamma(t')$$

defined by adding identities $\rho_i = \mathrm{id}$ and splitting γ_i in two parts, at all additional points. These maps give $\{F_\gamma(t)\}_{t\in T}$ the structure of a direct system of sets. Its direct limit is denoted by F_γ. It is straightforward to see that the anchors α_r and α_l, the actions of Lemmas 3.6 and 3.7, and the $\mathrm{Mor}(\Gamma)$-action of Lemma 3.8 descent to F_γ.

3.2.4 Step 4: Smooth structure Next we equip F_γ with the structure of a smooth manifold. Consider a point $p_0 \in \mathcal{P}_x$ and choose an element $\xi_0 \in F_\gamma$ with $\alpha_l(\xi_0) = p_0$. Such an element ξ_0 exists: choose $t \in T$ such that there exist sections $\sigma_i : U_i \longrightarrow \mathrm{Obj}(\mathcal{P})$ defined on open sets U_i with $\gamma([t_{i-1}, t_i]) \subseteq U_i$. By [13, Lemma 3.1.6] there exist $\rho \in \mathrm{Mor}(\mathcal{P})$ and $g \in G$ such that $t(\rho) = \sigma_1(x)$ and $R(s(\rho), g^{-1}) = p_0$. We set $\rho_0 := R(\rho, g^{-1})$ and $\gamma'_1 := R(\sigma_1(\gamma|_{[t_0,t_1]}), g^{-1})$. By Proposition 3.1 (a) there exists γ_1 such that $\pi(\gamma_1) = \pi(\gamma'_1)$ and $\gamma_1(t_0) = \gamma'_1(t_0)$. We set $p_1 := \gamma_1(t_1)$ and proceed in the same way until $i = n$. We end up with an element $\xi_0 = \rho_n * ... * \gamma_1 * \rho_0$ with $\alpha_l(\xi_0) = s(\rho_0) = p_0$.

Given ξ_0 we construct an open neighborhood U of p_0 with a local section against α_l. Choose an open neighborhood $\tilde{U} \subseteq \mathrm{Obj}(\mathcal{P}) \times_M \mathrm{Obj}(\mathcal{P})$ of (p_0, p_0) together with a transition span ρ and a transition function g ([13, Lemma 3.1.6]). We consider the smooth map $i_{p_0} : \mathrm{Obj}(\mathcal{P}_x) \longrightarrow \mathrm{Obj}(\mathcal{P}) \times_M \mathrm{Obj}(\mathcal{P})$ with $i_{p_0}(p) := (p_0, p)$, and define $U := i_{p_0}^{-1}(\tilde{U}) \subseteq \mathrm{Obj}(\mathcal{P}_x)$. We define a map $\sigma_{\xi_0,\rho,g} : U \longrightarrow F_\gamma$ by setting

$$\sigma_{\xi_0,\rho,g}(p) := (\rho(p_0, p)^{-1} \circ \xi_0) \cdot (1, g(p_0, p)^{-1}).$$

By Lemmas 3.6 and 3.8 we have $\alpha_l(\sigma_{\xi_0,\rho,g}(p)) = p$, i.e. $\sigma_{\xi_0,\rho,g}$ is a section against α_l. It determines a map

$$\phi_{\xi_0,\rho,g} : U \;{}_{\alpha_r\circ\sigma_{\xi_0,\rho,g}}\!\times_t \mathrm{Mor}(\mathcal{P}_y) \longrightarrow \alpha_l^{-1}(U) : (p, \tilde{\rho}) \longmapsto \sigma_{\xi_0,\rho,g}(p) \circ \tilde{\rho},$$

which is a bijection due to Lemma 3.7.

Lemma 3.9. *There exists a unique smooth manifold structure on F_γ such that all bijections $\phi_{\xi_0,\rho,g}$ are smooth.*

Proof. The bijections $\phi_{\xi_0,\rho,g}$ induce a topology on F_γ, which is Hausdorff and second countable. It remains to prove that the "transition functions" are smooth.

We write $\sigma := \sigma_{\xi_0,\rho,g}$ and consider another section $\sigma' := \sigma_{\xi'_0,\rho',g'} : U' \longrightarrow F_\gamma$ constructed in the same way around $p'_0 = \alpha_l(\xi'_0)$, such that $W := U \cap U'$ is non-empty. The transition function is the unique map $\tilde{\rho} : W \longrightarrow \mathrm{Mor}(\mathcal{P})$ with

$$\sigma_{\xi'_0,\rho',g'}(p) = \sigma_{\xi_0,\rho,g}(p) \circ \tilde{\rho}(p) \tag{10}$$

for all $p \in W$. In order to show that $\tilde{\rho}$ is smooth, we compute it explicitly. We fix $q \in W$. By Lemma 3.7 there exists a unique $\tilde{\rho}_q \in \mathrm{Mor}(\mathcal{P})$ such that $\sigma(q) = \sigma'(q) \circ \tilde{\rho}_q$. With [13, Lemma 3.1.4] there exists a smooth map $h : W \longrightarrow H$ such that

$$R(\rho(p_0, p), (h(p), g(p_0, p)^{-1} g(q, p) g(p_0, q))) = \rho(p_0, q) \circ R(\rho(q, p), g(p_0, q)) \tag{11}$$

$$t(h(p))g(p_0,p)^{-1}g(q,p)g(p_0,q) = 1 \tag{12}$$

for all $p \in W$. The definition of σ and Eq. (11) imply that

$$\sigma(p) = (R(\rho(q,p)^{-1}, (\alpha(g(p_0,q),h(p)),1)) \circ \sigma'(q) \circ \tilde{\rho}_q) \cdot (1, g(p_0,q)g(p_0,p)^{-1}). \tag{13}$$

Analogously, for the primed quantities, there exists a smooth map $h' : W \longrightarrow H$ with

$$\sigma'(p) = (R(\rho'(q,p)^{-1}, (\alpha(g'(p_0',q),h'(p)),1)) \circ \sigma'(q)) \cdot (1, g'(p_0',q)g'(p_0',p)^{-1}). \tag{14}$$

Again by [13, Lemma 3.1.4] there exists another smooth map $\eta : W \longrightarrow H$ such that

$$R(\rho(q,p), (\eta(p), g(q,p)^{-1}g'(q,p))) = \rho'(q,p).$$

Solving for $\rho(q,p)^{-1}$ and substituting in Eq. (13) gives

$$\sigma(p) = (R(\rho'(q,p)^{-1}, (\eta(p)\alpha(g(p_0,q),h(p)),1)) \circ \sigma'(q) \circ \tilde{\rho}_q) \cdot (1, g(p_0,q)g(p_0,p)^{-1}).$$

Forcing Eq. (14) into this expression yields

$$\sigma(p) = (\sigma'(p) \circ R(\tilde{\rho}_q, g'(p_0',q)g'(p_0',p)^{-1})) \cdot (h(p), t(h(p))^{-1}),$$

where $h(p) := \alpha(g'(p_0',p)g'(p_0',q)^{-1}, \alpha(g'(p_0',q),h'(p)^{-1})\eta(p)\alpha(g(p_0,q),h(p)))$. Using the compatibility of the Γ-action with the right $\mathcal{P}_y$-action, we can write

$$\sigma(p) = \sigma'(p) \circ (R(\tilde{\rho}_q, (1, g'(p_0',q)g'(p_0',p)^{-1}) \cdot (h(p), t(h(p))^{-1})).$$

This is an explicit expression for the transition function $\tilde{\rho}$, and it depends smoothly on p. ☐

Proposition 3.10. *The smooth manifold F_γ together with the anchor maps α_l and α_r, the actions of Lemmas 3.6 and 3.7, and the* $\mathrm{Mor}(\Gamma)$*-action of Lemma 3.8, define a Γ-equivariant anafunctor $F_\gamma : \mathcal{P}_x \longrightarrow \mathcal{P}_y$,*

Proof. In a chart ϕ, we have $\alpha_l \circ \phi = \mathrm{pr}_1$ and $\alpha_r \circ \phi = s \circ \mathrm{pr}_2$ hence α_l and α_r are smooth and surjective submersions. The right $\mathcal{P}_y$-action is in a chart $\phi(p,\tilde{\rho}) \circ \rho' = \sigma(p) \circ \tilde{\rho} \circ \rho' = \phi(p, \tilde{\rho} \circ \rho')$, hence it is smooth. We consider the smooth bijection $F_\gamma \, {}_{\alpha_r}\!\times_t \mathrm{Mor}(\mathcal{P}_y) \longrightarrow F_\gamma \, {}_{\alpha_l}\!\times_{\alpha_l} F_\gamma$; that its inverse is smooth is – in charts – precisely the smoothness of the transition function $\tilde{\rho}$ of Lemma 3.9. Thus, F_γ is a principal $\mathcal{P}_y$-bundle over $\mathcal{P}_x$.

For the left $\mathcal{P}_x$-action, consider a section $\sigma_{\xi_0,\rho,g}$ defined in an open neighborhood U around p_0, and a morphism $\rho_0 \in \mathrm{Mor}(\mathcal{P}_x)$ such that $s(\rho_0) = p_0$. We set $p_0' := t(\rho_0)$ and $\xi_0' := \rho_0 \circ \xi_0$. Choose a transition span $\rho' : \tilde{U}' \longrightarrow \mathrm{Mor}(\mathcal{P}_x)$ with transition function g' defined in an open neighborhood $\tilde{U}' \subseteq \mathrm{Obj}(\mathcal{P}_x) \times_M \mathrm{Obj}(\mathcal{P}_x)$ of (p_0,p_0). This makes up another section $\sigma_{\xi_0',\rho',g'}$ defined in an open neighborhood U' of p_0'. Let $V := s^{-1}(U) \cap t^{-1}(U') \subseteq \mathrm{Mor}(\mathcal{P}_x)$. Using [13, Lemma 3.1.4] there exists a unique smooth map $h : V \longrightarrow H$ such that

$$R(\rho'(p_0',t(\eta))^{-1} \circ \rho_0, (h(\eta), g(p_0,s(\eta))^{-1})) = \eta \circ R(\rho(p_0,s(\eta))^{-1}, (1, g(p_0,s(\eta))^{-1})) \tag{15}$$

$$t(h(\eta))g(p_0,s(\eta))^{-1} = g'(p_0',t(\eta))^{-1}, \tag{16}$$

for all $\eta \in V$. Then we have

$$s(\rho_2) = R(t(\rho(p_0,s(\eta))), g(p_0,s(\eta))^{-1}) = R(p_0, g(p_0,s(\eta))^{-1}) = R(s(\rho_1), g(p_0,s(\eta))^{-1})$$

$$t(\rho_2) = t(\eta) = R(s(\rho'(p_0', t(\eta))), g'(p_0', t(\eta))^{-1}) = R(t(\rho_1), g'(p_0', t(\eta))^{-1}).$$

By [13, Lemma 3.1.4] and there exists a unique $h_\eta \in H$ such that

$$R(\rho_1, (h_\eta, g(p_0, s(\eta))^{-1})) = \rho_2 \quad \text{and} \quad t(h_\eta)g(p_0, s(\eta))^{-1} = g'(p_0', t(\eta))^{-1},$$

and the map $h : V \longrightarrow H : \eta \longmapsto h_\eta$ is smooth. Using these formulas, one can check that the left action is given in charts by $\eta \circ \phi_{\xi_0,\rho,g}(p, \tilde{\rho}) = \phi_{\xi_0',\rho',g'}(t(\eta), R(\tilde{\rho}, (h(\eta), g(p_0, p)^{-1})))$, and is hence smooth.

It remains to verify the smoothness of the $\mathrm{Mor}(\Gamma)$-action. Consider again a section $\sigma_{\xi_0,\rho,g}$ defined in an open neighborhood U of a point p_0, and a morphism $(h, g) \in \mathrm{Mor}(\Gamma)$. We recall that ρ is a transition span defined in an open set $\tilde{U} \subseteq \mathrm{Obj}(\mathcal{P}_x) \times_M \mathrm{Obj}(\mathcal{P}_x)$, and that $U = i_{p_0}^{-1}(\tilde{U})$. Let $g_0 := t(h)g$ and $p_0' := R(p_0, g_0)$. Choose open neighborhoods $V \subseteq \mathrm{Obj}(\Gamma)$ of g_0 and $\tilde{U}' \subseteq \mathrm{Obj}(\mathcal{P}_x) \times_M \mathrm{Obj}(\mathcal{P}_x)$ of (p_0', p_0') such that $(R(x', \tilde{g}^{-1}), R(y', \tilde{g}^{-1})) \in \tilde{U}$ and $(p_0, R(p, \tilde{g}g_0^{-1})) \in \tilde{U}$ for all $(p_0, p) \in \tilde{U}'$, $(x', y') \in \tilde{U}'$ and $\tilde{g} \in V$. Using [13, Lemma 3.1.4] one can construct a smooth map $h : U \times V \longrightarrow H$ such that parameterizes the dependence in the second argument of ρ under the action be group elements of the form $\tilde{g}g_0^{-1}$ for $\tilde{g} \in V$, in the sense that

$$R(\rho(p_0, p), (h(p, \tilde{g}), g(p_0, p)^{-1}\tilde{g}g_0^{-1}g(p_0, R(p, \tilde{g}g_0^{-1})))) = \rho(p_0, R(p, \tilde{g}g_0^{-1})) \tag{17}$$

$$t(h(p, \tilde{g}))g(p_0, p)^{-1}\tilde{g}g_0^{-1}g(p_0, R(p, \tilde{g}g_0^{-1})) = 1. \tag{18}$$

Next we translate the transition span ρ along g_0, and obtain another transition span $\rho' : \tilde{U}' \longrightarrow \mathrm{Mor}(\mathcal{P}_x)$ with transition function $g' : \tilde{U}' \longrightarrow G$ by setting

$$\rho'(x', y') := R(\rho(R(x', g_0^{-1}), R(y', g_0^{-1})), (h, g)).$$
$$g'(x', y') := g_0^{-1}g(R(x', g_0^{-1}), R(y', g_0^{-1}))g$$

We set $\xi_0' := \xi_0 \cdot (h, g)$; this satisfies $\alpha_l(\xi_0') = R(\alpha_l(\xi_0), g_0) = p_0'$. Now we have defined another section $\sigma_{\xi_0',\rho',g'}$ in a neighborhood $U' := i_{p_0'}^{-1}(\tilde{U}')$ of p_0'. We let $\tilde{V} := t^{-1}(V) \subseteq \mathrm{Mor}(\Gamma)$. Now, the action in charts of $(\tilde{h}, \tilde{g}) \in \tilde{V}$ is given by $\phi_{\xi_0,\rho,g}(p, \tilde{\rho}) \cdot (\tilde{h}, \tilde{g}) = \phi_{\xi_0',\rho',g'}(p', \tilde{\rho}')$ with $p' := R(p, \tilde{g}_0)$, and

$$\tilde{\rho}' := R(\tilde{\rho}, (\tilde{h}\alpha(\tilde{g}g_0^{-1}g(p_0, R(p, \tilde{g}_0g_0^{-1}))^{-1}, h(p, \tilde{g}_0)), \tilde{g}g_0^{-1}g(p_0, R(p, \tilde{g}_0g_0^{-1}))^{-1})),$$

where $\tilde{g}_0 := t(\tilde{h})\tilde{g}$. Both expressions depend smoothly on p, $\tilde{\rho}$, $\tilde{h}$, and $\tilde{g}$; this shows the smoothness. □

Remark 3.11. If Γ is topologically discrete (in the sense that objects and morphisms form 0-dimensional manifolds), then a principal Γ-2-bundle $\mathcal{P}$ is an example of a *2-covering space* in the sense of [8, Definition 4.47]. Further, $\mathcal{P}$ admits precisely one connection Ω with all components zero. In general, 2-covering space have a path lifting property [8, Prop. 4.63]. In case of a principal 2-bundle for a topologically trivial 2-group Γ, the path lifting property coincides with our result that $\alpha_l : F_\gamma \longrightarrow \mathcal{P}_{\gamma(0)}$ is surjective, for every path γ (Proposition 3.10). The path lifting property implies that the fibres of a 2-covering space over points in the same connected component are pairwise weakly equivalent [8, Prop. 4.66]. In case of a principal Γ-2-bundle, this follows abstractly from the definition (every fibre is weakly equivalent to Γ), and a concrete equivalence between $\mathcal{P}_x$ and $\mathcal{P}_y$ is given by F_γ (for $x = \gamma(0)$ and $y = \gamma(1)$), see Corollary 6.4.

3.3 Compatibility with path composition In this section we describe the compatibility of the parallel transport along paths with the composition of paths. In the transport 2-functor formalism described in Section 6.2 they constitute the functoriality of the 2-functor on the level of 1-morphisms.

Before we come to path composition, we look at the constant path id_x at $x \in M$. We define a Γ-equivariant transformation

$$u_x : \mathrm{id}_{\mathcal{P}_x} \Longrightarrow F_{\mathrm{id}_x},$$

which expresses the fact that the parallel transport along a constant path id_x is canonically 2-isomorphic to the identity 2-functor on the fibre $\mathcal{P}_x$. We define u_x using Remark 2.2 (d). The underlying smooth map $\tilde{u}_x : \mathrm{Obj}(\mathcal{P}_x) \longrightarrow F_{\mathrm{id}_x}$ is defined by $\tilde{u}_x(p) := \mathrm{id}_p$, where id_p denotes the constant path at p. Verifying (T1) to (T3) are straightforward calculations; alone in (T2) one has to use once the equivalence relation on F_{id_x}.

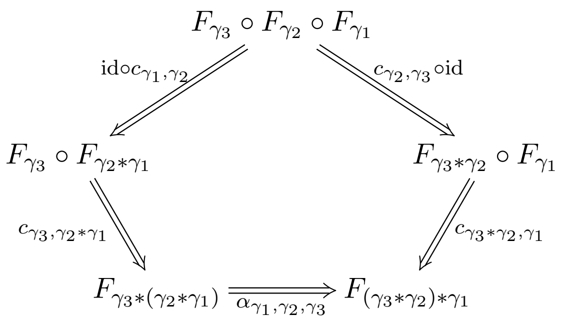
Two paths $\gamma_1, \gamma_2 : [0,1] \longrightarrow M$ are *composable*, if $\gamma_1(1) = \gamma_2(0)$ and the usual path concatenation $\gamma_2 * \gamma_1$ is smooth. In the following, we will often assume composability without explicitly mentioning it; at no place we use piecewise smoothness or any other regularity. We construct a transformation

$$c_{\gamma_1,\gamma_2} : F_{\gamma_2} \circ F_{\gamma_1} \Longrightarrow F_{\gamma_2 * \gamma_1},$$

which expresses the fact that the parallel transport along a composite path is canonically 2-isomorphic to the composition of the separate parallel transports. In order to define c_{γ_1,γ_2} we consider $\xi_1 \in F_{\gamma_1}$ and $\xi_2 \in F_{\gamma_2}$ such that $\alpha_r(\xi_1) = \alpha_l(\xi_2)$, i.e. $(\xi_1, \xi_2) \in F_{\gamma_2} \circ F_{\gamma_1}$. Its image under c_{γ_1,γ_2} is the element $\xi_2 * \xi_1 \in F_{\gamma_2 * \gamma_1}$ obtained by reparameterizing all path pieces, and composing the last morphism of ξ_1 with the first of ξ_2. This is obviously anchor-preserving and action-preserving, and preservation of the $\mathrm{Mor}(\Gamma)$-action can easily be checked.

If γ_1, γ_2, and γ_3 are paths, and $\gamma_3 * (\gamma_2 * \gamma_1)$ and $(\gamma_3 * \gamma_2) * \gamma_1$ are again paths (i.e. smooth), then the canonical reparameterization between $\gamma_3 * (\gamma_2 * \gamma_1)$ and $(\gamma_3 * \gamma_2) * \gamma_1$ induces an obvious transformation $\alpha_{\gamma_1,\gamma_2,\gamma_3} : F_{\gamma_3 * (\gamma_2 * \gamma_1)} \Longrightarrow F_{(\gamma_3 * \gamma_2) * \gamma_1}$. The following coherence property follows then directly from the definition of c_{γ_1,γ_2}.

Proposition 3.12. *Let $\mathcal{P}$ be a principal Γ-2-bundle with connection. Then, the following diagram commutes:*



Next we observe a compatibility condition between the transformations c_{γ_1,γ_2} and u_x. For this purpose, we first identify transformations $l_\gamma : F_\gamma \Longrightarrow F_{\gamma * \mathrm{id}_x}$ and $r_\gamma : F_\gamma \Longrightarrow F_{\mathrm{id}_y * \gamma}$ associated to a path $\gamma : x \longrightarrow y$. Indeed, given $\xi \in F_\gamma$, we reparameterize all path pieces by $t \longmapsto \frac{1}{2} + \frac{1}{2}t$, and add (using the formal composition $*$) the constant path $[0, \frac{1}{2}] \ni t \longmapsto \alpha_l(\xi)$ at the beginning; this gives an element of $F_{\gamma * \mathrm{id}_x}$, and defines the transformation l_γ. The transformation r_γ is defined analogously. The following result follows directly from the definitions.

Proposition 3.13. *Let $\mathcal{P}$ be a principal Γ-2-bundle with connection. For every path $\gamma : x \longrightarrow y$ the following diagrams commute:*

$$\begin{array}{ccc} F_\gamma \circ \mathrm{id}_{\mathcal{P}_x} & =\!=\!=\!= & F_\gamma \\ {\scriptstyle \mathrm{id}\circ u_x}\Big\Downarrow & & \Big\Downarrow{\scriptstyle l_\gamma} \\ F_\gamma \circ F_{\mathrm{id}_x} & \xRightarrow[c_{\mathrm{id}_x,\gamma}]{} & F_{\gamma * \mathrm{id}_x} \end{array} \quad \textit{and} \quad \begin{array}{ccc} \mathrm{id}_{\mathcal{P}_y} \circ F_\gamma & =\!=\!=\!= & F_\gamma \\ {\scriptstyle u_y\circ \mathrm{id}}\Big\Downarrow & & \Big\Downarrow{\scriptstyle r_\gamma} \\ F_{\mathrm{id}_y} \circ F_\gamma & \xRightarrow[c_{\gamma,\mathrm{id}_y}]{} & F_{\mathrm{id}_y * \gamma} \end{array}$$

3.4 Naturality with respect to bundle morphisms Suppose $J : \mathcal{P} \longrightarrow \mathcal{P}'$ is a 1-morphism between principal Γ-2-bundles with connections Ω and Ω', respectively, equipped with a connective, connection-preserving Ω'-pullback ν. Let $\gamma : [0,1] \longrightarrow M$ be a path with $x := \gamma(0)$ and $y := \gamma(1)$. We construct a Γ-equivariant transformation

$$\begin{array}{ccc} \mathcal{P}_x & \xrightarrow{F_\gamma} & \mathcal{P}_y \\ {\scriptstyle J_x}\Big\downarrow & J_\gamma \; \swarrow\!\!\!\swarrow & \Big\downarrow{\scriptstyle J_y} \\ \mathcal{P}'_x & \xrightarrow[F'_\gamma]{} & \mathcal{P}'_y \end{array}$$

relating the parallel transport F_γ in $\mathcal{P}$ with the parallel transport F'_γ in $\mathcal{P}'$. Involving the definition of composition of anafunctors ([13, Remark 2.3.2 (a)]) J_γ is induced by a map

$$J_\gamma : F_\gamma \, {}_{\alpha_r}\!\times_{\alpha_l} J_y \longrightarrow J_x \, {}_{\alpha_r}\!\times_{\alpha_l} F'_\gamma,$$

which we define in the following. We start with the following terminology: a *horizontal lift of $\xi \in F_\gamma$ to J* is a collection $\tilde{\xi} = (n, t, \{\rho_i\}_{i=0}^n, \{\tilde{\gamma}_i\}_{i=1}^n)$ consisting of $n \in \mathbb{N}$, a subdivision $t \in T_n$, morphisms $\rho_i \in \mathrm{Mor}(\mathcal{P}_x)$ and horizontal paths $\tilde{\gamma}_i : [t_{i-1}, t_i] \longrightarrow J$ such that $\alpha_l(\tilde{\gamma}_i(t_i)) = s(\rho_i)$ and $\alpha_l(\tilde{\gamma}_i(t_{i-1})) = t(\rho_{i-1})$ for $1 \leq i \leq n$, and

$$\xi \sim \rho_n * \alpha_l(\tilde{\gamma}_n) * \rho_{n-1} * \ldots * \alpha_l(\tilde{\gamma}_1) * \rho_0.$$

This means, in particular, that the paths $\alpha_l(\tilde{\gamma}_1)$ are horizontal. It is easy to see, e.g. using Propositions 3.4 (a) and 3.4 (e), that every $\xi \in F_\gamma$ admits horizontal lifts.

If $j \in J$ such that $\alpha_r(\xi) = \alpha_l(j)$, then the *$j$-target* of a horizontal lift $\tilde{\xi}$ is an element $(j', \xi') \in J_x \, {}_{\alpha_r}\!\times_{\alpha_l} F'_\gamma$ defined in the following way. We set $\gamma'_i := \alpha_r(\tilde{\gamma}_i)$. Since $\tilde{\gamma}_i$ and $\alpha_l(\tilde{\gamma}_i)$ are horizontal, γ'_i is horizontal by Proposition 3.4 (b). We proceed for an index $1 \leq i < n$, and note that $\alpha_l(\rho_i \circ \tilde{\gamma}_i(t_i)) = \alpha_l(\tilde{\gamma}_{i+1}(t_i))$. Since $\alpha_l : J \longrightarrow \mathrm{Obj}(\mathcal{P})$ is a principal $\mathcal{P}'$-bundle, there exists a unique $\rho'_i \in \mathrm{Mor}(\mathcal{P}')$ such that $\rho_i \circ \tilde{\gamma}_i(t_i) = \tilde{\gamma}_{i+1}(t_i) \circ \rho'_i$, and we get $t(\rho'_i) = \gamma'_{i+1}(t_i)$ and $s(\rho'_i) = \gamma'_i(t_i)$. The case $i = n$ is treated separately involving the element j. We have $\alpha_l(\rho_n \circ \tilde{\gamma}_n(1)) = \alpha_l(j)$. Hence, there exists a unique $\rho'_n \in \mathrm{Mor}(\mathcal{P}')$ such that $\rho_n \circ \tilde{\gamma}_n(1) = j \circ \rho'_n$, satisfying $s(\rho'_n) = \gamma'_n(t_n)$. The relations we have collected assert that we can combine the morphisms ρ'_i and paths γ'_i and set $\xi' := \rho'_n * \gamma'_n * \ldots * \gamma'_1 * \mathrm{id} \in F_{\gamma'}$. Finally, we put $j' := \rho_0^{-1} \circ \tilde{\gamma}_1(0) \in J_x$. The pair (j', ξ') is by definition the j-target of $\tilde{\xi}$.

Lemma 3.14. *The j-target is independent of the horizontal lift: if $\tilde{\xi}_1$ and $\tilde{\xi}_2$ are horizontal lifts of ξ and $j \in J$ with $\alpha_r(\xi) = \alpha_l(j)$, then the j-targets of $\tilde{\xi}_1$ and $\tilde{\xi}_2$ coincide.*

Proof. If the lifts are $\tilde{\xi}_1 = (\{\rho_{1,i}\}, \{\tilde{\gamma}_{1,i}\})$ and $\tilde{\xi}_2 = (\{\rho_{2,i}\}, \{\tilde{\gamma}_{2,i}\})$, then we have

$$\rho_{1,n} * \alpha_l(\tilde{\gamma}_{1,n}) * \rho_{1,n-1} * ... * \alpha_l(\tilde{\gamma}_{1,1}) * \rho_{1,0} \sim \rho_{2,n} * \alpha_l(\tilde{\gamma}_{2,n}) * \rho_{2,n-1} * ... * \alpha_l(\tilde{\gamma}_{2,1}) * \rho_{2,0}.$$

Thus, there exist horizontal paths ϕ_i in $\mathrm{Mor}(\mathcal{P}_x)$ with $s(\phi_i) = \alpha_l(\tilde{\gamma}_{1,i})$ and $t(\phi_i) = \alpha_l(\tilde{\gamma}_{2,i})$, as well as

$$\phi_1(0) \circ \rho_{1,0} = \rho_{2,0} \quad , \quad \rho_{1,n} = \rho_{2,n} \circ \phi_n(1) \quad \text{and} \quad \phi_{i+1}(t_i) \circ \rho_{1,i} = \rho_{2,i} \circ \phi_i(t_i). \tag{19}$$

We have $\alpha_l(\phi_i(t) \circ \tilde{\gamma}_{1,i}(t)) = \alpha_l(\tilde{\gamma}_{2,i}(t))$, so that there exist unique paths ϕ'_i in $\mathrm{Mor}(\mathcal{P}_y)$ with $t(\phi'_i) = \alpha_r(\tilde{\gamma}_{2,i}) = \gamma'_{2,i}$ and $s(\phi'_i) = \alpha_r(\tilde{\gamma}_{1,i}) = \gamma'_{1,i}$, such that

$$\phi_i(t) \circ \tilde{\gamma}_{1,i}(t) = \tilde{\gamma}_{2,i}(t) \circ \phi'_i(t) \tag{20}$$

By Propositions 3.4 (c) and 3.4 (d) it follows that ϕ'_i is horizontal. Next we collect the necessary identities Eqs. (21) to (23) that prove that the paths ϕ'_i constitute an equivalence between ξ'_1 and $\phi'_1(0)^{-1} \circ \xi'_2$. We consider for $1 \leq i < n$ the defining relations

$$\rho_{1,i} \circ \tilde{\gamma}_{1,i}(t_i) = \tilde{\gamma}_{1,i+1}(t_i) \circ \rho'_{1,i} \quad \text{and} \quad \rho_{2,i} \circ \tilde{\gamma}_{2,i}(t_i) = \tilde{\gamma}_{2,i+1}(t_i) \circ \rho'_{2,i}$$

for $\rho'_{1,i}$ and $\rho'_{2,i}$. Combining with Eqs. (19) and (20) we get

$$\rho'_{2,i} \circ \phi'_i(t_i) = \phi'_{i+1}(t_i) \circ \rho'_{1,i}. \tag{21}$$

At $i = 0$ we have $\rho'_{1,0} = \rho'_{2,0} = \mathrm{id}$. But $\phi'_1(0)^{-1} \circ \xi'_2$ has as its first morphism not $\rho'_{2,0}$ but $\rho''_{2,0} = \rho'_{2,0} \circ \phi'_1(0)$, so that we have

$$\rho'_{1,0} \circ \phi'_1(0) = \rho''_{2,0} \tag{22}$$

At $i = n$ the defining relations are $\rho_{1,n} \circ \tilde{\gamma}_{1,n}(1) = j \circ \rho'_{1,n}$ and $\rho_{2,n} \circ \tilde{\gamma}_{2,n}(1) = j \circ \rho'_{2,n}$. Combining with with Eqs. (19) and (20) gives

$$\rho'_{2,n} \circ \phi'_n(1) = \rho'_{1,n}. \tag{23}$$

Finally, we have $\rho_{1,0} \circ j'_1 = \tilde{\gamma}_{1,1}(0)$ and $\rho_{2,0} \circ j'_2 = \tilde{\gamma}_{2,1}(0)$. Combining with Eqs. (19) and (20) we get $j'_2 \circ \phi'_1(0) = j'_1$. Thus, we have

$$(j'_1, \xi'_1) = (j'_2 \circ \phi'_1(0), \phi'_1(0)^{-1} \circ \xi'_2) = (j'_2, \xi'_2);$$

this shows the claim. □

By Lemma 3.14 we have a well-defined map

$$J_\gamma : F_\gamma \, {}_{\alpha_r}\!\times_{\alpha_l} J_y \longrightarrow J_x \, {}_{\alpha_r}\!\times_{\alpha_l} F'_\gamma : (\xi, j) \longmapsto (j', \xi').$$

Lemma 3.15. *The map J_γ induces a transformation $J_y \circ F_\gamma \Longrightarrow F'_\gamma \circ J_x$.*

Proof. Again consulting [13, Remark 2.3.2 (a)] we have to check first that

$$J_\gamma(\xi \circ \rho, \rho^{-1} \circ j) = J_\gamma(\xi, j).$$

Let $\tilde{\xi} = (\{\rho_i\}, \{\tilde{\gamma}_i\})$ be a lift ξ with j-target (j', ξ'). Then, a lift $\tilde{\zeta}$ of $\xi \circ \rho$ is obtained from $\tilde{\xi}$ by only changing ρ_n to $\tilde{\rho}_n := \rho^{-1} \circ \rho_n$. We compute the j-target (j', ξ') of $\tilde{\xi}$ and the $(\rho^{-1} \circ j)$-target

(j'', ζ') of $\tilde{\zeta}$. The only difference is at their last morphisms ρ'_n and $\tilde{\rho}'_n$, whose defining identities are $\rho_n \circ \tilde{\gamma}_n(1) = j \circ \rho'_n$ and $\tilde{\rho}_n \circ \tilde{\gamma}_n(1) = \rho^{-1} \circ j \circ \tilde{\rho}'_n$, showing that $\tilde{\rho}'_n = \rho'_n$ and thus $\xi' = \zeta'$. The equality $j' = j''$ is obvious from their defining identities; this shows the claim. Now we have a well-defined map $J_y \circ F_\gamma \Longrightarrow F'_\gamma \circ J_x$ and have to check that it is a Γ-equivariant transformation.

That J_γ is anchor-preserving is straightforward to see. It also respects the actions: for the right action we have to show that

$$J_\gamma(\xi, j \circ \rho) = (j', \xi' \circ \rho),$$

where (j', ξ') is the j-target of a horizontal lift $\tilde{\xi} = (\{\rho_i\}, \{\tilde{\gamma}_i\})$ of ξ. The same $\tilde{\xi}$ is also a horizontal lift of ξ, and the only difference between its j-target and its $(j \circ \rho)$-target is at the last morphisms, where we get instead of ρ'_n the morphism $\rho''_n = \rho^{-1} \circ \rho'_n$. Thus, its $(j \circ \rho)$-target is $(j', \xi' \circ \rho)$, as claimed. For the left action we have to show

$$J_\gamma(\rho \circ \xi, j) = (\rho \circ j', \xi').$$

We choose for $\rho \circ \xi$ the horizontal lift $\tilde{\xi}$ with only ρ_0 changed to $\tilde{\rho}_0 = \rho_0 \circ \rho^{-1}$. Correspondingly, j' changes to $\tilde{j}' = \rho \circ j'$, while ξ' remains unchanged. This shows the claim.

We check that the $\mathrm{Mor}(\Gamma)$-action is preserved, which is equivalent to the identity

$$J_\gamma(\xi \cdot (h, g), j \cdot \mathrm{id}_g) = (j' \cdot (h, g), \xi' \cdot g)$$

see [13, Remark 2.4.2 (a)]. Here we have again fixed a choice $\tilde{\xi} = (\{\rho_i\}, \{\tilde{\gamma}_i\})$ of a horizontal lift of ξ, and defined (j', ξ') as the j-target of $\tilde{\xi}$. For

$$\xi \cdot (h, g) = R(\rho_n, g) * R(\gamma_n, g) * ... * R(\gamma_1, g) * R(\rho_0, (h^{-1}, t(h)g))$$

we choose the horizontal lift $(\{\tilde{\rho}_i\}, \{\tilde{\gamma}_i \cdot \mathrm{id}_g\})$ with $\tilde{\rho}_i := R(\rho_i, g)$ for $1 \leq i \leq n$ and $\tilde{\rho}_0 := R(\rho_0, (h^{-1}, t(h)g))$. We compute its $(j \cdot \mathrm{id}_g)$-target. We obtain the paths $R(\gamma'_i, g)$ and the morphisms $R(\rho'_i, g)$, and hence $\xi' \cdot \mathrm{id}_g$. The change in $\tilde{\rho}_0$ only enters the defining identity for j' from $\rho_0 \circ j' = \tilde{\gamma}_1(0)$ to $R(\rho_0, (h^{-1}, t(h)g)) \circ \tilde{j}' = \tilde{\gamma}_1(0) \cdot \mathrm{id}_g$, ending up with $\tilde{j}' = j' \cdot (h, g)$; this shows the claim.

Finally, we check that our map J_γ is smooth. We consider a chart of $F_\gamma \,{}_{\alpha_r}\!\times_{\alpha_l} J_y$,

$$U \,{}_{\alpha_r \circ \sigma_{\xi_0, \rho, g}}\!\times_t \mathrm{Mor}(\mathcal{P}_y) \,{}_s\!\times_{\alpha_l} J_y \longrightarrow \alpha_l^{-1}(U) \,{}_{\alpha_r}\!\times_{\alpha_l} J_y : (p, \tilde{\rho}, j) \longmapsto (\sigma_{\xi_0, \rho, g}(p) \circ \tilde{\rho}, j),$$

where $\sigma_{\xi_0, \rho, g}$ is the smooth section defined in Section 3.2. Using the approved compatibility with the various actions, we obtain

$$J_\gamma(\sigma_{\xi_0, \rho, g}(p) \circ \tilde{\rho}, j) = (\rho(p_0, p)^{-1} \circ J_\gamma(\xi_0, (\tilde{\rho} \circ j) \cdot (1, g(p_0, p)))) \cdot (1, g(p_0, p)^{-1}). \tag{24}$$

Now let $\tilde{\xi}$ be a horizontal lift for ξ_0. Let (j', ξ') be its j-target (note that $j = (\tilde{\rho} \circ j) \cdot (1, g(p_0, p))$ for $\tilde{\rho} = \mathrm{id}$ and $p = p_0$). Now we compute its $((\tilde{\rho} \circ j) \cdot (1, g(p_0, p)))$-target for general $\tilde{\rho}$ and p. The change does not affect j', and we denote it by $(j', \xi'(p, \tilde{\rho}, j))$, with $\xi'(p_0, \mathrm{id}, j) = \xi'$. In fact, the change only affects the last morphism $\rho'_n(p, \tilde{\rho}, j)$ of $\xi'(p, \tilde{\rho}, j)$. Its defining identity is

$$\rho_n \circ \tilde{\gamma}_n(1) = (\tilde{\rho} \circ j) \cdot (1, g(p_0, p)) \circ \rho'_n(p, \tilde{\rho}, j).$$

Since $\alpha_l : J \longrightarrow \mathrm{Obj}(\mathcal{P}_x)$ is a principal $\mathcal{P}_y$-bundle, this shows that $\rho'_n(p, \tilde{\rho}, j)$ depends smoothly on all parameters. We can write $\xi'(p, \tilde{\rho}, j) = \xi' \circ (\rho'_n \circ \rho'_n(p, \tilde{\rho}, j)^{-1})$. Thus,

$$J_\gamma(\xi_0, (\tilde{\rho} \circ j) \cdot (1, g(p_0, p))) = (j', \xi' \circ (\rho'_n \circ \rho'_n(p, \tilde{\rho}, j)^{-1})) = (j', \xi') \circ (\rho'_n \circ \rho'_n(p, \tilde{\rho}, j)^{-1}).$$

Inserting into Eq. (24) gives our final result for the map J_γ in above chart:

$$J_\gamma(\sigma_{\xi_0,\rho,g}(p) \circ \tilde{\rho}, j) = (\rho(p_0, p)^{-1} \circ (j', \xi') \circ (\rho'_n \circ \rho'_n(p, \tilde{\rho}, j)^{-1})) \cdot (1, g(p_0, p)^{-1}).$$

The right hand side is an expression of smooth functions in $(p, \tilde{\rho}, j)$ using operations that are smooth by Proposition 3.10; thus, it is smooth. □

Remark 3.16. Suppose $\phi : \mathcal{P} \longrightarrow \mathcal{P}'$ is a fibre-preserving, smooth, Γ-equivariant functor such that $\Omega = \phi^*\Omega'$. Then, there is a well-defined map $\phi : F_\gamma \longrightarrow F'_\gamma$ defined by associating to $\xi = \rho_n * \gamma_n * ... * \rho_0 \in F_\gamma$ the element $\phi(\xi) := \phi(\rho_n) * \phi(\gamma_n) * ... * \phi(\rho_0) \in F'_\gamma$. Now let $J = \mathrm{Obj}(\mathcal{P})\,_\phi\!\times_t \mathrm{Mor}(\mathcal{P}')$ be the associated anafunctor equipped with its canonical Ω'-pullback ν, see Remark 2.5. Then, the transformation

$$J_\gamma : F_\gamma\, {}_{\alpha_r}\!\times_{\alpha_l} J_y \longrightarrow J_x\, {}_{\alpha_r}\!\times_{\alpha_l} F'_\gamma$$

can be expressed in terms of the functor ϕ by the formula

$$J_\gamma(\xi, (p, \rho')) = ((p', \mathrm{id}_{\phi(p')}), \phi(\xi) \circ \rho'),$$

where $p = \alpha_r(\xi)$ and $p' = \alpha_l(\xi)$. Indeed, a horizontal lift $\tilde{\xi} = (\{\rho_i\}, \{\tilde{\gamma}_i\})$ of ξ is obtained by choosing a representative $\xi = \rho_n * \gamma_n * ... * \rho_0$ and setting $\tilde{\gamma}_i := (\gamma_i, \mathrm{id}_{\phi(\gamma_i)})$. By Remark 3.3 and Proposition 3.2 (b), $\tilde{\gamma}_i$ is horizontal with respect to ν. We compute the (p, ρ')-target (j', ξ') of $\tilde{\xi}$: first we obtain $\gamma'_i = \phi(\gamma_i)$. It is then straightforward to check that $\rho'_i = \phi(\rho_i)$ (for $1 \le i < n$) as well as $\rho'_n = \rho'^{-1} \circ \phi(\rho_n)$, so that

$$\xi' = (\rho'^{-1} \circ \phi(\rho_n)) * \phi(\gamma_n) * ... * \phi(\gamma_1) * \mathrm{id} = \phi(\rho_0) \circ \phi(\xi) \circ \rho'.$$

Finally, we obtain $j' = (\alpha_l(\xi), \phi(\rho_0)^{-1})$. The result is

$$(j', \xi') = ((\alpha_l(\xi), \phi(\rho_0)^{-1}), \phi(\rho_0) \circ \phi(\xi) \circ \rho') = ((\alpha_l(\xi), \mathrm{id}), \phi(\xi) \circ \rho'),$$

as claimed.

Now we suppose that $\kappa = (\kappa_0, \kappa_1)$ shifts the canonical Ω'-pullback ν to another connection-preserving and connective pullback ν^κ. The connections Ω and Ω' are then related by the formulas of [13, Remark 5.2.10 (e)]. Our lifts $\tilde{\gamma}_i = (\gamma_i, \mathrm{id}_{\phi(\gamma_i)})$ are no longer horizontal with respect to ν^κ. By Proposition 3.4 (e) there exist unique paths $h_i : [t_{i-1}, t_i] \longrightarrow H$ with $h_i(t_{i-1}) = 1$ such that $\tilde{\gamma}_i^\kappa := \tilde{\gamma}_i \cdot (h_i, t(h_i)^{-1}) = (\gamma_i, R(\mathrm{id}_{\phi(\gamma_i)}, (h_i, t(h_i)^{-1})))$ is horizontal. From Remark 3.3 one can conclude that h_i solves the initial value problem

$$\dot{h}_i(t) = -h_i(t)\kappa_0(\dot{\gamma}_i(t)) \quad \text{and} \quad h_i(t_{i-1}) = 1. \tag{25}$$

Since $\alpha_l(\tilde{\gamma}_i^\kappa) = \gamma_i$ it is clear that $\tilde{\xi}^\kappa = (\{\rho_i\}, \{\tilde{\gamma}_i^\kappa\})$ is a horizontal lift of ξ. We compute the (p, ρ')-target (j', ξ') of $\tilde{\xi}^\kappa$. We get $\gamma'_i = R(\phi(\gamma_i), t(h_i)^{-1})$, and for the morphisms we get $\rho'_i = R(\phi(\rho_i), (h_i(t_i), t(h_i(t_i))^{-1}))$ for $1 \le i < n$, $\rho'_n = \rho'^{-1} \circ R(\phi(\rho_n), (h_n(1), t(h_n(1))^{-1}))$, and $j' = (p, \phi(\rho_0)^{-1})$. Summarizing, we have

$$J_\gamma(\xi, (p, \rho')) = ((p', \mathrm{id}_{\phi(p')}), \phi^\kappa(\xi) \circ \rho'),$$

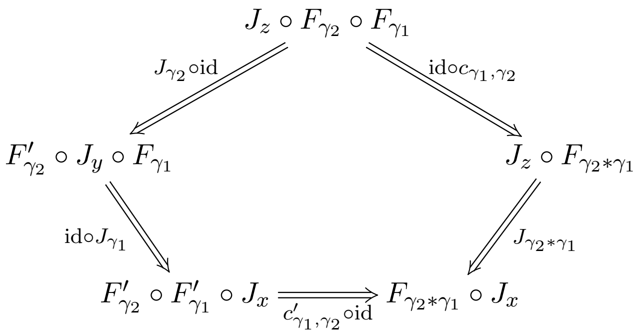
where $p' := \alpha_l(\xi)$ and we now define

$$\phi^\kappa(\xi) := \rho''_n * \gamma'_n * \rho''_{n-1} * ... * \gamma'_1 * \rho''_0$$

from the following components: γ'_i is given by above formulae, with h_i determined by Eq. (25), $\rho''_i := R(\phi(\rho_i), (h_i(t_i), t(h_i(t_i))^{-1}))$ for $1 \le i \le n$, and $\rho''_0 := \phi(\rho_0)$.

The following result describes in which way the transformation J_γ is compatible with path composition. In the 2-functor formalism described in Section 6.2 it is one of the axioms of a pseudonatural transformations between the transport 2-functors associated to the two principal Γ-2-bundles.

Proposition 3.17. *Let $J : \mathcal{P}_1 \longrightarrow \mathcal{P}_2$ be a 1-morphism in $2\text{-}\mathcal{B}un^{\nabla}_{\Gamma}(M)$. The associated transformation J_γ is compatible with path composition in the sense that the following diagram commutes for each pair of composable paths $\gamma_1 : x \longrightarrow y$ and $\gamma_2 : y \longrightarrow z$:*



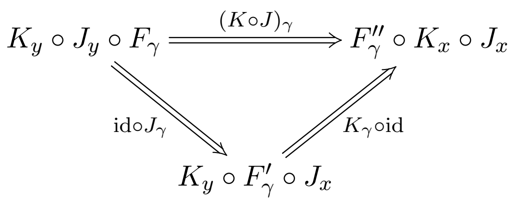
Proof. We consider an element $(\xi_1, \xi_2, j) \in F_{\gamma_1 \alpha_r}\!\times_{\alpha_l} F_{\gamma_2 \alpha_r}\!\times_{\alpha_l} J_z$. We choose separately horizontal lifts $\tilde{\xi}_1$ and $\tilde{\xi}_2$ of ξ_1 and ξ_2 to J. Let (j', ξ'_2) be the j-target of $\tilde{\xi}_2$, and let (j'', ξ'_1) be the j'-target of $\tilde{\xi}_1$. Then, going counter-clockwise, we have

$$(\xi_1, \xi_2, j) \longmapsto (\xi_1, j', \xi'_2) \longmapsto (j'', \xi'_1, \xi'_2) \longmapsto (j'', \xi'_2 * \xi'_1).$$

Under the obvious reparameterization and renumbering one can combine the horizontal lifts $\tilde{\xi}_1$ and $\tilde{\xi}_2$ to a lift $\widetilde{\xi_2 * \xi_1}$ of $c_{\gamma_1,\gamma_2}(\xi_1, \xi_2) = \xi_2 * \xi_1$. A straightforward computation shows that its j-target is $(j'', \xi'_2 * \xi'_1)$. This shows that the diagram is commutative. □

Further, J_γ is compatible with the composition of bundle morphisms in the following sense:

Proposition 3.18. *Suppose $J : \mathcal{P} \longrightarrow \mathcal{P}'$ and $K : \mathcal{P}' \longrightarrow \mathcal{P}''$ are 1-morphisms in $2\text{-}\mathcal{B}un^{\nabla}_{\Gamma}(M)$. Then, the following diagram commutes for each path $\gamma : x \longrightarrow y$:*



Proof. We start with an element (ξ, j, k) in $F_{\gamma\ \alpha_r}\!\times_{\alpha_l} J_{y\ \alpha_r}\!\times_{\alpha_l} K_y$. Let $\tilde{\xi} = (\{\rho_i^J\}, \{\tilde{\gamma}_i^J\})$ be a horizontal lift of ξ to J, and let (ξ', j') be its j-target. Let $\tilde{\xi}' = (\{\rho_i^K\}, \{\tilde{\gamma}_i^K\})$ be a horizontal lift of ξ' to K, and let (k', ξ'') be its k-target. Then, going counter-clockwise results in (j', k', ξ''). In order to compute the clock-wise direction, we notice that $(\{\rho_i^J\}, \{(\tilde{\gamma}_i^J, \tilde{\gamma}_i^K)\})$ is a horizontal lift of ξ to $K \circ J$, using the definition of the pullback on the composition $K \circ J$ (see [13, Lemma 4.3.5 (a)]). A straightforward computation shows that its (j, k)-target is (j', k', ξ''). □

Finally, there is a compatibility condition with 2-morphisms, which is responsible for an axiom of a modification in the 2-functor formalism of Section 6.2.

Proposition 3.19. *Suppose $J, J' : \mathcal{P} \longrightarrow \mathcal{P}'$ are 1-morphisms in $2\text{-}\mathcal{B}un^{\nabla}_{\Gamma}(M)$, and $f : J \Longrightarrow J'$ is a connection-preserving 2-morphism. Then, the following diagram commutes for each path $\gamma : x \longrightarrow y$:*

$$\begin{array}{ccc} J_y \circ F_\gamma & \overset{J_\gamma}{\Longrightarrow} & F'_\gamma \circ J_x \\ {\scriptstyle f_y \circ \mathrm{id}} \Big\Downarrow & & \Big\Downarrow {\scriptstyle \mathrm{id} \circ f_x} \\ J'_y \circ F_\gamma & \underset{J'_\gamma}{\Longrightarrow} & F'_\gamma \circ J'_x . \end{array}$$

Proof. Suppose $(\xi, j) \in F_{\gamma\,\alpha_r}\!\times_{\alpha_l} J_y$. Let $(\{\rho_i\}, \{\tilde\gamma_i\})$ be a horizontal lift of ξ to J, and let (j', ξ') be its j-target. Then, $(\{\rho_i\}, \{f(\tilde\gamma_i)\})$ is obviously a horizontal lift of ξ to J', and it is easy to show that its $f(j)$-target is $(f(j'), \xi')$. This shows commutativity. □

3.5 Naturality with respect to pullback Suppose $\mathcal{P}$ is a principal Γ-2-bundle over N with connection Ω, $f : M \longrightarrow N$ is a smooth map, and $\gamma : [0,1] \longrightarrow M$ is a path. We denote by $\mathcal{P}' := f^*\mathcal{P}$ the pullback bundle, obtain a Γ-equivariant smooth functor $\tilde f : \mathcal{P}' \longrightarrow \mathcal{P}$, and $\Omega' := \tilde f^*\Omega$ is a connection on $\mathcal{P}'$. We construct a transformation

$$\begin{array}{ccc} \mathcal{P}'_x & \xrightarrow{F'_\gamma} & \mathcal{P}_y \\ {\scriptstyle \tilde f_x} \Big\downarrow & \overset{\tilde f_\gamma}{\swarrow\!\!\swarrow} & \Big\downarrow {\scriptstyle \tilde f_y} \\ \mathcal{P}_{f(x)} & \xrightarrow[F_{f\circ\gamma}]{} & \mathcal{P}_{f(y)} \end{array}$$

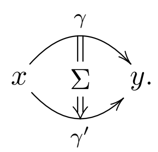
We first recall that $\mathrm{Obj}(\mathcal{P}') = M\,{}_f\!\times_\pi \mathrm{Obj}(\mathcal{P})$ and $\mathrm{Mor}(\mathcal{P}') = M\,{}_f\!\times_{\tilde\pi} \mathrm{Mor}(\mathcal{P})$, using that $\pi : \mathrm{Obj}(\mathcal{P}) \longrightarrow M$ and $\tilde\pi : \mathrm{Mor}(\mathcal{P}) \longrightarrow M$ (defined as $\tilde\pi = \pi \circ t = \pi \circ s$) are submersions. We can hence canonically identify $\mathcal{P}'_x = \mathcal{P}_{f(x)}$, so that the functor $\tilde f_x : \mathcal{P}'_x \longrightarrow \mathcal{P}_{f(x)}$ is just the identity. It remains to construct a transformation

$$\tilde f_\gamma : F'_\gamma \Longrightarrow F_{f\circ\gamma}.$$

Suppose $\xi \in F'_\gamma$, i.e. $\xi' = \rho_n * \gamma_n * ... * \rho_1 * \gamma_1 * \rho_0$, with γ_i horizontal paths in $\mathcal{P}'$ such that $\pi'(\gamma_i(t)) = \gamma(t)$. It is clear that $\tilde f \circ \gamma_i$ are horizontal paths in $\mathcal{P}$ with $\pi((\tilde f \circ \gamma_i)(t)) = (f \circ \gamma)(t)$. Thus, $\tilde f_\gamma(\xi) := \tilde f(\rho_n) * (\tilde f \circ \gamma_n) * ... * \tilde f(\rho_0) \in F_{f\circ\gamma}$. It is straightforward to show that this definition indeed defines a Γ-equivariant transformation.

4. Parallel transport along bigons

A *bigon* in a smooth manifold M is a smooth, fixed-ends homotopy between two paths γ and γ' with common end-points x and y. More precisely, a bigon is a smooth map $\Sigma : [0,1]^2 \longrightarrow M$ such that $\Sigma(s,0) = x$ and $\Sigma(s,1) = y$ for all $s \in [0,1]$, and $\gamma(t) = \Sigma(0,t)$ and $\gamma'(t) := \Sigma(1,t)$ for all $t \in [0,1]$. We use the notation $\Sigma : \gamma \Longrightarrow \gamma'$, and the instructive picture



Bigons represent directed pieces of surfaces, along which we are going to define parallel transport.

Let $\mathcal{P}$ be a principal Γ-bundle over M with a fake-flat connection Ω. That parallel transport along surfaces can only be defined for *fake-flat* connections is a well-known phenomenon [11]. In this section we define for each bigon $\Sigma : \gamma \Longrightarrow \gamma'$ a Γ-equivariant transformation

$$\varphi_\Sigma : F_\gamma \longrightarrow F_{\gamma'}$$

between the parallel transports along γ and γ'. For this purpose, we first introduce in Section 4.1 the notion of a horizontal lift of a bigon to the total space of $\mathcal{P}$. In Section 4.2 we give a complete definition of the transformation φ_Σ. In Sections 4.3 to 4.5 we derive several properties of φ_Σ with respect to the composition of bigons, 1-morphisms between principal 2-bundles, and pullback.

4.1 Horizontal lifts of bigons Let $\mathcal{P}$ be a principal Γ-2-bundle over M together with a fake-flat connection Ω.

Definition 4.1. Let $\Sigma : \gamma \Longrightarrow \gamma'$ be a bigon and $\xi \in F_\gamma$. A *horizontal lift* of Σ with source ξ is a tuple $(n, t, \{\Phi_i\}_{i=1}^n, \{\rho_i\}_{i=0}^n, \{g_i\}_{i=1}^n)$ consisting of $n \in \mathbb{N}$, a subdivision $t \in T_n$ and smooth maps

- $\Phi_i : [0,1] \times [t_{i-1}, t_i] \longrightarrow \mathrm{Obj}(\mathcal{P})$
- $\rho_i : [0,1] \longrightarrow \mathrm{Mor}(\mathcal{P})$ with ρ_0 and ρ_n constant
- $g_i : [0,1] \longrightarrow G$ with $g_i(0) = 1$

such that the following conditions are satisfied:

(a) Φ_i is a lift of Σ, i.e., $\pi \circ \Phi_i = \Sigma|_{[0,1]\times[t_{i-1},t_i]}$ for all for $1 \leq i \leq n$

(b) $t(\rho_i(s)) = \Phi_{i+1}(s, t_i)$ for all $0 \leq i < n$ and $s(\rho_i(s)) = R(\Phi_i(s, t_i), g_i(s))$ for all $1 \leq i \leq n$.

(c) The paths $\gamma_i'(t) := \Phi_i(1, t)$, $\nu_i(s) := \Phi_i(s, t_{i-1})$ and ρ_i are horizontal for all $1 \leq i \leq n$.

(d) $\xi = \rho_n * \gamma_n * ... * \gamma_1 * \rho_0$ with $\gamma_i(t) := \Phi_i(0, t)$ and $\rho_i := \rho_i(0)$

We begin with "small" bigons: a bigon $\Sigma : \gamma \Longrightarrow \gamma'$ is called *small*, if there exist $n \in \mathbb{N}$, $t \in T_n$ and sections $\sigma_i : U_i \longrightarrow \mathrm{Obj}(\mathcal{P})$ defined on open sets U_i such that

$$\Sigma(\{(s,t) \mid t_{i-1} \leq t \leq t_i, 0 \leq s \leq 1\}) \subseteq U_i.$$

Lemma 4.2. *For every small bigon $\Sigma : \gamma \Longrightarrow \gamma'$ and every $\xi \in F_\gamma$ there exists a horizontal lift with source ξ.*

Proof. We choose for $1 \leq i \leq n$ sections $\sigma_i : U_i \longrightarrow \mathrm{Obj}(\mathcal{P})$, and fr $1 \leq i < n$ transition spans $\sigma_{i,i+1} : U_i \cap U_j \longrightarrow \mathrm{Mor}(\mathcal{P})$ along (σ_i, σ_{i+1}) together with transition functions $g_{i,i+1} : U_i \cap U_{i+1} \longrightarrow G$. These choices can successively be adjusted such that $g_{i,i+1}(\Sigma(0, t_i)) = 1$.

We set, for $1 \leq i \leq n$, $\Phi_i := \sigma_i \circ \Sigma|_{[0,1]\times[t_{i-1},t_i]}$; this satisfies (a). We also set, for $1 \leq i < n$,

$$\rho_i(s) := R(\sigma_{i,i+1}(\Sigma(s,t_i))^{-1}, g_{i,i+1}(\Sigma(s,t_i))^{-1}) \quad \text{and} \quad g_i(s) := g_{i,i+1}(\Sigma(s,t_i))^{-1};$$

above adjustment achieves $g_i(0) = 1$. Further, we set $\rho_0(s) := \mathrm{id}_{\sigma_1(x)}$, $\rho_n(s) := \mathrm{id}_{\sigma_n(y)}$ and $g_n(s) := 1$. This satisfies (b) by definition of a transition span. Next we perform some modifications. Let $\xi = \rho_n' * \gamma_n' * ... * \gamma_1' * \rho_0'$ be a representative. We choose for $1 \leq i \leq n$ paths $\tilde{\rho}_i : [t_{i-1}, t_i] \longrightarrow \mathrm{Mor}(\mathcal{P})$ and $\tilde{g}_i : [t_{i-1}, t_i] \longrightarrow G$ such that $s(\tilde{\rho}_i(t)) = \gamma_i'(t)$ and $t(\tilde{\rho}_i(t)) = R(\Phi_i(0,t), \tilde{g}_i(t))$. By Proposition 3.2 (a) there exist a unique path $h_i : [t_{i-1}, t_i] \longrightarrow H$ with $h_i(t_{i-1}) = 1$ such that $\tilde{\rho}_i^{hor} := R(\tilde{\rho}_i, (h_i, 1))$ is horizontal. Successively, this data can be arranged such that $t(h_i(t_i))^{-1}\tilde{g}_i(t_i)^{-1}\tilde{g}_{i+1}(t_i) = 1$. We define:

$$\Phi_i'(s,t) := R(\Phi_i(s,t), \tilde{g}_i(t)t(h_i(t))) \qquad \text{for } 1 \leq i \leq n$$

$$\begin{aligned} \rho_i'(s) &:= R(\rho_i(s), \tilde g_{i+1}(t_i)) && \text{for } 0 \leq i < n \\ \rho_n'(s) &:= \rho_n(s) \\ g_i'(s) &:= t(h_i(t_i))^{-1}\tilde g_i(t_i)^{-1} g_i(s)\tilde g_{i+1}(t_i) && \text{for } 1 \leq i < n \\ g_n'(s) &:= t(h_n(t_n))^{-1}\tilde g_n(t_n)^{-1} g_n(s) \end{aligned}$$

This modification does not affect (a) and still satisfies (b). Since

$$s(\tilde\rho_i^{hor}(t)) = \gamma_i'(t) \quad \text{and} \quad t(\tilde\rho_i^{hor}(t)) = \Phi_i'(0,t),$$

we can apply the equivalence relation to the horizontal paths $\tilde\rho_i^{hor}$, so that $\xi = \rho_n * \gamma_n * ... * \gamma_1 * \rho_0$ with $\gamma_i(t) := \Phi_i'(0,t)$ and some new $\rho_0, ..., \rho_n \in \mathrm{Mor}(\mathcal{P})$. This makes up the first part of (d); we have not yet achieved that $\rho_i = \rho_i'(0)$. Note that by (b) we have $t(\rho_i) = \gamma_{i+1}(0) = t(\rho_i'(0))$ for $0 \leq i < n$ and $s(\rho_i) = \gamma_i(1) = s(\rho_i'(0))$ for $1 \leq i \leq n$. By [13, Lemma 3.1.4] there exist, for $1 \leq i < n$ unique $h_i \in H$ with $t(h_i) = 1$ such that $\rho_i = R(\rho_i'(0), (h, 1))$. We set $\rho_i''(s) := R(\rho_i'(s), (h_i, 1))$. At the endpoints we define $\rho_0''(s) := \rho_0$ and $\rho_n''(s) := \rho_n$; this still satisfies (b). Now $\{\Phi_i', \rho_i'', g_i'\}$ satisfy (a), (b) and (d).

Next we look for the first part of (c), horizontality of the γ_i'. By Proposition 3.1 (a) there exist, for $1 \leq i \leq n$ smooth maps $\tilde g_i : [t_{i-1}, t_i] \longrightarrow G$ with $\tilde g_i(t_{i-1}) = 1$ and $t \longmapsto R(\Phi_i(1,t), \tilde g_i(t))$ horizontal. We define $\varphi_i : [0,1] \times [t_{i-1}, t_i] \longrightarrow G$ by $\varphi_i(s,t) := \tilde g_i(s(t - t_{i-1}) + t_{i-1})$. We put

$$\Phi_i''(s,t) := R(\Phi_i'(s,t), \varphi_i(s,t)) \quad , \quad \rho_i'' := \rho_i' \quad \text{and} \quad g_i'' := \tilde g_i^{-1} \cdot g_i'.$$

Obviously, (a) is not affected. For (b) we check

$$t(\rho_i''(s)) = t(\rho_i'(s)) = \Phi_{i+1}'(s, t_i) = \Phi_{i+1}''(s, t_i)$$

since $\varphi_{i+1}(s, t_i) = \tilde g_{i+1}(0) = 1$, and

$$s(\rho_i''(s)) = s(\rho_i'(s)) = R(\Phi_i'(s,t_i), g_i'(s)) = R(\Phi_i'(s,t_i), \tilde g_i(s)^{-1}\varphi_i(s,t_i) g_i'(s)) = R(\Phi_i''(s,t_i), g_i''(s)).$$

Since $\varphi_i(0,t) = 1$, (d) is also not affected, and the new γ_i' are horizontal.

Next we look for horizontality of the ν_i. There exist paths $\tilde g_i : [0,1] \longrightarrow G$ with $\tilde g_i(0) = 1$ and $s \longmapsto R(\nu_i(s), \tilde g_i(s))$ horizontal. We set $\rho_n' := \rho_n$ and

$$\Phi_i'(s,t) := R(\Phi_i(s,t), \tilde g_i(s)) \quad , \quad \rho_i'(s) := R(\rho_i(s), \tilde g_{i+1}(s)) \quad \text{and} \quad g_i'(s) := \tilde g_i(s)^{-1} g_i(s) \tilde g_{i+1}(s).$$

Obviously, (a) is not affected. For (b) we check

$$\begin{aligned} t(\rho_i'(s)) &= R(t(\rho_i(s)), \tilde g_{i+1}(s)) = R(\Phi_{i+1}(s, t_i), \tilde g_{i+1}(s)) = \Phi_{i+1}'(s, t_i) \\ s(\rho_i'(s)) &= R(s(\rho_i(s)), \tilde g_{i+1}(s)) = R(\Phi_i(s,t_i), g_i(s)\tilde g_{i+1}(s)) = R(\Phi_i'(s,t_i), g_i'(s)). \end{aligned}$$

Further, since ν_1 is constant, we have $\tilde g_1 = 1$, meaning that ρ_0' remains constant. Since $\tilde g_i(0) = 1$, (d) is not affected, and since γ_i' is shifted by constant $\tilde g_i(1)$, horizontality of γ_i' is not spoiled.

Finally, we look for horizontality of the ρ_i. By Proposition 3.2 (a) there exist paths $\tilde h_i : [0,1] \longrightarrow H$ with $\tilde h_i(0) = 1$ and $s \longmapsto R(\rho_i(s), (\tilde h_i(s), 1))$ horizontal. By Proposition 3.2 (f) also $s \longmapsto R(\rho_i(s), (\tilde h_i(s), t(\tilde h_i(s)^{-1})))$ is horizontal. We set

$$\Phi_i' := \Phi_i \quad , \quad \rho_i'(s) := R(\rho_i(s), (\tilde h_i(s), t(\tilde h_i(s)^{-1}))) \quad \text{and} \quad g_i'(s) := g_i(s) t(\tilde h_i(s))^{-1}.$$

Obviously, (a) is not affected. For (b) we check

$$t(\rho_i'(s)) = t(\rho_i(s)) = \Phi_{i+1}(s,t_i) = \Phi_{i+1}'(s,t_i)$$
$$s(\rho_i'(s)) = R(s(\rho_i(s)), t(\tilde{h}_i(s))^{-1}) = R(\Phi_i(s,t_i), g_i(s)t(\tilde{h}_i(s))^{-1}) = R(\Phi_i'(s,t_i), g_i'(s)).$$

Since $\tilde{h}_i(0) = 1$, (d) is not affected, and since Φ_i is unchanged, horizontality of γ_i' and μ_i' persists. □

Next we define the target of a horizontal lift of a small bigon. We set $\mu_i(s) := \Phi_i(s,t_i)$; then we can reformulate (b) as:

$$s(\rho_i) = R(\mu_i, g_i) \text{ for all } 1 \leq i \leq n \quad \text{ and } \quad t(\rho_i) = \nu_{i+1} \text{ for all } 0 \leq i < n.$$

We consider a bigon-parameterization Σ_i of Φ_i, see Remark C.2, and the associated surface-ordered exponential $h_i := soe_\Omega(\Sigma_i) \in H$ defined in Section C.

Lemma 4.3. *We have $t(h_i) = g_i(1)^{-1}$ for all $1 \leq i \leq n$.*

Proof. We have $\Sigma_i : \tilde{\mu}_i * \tilde{\gamma}_i \Longrightarrow \tilde{\gamma}_i' * \tilde{\nu}_i$, where $\tilde{\mu}_i$ is thin homotopic to μ_i, and similar for the other paths. Since ν_i, γ_i and γ_i' are horizontal, we have by Proposition C.1 (b) $t(h_i)poe_{\Omega^a}(\mu_i) = 1$. Since ρ_i and $t(\rho_i)$ are horizontal, $s(\rho_i)$ is horizontal by Proposition 3.2 (g). Together with Proposition B.2 (a) we obtain

$$1 = poe_{\Omega^a}(s(\rho_i)) = poe_{\Omega^a}(R(\mu_i, g_i)) = g_i(1)^{-1} poe_{\Omega^a}(\mu_i).$$ □

We define

$$\rho_0' := \rho_0(1) \quad \text{ and } \quad \rho_i' := R(\rho_i(1), (h_i^{-1}, g_i(1)^{-1})). \tag{26}$$

It is straightforward to check that this gives an element $\xi' := \rho_n' * ... * \gamma_1' * \rho_0'$ in $F_{\gamma'}$, which we call the *target* of the horizontal lift.

Lemma 4.4. *If $\Sigma : \gamma \Longrightarrow \gamma'$ is a small bigon and $\xi \in F_\gamma$, then the target of a horizontal lift of Σ with source ξ is independent of the choice of the horizontal lift.*

Proof. Let $(n, t, \Phi_i, \rho_i, g_i)$ and $(\tilde{n}, \tilde{t}, \tilde{\Phi}_i, \tilde{\rho}_i, \tilde{g}_i)$ be horizontal lifts of Σ with source ξ. First of all, we can assume that $\tilde{n} = n$ and $\tilde{t} = t$, since we can introduce new points $t_{i-1} < t' < t_i$ and then cut a horizontal lift at t' (it is easy to see that one can arrange the new path $s \longmapsto \Phi_i(s,t')$ to be horizontal by compensating with the map g_i').

Next, we note that we have two sections Φ_i and $\tilde{\Phi}_i'$ into Obj($\mathcal{P}$) along $\Sigma|_{[0,1]\times[t_{i-1},t_i]}$. By [13, Lemma 3.1.6] they admit a transition span Ψ_i with transition function G_i. In the following we show that we can assume a couple of properties for Ψ_i and G_i.

The condition that our horizontal lifts have the same source, ξ, means that there exist horizontal path $\eta_i : [t_{i-1}, t_i] \longrightarrow$ Mor($\mathcal{P}$) with $s(\eta_i(t)) = \Phi_i(0,t)$ and $t(\eta_i(t)) = \tilde{\Phi}_i(0,t)$ and $\tilde{\rho}_i \circ \eta_i(t_i) = \eta_{i+1}(t_i) \circ \rho_i$. Comparing the transition spans $(\eta_i, 1)$ with $(\Psi_i(0,-), G_i(0,-))$ we obtain by [13, Lemma 3.1.6] a smooth map $h_i : [t_{i-1}, t_i] \longrightarrow H$ such that $R(\Psi_i(0,t), (h_i(t), G_i(0,t)^{-1}) = \eta_i(t)$ and $t(h_i(t)) = G_i(0,t)$. We consider

$$\Psi_i'(s,t) := R(\Psi_i(s,t), (h_i(t), G_i(0,t)^{-1}) \quad \text{ and } \quad G_i'(s,t) := G_i(s,t)t(h_i(t))^{-1},$$

which satisfy $t(\Psi_i'(s,t)) = \tilde{\Phi}_i(s,t)$ and $s(\Psi_i'(s,t)) = R(\Phi_i(s,t), G_i'(s,t))$. This shows that we can always choose our transition spans Ψ_i such that $t \longmapsto \Psi_i(0,t)$ is horizontal, $G_i(0,t) = 1$ and

$$\tilde{\rho}_i \circ \Psi_i(0,t_i) = \Psi_{i+1}(0,t_i) \circ \rho_i. \tag{27}$$

Next we consider the path $s \longmapsto \Psi_i(s,t_{i-1})$. By Proposition 3.2 (a) there exist paths $h_i : [0,1] \longrightarrow H$ with $h_i(0)=1$ such that $R(\Psi_i(-,t_{i-1}),(h_i,1))$ is horizontal. We consider

$$\Psi_i'(s,t) := R(\Psi_i(s,t),(h_i(s),t(h_i(s))^{-1})) \quad\text{and}\quad G_i'(s,t) := G_i(s,t)t(h_i(s))^{-1},$$

this gives by Proposition 3.2 (f) a horizontal path, satisfying $t(\Psi_i'(s,t)) = \tilde\Phi_i(s,t)$ and $s(\Psi_i'(s,t)) = R(\Phi_i(s,t),G_i'(s,t))$. Since the quantities at $s=0$ are unchanged, we can add the horizontality of $s \longmapsto \Psi_i(s,t_{i-1})$ to our assumptions.

Finally, we consider the path $t \longmapsto \Psi_i(1,t)$. By Proposition 3.2 (a) there exists a path $h_i : [0,1] \longrightarrow H$ with $h_i(0)=1$ such that $R(\Psi_i(1,-),(h_i,1))$ is horizontal. We consider

$$\begin{aligned}\Psi_i'(s,t) &:= R(\Psi_i(s,t),(h_i(s\tfrac{t-t_{i-1}}{t_i-t_{i-1}}),t(h_i(s\tfrac{t-t_{i-1}}{t_i-t_{i-1}}))^{-1}))\\ G_i'(s,t) &:= G_i(s,t)t(h_i(s\tfrac{t-t_{i-1}}{t_i-t_{i-1}}))^{-1},\end{aligned}$$

satisfying $t(\Psi_i'(s,t)) = \tilde\Phi_i(s,t)$ and $s(\Psi_i'(s,t)) = R(\Phi_i(s,t),G_i'(s,t))$. Since the quantities at $s=0$ and $t=t_{i-1}$ are unchanged, we can add the horizontality of $t \longmapsto \Psi_i(1,t)$ to our assumptions.

We continue with choices Ψ_i and G_i satisfying all assumptions collected above. We notice the following: since $\Psi_i(-,t_{i-1})$ is horizontal, and $t(\Psi_i(-,t_{i-1})) = \tilde\Phi_i(-,t_{i-1}) = \tilde\nu_i$ is horizontal, we have by Proposition 3.2 (g) that $s(\Psi_i(-,t_{i-1})) = R(\nu_i,G_i(-,t_{i-1}))$ is horizontal, too. But since ν_i itself is horizontal, it follows that $s \longmapsto G_i(s,t_{i-1})$ is constant, i.e. $G_i(s,t_{i-1})=1$. With the same argument, we have that $t \longmapsto G_i(1,t)$ is constant, i.e. $G_i(1,t)=1$.

Next we consider bigon-parameterizations $\tilde\Psi_i : \beta_i \Longrightarrow \beta_i'$ of Ψ_i and $\tilde G_i : \mu_i \Longrightarrow \mu_i'$ of G_i, see Remark C.2. We have by Propositions B.8 (d) and B.8 (f)

$$\begin{aligned}h_\Omega(\beta_i) &= h_\Omega(\Psi_i(-,t_i))\cdot\alpha(poe_{\Omega^a}(s(\Psi_i(-,t_i))),h_\Omega(\Psi_i(0,-))) = h_\Omega(\Psi_i(-,t_i))\\ h_\Omega(\beta_i') &= h_\Omega(\Psi_i(1,-))\cdot\alpha(poe_{\Omega^a}(s(\Psi_i(1,-))),h_\Omega(\Psi_i(-,t_{i-1}))) = 1\end{aligned}$$

Further, we have $soe_\Omega(R(s(\tilde\Psi_i),\tilde G_i^{-1})) = h_i$ and $soe_\Omega(t(\tilde\Psi_i)) = \tilde h_i$. Now Proposition C.4 implies

$$h_i \cdot h_\Omega(\beta_i)^{-1} = \tilde h_i. \tag{28}$$

We notice that

$$\begin{aligned}\Psi_{i+1}(s,t_i)\circ\rho_i(s) &\quad\text{and}\quad g_i(s)\\ \tilde\rho_i(s)\circ R(\Psi_i(s,t_i),\tilde g_i(s)) &\quad\text{and}\quad G_i(s,t_i)\tilde g_i(s)\end{aligned}$$

are two transition spans with transitions functions along $s \longmapsto (\tilde\Phi(s,t_i),\Phi(s,t_i))$. Hence, by [13, Lemma 3.1.6] there exists a unique path $\eta : [0,1] \longrightarrow H$ with

$$\begin{aligned}g_i(s)t(\eta(s)) &= G_i(s,t_i)\tilde g_i(s)\\ R(\tilde\rho_i(s)\circ R(\Psi_i(s,t_i),\tilde g_i(s)),(\eta(s),t(\eta(s))^{-1})) &= \Psi_{i+1}(s,t_i)\circ\rho_i(s)\end{aligned} \tag{29}$$

From Eq. (27) we conclude that $\eta(0)=1$. Eq. (29) is equivalent to:

$$R(\Psi_i(s,t_i),G_i(s,t_i)^{-1}) = R(\tilde\rho_i(s)^{-1}\circ\Psi_{i+1}(s,t_i)\circ\rho_i(s),(\eta(s)^{-1},g_i(s)^{-1})).$$

In their dependence on s, this is an equality between two paths in $\mathrm{Mor}(\mathcal{P})$. We compute h_Ω on both sides. On the left, we obtain $h_\Omega(\beta_i)$ via Proposition B.8 (a). On the right we compute:

$$h_\Omega(R(\tilde\rho_i(s)^{-1}\circ\Psi_{i+1}(s,t_i)\circ\rho_i(s),(\eta(s)^{-1},g_i(s)^{-1}))$$

$$= h_\Omega(R(R(\tilde{\rho}_i(s)^{-1} \circ \Psi_{i+1}(s,t_i) \circ \rho_i(s), (\eta(s)^{-1}, 1)), g_i(s)^{-1}))$$

$$\overset{\text{Proposition B.8 (a)}}{\downarrow}$$
$$= \alpha(g_i(1), h_\Omega(R(\tilde{\rho}_i(s)^{-1} \circ \Psi_{i+1}(s,t_i) \circ \rho_i(s), (\eta(s)^{-1}, 1)), 1))$$

$$\overset{\text{Proposition B.8 (g)}}{\downarrow}$$
$$= \alpha(g_i(1), \eta(1)).$$

In the last step we have used that $\tilde{\rho}_i(s)^{-1} \circ \Psi_{i+1}(s,t_i) \circ \rho_i(s)$ is horizontal (Proposition 3.2 (d)) and has horizontal source $s(\rho_i)$ (since ρ_i is horizontal and $t(\rho_i) = \nu_{i+1}$ is horizontal, see Proposition 3.2 (g)). Summarizing, we have $\eta(1) = \alpha(g_i(1)^{-1}, h_\Omega(\beta_i))$. Thus, we get from Eq. (29):

$$\tilde{\rho}_i(1) \circ R(\Psi_i(1,t_i), (h_\Omega(\beta_i, \mu_i), g_i(1))) = \Psi_{i+1}(1,t_i) \circ \rho_i(1). \tag{30}$$

Now we consider the paths $t \longmapsto \Psi_i'(t) := \Psi_i(1,t)$ in $\mathrm{Mor}(\mathcal{P})$ which are horizontal and have $t(\Psi_i') = \tilde{\gamma}_i'$ and $s(\Psi_i') = \gamma_i'$. We claim that they establish an equivalence between the targets of the two horizontal lifts. This is confirmed by the following calculation:

$$\tilde{\rho}_i' \circ \Psi_i'(t_i) = \tilde{\rho}_i(1) \circ R(\Psi_i(1,t_i), (\tilde{h}_i^{-1}, 1))$$

$$\overset{\text{Eq. (28)}}{\downarrow}$$
$$= \tilde{\rho}_i(1) \circ R(\Psi_i(1,t_i), (h_\Omega(\beta_i) h_i^{-1}, 1))$$
$$= R(\tilde{\rho}_i(1) \circ R(\Psi_i(1,t_i), (h_\Omega(\beta_i), g_i(1))), (h_i^{-1}, g_i(1)^{-1}))$$

$$\overset{\text{Eq. (30)}}{\downarrow}$$
$$= R(\Psi_{i+1}(1,t_i) \circ \rho_i(1), (h_i^{-1}, g_i(1)^{-1}))$$
$$= \Psi_{i+1}'(t_i) \circ \rho_i'.$$

□

4.2 Definition of parallel transport along bigons The results of the previous section show that choosing a horizontal lift of a small bigon and computing its target establishes a well-defined map $\varphi_\Sigma^{small} : F_\gamma \longrightarrow F_{\gamma'}$. Before we extend it to arbitrary bigons, we discuss some properties.

Lemma 4.5. *The map φ_Σ^{small} has the following properties:*

(a) It preserves the anchors α_l and α_r.
(b) It is equivariant with respect to the left $\mathcal{P}_x$-action and the right $\mathcal{P}_y$-action.
(c) It is equivariant with respect to the $\mathrm{Mor}(\Gamma)$*-action.*
(d) It is smooth.

Proof. (a) is straightforward to check using that ρ_0 and ρ_n are constant. In (b) is even more obvious. In (c) we have to prove coincidence between

$$\varphi_\Sigma^{small}(\rho_n * \gamma_n * \ldots * \gamma_1 * \rho_0) \cdot (h,g) = R(\rho_n', g) * R(\gamma_n', g) * \ldots * R(\gamma_1', g) * R(\rho_0', (h^{-1}, t(h)g))$$

and

$$\varphi_\Sigma^{small}((\rho_n * \gamma_n * \ldots * \gamma_1 * \rho_0) \cdot (h,g))$$
$$= \varphi_\Sigma^{small}(R(\rho_n, g) * R(\gamma_n, g) * \ldots * R(\gamma_1, g) * R(\rho_0, (h^{-1}, t(h)g))).$$

Let a horizontal lift of $\rho_n * \gamma_n * ... * \gamma_1 * \rho_0$ consist of Φ_i, ρ_i and g_i. We consider for $1 \leq i \leq n$ the data of $\tilde{\Phi}_i := R(\Phi_i, g)$, $\tilde{\rho}_i := R(\rho_i, g)$ and $\tilde{g}_i := g^{-1}g_i g$, as well as $\tilde{\rho}_0 := R(\rho_0, (h^{-1}, t(h)g))$. It is straightforward to check that this is a horizontal lift of $(\rho_n * \gamma_n * ... * \gamma_1 * \rho_0) \cdot (h, g)$. The target of (Φ_i, ρ_i, g_i) consists by definition of the paths $\gamma_i'(t) := \Phi_i(1,t)$ and the morphisms $\rho_0' := \rho_0(1)$ and $\rho_i' := R(\rho_i(1), (h_i^{-1}, g_i(1)^{-1}))$, where $h_i := soe_\Omega(\Sigma_i)$ and Σ_i is a bigon-parameterization of Φ_i. Now we compute the target of $(\tilde{\Phi}_i, \tilde{\rho}_i, \tilde{g}_i)$. We have $\tilde{\gamma}_i'(t) = \tilde{\Phi}_i(1,t) = R(\Phi_i(1,t), g) = R(\gamma_i'(t), g)$. We use the bigon-parameterization $\tilde{\Sigma}_i(s,t) := R(\Sigma_i(s,t), g)$ and $\tilde{h}_i := soe_\Omega(\tilde{\Sigma}_i)$. By Corollary C.5 we get $\tilde{h}_i = \alpha(g^{-1}, h_i)$. Then, a short calculation shows that $\tilde{\rho}_0' = R(\rho_0', (h^{-1}, t(h)g))$ and $\tilde{\rho}_i' = R(\rho_i', g)$. This shows the required coincidence. For (d) we consider a chart $\phi_{\xi_0,\rho,g}$ of F_γ, and a chart $\phi_{\xi_0',\rho,g}$ of $F_{\gamma'}$ with $\xi_0' := \varphi_\Sigma^{small}(\xi_0)$. In these charts, φ_Σ^{small} is the identity, as one can see using (b) and (c); in particular, it is smooth. □

Next we extend φ_Σ^{small} to arbitrary bigons. For an arbitrary bigon $\Sigma : \gamma \Longrightarrow \gamma'$ there exists a subdivision $s \in T_n$ (i.e., $0 = s_0 < ... < s_n = 1$) such that the pieces $\Sigma_i(s,t) := \Sigma((s_i - s_{i-1})s + s_{i-1}, t)$ are small. We define

$$\varphi_\Sigma(s) := \varphi_{\Sigma_n}^{small} \circ ... \circ \varphi_{\Sigma_1}^{small}.$$

Lemma 4.6. *The map $\varphi_\Sigma(s)$ is independent of the choice of s.*

Proof. It suffices to prove that, for a small bigon Σ, $\varphi_\Sigma(s) = \varphi_\Sigma(s')$, where $s \in T_1$ and $s' \in T_2$ with $0 = s_0' < s_1' < s_2' = 1$. Thus, we have to show that

$$\varphi_\Sigma^{small} = \varphi_{\Sigma_2}^{small} \circ \varphi_{\Sigma_1}^{small}, \tag{31}$$

where Σ_1 and Σ_2 are (reparameterizations of) $\Sigma|_{[0,s_1']\times[0,1]}$ and $\Sigma|_{[s_1',1]\times[0,1]}$, respectively. We choose a horizontal lift $(n, t, \Phi_i, \rho_i, g_i)$ of Σ with source $\xi \in F_\gamma$. By a slight modification of the arguments in the proof of Lemma 4.2 we can assume that the paths $\gamma_i'(t) := \Phi_i(s_1', t)$ are horizontal. We consider the elements $h_i \in H$ and $\xi'' := \rho_n'' * \gamma_n'' * ... * \gamma_1'' * \rho_0''$ with $\gamma_i''(t) := \Phi_i(1,t)$, $\rho_0'' := \rho_0(1)$, and $\rho_i'' := R(\rho_i(1), (h_i^{-1}, g_i(1)^{-1}))$. By restriction of all parameters s to $0 = s_0' \leq s \leq s_1'$ and reparameterization to $[0,1]$, we obtain a horizontal lift $(n, t, \Phi_1^1, \rho_i^1, g_i^1)$ of Σ_1 with source ξ. We consider the elements $h_i^1 \in H$ and the target $\xi' = \rho_n' * \gamma_n' * ... * \gamma_1' * \rho_0'$ with $\gamma_i'(t) := \Phi_i^1(1,t)$, $\rho_0' := \rho_0^1(1)$, and $\rho_i' := R(\rho_i^1(1), ((h_i^1)^{-1}, g_i^1(1)^{-1}))$. Let $(\Phi_i^2, \tilde{\rho}_i^2, \tilde{g}_i^2)$ denote the restriction of (Φ_i, ρ_i, g_i) to $s_1' \leq s \leq s_2' = 1$. Define the following modification:

$$\rho_0^2(s) := \tilde{\rho}_0^2(s) \quad , \quad \rho_i^2(s) := R(\tilde{\rho}_i^2(s), ((h_i^1)^{-1}, g_i^1(1)^{-1})) \quad \text{and} \quad g_i^2(s) := \tilde{g}_i^2 g_i^1(1)^{-1}.$$

Then, $(n, t, \Phi_i^2, \rho_i^2, g_i^2)$ is a horizontal lift of Σ_2 with source ξ'. We consider again the corresponding elements $h_i^2 \in H$. We have $g_i^2(1) = g_i(1)g_i^1(1)^{-1}$ and

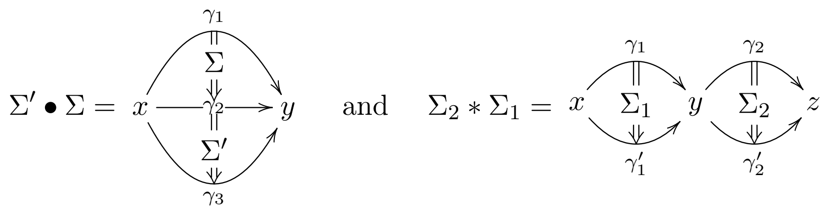
$$h_i = h_i^2\alpha(poe_{\Omega^a}(\mu_i), h_i^1) = h_i^2\alpha(g_i^2(1), h_i^1) = h_i^2\alpha(t(h_i^2)^{-1}, h_i^1) = h_i^1 h_i^2.$$

Then we obtain $\Phi_i^2(1,t) = \gamma_i''(t)$, $\rho_0^2(1) = \rho_0''$ and $R(\rho_i^2(1), ((h_i^2)^{-1}, g_i^2(1)^{-1})) = \rho_i''$. This shows Eq. (31). □

By Lemma 4.6, we simply write φ_Σ for any of the maps $\varphi_\Sigma(s)$, and summarize the properties of Lemma 4.5 as follows.

Proposition 4.7. *The map $\varphi_\Sigma : F_\gamma \longrightarrow F_{\gamma'}$ is a Γ-equivariant transformation.*

4.3 Compatibility with bigon composition Bigons can be composed in two ways, vertically and horizontally, which can most easily be described by a picture:



A more detailed description of bigon composition can be found in [10, Section 2.1]. The content of the following two propositions is that parallel transport along bigons is compatible with these two compositions. In the transport 2-functor formalism described in Section 6.2 they prove the functoriality of the 2-functor on the level of 2-morphisms.

Proposition 4.8. *Let $\mathcal{P}$ be a principal Γ-2-bundle with fake-flat connection. Suppose $\Sigma : \gamma_1 \Longrightarrow \gamma_2$ and $\Sigma' : \gamma_2 \Longrightarrow \gamma_3$ are vertically composable bigons. Then,*

$$\varphi_{\Sigma_2} \bullet \varphi_{\Sigma_1} = \varphi_{\Sigma_2 \bullet \Sigma_1} \quad \text{and} \quad \varphi_{\mathrm{id}_\gamma} = \mathrm{id}_{F_\gamma}.$$

Proof. This follows immediately from the definition of φ_Σ. □

Proposition 4.9. *Let $\mathcal{P}$ be a principal Γ-2-bundle with fake-flat connection. Suppose $\Sigma_1 : \gamma_1 \Longrightarrow \gamma_1'$ and $\Sigma_2 : \gamma_2 \Longrightarrow \gamma_2'$ are horizontally composable bigons. Then, the following diagram is commutative:*

$$\begin{array}{ccc} F_{\gamma_2} \circ F_{\gamma_1} & \xRightarrow{\varphi_{\Sigma_2} \circ \varphi_{\Sigma_1}} & F_{\gamma_2'} \circ F_{\gamma_1'} \\ {\scriptstyle c_{\gamma_1,\gamma_2}} \Big\Downarrow & & \Big\Downarrow {\scriptstyle c_{\gamma_1',\gamma_2'}} \\ F_{\gamma_2 * \gamma_1} & \xRightarrow[\varphi_{\Sigma_2 * \Sigma_1}]{} & F_{\gamma_2' * \gamma_1'} \end{array}$$

Proof. Given $(\xi_1, \xi_2) \in F_{\gamma_1} \,{}_{\alpha_r}\!\times_{\alpha_l} F_{\gamma_2}$ we choose horizontal lifts $(\Phi_i^1, \rho_i^1, g_i^1)$ and $(\Phi_i^2, \rho_i^2, g_i^2)$ of Σ^1 and Σ^2 with sources ξ_1 and ξ_2, respectively. We have $t(\rho_0^2) = \Phi_1^2(s, 0)$ and $s(\rho_n^1) = R(\Phi_n^1(s, 1), g_n^1(s))$, as well as $s(\rho_0^2) = \alpha_l(\xi_2) = \alpha_r(\xi_1) = t(\rho_n^1)$, by Definition 4.1. It is now obvious that under the usual reparameterizations of paths, the collection consisting of the families $(\Phi_1^1, ..., \Phi_n^1, \Phi_1^2, ..., \Phi_n^2)$, $(\rho_0^1, ..., \rho_{n-1}^1, \rho_0^2 \circ \rho_n^1, \rho_1^2, ..., \rho_n^2)$ and $(g_1^1, ..., g_n^1, g_1^2, ..., g_n^2)$ is a horizontal lift of $c_{\gamma_1,\gamma_2}(\xi_1, \xi_2)$. Computing the separate targets, we get from Eq. (26) $\rho_0^{k\prime} := \rho_0^k$ and $\rho_i^{k\prime} := R(\rho_i^k(1), ((h_i^k)^{-1}, g_i^k(1)^{-1}))$, giving us $\xi_k' = \rho_n^{k\prime} * \gamma_i^k * ... * \rho_0^{k\prime}$. For the target of the combined lift, we obtain in the middle the morphism

$$R(\rho_0^2 \circ \rho_n^1, ((h_n^1)^{-1}, g_n^1(1)^{-1})) = \rho_0^{2\prime} \circ \rho_n^{1\prime}.$$

This shows that $\varphi_{\Sigma_2 * \Sigma_1}(c_{\gamma_1,\gamma_2}(\xi_1, \xi_2)) = c_{\gamma_1',\gamma_2'}(\xi_1', \xi_2')$. □

4.4 Naturality with respect to bundle morphisms In this section we compare the parallel transports along a bigon in two isomorphic principal Γ-2-bundles. In the 2-functor formalism of Section 6.2, this is one axiom for a pseudonatural transformation associated to J.

Proposition 4.10. *Suppose $J : \mathcal{P} \longrightarrow \mathcal{P}'$ is a 1-morphism in $2\text{-}\mathcal{B}un_\Gamma^{\nabla^{ff}}(M)$. Let $\Sigma : \gamma_1 \Longrightarrow \gamma_2$ be a bigon between paths γ_1, γ_2 with $x := \gamma_1(0) = \gamma_2(0)$ and $y = \gamma_1(1) := \gamma_2(1)$. Let J_{γ_1} and*

J_{γ_2} be the transformations associated to J defined in Section 3.4. Then, the following diagram is commutative:

$$\begin{array}{ccc} J_y \circ F_{\gamma_1} & \xRightarrow{\mathrm{id}\circ\varphi_\Sigma} & J_y \circ F_{\gamma_2} \\ {\scriptstyle J_{\gamma_1}}\Big\Downarrow & & \Big\Downarrow{\scriptstyle J_{\gamma_2}} \\ F'_{\gamma_1} \circ J_x & \xRightarrow[\varphi'_\Sigma\circ\mathrm{id}]{} & F'_{\gamma_2} \circ J_x \end{array}$$

Proof. We start with $(\xi_1, j) \in F_{\gamma_1}\, {}_{\alpha_r}\!\times_{\alpha_l} J_y$. Let $(\{\Phi_i\}, \{\rho_i\}, \{g_i\})$ be a horizontal lift of Σ to $\mathcal{P}$ with source ξ_1, and let ξ_2 be its target. Let $\tilde{\xi}_1 = (\{\tilde{\rho}_i\}, \{\tilde{\gamma}_i\})$ be a horizontal lift of ξ_1 to J, and let (j', ξ'_1) be its j-target. We can assume that the induced representatives for ξ_1 coincide, i.e. $\alpha_l(\tilde{\gamma}_i) = \Phi_i(0, -)$ and $\rho_i(0) = \tilde{\rho}_i$, and $j' := \tilde{\rho}_0^{-1} \circ \tilde{\gamma}_1(0)$. Finally, let $(\{\Phi'_i\}, \{\rho'_i\}, \{g'_i\})$ be a horizontal lift of Σ to $\mathcal{P}'$ with source ξ'_1, and let ξ'_2 be its target. We can assume that the induced representatives for ξ'_1 coincide, i.e. the path pieces of ξ'_1 are $\alpha_r(\tilde{\gamma}_i) = \Phi'_i(0, -)$, the the jumps $\rho'_i(0)$ satisfy $\rho'_0(0) = \mathrm{id}$ and $\tilde{\rho}_i \circ \tilde{\gamma}_i(t_i) = \tilde{\gamma}_{i+1}(t_i) \circ \rho'_i(0)$ for $1 \leq i < n$ and $\tilde{\rho}_n \circ \tilde{\gamma}_n(1) = j \circ \rho'_n(0)$. We have to prove that $J_{\gamma_2}(\xi_2, j) = (j', \xi'_2)$. For this purpose we provide a horizontal lift $\tilde{\xi}_2$ of ξ_2 to J with j-target (j', ξ'_2).

Due to Proposition 4.8 it suffices to discuss small bigons. We can even assume by [13, Lemma 3.1.9] that the image of $\Phi_i \times \Phi'_i$ is contained in an open subset $V_i \subseteq \mathrm{Obj}(\mathcal{P}) \times_M \mathrm{Obj}(\mathcal{P}')$ that supports a transition spans $\tau_i : V_i \longrightarrow J$, with transition functions p_i. We define $\Psi_i : [0,1] \times [t_{i-1}, t_i] \longrightarrow J$ by $\Psi_i(s,t) := \tau_i(\Phi_i(s,t), \Phi'_i(s,t))$ and similarly $G_i(s,t) := p_i(\Phi_i(s,t), \Phi'_i(s,t))$; these satisfy $\alpha_l(\Psi_i(s,t)) = \Phi_i(s,t)$ and $\alpha_r(\Psi_i(s,t)) = R(\Phi'_i(s,t), G_i(s,t))$. After performing several adjustments analogously to the ones of Lemma 4.2 we can assume that $\Psi_i(0,t) = \tilde{\gamma}_i(t)$; in particular, $t \longmapsto \Psi_i(0,t)$ is horizontal, and we can assume that $s \longmapsto \Psi_i(s,0)$ and $t \longmapsto \Psi_i(1,t)$ are horizontal. Since the left anchors of these three paths are horizontal, their right anchors are also horizontal by Proposition 3.4 (b). But since the corresponding three paths in $\Phi'_i(s,t)$ are horizontal, too, it follows from the uniqueness of Proposition 3.1 (a) that $G_i(0,t) = G_i(1,t) = G_i(s, t_{i-1}) = 1$.

We use this in the following way. We write $\xi_2 = \zeta_n * \beta_n * \ldots * \beta_1 * \zeta_0$, and obtain from the definition of ξ_2 as the target of the chosen horizontal lift $\beta_i(t) = \Phi_i(1,t)$, $\zeta_0 := \rho_0(1)$ and $\zeta_i := R(\rho_i(1), (h_i^{-1}, g_i(1)^{-1}))$, where h_i is the surface-ordered exponential of a bigon-parameterization of Φ_i. We define $\tilde{\gamma}'_i(t) := \Psi_i(1,t)$. Then, $\tilde{\xi}_2 := (\{\zeta_i\}, \{\tilde{\gamma}'_i\})$ is a horizontal lift of ξ_2 to J. It remains to prove that its j-target is (j', ξ'_2). There are three parts: the path pieces of ξ'_2, the morphisms pieces, and the element j'. First, the paths are $\alpha_r(\tilde{\gamma}'_i) = \alpha_r(\Psi_i(1,-)) = R(\Phi'_i(1,t), G_i(1,t)) = \Phi'_i(1,t)$; these are indeed the paths of ξ'_2.

Second, the morphisms ζ'_i are characterized by $\zeta'_0 = \mathrm{id}$ and $\zeta_i \circ \tilde{\gamma}'_i(t_i) = \tilde{\gamma}'_{i+1}(t_i) \circ \zeta'_i$ (for $1 \leq i < n$) and $\zeta_n \circ \tilde{\gamma}'_n(1) = j \circ \zeta'_n$. We have to show that they coincide with the result of computing the target of the horizontal lift of Σ to $\mathcal{P}'$, namely

$$\zeta'_0 := \rho'_0(1) \quad \text{and} \quad \zeta'_i := R(\rho'_i(1), (h'^{-1}_i, g'_i(1)^{-1})),$$

where h'_i is the surface-ordered exponential of a bigon-parameterization of Φ'_i. We have $\zeta'_0 = \rho'_0(1) = \rho'_0(0) = \mathrm{id}$ at the beginning. For the pieces in the middle, let $\tau : [0,1] \longrightarrow \mathrm{Mor}(\mathcal{P}')$ be the unique path such that

$$R(\rho_i(s), g_i(s)^{-1}) \circ \Psi_i(s, t_i) \circ \tau(s) = (\Psi_{i+1}(s, t_i) \circ \rho'_i(s)) \cdot \mathrm{id}_{g_i(s)^{-1}}. \tag{32}$$

It is straightforward to check using [13, Lemma 3.1.4] that

$$\tau(s) = R(\mathrm{id}_{\Phi'_i(s,t_i)}, (\eta(s), g'_i(s) g_i(s)^{-1}))$$

for a unique smooth map $\eta : [0,1] \longrightarrow H$ with $t(\eta(s)) = G_i(s,t_i)g_i(s)g_i'(s)^{-1}$. At $s = 0$, Eq. (32) results by construction in $\tau(0) = \mathrm{id}$, i.e. $\eta(0) = 1$. We claim that

$$\eta(1) = h_i^{-1}h_i'. \tag{33}$$

Using this claim we get

$$\begin{aligned}
\zeta_i \circ \tilde{\gamma}_i'(t_i) &= R(\rho_i(1), (h_i^{-1}, g_i(1)^{-1})) \circ \Psi_i(1,t_i) \\
&\overset{\text{Eq. (33)}}{=} (R(\rho_i(1), g_i(1)^{-1}) \circ \Psi_i(1,t_i) \circ \tau(1)) \cdot (\alpha(g_i(1), h_i'^{-1}), g_i(1)g_i'(1)^{-1}) \\
&\overset{\text{Eq. (32)}}{=} (\Psi_{i+1}(1,t_i) \circ \rho_i'(1)) \cdot (1, g_i(1)^{-1}) \cdot (\alpha(g_i(1), h_i'^{-1}), g_i(1)g_i'(1)^{-1}) \\
&= \Psi_{i+1}(1,t_i) \circ R(\rho_i'(1), (h_i'^{-1}, g_i'(1)^{-1})) \\
&= \tilde{\gamma}_{i+1}'(t_i) \circ \zeta_i';
\end{aligned}$$

this proves the desired property of the ζ_i'. In order to prove Eq. (33) we write Eq. (32) in the equivalent form

$$\lambda_{lhs} := \Psi_i(s,t_i) \cdot G_i(s,t_i)^{-1}g_i'(s) = (\rho_i(s)^{-1} \circ \Psi_{i+1}(s,t_i) \circ \rho_i'(s)) \cdot (\alpha(g_i'(s)^{-1}, \eta(s)^{-1}), 1) =: \lambda_{rhs}$$

This is an equality between two paths λ_{lhs} and λ_{rhs} in J. We compute the path-ordered exponential poe_{ν_0} separately on both sides. On the right hand side, the path $s \longmapsto \rho_i(s)^{-1} \circ \Psi_{i+1}(s,t_i) \circ \rho_i'(s)$ is horizontal by Proposition 3.4 (c) and (d). Its right anchor is $s(\rho_i')$; since ρ_i' and $t(\rho_i')$ are horizontal by Definition 4.1, this is horizontal by Proposition 3.2 (g). Hence, by Proposition B.3 $poe_{\nu_0}(\lambda_{rhs}) = \alpha(g_i'(1)^{-1}, \eta(1))$.

On the left hand side, the right anchor of λ_{lhs} is again $s(\rho_i')$ and thus horizontal. Hence, by Proposition B.11 we have $poe_{\nu_0}(\lambda_{lhs}) = \alpha(g_i'(1), h_\nu(\lambda_{lhs} \cdot g_i'^{-1}))$. Let $\Sigma_i : \lambda_i \Longrightarrow \lambda_i'$ be a bigon-parameterization for Ψ_i, where $\lambda_i = \Psi_i(-,t_i) \circ \Psi_i(0,-)$ and $\lambda_i' = \Psi_i(1,-) \circ \Psi_i(-,0)$ up to thin homotopy, and let $\Theta_i : \gamma_i \Longrightarrow \gamma_i'$ be a bigon-parameterization of G_i with analogous γ_i and γ_i'. Then, $\alpha_l(\Sigma_i)$ is a bigon-parameterization for Φ_i and $R(\alpha_r(\Sigma_i), \Theta_i^{-1})$ is one for Φ_i'. Now Proposition C.6 implies

$$h_i' \cdot h_\nu(\lambda_i \cdot \gamma_i^{-1})^{-1} = h_\nu(\lambda_i' \cdot \gamma_i'^{-1})^{-1} \cdot h_i.$$

Note that λ_i' is horizontal with horizontal right anchor. By Proposition B.11 we get $h_\nu(\lambda_i' \cdot \gamma_i'^{-1}) = \alpha(\gamma_i'(1), poe_{\nu_0}(\lambda_i')) = 1$. The same applies to the first half of the path λ_i; hence by Proposition B.4 (a) we have

$$h_\nu(\lambda_i \cdot \gamma_i^{-1}) = h_\nu(\Psi_i(-,t_i)G_i(-,t_i)) = h_\nu(\lambda_{lhs} \cdot g_i'^{-1}).$$

Summarizing collected identities, we obtain $poe_{\nu_0}(\lambda_{lhs}) = \alpha(g_i'(1), h_i^{-1}h_i')$. Equating with the result of the right hand side yields the claim Eq. (33).

Third, for $i = n$, we obtain from Proposition C.6, similarly as above, $h_n' = h_n$, and we have $g_n(1) = g_n'(1) = 1$. Using this it is straightforward to show that $\zeta_n \circ \tilde{\gamma}_n'(1) = j \circ R(\rho_n'(1), (h_n'^{-1}, 1))$; this is the correct characterization for ζ_n'. Third, we show that the element j' is reproduced:

$$\zeta_0^{-1} \circ \tilde{\gamma}_1'(0) = \rho_0(1)^{-1} \circ \Psi_1(1,0) = \rho_0(0)^{-1} \circ \Psi_1(0,0) = \tilde{\rho}_0^{-1} \circ \tilde{\gamma}_1(0) = j'.$$

This completes the proof. □

4.5 Naturality with respect to pullback Suppose $\mathcal{P}$ is a principal Γ-2-bundle over N with fake-flat connection Ω, and $f : M \longrightarrow N$ is a smooth map. We denote by $\mathcal{P}' := f^*\mathcal{P}$ the pullback bundle, obtain a Γ-equivariant smooth functor $\tilde{f} : \mathcal{P}' \longrightarrow \mathcal{P}$, and $\Omega' := \tilde{f}^*\Omega$ is a connection on $\mathcal{P}'$. We recall from Section 3.5 that we have associated to each path $\gamma : x \longrightarrow y$ a Γ-equivariant transformation $\tilde{f}_\gamma : \tilde{f}_y \circ F'_\gamma \Longrightarrow F_{f(\gamma)} \circ \tilde{f}_x$.

Proposition 4.11. *Suppose $\mathcal{P}$ is a principal Γ-2-bundle over N with fake-flat connection Ω, and $f : M \longrightarrow N$ is a smooth map. Let $\Sigma : \gamma_1 \Longrightarrow \gamma_2$ be a bigon in M with $x := \gamma_1(0) = \gamma_2(0)$ and $y := \gamma_1(1) = \gamma_2(1)$. Let F'_γ and φ'_Σ denote the parallel transport in the pullback bundle $f^*\mathcal{P}$. Then, the following diagram is commutative:*

$$\begin{array}{ccc} \tilde{f}_y \circ F'_{\gamma_1} & \xRightarrow{\mathrm{id}\circ\varphi'_\Sigma} & \tilde{f}_y \circ F'_{\gamma_2} \\ {\scriptstyle \tilde{f}_{\gamma_1}}\Big\Downarrow & & \Big\Downarrow{\scriptstyle \tilde{f}_{\gamma_2}} \\ F_{f(\gamma_1)} \circ \tilde{f}_x & \xRightarrow[\varphi_{f(\Sigma)}\circ\mathrm{id}]{} & F_{f(\gamma_2)} \circ \tilde{f}_x \end{array}$$

Proof. Like in Section 3.5 we identify canonically $\mathcal{P}'_x = \mathcal{P}_{f(x)}$ so that $\tilde{f}_x = \mathrm{id}$, and $\tilde{f}_\gamma : F'_\gamma \Longrightarrow F_{f(\gamma)}$ is given by $\rho_n * \gamma_n * ... * \rho_0 \longmapsto \tilde{f}(\rho_n) * \tilde{f}(\gamma_n) * ... * \tilde{f}(\rho_0)$. Suppose we have a horizontal lift $(\{\Phi_i\}, \{\rho_i\}, \{g_i\})$ of Σ to $\mathcal{P}'$ with source ξ. Let ξ' be its target, so that $\xi' = \varphi'_\Sigma(\xi)$. Since $\Omega' = \tilde{f}^*\Omega$ and $\tilde{f}$ is Γ-equivariant it is clear that $(\{\tilde{f} \circ \Phi_i\}, \{\tilde{f} \circ \rho_i\}, \{g_i\})$ is a horizontal lift of $f(\Sigma)$ to $\mathcal{P}$ with source $\tilde{f}_\gamma(\xi)$. Using the naturality of the surface ordered exponential under pullbacks (Proposition C.1 (e)), its target is $\tilde{f}_\gamma(\xi')$; this shows that commutativity of the diagram. □

5. Backwards compatibility

We exhibit our constructions of Sections 3 and 4 for two particular classes of principal Γ-2-bundles: trivial 2-bundles and 2-bundles induced from ordinary principal bundles.

5.1 Trivial principal 2-bundles It is certainly important to see what the parallel transport constructions of Sections 3 and 4 reduce to in case of the trivial bundle. Also, we will need the results of this section in the proofs in Sections 6.2 and 6.3.

In the following remark we relate (in a functorial way) Γ-connections on M to connections on the trivial principal Γ-2-bundle. The next remark identifies what that relation is over a one-point-manifold.

Remark 5.1. Let $\mathcal{I} := M_{dis} \times \Gamma$ be the trivial bundle. We summarize three constructions of [13, Section 5.4]; also see Section A for the categorical structure of Γ-connections:

(a) Every (fake-flat) Γ-connection (A, B) on M defines a (fake-flat) connection $\Omega_{A,B}$ on $\mathcal{I}$; we denote by $\mathcal{I}_{A,B}$ the trivial bundle equipped with that connection. In more detail, we have

$$\Omega^a_{A,B} = \mathrm{Ad}_g^{-1}(p^*A) + g^*\theta \ , \quad \Omega^b_{A,B} = (\alpha_{g^{-1}})_*((\tilde{\alpha}_h)_*(p^*A) + h^*\theta) \quad \text{and} \quad \Omega^c_{A,B} = -(\alpha_{g^{-1}})_*(p^*B),$$

where g, h and p denote the projections to G, H, and M, respectively.

(b) Every gauge transformation (g, φ) between Γ-connections (A, B) and (A', B') on M defines a 1-morphism $J_{g,\varphi} : \mathcal{I}_{A,B} \longrightarrow \mathcal{I}_{A',B'}$ in $2\text{-}\mathcal{B}un^\nabla_\Gamma(M)$. If the Γ-connections are fake-flat, this is a 1-morphism in $2\text{-}\mathcal{B}un^{\nabla^{ff}}_\Gamma(M)$.

(c) Every gauge 2-transformation a between gauge transformations (g_1, φ_1) and (g_2, φ_2) defines a 2-morphism $f_a : J_{g_1,\varphi_1} \Longrightarrow J_{g_2,\varphi_2}$.

(d) By [13, Proposition 5.4.4], (a) to (c) yield a 2-functor $L_M^{ff} : \mathcal{C}on_\Gamma^{ff}(M) \longrightarrow 2\text{-}\mathcal{B}un_\Gamma^{\nabla ff}(M)$.

Remark 5.2. We reduce the structure of Remark 5.1 to the one-point manifold $M = *$. It is easy to see that $\mathcal{C}on_\Gamma^{ff}(*) = \mathcal{C}on_\Gamma(*) = B\Gamma$, the delooping of Γ: this bigroupoid has a single object, whose Hom-groupoid is Γ. In order to identify $2\text{-}\mathcal{B}un_\Gamma^{\nabla ff}(*)$ we fix the following definition: a Γ-*torsor* is a Lie groupoid $\mathcal{P}$ together with a smooth right Γ-action R of Γ on $\mathcal{P}$ such that the functor

$$\tau := (\mathrm{pr}_1, R) : \mathcal{P} \times \Gamma \longrightarrow \mathcal{P} \times \mathcal{P}$$

is a weak equivalence. The bicategory $\Gamma\text{-}\mathcal{T}or$ is the full sub-bicategory of the bicategory of Lie groupoids with smooth Γ-action. Then we have $2\text{-}\mathcal{B}un_\Gamma^{\nabla ff}(*) = 2\text{-}\mathcal{B}un_\Gamma(*) = \Gamma\text{-}\mathcal{T}or$.

(a) The canonical Γ-torsor is $\mathcal{P} := \Gamma$ with R given by the 2-group structure.

(b) For every $g \in G$ there is a 1-morphism $i_g : \Gamma \longrightarrow \Gamma$ in $\Gamma\text{-}\mathcal{T}or$, which can be given as a smooth functor: we set $i_g(g') := gg'$ and $i_g(h, g') := (\alpha(g, h), gg')$. This is a functor and strictly equivariant with respect to the Γ-action. Hence it induces a 1-morphism in $\Gamma\text{-}\mathcal{T}or$.

(c) For every $(h, g) \in \mathrm{Mor}(\Gamma)$ we define a natural transformation $i_{(h,g)} : i_g \Longrightarrow i_{t(h)g}$ whose component at g' is $i_{(h,g)}(g') := (h, gg')$. The natural transformation $i_{(g,h)}$ is Γ-equivariant in the sense that

$$i_{(h,g)}(g'g'') = i_{(h,g)}(g') \cdot \mathrm{id}_{g''}.$$

Hence we can regard it as a 2-morphism in $\Gamma\text{-}\mathcal{T}or$.

(d) It is straightforward to verify directly that (a) to (c) form a (strict) 2-functor $i : B\Gamma \longrightarrow \Gamma\text{-}\mathcal{T}or$. Further it is easy to check that under the identification of $\mathcal{I}_x = \{x\} \times \Gamma \cong \Gamma$ the restriction of the 2-functor L_M^{ff} to $M = *$ is exactly i. Finally, one can show that i is an equivalence of bicategories.

Now we start to identify the parallel transport along a path $\gamma : x \longrightarrow y$ in the trivial principal Γ-2-bundle $\mathcal{I}_{A,B}$, where (A, B) is a Γ-connection. We show that the anafunctor F_γ is canonically 2-isomorphic to (the anafunctor induced by) the functor $i_{poe_A(\gamma)}$ of Remark 5.2 (b), where $poe_A(\gamma) \in G$ is the path-ordered exponential of A along γ, see Section B. For this purpose we define a Γ-equivariant transformation

$$\eta_\gamma : i_{poe_A(\gamma)} \Longrightarrow F_\gamma. \tag{34}$$

For simplicity we set $g := poe_A(\gamma)$. We define η_γ using Remark 2.2 (d); the underlying smooth map $\tilde{\eta}_\gamma : G \longrightarrow F_\gamma$ is defined as follows. Let $\kappa : [0, 1] \longrightarrow G$ be the solution of the initial value problem

$$\dot{\kappa}(t) = -A(\dot{\gamma}(t))\kappa(t) \quad \text{and} \quad \kappa(0) = 1, \tag{35}$$

so that $g = \kappa(1)$. Consider the path (γ, κ) in $\mathrm{Obj}(\mathcal{P}) = M \times G$. It is horizontal:

$$\Omega_{A,B}^a(\dot{\gamma}(t), \dot{\kappa}(t)) = \mathrm{Ad}_{\kappa(t)}^{-1}(A(\dot{\gamma}(t))) + \theta(\dot{\kappa}(t)) = 0.$$

For $g' \in G$ the path $(\gamma, \kappa g')$ is then horizontal, too, by Proposition 3.1 (b). Thus, we obtain an element $\xi_{g'} := \mathrm{id}_{(y,gg')} * (\gamma, \kappa g') * \mathrm{id}_{(x,g')} \in F_\gamma$. We set $\tilde{\eta}_\gamma(g') := \xi_{g'}$.

Lemma 5.3. *The map $\tilde{\eta}_\gamma$ satisfies (T1) to (T3).*

Proof. We have $\alpha_l(\xi_{g'}) = (x, g')$ and $\alpha_r(\xi_{g'}) = (y, gg')$, this is (T1). For morphisms $\alpha \in \mathrm{Mor}(\mathcal{P}_x) = \mathrm{Mor}(\Gamma)$ (with $s(\alpha) = g'$) and $\beta \in \mathrm{Mor}(\mathcal{P}_y) = \mathrm{Mor}(\Gamma)$ (with $t(\beta) = gg'$) we have

$$\alpha \circ \tilde{\eta}_\gamma(g') \circ \beta = \beta^{-1} * (\gamma, \kappa g') * \alpha^{-1} \sim (\beta^{-1} \circ i_g(\alpha)^{-1}) * (\gamma, \kappa t(\alpha)) * \mathrm{id} = \tilde{\eta}_\gamma(t(\alpha)) \circ i_g(\alpha) \circ \beta,$$

where $\sim$ denotes one application of the equivalence relation on F_γ, performed as follows. Consider the path $\rho = (\gamma, \mathrm{id}_\kappa \cdot \alpha)$ in $\mathrm{Mor}(\mathcal{P})$ satisfying $s(\rho) = (\gamma, \kappa g')$ and $t(\rho) = (\gamma, \kappa t(\alpha))$. It is horizontal by Propositions 3.2 (b) and 3.2 (f). Thus, $\mathrm{id} * (\gamma, \kappa g') * \alpha^{-1}$ is equivalent to

$$\begin{aligned}\rho(1)^{-1} * (\gamma, \kappa t(\alpha)) * (\rho(0) \circ \alpha^{-1}) &= (\mathrm{id}_g \cdot \alpha)^{-1} * (\gamma, \kappa t(\alpha)) * ((\mathrm{id}_1 \cdot \alpha) \circ \alpha^{-1}) \\ &= i_g(\alpha)^{-1} * (\gamma, \kappa t(\alpha)) * \mathrm{id}.\end{aligned}$$

This shows (T2). Finally, we have $\xi_{g'} \cdot \mathrm{id}_g = \xi_{g'g}$, this is (T3). □

Proposition 5.4. *Let (A, B) be a Γ-connection on M, and let F_γ denote the parallel transport in the associated trivial principal Γ-2-bundle $\mathcal{I}_{A,B}$. Then, the transformation $\eta_\gamma : i_{poe_A(\gamma)} \Longrightarrow F_\gamma$ is compatible with path composition: if γ_2 and γ_1 are composable paths, then we have*

$$\eta_{\gamma_2 * \gamma_1} = c_{\gamma_1,\gamma_2} \bullet (\eta_{\gamma_2} \circ \eta_{\gamma_1}),$$

where $c_{\gamma_1,\gamma_2} : F_{\gamma_2} \circ F_{\gamma_1} \Longrightarrow F_{\gamma_2 * \gamma_1}$ was defined in Section 3.3.

Proof. On the level of the corresponding smooth maps $\tilde{\eta}_\gamma$, the claim becomes

$$\tilde{\eta}_{\gamma_2 * \gamma_1}(g') = c_{\gamma_1,\gamma_2}(\tilde{\eta}_{\gamma_1}(g'), \tilde{\eta}_{\gamma_2}(g_1 g'))$$

for all $g' \in G$. Let $\kappa_1, \kappa_2 : [0, 1] \longrightarrow G$ be the solutions to the initial value problems Eq. (35) corresponding to γ_1 and γ_2, respectively, so that $\tilde{\eta}_{\gamma_i}(g') = \mathrm{id} * (\gamma_i, \kappa_i g') * \mathrm{id}$. Then, $\tilde{\kappa} := \kappa_2 g_1 * \kappa_1$ (composition of paths in G) is the solution for $\gamma_2 * \gamma_1$, i.e. $\tilde{\eta}_{\gamma_2 * \gamma_1}(g') = \mathrm{id} * (\gamma_2 * \gamma_1, \tilde{\kappa} g') * \mathrm{id} \in F_{\gamma_2 * \gamma_1}$. In the direct limit definition of $F_{\gamma_2 * \gamma_1}$, this is equivalent to $\mathrm{id} * (\gamma_2, \tilde{\kappa}_2 g_1 g') * \mathrm{id} * (\gamma_1, \kappa_1 g') * \mathrm{id}$, which is precisely $c_{\gamma_1,\gamma_2}(\tilde{\eta}_{\gamma_1}(g'), \tilde{\eta}_{\gamma_2}(g_1 g'))$. □

Remark 5.5. Let $\mathcal{P}$ be a principal Γ-bundle with a connection Ω, (A, B) be a Γ-connection on M, and $J : \mathcal{I}_{A,B} \longrightarrow \mathcal{P}$ be a 1-morphism in 2-$\mathcal{B}un_\Gamma^\nabla(M)$. Such "trivializations" always exist locally. Combining the transformation η_γ with the transformation J_γ from Section 3.4 we obtain a transformation

$$\begin{array}{ccc} \Gamma & \xrightarrow{\;i_{poe_A(\gamma)}\;} & \Gamma \\ {\scriptstyle J_x}\downarrow & \swarrow\!\!\swarrow & \downarrow{\scriptstyle J_y} \\ \mathcal{P}_x & \xrightarrow[\;F_\gamma\;]{} & \mathcal{P}_y \end{array}$$

In this sense, parallel transport in any principal Γ-2-bundle is – locally – multiplication with the path-ordered exponential of a local connection 1-form A along the path.

Suppose (A, B) and (A', B') are Γ-connections on M and (g, φ) is a gauge transformation. By Remark 5.1 (b) there is a 1-morphism $J := J_{g,\varphi} : \mathcal{I}_{A,B} \longrightarrow \mathcal{I}_{A',B'}$ in 2-$\mathcal{B}un_\Gamma^\nabla(M)$. It is induced from a smooth functor ϕ_g, whose restriction to a point x is the functor $i_{g(x)}$ determined by the gauge transformation g and Remark 5.2 (b). Thus, we have $J_x = i_{g(x)}$. According to Section 3.4, J determines a transformation $J_\gamma : J_y \circ F_\gamma \Longrightarrow F'_\gamma \circ J_x$ for each path $\gamma : x \longrightarrow y$. The goal of the following proposition is to determine J_γ in the present case of $J = J_{g,\varphi}$.

We consider $h_{g,\varphi}(\gamma) \in H$ explained in Section C. By Proposition B.4 (b) it satisfies

$$poe_{A'}(\gamma) \cdot g(x) = t(h_{g,\varphi}(\gamma))^{-1} \cdot g(y) \cdot poe_A(\gamma). \tag{36}$$

In other words, $\alpha_{g,\varphi}(\gamma) := (h_{g,\varphi}(\gamma)^{-1}, g(y)poe_A(\gamma)) \in H \times G = \mathrm{Mor}(\Gamma)$ is a morphism with source $g(y)poe_A(\gamma)$ and target $poe_{A'}(\gamma)g(x)$. Associated to $a_{g,\varphi}(\gamma)$ is by Remark 5.2 (c) a natural transformation

$$i_{\alpha_{g,\varphi}(\gamma)} : i_{g(y)poe_A(\gamma)} \Longrightarrow i_{poe_{A'}(\gamma)g(x)}.$$

The following proposition shows that $i_{a_{g,\varphi}(\gamma)}$ corresponds to J_γ under the transformation of Eq. (34).

Proposition 5.6. *Let $(g, \varphi) : (A, B) \longrightarrow (A', B')$ be a gauge transformation between Γ-connections, and let $J : \mathcal{I}_{A,B} \longrightarrow \mathcal{I}_{A',B'}$ be the associated 1-morphism in $2\text{-}\mathcal{B}un^{\nabla}_{\Gamma}(M)$. For every path $\gamma : x \longrightarrow y$ the diagram*

$$\begin{array}{ccc}
i_{g(y)poe_A(\gamma)} & \xRightarrow{\;i_{\alpha_{g,\varphi}(\gamma)}\;} & i_{poe_{A'}(\gamma)g(x)} \\
\Vert & & \Vert \\
i_{g(y)} \circ i_{poe_A(\gamma)} & & i_{poe_{A'}(\gamma)} \circ i_{g(x)} \\
{\scriptstyle \mathrm{id}\circ\eta_\gamma}\Big\Downarrow & & \Big\Downarrow{\scriptstyle \eta'_\gamma\circ\mathrm{id}} \\
J_y \circ F_\gamma & \xRightarrow[J_\gamma]{} & F'_\gamma \circ J_x,
\end{array}$$

is commutative, where F_γ, η_γ and F'_γ, η'_γ denote the parallel transports and the transformations of Eq. (34) for $\mathcal{I}_{A,B}$ and $\mathcal{I}_{A',B'}$, respectively.

Proof. The diagram is an equality between two transformations from a smooth functor to an anafunctor. We express them under the correspondence of Remark 2.2 (d), getting counter-clockwise the smooth map

$$g' \longmapsto J_\gamma(\eta_\gamma(g'), (y, poe_A(\gamma)g', \mathrm{id}_{g(y)poe_A(\gamma)g'})) \tag{37}$$

and clockwise the smooth map

$$g' \longmapsto ((x, g', \mathrm{id}_{g(x)g'}), \tilde{\eta}'_\gamma(g(x)g')) \circ i_{\alpha_{g,\varphi}(\gamma)}(g'). \tag{38}$$

We show that both expressions coincide. In the clockwise direction, we employ the definition of $\eta_{\gamma'}$ and obtain after some straightforward manipulations

$$((x, g', \mathrm{id}_{g(x)g'}), ((y, h_{g,\varphi}(\gamma), poe_{A'}(\gamma)g(x)g') * (\gamma, \kappa' g(x)g') * \mathrm{id}_{(x,g(x)g')}). \tag{39}$$

Counter-clockwise, we write $\xi_{g'} := \eta_\gamma(g') = \mathrm{id}_{(y,\kappa(1)g')} * (\gamma, \kappa g') * \mathrm{id}_{(x,g')}$. The result of J_γ will be computed using Remark 3.16 and the following facts about $J = J_{g,\varphi}$, which can be looked up in [13, Section 5.4]. The first fact is that J has an underlying functor ϕ_g, and the second fact is that the canonical $\Omega_{A',B'}$-pullback on J is shifted by a pair of forms (φ_0, φ_1), with $\varphi_0 := (\alpha_{\mathrm{pr}_G^{-1} \cdot g^{-1}})_*(\mathrm{pr}_M^* \varphi)$, see [13, Eq. 5.4.4]. Now, Remark 3.16 implies

,

$$J_\gamma(\xi_{g'}, (y, poe_A(\gamma)g', \mathrm{id}_{g(y)poe_A(\gamma)g'})) = ((x, g', \mathrm{id}_{g(x)g'}), \phi_g^{\varphi}(\xi_{g'})), \tag{40}$$

where

$$\phi_g^\varphi(\xi_{g'}) = (y, (\alpha(g(y)\kappa(1)g', \tilde{h}(1)), g(y)\kappa(1)g't(\tilde{h}(1))^{-1})) * (\gamma, g(\gamma)\kappa g't(\tilde{h})^{-1}) * \mathrm{id}_{(x,g(x)g')},$$

and $\tilde{h}$ is the solution to the initial value problem

$$\partial_t\tilde{h}(t) = -\tilde{h}(t)\varphi_0(\dot{\gamma}(t), \dot{\kappa}(t)g') \quad \text{and} \quad \tilde{h}(0) = 1. \tag{41}$$

The key to the proof that Eqs. (39) and (40) coincide is to understand the relation between κ, κ, and $\tilde{h}$. The relation between κ and κ' is established by the gauge transformation, which gives $\mathrm{Ad}_g(A) - g^*\bar{\theta} = A' + t_*(\varphi)$. From the proof of [11, Lemma 2.18] we have

$$\kappa'(t) = t(h(t))^{-1}g(\gamma(t))\kappa(t)g(x)^{-1} \tag{42}$$

where $h : [0,1] \longrightarrow H$ is a smooth map such that the pair (h, κ') solves the initial value problem

$$\partial_t(h(t), \kappa'(t)) = -(\varphi(\partial_t\gamma(t)), A'(\partial_t\gamma(t))) \cdot (h(t), \kappa'(t)) \quad \text{and} \quad h(0) = 1,\ \kappa'(0) = 1.$$

Splitting this into components, one obtains as an equivalent characterization that h solves the initial value problem

$$h(t)^{-1}\partial h_t(t) = -\mathrm{Ad}_{h(t)}^{-1}(\varphi(\partial_t\gamma(t))) + (\tilde{\alpha}_{h(t)})_*(\partial_t\kappa'(t)\kappa'(t)^{-1}) \quad \text{and} \quad h(0) = 1. \tag{43}$$

By construction, $\kappa(1) = poe_A(\gamma)$, $\kappa'(1) = poe_{A'}(\gamma)$, and $h(1) = h_{g,\varphi}(\gamma)$. Evaluating at $t = 1$, Eq. (42) implies Eq. (36). We claim that

$$\tilde{h} = \alpha(g'^{-1}\kappa^{-1}g(\gamma)^{-1}, h). \tag{44}$$

Given that claim, we have coincidence of Eqs. (39) and (40), established by the two equalities

$$(\gamma, g(\gamma)\kappa g't(\tilde{h})^{-1}) = (\gamma, \kappa' g(x)g')$$
$$(\alpha(g(y)\kappa(1)g', \tilde{h}(1)), g(y)\kappa(1)g't(\tilde{h}(1))^{-1}) = (h_{g,\varphi}(\gamma), poe_{A'}(\gamma)g(x)g'),$$

which can easily be deduced from Eq. (44). It remains to prove the claim, Eq. (44). For this purpose we prove that $\tilde{h}$ as defined in Eq. (44) solves the initial value problem Eq. (41). We have $\tilde{h}(0) = 1$ and obtain

$$\partial_t\tilde{h}(t) = (\alpha_{h(t)})_*(g'^{-1}\partial_t\kappa(t)^{-1}g(\gamma(t))^{-1} + g'^{-1}\kappa(t)^{-1}\partial_t g(\gamma(t))^{-1}) + (\alpha_{g'^{-1}\kappa(t)^{-1}g(\gamma(t))^{-1}})_*(\partial_t h(t)).$$

Taking derivative in the inverse of Eq. (42) gives

$$\begin{aligned} g'^{-1}g(x)^{-1}\partial_t\kappa'(t)^{-1}t(h(t))^{-1} = g'^{-1}\partial_t\kappa(t)^{-1}g(\gamma(t))^{-1} + g'^{-1}\kappa(t)^{-1}\partial_t g(\gamma(t))^{-1} \\ + g'^{-1}\kappa(t)^{-1}g(\gamma(t))^{-1}t_*(\partial_t h(t)h(t)^{-1}) \end{aligned} \tag{45}$$

Using Eq. (45) the differential equation of Eq. (41) follows. □

We continue our discussion of parallel transport in the trivial principal Γ-bundle $\mathcal{I}_{A,B}$ with the parallel transport along bigons, now assuming that (A, B) is fake-flat.

Proposition 5.7. *Let (A, B) be a fake-flat Γ-connection on M. For a bigon $\Sigma : \gamma \Longrightarrow \gamma'$ we let $\varphi_\Sigma : F_\gamma \Longrightarrow F_{\gamma'}$ denote the parallel transport in the associated trivial principal Γ-2-bundle $\mathcal{I}_{A,B}$. We set $g := poe_A(\gamma)$, $g' := poe_A(\gamma')$ and $h := soe_{A,B}(\Sigma)$. Then, the diagram*

$$\begin{array}{ccc} i_g & \overset{\eta_\gamma}{\Longrightarrow} & F_\gamma \\ {\scriptstyle i_{h,g}}\Big\Downarrow & & \Big\Downarrow{\scriptstyle \varphi_\Sigma} \\ i_{g'} & \underset{\eta_{\gamma'}}{\Longrightarrow} & F_{\gamma'} \end{array}$$

is commutative, where i_g and $i_{g'}$ are the functors of Remark 5.2 (b) and $i_{g,h}$ is the natural transformation of Remark 5.2 (c).

Proof. The diagram is an equality between transformations from a functor to an anafunctor. In terms of the corresponding smooth maps of Remark 2.2 (d) the commutativity means $\varphi_\Sigma(\tilde{\eta}_\gamma(\tilde{g})) = \tilde{\eta}_{\gamma'}(\tilde{g}) \circ i_{h,g}(\tilde{g})$. We recall that in order to compute $\tilde{\eta}_\gamma$ and $\tilde{\eta}_{\gamma'}$ we have the paths κ and κ'. In fact, since γ and γ' are homotopic via the bigon Σ, there is a family κ_s of paths such that $\kappa_0 = \kappa$ and $\kappa_1 = \kappa'$.

We construct a horizontal lift of Σ with source $\tilde{\eta}_\gamma(\tilde{g}) = \mathrm{id}_{(y,g\tilde{g})} * (\gamma, \kappa\tilde{g}) * \mathrm{id}_{(x,\tilde{g})}$, in the sense of Definition 4.1. It is given by $\Phi_1 : [0,1]^2 \longrightarrow M \times G$ defined by $\Phi_1(s,t) := (\Sigma(s,t), \kappa_s(t)\tilde{g})$, $\rho_0 = \mathrm{id}_{(x,\tilde{g})}$, $\rho_1 = \mathrm{id}_{(y,g\tilde{g})}$ and $g_1(s) := \tilde{g}^{-1}\kappa_s(1)^{-1}g\tilde{g}$, and all other data trivial. Its target is $\varphi_\Sigma(\tilde{\eta}_\gamma(\tilde{g})) = \rho_1' * (\gamma', \kappa'\tilde{g}) * \mathrm{id}_{(x,\tilde{g})}$ where ρ_1' has to be determined. We claim that $\rho_1' = (h, g\tilde{g})^{-1}$; given this claim we have

$$\varphi_\Sigma(\tilde{\eta}_\gamma(\tilde{g})) = \rho_1' * (\gamma', \kappa'\tilde{g}) * \mathrm{id}_{(x,\tilde{g})} = (\mathrm{id}_{(y,g'\tilde{g})} * (\gamma', \kappa'\tilde{g}) * \mathrm{id}_{(x,\tilde{g})}) \circ (h, g\tilde{g}) = \tilde{\eta}_{\gamma'}(\tilde{g}) \circ i_{h,g}(\tilde{g});$$

this proves the commutativity of the diagram.

In order to prove the claim, we recall the definition of the target in Eq. (26), resulting in $\rho_1' := R(\mathrm{id}_{(y,g\tilde{g})}, (h_1^{-1}, g_1(1)^{-1}))$. In order to compute h_1 we have to choose a bigon-parameterization Σ_1 of Φ_1. It will suffice to chose a bigon-parameterization Ξ of $(s,t) \longmapsto \kappa_s(t)\tilde{g}$, so that $\Xi : g\tilde{g}g_1^{-1} * \kappa\tilde{g} \Longrightarrow \kappa'\tilde{g} * \mathrm{id}_{\tilde{g}}$ is a bigon in G. Then we may choose $\Sigma_1 := (\Sigma, \Xi) = R((\Sigma, 1), \Xi)$. The canonical section $s : x \longmapsto (x, 1)$ satisfies $s^*\Omega^a = A$ and $s^*\Omega^c = -B$; thus Proposition C.1 (e) gives

$$soe_\Omega(\Sigma, 1) = soe_\Omega(s(\Sigma)) = soe_{A,B}(\Sigma). \tag{46}$$

Now we obtain

$$h_1 := soe_\Omega(\Sigma_1) \overset{\text{Corollary C.5}}{=} \alpha((g'\tilde{g})^{-1}, soe_\Omega(\Sigma, 1)) \overset{\text{Eq. (46)}}{=} \alpha(\tilde{g}^{-1}g'^{-1}, soe_{A,B}(\Sigma)) \overset{\text{Proposition C.1 (b)}}{=} \alpha(\tilde{g}^{-1}g^{-1}, h).$$

Now a straightforward computation shows the claim. □

5.2 Ordinary principal bundles Consider an ordinary principal G-bundle P over M with connection $\omega \in \Omega^1(P, \mathfrak{g})$. As discussed in [13, Example 5.1.11] the action groupoid $P/\!\!/H$ for the right H-action on P induced via $t : H \longrightarrow G$ is a principal Γ-2-bundle over M, and it is equipped with a connection Ω induced by ω. For a point $x \in M$ we have $(P/\!\!/H)_x = P_x/\!\!/H$. For a path $\gamma : x \longrightarrow y$ in M, we have the ordinary parallel transport map $\tau_\gamma : P_x \longrightarrow P_y$. It is G-equivariant, hence H-equivariant, and thus induces a smooth functor

$$\phi_\gamma : P_x/\!\!/H \longrightarrow P_y/\!\!/H$$

between action groupoids. It is straightforward to check that it is Γ-equivariant. We claim that there exists a canonical Γ-equivariant transformation

$$f_\gamma : J_{\phi_\gamma} \Longrightarrow F_\gamma$$

between the anafunctor induced by ϕ_γ and F_γ. We construct f_γ using Remark 2.2 (d); the underlying smooth map $\tilde{f}_\gamma : P_x \longrightarrow F_\gamma$ is defined by $\tilde{f}_\gamma(p) := \mathrm{id}_{\phi_\gamma(p)} * \tilde{\gamma}_p * \mathrm{id}_p$, where $\tilde{\gamma}_p$ is the unique horizontal lift of γ with initial point p.

Lemma 5.8. *The map $\tilde{f}_\gamma$ satisfies (T1) to (T3).*

Proof. (T1) is obvious. For (T2) we compute for $\alpha = (p, h) \in P \times H = \mathrm{Mor}(P_x/\!/H)$:

$$\begin{aligned} \alpha \circ \tilde{f}_\gamma(p) \circ \beta &= \beta^{-1} * \tilde{\gamma}_p * \alpha^{-1} \\ &\sim (\beta^{-1} \circ (\tilde{\gamma}_p(1), h)^{-1}) * \tilde{\gamma}_{t(\alpha)} * (\alpha \circ \alpha^{-1}) \\ &= (\beta^{-1} \circ \phi_\gamma(p, h)^{-1}) * \tilde{\gamma}_{t(\alpha)} * \mathrm{id}_{t(\alpha)} \\ &= \tilde{f}_\gamma(t(\alpha)) \circ \phi_\gamma(\alpha) \circ \beta \end{aligned}$$

Here we have applied the equivalence relation in F_γ to the path $\rho(t) := (\tilde{\gamma}_p(t), h) \in P \times H$, which is horizontal: following [13, Example 5.1.11] we have $\Omega^b = (\tilde{\alpha}_{\mathrm{pr}_H})_*(\mathrm{pr}_P^*\omega) + \mathrm{pr}_H^*\theta$ and hence $\Omega^b(\dot{\rho}(t)) = 0$. Finally, (T3) is a straightforward calculation. ☐

Summarizing, in the principal Γ-2-bundle $P/\!/H$, parallel transport along a path γ is given, up to canonical isomorphism of Γ-equivariant anafunctors, by the smooth functor ϕ_γ. It is obvious that this identification is compatible with pullbacks, bundle morphisms, and path composition.

Proposition 5.9. *Let P be a principal G-bundle with flat connection ω. Let $\Sigma : \gamma_0 \Longrightarrow \gamma_1$ be a bigon. The diagram*

$$\begin{array}{ccc} J_{\phi_{\gamma_0}} & = & J_{\phi_{\gamma_1}} \\ {\scriptstyle f_{\gamma_0}}\Big\Downarrow & & \Big\Downarrow{\scriptstyle f_{\gamma_1}} \\ F_{\gamma_0} & \underset{\varphi_\Sigma}{\Longrightarrow} & F_{\gamma_1} \end{array}$$

is commutative, where $\varphi_\Sigma : F_{\gamma_0} \Longrightarrow F_{\gamma_1}$ denotes the parallel transport in the principal Γ-2-bundle $P/\!/H$. In particular, φ_Σ only depends on γ_0 and γ_1 but not on the bigon Σ.

Proof. Since ω is flat, the induced connection Ω on $P/\!/H$ is fake-flat. Further, since the parallel transport of a flat connection only depends on the homotopy class of the path, we have $\tau_{\gamma_0} = \tau_{\gamma_1}$, thus $\phi_{\gamma_0} = \phi_{\gamma_1}$, and in turn $J_{\phi_{\gamma_0}} = J_{\phi_{\gamma_1}}$. In order to prove commutativity, we specify a horizontal lift of Σ in the sense of Definition 4.1, with source $\tilde{f}_{\gamma_0}(p)$ for some $p \in P_x$. Let $\gamma_s : [0,1] \longrightarrow M$ be defined by $\gamma_s(t) := \Sigma(s,t)$, and let $\tilde{\gamma}_{s,p}$ be the unique horizontal lift into P of γ_s with initial point p. Let $\Phi(s,t) := \tilde{\gamma}_{s,p}(t)$. Because ω is flat, we have $\Phi(s,1) = q$ for some constant point $q \in P_y$. Taking all other data trivial, Φ is indeed a horizontal lift of Σ with source $\tilde{f}_{\gamma_0}(p)$. Since $\Omega^c = 0$, we have $soe_\Omega(\Phi) = 1$ by Proposition C.1 (f); hence, the target of this horizontal lift is $\tilde{\gamma}_{1,p} = \tilde{f}_{\gamma_1}(p)$. ☐

6. The parallel transport 2-functor

In this section we prove the main result of this article, namely that the parallel transport constructions of Sections 3 and 4 fit in the axiomatic framework of transport 2-functors. This framework is formulated for *thin homotopy classes* of paths and bigons. In Section 6.1 we provide a way to push our constructions into the setting of thin homotopy classes. In Section 6.2 we show that the various properties we have proved in Sections 3 and 4 show that parallel transport is a 2-functor, and in Section 6.3 we show that this 2-functor is a *transport* 2-functor.

6.1 Thin homotopy invariance We study the dependence of the parallel transports along paths and bigons under thin homotopies, i.e. smooth homotopies with non-maximal rank. Here, by rank of a smooth map we mean the supremum of the rank of its differential over all points. All kinds of reparameterizations are special cases of thin homotopies.

Two bigons $\Sigma, \Sigma' : \gamma \Longrightarrow \gamma'$ between paths $\gamma, \gamma' : x \longrightarrow y$ are called *homotopic*, if there exists a smooth homotopy $h : [0,1]^3 \longrightarrow M$ (i.e., $h(0,s,t) = \Sigma(s,t)$ and $h(1,s,t) = \Sigma'(s,t)$) that fixes all boundaries, i.e. $h(r,s,0) = x$, $\Sigma(r,s,1) = y$, $h(r,0,t) = \gamma(t)$ and $h(r,1,t) = \gamma'(t)$ for all $r,s,t \in [0,1]$. Two bigons are called *thin homotopic*, if they are homotopic by a homotopy of rank *less* than 3.

Proposition 6.1. *Let $\mathcal{P}$ be a principal Γ-2-bundle with fake-flat connection. Then, the parallel transport along bigons depends only on the thin homotopy class of the bigon, i.e., if $\Sigma, \Sigma' : \gamma \Longrightarrow \gamma'$ are thin homotopic bigons, then $\varphi_\Sigma = \varphi_{\Sigma'}$.*

Proof. We first note that every thin homotopy can be split into finitely many small ones, so that it suffices to prove the claim for a small thin homotopy. By "small" we mean that there exist $n \in \mathbb{N}$, $t \in T_n$ and sections $\sigma_i : U_i \longrightarrow \mathrm{Obj}(\mathcal{P})$ defined on open sets U_i such that

$$h(\{(r,s,t) \mid t_{i-1} \leq t \leq t_i, 0 \leq r,s \leq 1\}) \subseteq U_i.$$

We think of $\Sigma^r := h(r,-,-)$ as a smooth family of small bigons. We claim that we can consistently choose a smooth family of horizontal lifts with a common source $\xi \in F_\gamma$. This means, there exist $\Phi_i^r : [0,1] \times [t_{i-1}, t_i] \longrightarrow \mathrm{Obj}(\mathcal{P})$, $\rho_i^r : [0,1] \longrightarrow \mathrm{Mor}(\mathcal{P})$, and $g_i^r : [0,1] \longrightarrow G$ depending smoothly on r (this means, for instance, that $[0,1] \times [0,1] \times [t_{i-1}, t_i] \longrightarrow \mathrm{Obj}(\mathcal{P}) : (r,s,t) \longmapsto \Phi_i^r(s,t)$ is smooth), such that $(\Phi_i^r, \rho_i^r, g_i^r)$ is a horizontal lift of Σ^r with source ξ, and $\Phi_i^r(s,t)$ has rank less than 3. Additionally, we can require $\Phi_i^r(0,t)$ and $\rho_i^r(0)$ are independent of r, and we require that there exist smooth maps $k_i : [0,1] \longrightarrow G$, denoted k_i^r, such that $k_i^0 = 1$ and $\Phi_i^r(1,t) = R(\Phi_i^0(1,t), (k_i^r)^{-1})$. This claim can be proved by repeating the proof of Lemma 4.2 in families.

We remark that $\rho_i^0(1)$ and $\rho_i^r(1)$ satisfy

$$R(t(\rho_i^0(1)), (k_{i+1}^r)^{-1}) = R(\Phi_{i+1}^0(1, t_{i-1}), (k_{i+1}^r)^{-1}) = \Phi_{i+1}^r(1, t_{i-1}) = t(\rho_i^r(1))$$
$$R(s(\rho_i^0(1)), g_i^0(1)^{-1}(k_i^r)^{-1} g_i^r(1)) = R(\Phi_i^0(1, t_i), (k_i^r)^{-1} g_i^r(1)) = R(\Phi_i^r(1, t_i), g_i^r(1)) = s(\rho_i^r(1))$$

By [13, Lemma 3.1.4], there exist unique $\eta_i^r \in H$, smoothly depending on r, with $\eta_i^0 = 1$ such that

$$\rho_i^r(1) = R(\rho_i^0(1), (\eta_i^r, g_i^0(1)^{-1}(k_i^r)^{-1} g_i^r(1))) \quad \text{and} \quad t(\eta_i^r) g_i^0(1)^{-1}(k_i^r)^{-1} g_i^r(1) = (k_{i+1}^r)^{-1}. \tag{47}$$

Next we perform the following pre-computations (recall the notion of a bigon-parameterization from Remark C.2):

(a) Consider a bigon-parameterization $\Theta : \zeta_i \Longrightarrow \zeta_i'$ in G of $(r,t) \longmapsto (k_i^r)^{-1}$ with $\zeta_i = (k_i^-)^{-1} * \mathrm{id}_1$ and $\zeta_i' = \mathrm{id}_1 * (k_i^-)^{-1}$. Then, $\Xi_i := R(\mathrm{id}_{\gamma_i'^0}, \Theta)$ is a bigon-parameterization of $(r,t) \longmapsto \Phi_i^r(1,t)$, going between the paths $\kappa_i(r) := R(\gamma_i'^0(t_i), (k_i^r)^{-1})$ and $\kappa_i'(r) := R(\gamma_i'^0(t_{i-1}), (k_i^r)^{-1})$. By Corollary C.5 we have

$$soe_\Omega(\Xi_i) = soe_\Omega(R(\mathrm{id}_{\gamma_i'^0}, \Theta)) = \alpha(\zeta_i(1)^{-1}, soe_\Omega(\mathrm{id}_{\gamma_i'^0})) = 1.$$

(b) We note that $r \longmapsto \Phi_i^r(s,t)$ is a homotopy between bigons

$$(\mathrm{id}_{\gamma_i'^1} \circ \Psi_i') \bullet (\Xi_i \circ \mathrm{id}_{\nu_i^0}) \bullet (\mathrm{id}_{\kappa_i} \circ \Sigma_i^0) \quad \text{and} \quad \Sigma_i^1 \bullet (\Psi_i \circ \mathrm{id}_{\gamma_i}),$$

both going from $\kappa_i \circ \mu_i^0 \circ \gamma_i$ to $\gamma_i'^1 \circ \nu_i^1$. Since $\Phi_i^r(s,t)$ is thin, the surface-ordered exponentials of both bigons coincide (Proposition C.1 (a)). Using Propositions C.1 (b) and C.1 (c) we get $\psi_i' \cdot \alpha(k_i^1, h_i^0) = h_i^1 \cdot \psi_i$. With Eq. (50) we can rewrite this as

$$h_i^0 \cdot \psi_i' = h_i^1 \cdot \psi_i \tag{48}$$

(c) We consider bigon-parameterizations $\Upsilon_i : \rho_i^-(1) * \rho_i^0 \Longrightarrow \rho_i^1$ in $\mathrm{Mor}(\mathcal{P})$ of $(r,s) \longmapsto \rho_i^r(s)$ and $\Theta_i : g_i^-(1) * g_i^0 \Longrightarrow g_i^1$ in G of $(r,s) \longmapsto g_i^r(s)$. Note that $\Psi_{i+1} := t(\Upsilon_i) : \kappa_{i+1} \circ \mu_{i+1}^0 \Longrightarrow \mu_{i+1}^1$ is a bigon-parameterization of $(r,s) \longmapsto \Phi_{i+1}^r(s, t_{i+1})$, and that $\Psi_i' := R(s(\Upsilon_i), \Theta_i^{-1}) : \kappa_i' \circ \nu_i^0 \Longrightarrow \nu_i^1$ is a bigon-parameterizations of $(r,s) \longmapsto \Phi_i^r(s, t_{i-1})$. We set $\psi_i' := soe_\Omega(\Psi_i')$ and $\psi_i := soe_\Omega(\Psi_i)$. We compute the quantity h_Ω of the source and target paths of Υ_i. Since ρ_i^r are horizontal with horizontal target ν_{i+1}^r (and hence also horizontal source), we have by Proposition B.8 (g) $h_\Omega(\rho_i^r) = 1$. Further, we calculate

$$\begin{aligned} h_\Omega(\rho_i^-(1)) &\overset{\text{Eq. (47)}}{=} h_\Omega(R(\rho_i^0(1), (\eta_i^r, g_i^0(1)^{-1}(k_i^r)^{-1} g_i^r(1)))) \\ &= h_\Omega(R(R(\rho_i^0(1), (\eta_i^r, 1)), (1, g_i^0(1)^{-1}(k_i^r)^{-1} g_i^r(1))))) \\ &\overset{\text{Proposition B.8 (a)}}{=} \alpha(g_i^1(1)^{-1} k_i^1 g_i^0(1), h_\Omega(R(\rho_i^0(1), (\eta_i^r, 1)), 1)) \\ &\overset{\text{Proposition B.8 (f)}}{=} \alpha(g_i^1(1)^{-1} k_i^1 g_i^0(1), (\eta_i^1)^{-1}) \end{aligned}$$

Now, Proposition C.4 gives

$$\psi_i \cdot \alpha(k_i^1 g_i^0(1), \eta_i^1) = \alpha(g_i^1(1), \psi_{i+1}'). \tag{49}$$

(d) Since ν_i^0 and ν_i^1 are horizontal, we have from Proposition B.2 (a)

$$1 = t(\psi_i') poe_{\Omega^a}(\kappa_i') = t(\psi_i') k_i^1. \tag{50}$$

Summarizing our pre-calculations, we obtain:

$$\begin{aligned} \eta_i^1 h_i^0 \psi_i' (h_i^1)^{-1} &\overset{\text{Eq. (48)}}{=} h_i^1 \psi_i \psi_i'^{-1} (h_i^0)^{-1} \eta_i^1 h_i^0 \psi_i' (h_i^1)^{-1} \\ &\overset{\text{Eq. (50)}}{=} h_i^1 \psi_i \alpha(k_i^1, (h_i^0)^{-1} \eta_i^1 h_i^0)(h_i^1)^{-1} \end{aligned}$$

$$\overset{\text{Lemma 4.3}}{\downarrow}{=} \alpha(g_i^1(1)^{-1}, \psi_i)\alpha(g_i^1(1)^{-1}k_i^1 g_i^0(1), \eta_i^1)$$
$$\overset{\text{Eq. (49)}}{\downarrow}{=} \psi'_{i+1} \tag{51}$$

In order to show the statement of the proposition, we have to prove that the elements

$$\rho_n'^0 * \gamma_n'^0 * \ldots * \gamma_1'^0 * \rho_0'^0 \quad \text{and} \quad \rho_n'^1 * \gamma_n'^1 * \ldots * \gamma_1'^1 * \rho_0'^1$$

are equivalent, where $\rho_i'^r = R(\rho_i^r(1), ((h_i^r)^{-1}, g_i^r(1)^{-1}))$ according to the definition of the target of a horizontal lift. We consider the paths $\tilde{\rho}_i : [t_{i-1}, t_i] \longrightarrow \mathrm{Mor}(\mathcal{P})$ defined by $\tilde{\rho}_i(t) := R(\mathrm{id}_{\gamma_i'^0(t)}, (\psi_i', 1))$. They are horizontal by Propositions 3.2 (b) and 3.2 (e), and have $s(\tilde{\rho}_i) = \gamma_i'^0$ and $t(\tilde{\rho}_i) = \gamma_i'^1$ using Eq. (50). It remains to check that the paths $\tilde{\rho}_i$ convey the required equivalence:

$$\rho_i'^1 \circ \tilde{\rho}_i(t_i) \overset{\text{Eq. (47)}}{\downarrow}{=} R(R(\rho_i^0(1), (\eta_i^1, g_i^0(1)^{-1}(k_i^1)^{-1}g_i^1(1))), ((h_i^1)^{-1}, g_i^1(1)^{-1})) \circ R(\mathrm{id}_{\gamma_i'^0(t_i)}, (\psi_i', 1))$$
$$= R(\rho_i^0(1), (\eta_i^1\alpha(g_i^0(1)^{-1}(k_i^1)^{-1}, (h_i^1)^{-1})\alpha(g_i^0(1)^{-1}, \psi_i'), g_i^0(1)^{-1}))$$
$$\overset{\text{Eq. (50) and Lemma 4.3}}{\downarrow}{=} R(\rho_i^0(1), (\eta_i^1\alpha(t(h_i^0)t(\psi_i'), (h_i^1)^{-1})\alpha(t(h_i^0), \psi_i'), g_i^0(1)^{-1}))$$
$$= R(\rho_i^0(1), (\eta_i^1 h_i^0 \psi_i' (h_i^1)^{-1}(h_i^0)^{-1}, g_i^0(1)^{-1}))$$
$$\overset{\text{Eq. (51)}}{\downarrow}{=} R(\rho_i^0(1), (\psi_{i+1}'(h_i^0)^{-1}, g_i^0(1)^{-1}))$$
$$= \tilde{\rho}_{i+1}(t_i) \circ \rho_i'^0$$

□

Next we come to the paths, where the situation is more complicated. First of all, two paths $\gamma, \gamma' : x \longrightarrow y$ are called *thin homotopic*, if there exists a bigon $\Sigma : \gamma \Longrightarrow \gamma'$ of rank less than two. The complications arise because the anafunctors F_γ and $F_{\gamma'}$ associated to thin homotopic paths are not equal. The following proposition shows that they are canonically 2-isomorphic, which is the best that we can expect. The 2-isomorphism is the Γ-equivariant transformation φ_Σ associated to a thin homotopy $\Sigma : \gamma \Longrightarrow \gamma'$. The important point is that this does not depend on the choice of the bigon.

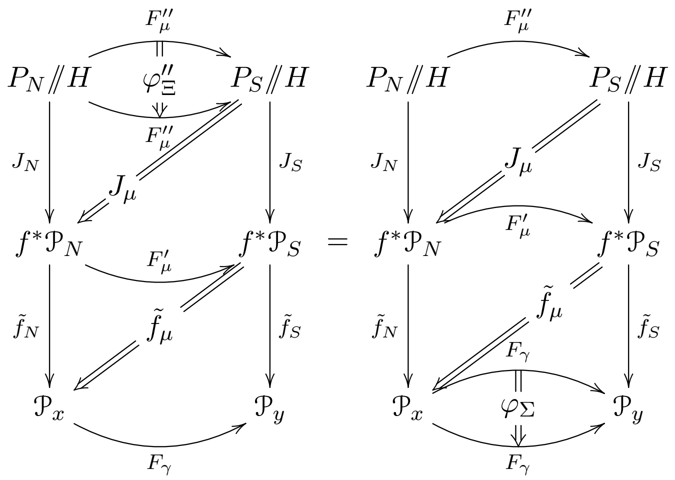
Proposition 6.2. *Let $\mathcal{P}$ be a principal Γ-2-bundle with fake-flat connection. Then, the parallel transport along a thin bigon is independent of the bigon, i.e., if $\Sigma, \Sigma' : \gamma_1 \Longrightarrow \gamma_2$ are bigons of rank less than two, then $\varphi_\Sigma = \varphi_{\Sigma'}$.*

Proof. We prove the equivalent statement that a thin homotopy $\Sigma : \gamma \Longrightarrow \gamma$ induces the identity $\varphi_\Sigma = \mathrm{id}_{F_\gamma}$. Proposition 6.1 allows us to change Σ within its thin homotopy class, so that we can assume that Σ has the following properties:

(a) There exists $\epsilon > 0$ such that $\Sigma(s,t) = x$ for $0 \leq t < \epsilon$ and $\Sigma(s,t) = y$ for $1 - \epsilon < t \leq 1$.
(b) There exists $\epsilon > 0$ such that $\Sigma(s,t) = \gamma(t)$ for all $0 \leq s < \epsilon$ and all $1 - \epsilon < s \leq 1$.

We define a smooth map $f : S^2 \longrightarrow M$ by $f(\vartheta, \varphi) := \Sigma(\frac{\varphi}{2\pi}, \frac{\vartheta}{\pi})$, where $0 \leq \vartheta \leq \pi$ and $0 \leq \varphi \leq 2\pi$ are spherical coordinates. This is well-defined due to (a) and smooth due to (b). Obviously

f has rank one. By [13, Corollary 5.3.6] there exists a principal G-bundle P over S^2 with flat connection ω, together with a 1-morphism $J : P /\!\!/ H \longrightarrow f^*\mathcal{P}$ equipped with a fake-flat, connective, connection-preserving pullback. We define the map $\Xi : [0,1] \longrightarrow S^2$ by $\Xi(s,t) := (2\pi t, \pi s)$. This is a bigon $\Xi : \mu \Longrightarrow \mu$, where μ is the $\varphi = 0$ meridian passing from north pole N ($\vartheta = 0$) to south pole ($\vartheta = 1$). We have $\gamma = f \circ \mu$ and $\Sigma = f \circ \Xi$. Combining Propositions 4.10 and 4.11 we obtain the following the "tin can" equation between transformations:



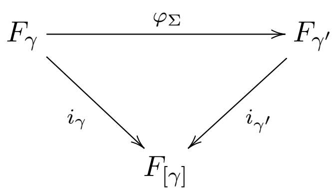
Here, F'_μ denotes the parallel transport along μ in $f^*\mathcal{P}$, and F''_μ denotes the parallel transport in $P /\!\!/ H$. From Proposition 5.9 we conclude that $\varphi''_\Xi = \mathrm{id}_{F''_\mu}$. Thus, $\varphi_\Sigma = \mathrm{id}_{F_\gamma}$. □

Let $[\gamma]$ be a thin homotopy class of paths. We define the set

$$F_{[\gamma]} := \left(\bigsqcup_{\gamma \in [\gamma]} F_\gamma \right) / \sim ,$$

where $\xi \in F_\gamma$ and $\xi' \in F_{\gamma'}$ are defined to be equivalent if $\xi' = \varphi_\Sigma(\xi)$ for some (and hence by Proposition 6.2 all) thin homotopy $\Sigma : \gamma \Longrightarrow \gamma'$. By Lemmas 4.5 (a) to 4.5 (c) it is clear that anchors and actions are well-defined on the set $F_{[\gamma]}$.

Lemma 6.3. *There exists a unique smooth manifold structure on $F_{[\gamma]}$ such that $F_{[\gamma]}$ is a Γ-equivariant anafunctor, the projections $i_\gamma : F_\gamma \longrightarrow F_{[\gamma]}$ are Γ-equivariant transformations, and the diagram*



is commutative for all thin bigons $\Sigma : \gamma \Longrightarrow \gamma'$

Proof. We consider the open cover $\{U_\gamma\}_{\gamma \in [\gamma]}$ of $\mathcal{P}_x$, where $U_\gamma := \mathrm{Obj}(\mathcal{P}_x)$. Over each open set U_γ we have the principal $\mathcal{P}_y$-bundle F_γ. Over each double overlap ($U_\gamma \cap U_{\gamma'} = \mathrm{Obj}(\mathcal{P}_x)$) we have a bundle isomorphism $\varphi_\Sigma : F_\gamma \longrightarrow F_{\gamma'}$, for some choice of a thin homotopy Σ. Over each triple overlap, these satisfy the cocycle condition due to Proposition 6.2. Our definition of $F_{[\gamma]}$ realizes the descend construction for the stack of principal $\mathcal{P}_y$-bundles over $\mathcal{P}_x$; hence $F_{[\gamma]}$ is a principal $\mathcal{P}_y$-bundle. Now, the remaining statements follow. □

Now we are in position to re-define parallel transport in a setting of thin homotopy classes of paths and bigons. To a thin homotopy class $[\gamma]$ of paths between x and y we associate the Γ-equivariant anafunctor

$$F_{[\gamma]} : \mathcal{P}_x \longrightarrow \mathcal{P}_y.$$

Two bigons $\Sigma : \gamma_0 \Longrightarrow \gamma_1$ and $\Sigma' : \gamma_0' \Longrightarrow \gamma_1'$ will now be called thin homotopic, if there exists a homotopy h between them of rank less than three, that fixes the endpoints x and y, and restricts to homotopies $h_0 : \gamma_0 \Longrightarrow \gamma_0'$ and $h_1 : \gamma_1 \Longrightarrow \gamma_1'$ of rank less than 2. This generalizes the relation introduced at the beginning of this section in that the bounding paths do not have to be equal but can be thin homotopic themselves. For a thin homotopy class $[\Sigma] : [\gamma_0] \Longrightarrow [\gamma_1]$ of bigons we define

$$\varphi_{[\Sigma]} = i_{\gamma_1} \circ \varphi_\Sigma \circ i_{\gamma_0}^{-1}.$$

It is straightforward to check using Proposition 6.1 and Lemma 6.3 that this definition is independent of the choice of the representative Σ. Similarly, the transformations u_x and c_{γ_1,γ_2} of Section 3.3 induce well-defined transformations

$$u_x : F_{[\mathrm{id}_x]} \Longrightarrow \mathrm{id}_{\mathcal{P}_x} \quad \text{and} \quad c_{[\gamma_1],[\gamma_2]} : F_{[\gamma_2]} \circ F_{[\gamma_1]} \Longrightarrow F_{[\gamma_2]\circ[\gamma_1]}.$$

At this point we have the following result.

Corollary 6.4. *If $\gamma : x \longrightarrow y$ is a path in M, then $F_{[\gamma]} : \mathcal{P}_x \longrightarrow \mathcal{P}_y$ is a weak equivalence.*

Proof. Let $\bar{\gamma}$ be the reversed path. Then, we have transformations $F_{[\bar{\gamma}]} \circ F_{[\gamma]} \cong F_{[\bar{\gamma}*\gamma]} \cong F_{[\mathrm{id}_x]} \cong \mathrm{id}_{\mathcal{P}_x}$ and, analogously, $F_{[\gamma]} \circ F_{[\bar{\gamma}]} \cong \mathrm{id}_{\mathcal{P}_y}$. □

Finally, suppose $J : \mathcal{P}_1 \longrightarrow \mathcal{P}_2$ is a 1-morphism in $2\text{-}\mathcal{B}un_\Gamma^{\nabla ff}(M)$. Then we define a transformation

$$J_{[\gamma]} := (i_\gamma \circ \mathrm{id}_{J_x}) \bullet J_\gamma \bullet (\mathrm{id}_{J_y} \circ i_\gamma^{-1}) : J_y \circ F_{[\gamma]} \Longrightarrow F'_{[\gamma]} \circ J_x.$$

Using Proposition 4.10 one can check that this definition is independent of the choice of the representative γ. Summarizing, all our definitions of Sections 3 and 4 persist under the passage to thin homotopy classes. In the following section we will see the main advantage of this passage, namely that it allows an organization in bicategories.

6.2 Organization in bicategories The path 2-groupoid of M is the 2-groupoid $\mathcal{P}_2(M)$ whose objects are the points of M, 1-morphisms are thin homotopy classes of paths in M, and 2-morphisms are thin homotopy classes of bigons in M. A detailed definition is in [10, Section 2.1]. In this subsection we assemble the parallel transports of the previous subsection into a 2-functor $\mathrm{tra}_\mathcal{P} : \mathcal{P}_2(M) \longrightarrow \Gamma\text{-}\mathcal{T}or$, where $\Gamma\text{-}\mathcal{T}or$ is the bicategory of Γ-torsors, see Remark 5.2. For the terminology of bicategories we refer to [12, Appendix A].

Proposition 6.5. *Let $\mathcal{P}$ be a principal Γ-2-bundle $\mathcal{P}$ with fake-flat connection Ω. Then, the assignments $x \longmapsto \mathcal{P}_x$, $[\gamma] \longmapsto F_{[\gamma]}$, and $[\Sigma] \longmapsto \varphi_{[\Sigma]}$ form a 2-functor*

$$\mathrm{tra}_\mathcal{P} : \mathcal{P}_2(M) \longrightarrow \Gamma\text{-}\mathcal{T}or$$

with unitors u_x and compositors $c_{[\gamma_1],[\gamma_2]}$.

Proof. There are four axioms to check, see, e.g. [12, Def. A.5]. Axiom (F1) is functoriality with respect to vertical composition; this is Proposition 4.8. Axiom (F2) is the compatibility with the horizontal composition; this is Proposition 4.9. Axioms (F3) and (F4) concern the coherence of compositors and unitors; these are Propositions 3.12 and 3.13. □

Example 6.6. Let $\mathcal{I}_{A,B}$ be the trivial principal Γ-2-bundle over M whose connection is induced from a fake-flat Γ-connection (A, B). We obtain from (A, B) the smooth 2-functor $F_{A,B} : \mathcal{P}_2(M) \longrightarrow B\Gamma$ (see Section D). Its composition $i(F_{A,B})$ with the 2-functor $i : B\Gamma \longrightarrow \Gamma\text{-}\mathcal{T}or$ of Remark 5.2 (d) is a "trivial" transport 2-functor only depending on the Γ-connection (A, B). On the other hand, we have the 2-functor $\mathrm{tra}_{\mathcal{I}_{A,B}}$ of Proposition 6.5. The two 2-functors are equivalent via a pseudonatural transformation

$$\eta_{A,B} : i(F_{A,B}) \longrightarrow \mathrm{tra}_{\mathcal{I}_{A,B}},$$

whose components are the assignments $x \longmapsto \mathrm{id}_\Gamma$ and $\gamma \longmapsto \eta_\gamma$ of Section 5.1. There are two axioms to check [12, Def. A.6]; these are precisely Propositions 5.4 and 5.7. This "computes" the parallel transport 2-functor of a connection on the trivial principal Γ-2-bundle.

Proposition 6.7. *Let $J : \mathcal{P}_1 \longrightarrow \mathcal{P}_2$ be a 1-morphism in 2-$\mathcal{B}un_\Gamma^{\nabla ff}(M)$. Then, the assignments $x \longmapsto J_x$ and $[\gamma] \longmapsto J_{[\gamma]}$ form a pseudonatural transformation*

$$\rho_J : \mathrm{tra}_{\mathcal{P}_1} \longrightarrow \mathrm{tra}_{\mathcal{P}_2}.$$

Proof. There are two axioms to check, see [12, Def. A.6]. Axiom (T1) is the compatibility with path composition; this is Proposition 3.17. Axiom (T2) is naturality with respect to 2-morphisms; this is Proposition 4.10. □

Example 6.8. Suppose fake-flat Γ-connections (A, B) and (A', B') are related by a gauge transformation (g, φ). On one side, we have a smooth pseudonatural transformation

$$\rho_{g,\varphi} : F_{A,B} \longrightarrow F_{A',B'},$$

see Section D. On the other side, we have a 1-morphism $J = J_{g,\varphi} : \mathcal{I}_{A,B} \longrightarrow \mathcal{I}_{A',B'}$ in 2-$\mathcal{B}un_\Gamma^{\nabla ff}(M)$ (Remark 5.1 (b)), to which Proposition 6.7 associates a pseudonatural transformation $\rho_J : \mathrm{tra}_{\mathcal{I}_{A,B}} \longrightarrow \mathrm{tra}_{\mathcal{I}_{A',B'}}$. We find a commutative diagram

$$\begin{array}{ccc} i(F_{A,B}) & \xrightarrow{\eta_{A,B}} & \mathrm{tra}_{\mathcal{I}_{A,B}} \\ {\scriptstyle i(\rho_{g,\varphi})}\downarrow & & \downarrow{\scriptstyle \rho_J} \\ i(F_{A',B'}) & \xrightarrow[\eta_{A',B'}]{} & \mathrm{tra}_{\mathcal{I}_{A',B'}} \end{array}$$

of pseudonatural transformations, where $\eta_{A,B}$ and $\eta_{A',B'}$ are the pseudonatural transformations of Example 6.6. Commutativity means that clockwise and counter-clockwise compositions have the same assignments to points and paths. Coincidence for points follows from the definition of $J_{g,\varphi}$, which has an underlying smooth functor ϕ_g. Coincidence for paths follows from Proposition 5.6. This "computes" the transformation between parallel transport 2-functors of trivial bundles with gauge equivalent connections.

Proposition 6.9. *Let $f : J \Longrightarrow J'$ be a 2-morphism in $2\text{-}\mathcal{B}un_\Gamma^{\nabla ff}(M)$. Then, the assignment $x \longmapsto f_x$ forms a modification*

$$\mathcal{A}_f : \rho_J \Longrightarrow \rho_{J'}.$$

Proof. There is only one axiom ([12, Definition A.8]); proved by Proposition 3.19. □

Example 6.10. Suppose we have two fake-flat Γ-connections (A, B) and (A', B'), two gauge transformations (g_1, φ_1) and (g_2, φ_2), and a gauge 2-transformation a between (g_1, φ_1) and (g_2, φ_2). Then we have a smooth modification

$$\mathcal{A}_a : \rho_{g_1,\varphi_1} \Longrightarrow \rho_{g_2,\varphi_2},$$

see Section D. On the other side, we have a 2-morphism $f_a : J_{g_1,\varphi_1} \Longrightarrow J_{g_2,\varphi_2}$ between the 1-morphisms associated to the gauge transformations, see Remark 5.1 (c). In turn, we obtain a modification $\mathcal{A}_{f_a} : \rho_{J_{g_1,\varphi_1}} \Longrightarrow \rho_{J_{g_2,\varphi_2}}$. Then we have a commutative diagram

$$\begin{array}{ccc} i(\rho_{g_1,\varphi_1}) & = & \eta_k^{-1} \circ \rho_{J_{g_1,\varphi_1}} \circ \eta_i \\ {\scriptstyle i(\mathcal{A}_a)} \Big\Downarrow & & \Big\Downarrow {\scriptstyle \mathrm{id} \circ \mathcal{A}_{f_a} \circ \mathrm{id}} \\ i(\rho_{g_2,\varphi_2}) & = & \eta_k^{-1} \circ \rho_{J_{g_2,\varphi_2}} \circ \eta_i \end{array}$$

of modifications between pseudonatural transformations between 2-functors from $\mathcal{P}_2(M)$ to $\Gamma\text{-}\mathcal{T}or$. Indeed, evaluating at a point x gives a 2-morphism in $\Gamma\text{-}\mathcal{T}or$, in fact between 1-morphisms that are smooth functors (not anafunctors). Thus, the diagram is, for each $x \in M$, an equality between natural transformations between functors from Γ to Γ. We compare its components at an object g, only using the given definitions:

$$\mathcal{A}_{f_a}(x)(g) = f_a|_x(g) = f_a(x, g) = (\mathrm{id}_x, (a(x), g_1(x)g)) = i_{a(x),g_1(x)}(g) = i(\mathcal{A}_a)(g).$$

This shows commutativity.

Theorem 6.11. *Propositions 6.5, 6.7 and 6.9 furnish a (strict) 2-functor*

$$\mathrm{tra} : 2\text{-}\mathcal{B}un_\Gamma^{\nabla ff}(M) \longrightarrow \mathcal{F}un(\mathcal{P}_2(M), \Gamma\text{-}\mathcal{T}or).$$

Proof. That the composition of 1-morphisms is respected is the content of Proposition 3.18. On the level of 2-morphisms, the 2-functor is just restriction to points (see Proposition 6.9); this clearly preserves horizontal and vertical composition. □

Remark 6.12. Examples 6.6, 6.8 and 6.10 can be interpreted as follows. The constructions of Remark 5.1 relating Γ-connections to trivial 2-bundles form a 2-functor

$$L^{ff} : \mathcal{C}on_\Gamma^{ff}(M) \longrightarrow 2\text{-}\mathcal{B}un_\Gamma^{\nabla ff}(M)$$

relating Γ-connections to connections on trivial Γ-2-bundles. The constructions of Section D form another 2-functor

$$\mathcal{P} : \mathcal{C}on_\Gamma^{ff}(M) \longrightarrow \mathcal{F}un(\mathcal{P}_2(M), B\Gamma)$$

relating Γ-connections to 2-functors on the path 2-groupoid. We have a pseudonatural equivalence

$$\mathrm{tra} \circ L^{ff} \cong i \circ \mathcal{P},$$

established by assigning $(A, B) \longmapsto \eta_{A,B}$ and $(g, \varphi) \longmapsto \mathrm{id}$. It expresses the fact that trivial principal 2-bundles have trivial (more precisely: canonically trivializable) parallel transport 2-functors.

6.3 The transport 2-functor formalism The transport 2-functor formalism [11] axiomatically specifies a sub-bicategory

$$\mathrm{Trans}_\Gamma(M, \Gamma\text{-}\mathcal{T}or) \subseteq \mathcal{F}un(\mathcal{P}_2(M), \Gamma\text{-}\mathcal{T}or)$$

of 2-functors, pseudonatural transformations, and modifications that are supposed to implement higher-dimensional parallel transport. Essentially, the axioms require that a transport 2-functor can locally be described by path-ordered and surface-ordered exponentials of Γ-connections. We will give more details in the proof of the following result.

Theorem 6.13. *The image of the 2-functor* tra *of Theorem 6.11 is contained in the sub-bicategory* $\mathrm{Trans}_\Gamma(M, \Gamma\text{-}\mathcal{T}or)$*, and hence induces a 2-functor*

$$\mathrm{tra} : 2\text{-}\mathcal{B}un_\Gamma^{\nabla ff}(M) \longrightarrow \mathrm{Trans}_\Gamma(M, \Gamma\text{-}\mathcal{T}or).$$

In other words, parallel transport in principal Γ*-2-bundles fits into the axiomatic framework for higher-dimensional parallel transport.*

One nice consequence of Theorem 6.13 is the following general result about transport 2-functors, see [11, Proposition 3.3.6].

Corollary 6.14. *If* Ω *is a flat connection on a principal* Γ*-2-bundle, then the parallel transport along bigons only depends on the homotopy class of the bigon.*

As a further consequence, the discussion of surface holonomy given in [11, Section 5] applies to principal Γ-2-bundles. In the remainder of this subsection we prove Theorem 6.13, split into Propositions 6.15, 6.17 and 6.19.

Proposition 6.15. *If* $\mathcal{P}$ *is a principal* Γ*-2-bundle over* M *with fake-flat connection, then the 2-functor* $\mathrm{tra}_\mathcal{P}$ *of Proposition 6.5 is a transport 2-functor with* $B\Gamma$*-structure.*

Remark 6.16. For the proof we extract and slightly reformulate the following results of [13, Propositions 5.4.6 & 5.4.9]. Let $\mathcal{P}$ be a principal Γ-2-bundle over M with fake-flat connection.

(a) Every point $x \in M$ has an open neighborhood $x \in U \subseteq M$ together with a Γ-connection (A, B) on U and a 1-morphism $T : \mathcal{I}_{A,B} \longrightarrow \mathcal{P}$ in $2\text{-}\mathcal{B}un_\Gamma^{\nabla ff}(M)$.

(b) Suppose $U \subseteq M$ is a contractible open set. If (A, B) and (A', B') are Γ-connections on U, and $T : \mathcal{I}_{A,B} \longrightarrow \mathcal{P}|_U$ and $T' : \mathcal{I}_{A',B'} \longrightarrow \mathcal{P}|_U$ are 1-morphisms in $2\text{-}\mathcal{B}un_\Gamma^{\nabla ff}(U)$, then there exists a gauge transformation $(g, \varphi) : (A, B) \longrightarrow (A', B')$ and a 2-morphism $\tilde\sigma : T'^{-1} \circ T \Longrightarrow J_{g,\varphi}$ in $2\text{-}\mathcal{B}un_\Gamma^{\nabla ff}(M)$.

(c) Suppose $U \subseteq M$ is an open set, and I is some (index) set. If, for each $i \in I$, (A_i, B_i) are Γ-connections on U, $T_i : \mathcal{I}_{A_i,B_i} \longrightarrow \mathcal{P}|_U$ are 1-morphisms in $2\text{-}\mathcal{B}un_\Gamma^{\nabla ff}(U)$, $(g_{ij}, \varphi_{ij}) : (A_i, B_i) \longrightarrow (A_j, B_j)$ are gauge transformations, and $\sigma_{ij} : T_j^{-1} \circ T_i \Longrightarrow J_{g_{ij},\varphi_{ij}}$ are 2-morphisms in $2\text{-}\mathcal{B}un_\Gamma^{\nabla ff}(U)$, then there exists a unique gauge 2-transformation

$$a_{ijk} : (g_{jk}, \varphi_{jk}) \circ (g_{ij}, \varphi_{ij}) \Longrightarrow (g_{ik}, \varphi_{ik}),$$

and a commutative diagram

$$\begin{array}{ccc} T_k^{-1} \circ T_j \circ T_j^{-1} \circ T_i & \xRightarrow{\sigma_{jk} \circ \sigma_{ij}} & J_{g_{jk},\varphi_{jk}} \circ J_{g_{ij},\varphi_{ij}} \\ {\scriptstyle \mathrm{id} \circ d_{T_j} \circ \mathrm{id}} \Downarrow & & \Downarrow {\scriptstyle f_{a_{ijk}}} \\ T_k^{-1} \circ T_i & \xRightarrow[\sigma_{ik}]{} & J_{g_{ik},\varphi_{ik}} \end{array} \tag{52}$$

where $d_F : J \circ J^{-1} \Longrightarrow \mathrm{id}$ stands for the canonical "death" transformation expressing the invertibility of an anafunctor J.

Proof of Proposition 6.15. The first step is to specify local trivializations. Consider an open set $U \subseteq M$ as in Remark 6.16 (a), a Γ-connection (A, B) and a 1-morphism $T : \mathcal{I}_{A,B} \longrightarrow \mathcal{P}|_U$. By Example 6.6 and Proposition 6.7 we obtain a pseudonatural transformation

$$\tau := \rho_T \circ \eta : i(F_{A,B}) \longrightarrow \mathrm{tra}_{\mathcal{P}}|_U. \tag{53}$$

We also have to fix a "weak inverse" of τ, and choose $\tau^{-1} := \eta^{-1} \circ \rho_{T^{-1}}$. Here it is important that η is "strictly invertible" because $\eta_x = \mathrm{id}$. Finally, we need to fix modifications $\mathcal{D}_\tau : \tau \circ \tau^{-1} \longrightarrow \mathrm{id}_{i(F_{A,B})}$ and $\mathcal{B}_\tau : \mathrm{id}_{\mathrm{tra}_{\mathcal{P}}|_U} \Longrightarrow \tau^{-1} \circ \tau$, which can be induced from the canonical transformations $d_T : T \circ T^{-1} \Longrightarrow \mathrm{id}_{\mathcal{I}_\Psi}$ and $b_T : \mathcal{P}|_U \Longrightarrow T^{-1} \circ T$ via Proposition 6.9.

In the second step we form an open cover $\{U_i\}_{i \in M}$ of M composed of open sets as above, with contractible double intersections. Over each open set U_i we choose a Γ-connection (A_i, B_i) and a 1-morphism $T_i : \mathcal{I}_{A_i,B_i} \longrightarrow \mathcal{P}|_{U_i}$, and consider the induced local trivialization $(\tau_i, \tau_i^{-1}, \mathcal{D}_i, \mathcal{B}_i)$. Now we have to extract descent data. The first descent datum are the 2-functors $F_{A_i,B_i} : \mathcal{P}_2(U_i) \longrightarrow B\Gamma$. The second descent datum are the pseudonatural transformations

$$\gamma_{ij} := \tau_j^{-1} \circ \tau_i : i(F_{A_i,B_i}) \longrightarrow i(F_{A_j,B_j})$$

between 2-functors $\mathcal{P}_2(U_i \cap U_j) \longrightarrow \Gamma\text{-}\mathcal{T}or$. The third descent datum consists of the modifications $\mathcal{B}_i : \mathrm{id}_{i(F_{A_i,B_i})} \Longrightarrow \gamma_{ii}$ and $\mathcal{F}_{ijk} : \gamma_{jk} \circ \gamma_{ij} \Longrightarrow \gamma_{ik}$ defined by

$$\gamma_{jk} \circ \gamma_{ij} = \tau_k^{-1} \circ \tau_j \circ \tau_j^{-1} \circ \tau_i \xRightarrow{\mathrm{id} \circ \mathcal{D}_j \circ \mathrm{id}} \tau_k^{-1} \circ \tau_i = \gamma_{ik}. \tag{54}$$

The third step is to show that all this descent data is smooth in a certain sense. For the 2-functors F_{A_i,B_i} this simply means that they have to be smooth, which is the case. For the pseudonatural transformation γ_{ij} it suffices to show that it factors through a *smooth* pseudonatural transformation $\tilde{\gamma}_{ij} : F_{A_i,B_i} \longrightarrow F_{A_j,B_j}$, i.e. $\gamma_{ij} \cong i(\tilde{\gamma}_{ij})$. We construct $\tilde{\gamma}_{ij}$ as follows. Since $U_i \cap U_j$ is contractible, there exist gauge transformations $(g_{ij}, \varphi_{ij}) : (A_i, B_i) \longrightarrow (A_j, B_j)$ and 2-morphisms $\sigma_{ij} : T_j^{-1} \circ T_i \Longrightarrow J_{ij}$, where we write $J_{ij} := J_{g_{ij},\varphi_{ij}}$ for short; see Remark 6.16 (b). We let $\tilde{\gamma}_{ij} := \rho_{g_{ij},\varphi_{ij}}$ be the smooth pseudonatural transformation (Section D). We define a modification $\mathcal{A}_{ij} : \gamma_{ij} \Longrightarrow i(\tilde{\gamma}_{ij})$ as follows:

$$\gamma_{ij} = \eta_j^{-1} \circ \rho_{T_j}^{-1} \circ \rho_{T_i} \circ \eta_i = \eta_j^{-1} \circ \rho_{T_j^{-1} \circ T_i} \circ \eta_i \xRightarrow{\mathrm{id} \circ \mathcal{A}_{\sigma_{ij}} \circ \mathrm{id}} \eta_j^{-1} \circ \rho_{J_{ij}} \circ \eta_i \overset{\substack{\text{Example 6.8} \\ \downarrow}}{=} i(\tilde{\gamma}_{ij}). \tag{55}$$

Finally, we have to verify the smoothness of the modifications $\mathcal{B}_i$ and $\mathcal{F}_{ijk}$. For this we have to show that there exist smooth modifications $\widetilde{\mathcal{B}}_i : \mathrm{id}_{F_{A_i,B_i}} \Longrightarrow \tilde{\gamma}_{ii}$ and $\widetilde{\mathcal{F}}_{ijk} : \tilde{\gamma}_{jk} \circ \tilde{\gamma}_{ij} \Longrightarrow \tilde{\gamma}_{ik}$ such that

$$\mathcal{B}_i = \mathcal{A}_{ii}^{-1} \bullet i(\widetilde{\mathcal{B}}_i) \quad \text{and} \quad \mathcal{F}_{ijk} = \mathcal{A}_{ik}^{-1} \bullet i(\widetilde{\mathcal{F}}_{ijk}) \bullet (\mathcal{A}_{jk} \circ \mathcal{A}_{ij}). \tag{56}$$

Without loss of generality we can assume that $g_{ii} = 1$, $\varphi_{ii} = 0$, and $\sigma_{ii} = b_{T_i}^{-1}$, so that $\tilde{\gamma}_{ii} = \mathrm{id}_{F_{A_i,B_i}}$ and $J_{ii} = \mathrm{id}_{\mathcal{I}_{A_i,B_i}}$. This shows $\mathcal{B}_i = \mathcal{A}_{ii}^{-1}$, i.e. $\widetilde{\mathcal{B}}_i := \mathrm{id}$ does the job. On a triple overlap $U_i \cap U_j \cap U_k$, we obtain via Remark 6.16 (c) a gauge 2-transformation

$$a_{ijk} : (g_{jk}, \varphi_{jk}) \circ (g_{ij}, \varphi_{ij}) \Longrightarrow (g_{ik}, \varphi_{ik}).$$

We let $\widetilde{\mathcal{F}}_{ijk} := \mathcal{A}_{a_{ijk}}$ be the smooth modification (Section D). We have a diagram

$$\begin{array}{ccccc}
\gamma_{jk}\circ\gamma_{ij} & & \xRightarrow{\mathcal{A}_{jk}\circ\mathcal{A}_{ij}} & & i(\tilde\gamma_{jk})\circ i(\tilde\gamma_{ij}) \\
 & \eta_k^{-1}\circ\rho_{T_k^{-1}}\circ\rho_{T_j}\circ\rho_{T_j^{-1}}\circ\rho_{T_i}\circ\eta_i & \xRightarrow{\mathrm{id}\circ\mathcal{A}_{\tilde\sigma_{jk}}\circ\mathcal{A}_{\tilde\sigma_{ij}}\circ\mathrm{id}} & \eta_k^{-1}\circ\rho_{J_{jk}}\circ\rho_{J_{ij}}\circ\eta_i & \\
\mathcal{F}_{ijk}\Big\Downarrow & \Big\Downarrow{\scriptstyle \mathrm{id}\circ\mathrm{id}\circ\mathcal{A}_{d_j}\circ\mathrm{id}\circ\mathrm{id}} & & \Big\Downarrow{\scriptstyle \mathrm{id}\circ\mathcal{A}_{\eta_{a_{ijk}}}\circ\mathrm{id}} & \Big\Downarrow i(\widetilde{\mathcal{F}}_{ijk}) \\
 & \eta_k^{-1}\circ\rho_{T_k^{-1}}\circ\rho_{T_i}\circ\eta_i & \xRightarrow[\mathrm{id}\circ\mathcal{A}_{\tilde\sigma_{ik}}\circ\mathrm{id}]{} & \eta_k^{-1}\circ\rho_{J_{ik}}\circ\eta_i & \\
\gamma_{ik} & & \xRightarrow[\mathcal{A}_{ik}]{} & & i(\tilde\gamma_{ik})
\end{array}$$

The subdiagrams at the top and at the bottom are commutative due to the definition of $\mathcal{A}_{ij}$ (Eq. (55)). The subdiagram on the left is the definition of $\mathcal{F}_{ijk}$ (Eq. (54)). The subdiagram on the right is the one of Example 6.10. The subdiagram in the middle is induced from the diagram of Eq. (52) and hence commutative. Thus, the whole diagram is commutative; this is the second equation in Eq. (56). ☐

Proposition 6.17. *If $J : \mathcal{P} \longrightarrow \mathcal{P}'$ is a 1-morphism in 2-$\mathcal{B}un_{\Gamma}^{\nabla ff}(M)$, then the pseudonatural transformation $\rho_J : \mathrm{tra}_{\mathcal{P}} \longrightarrow \mathrm{tra}_{\mathcal{P}'}$ of Proposition 6.7 is a 1-morphism between transport 2-functors.*

Remark 6.18. For the proof we extract and slightly reformulate the following results of [13, Propositions 5.4.6 & 5.4.9]. Suppose $J : \mathcal{P} \longrightarrow \mathcal{P}'$ is a 1-morphism in 2-$\mathcal{B}un_{\Gamma}^{\nabla ff}(M)$.

(a) Every point $x \in M$ has an open neighborhood $x \in U \subseteq M$ such that there exist Γ-connections (A, B) and (A', B') on U, 1-morphisms $T : \mathcal{I}_{A,B} \longrightarrow \mathcal{P}|_U$ and $T' : \mathcal{I}_{A',B'} \longrightarrow \mathcal{P}'|_U$, a gauge transformation $(h, \phi) : (A, B) \longrightarrow (A', B')$, and a 2-morphism $\tau : T'^{-1} \circ J \circ T \Longrightarrow J_{h,\phi}$ in 2-$\mathcal{B}un_{\Gamma}^{\nabla ff}(M)$.

(b) Suppose $U \subseteq M$ is open, and we have a diagram of Γ-connections and gauge transformations

$$\begin{array}{ccc}
(A_1, B_1) & \xrightarrow{(g,\varphi)} & (A_2, B_2) \\
{\scriptstyle (h_1,\phi_1)}\Big\downarrow & & \Big\downarrow{\scriptstyle (h_2,\phi_2)} \\
(A_1', B_1') & \xrightarrow[(g',\varphi')]{} & (A_2', B_2')
\end{array}$$

together with 1-morphisms $T_i : \mathcal{I}_{A_i,B_i} \longrightarrow \mathcal{P}|_U$ and $T_i' : \mathcal{I}_{A_i',B_i'} \longrightarrow \mathcal{P}'|_U$, 2-morphisms $\sigma : T_2^{-1} \circ T_1 \Longrightarrow J_{g,\varphi}$, $\sigma' : T_2'^{-1} \circ T_1' \Longrightarrow J_{g',\varphi'}$ and $\tau_i : T_i'^{-1} \circ J \circ T_i \Longrightarrow J_{h_i,\phi_i}$. Then, there exists a unique gauge 2-transformation

$$e : (h_2, \phi_2) \circ (g, \varphi) \Longrightarrow (g', \varphi') \circ (h_1, \phi_1),$$

and a commutative diagram

$$\begin{array}{ccc} T_j'^{-1}\circ J\circ T_j\circ T_j^{-1}\circ T_i & \xRightarrow{\tau_j\circ\sigma} & J_{h_j,\phi_j}\circ J_{g,\varphi} \\ {\scriptstyle \mathrm{id}\circ d^{-1}_{T'_i}\circ d_{T_j}\circ \mathrm{id}}\Big\Downarrow & & \Big\Downarrow{\scriptstyle \mathcal{A}_e} \\ T_j'^{-1}\circ T_i'\circ T_i'^{-1}\circ J\circ T_i & \xRightarrow[\sigma'\circ\tau_i]{} & J_{g',\varphi'}\circ J_{h_i,\phi_i}. \end{array}$$

Proof of Proposition 6.17. We choose an open cover $\{U_i\}_{i\in I}$ and over each open set the data of Remark 6.18 (a). We form the pseudonatural transformations τ_i and τ_i' for $\mathcal{P}|_{U_i}$ and $\mathcal{P}'|_{U_i}$, respectively, as in Eq. (53). We define the pseudonatural transformation

$$\lambda_i := \tau_i'^{-1}\circ\rho_J\circ\tau_i : i(F_{A_i,B_i}) \longrightarrow i(F_{A_i',B_i'}),$$

which is the first descent datum. The second decent datum is over double intersections; it is the modification $\mathcal{E}_{ij} : \lambda_j\circ\gamma_{ij} \Longrightarrow \gamma_{ij}'\circ\lambda_i$ defined by

$$\tau_j'^{-1}\circ\rho_J\circ\tau_j\circ\tau_j^{-1}\circ\tau_i \xRightarrow{\mathrm{id}\circ\mathrm{id}\circ\mathcal{D}_j\circ\mathrm{id}} \tau_j'^{-1}\circ\rho_J\circ\tau_i \xRightarrow{\mathrm{id}\circ\mathcal{D}_i^{-1}\circ\mathrm{id}\circ\mathrm{id}} \tau_j'^{-1}\circ\tau_i'\circ\tau_i'^{-1}\circ\rho_J\circ\tau_i\ .$$

For the first smoothness condition it suffices to show that λ_i factors through a *smooth* pseudonatural transformation $\tilde\lambda_i : F_{A_i,B_i} \longrightarrow F_{A_i',B_i'}$, i.e. $\lambda_i \cong i(\tilde\lambda_i)$. We construct $\tilde\lambda_i$ as follows. Using the gauge transformations (h_i,ϕ_i) of Remark 6.18 (a) we let $\tilde\lambda_i := \rho_{h_i,\phi_i}$ be the smooth pseudonatural transformation associated to (h_i,ϕ_i), see Section D. We obtain a modification $\mathcal{L}_i : \lambda_i \Longrightarrow i(\tilde\lambda_i)$ defined as:

$$\lambda_i = \eta_i'^{-1}\circ\rho_{T_i'^{-1}}\circ\rho_J\circ\rho_{T_i}\circ\eta_i = \eta_i'^{-1}\circ\rho_{T_i'^{-1}\circ J\circ T_i}\circ\eta_i \xRightarrow{\mathrm{id}\circ\mathcal{A}_{\tilde\tau_{ij}}\circ\mathrm{id}} \eta_i'^{-1}\circ\rho_{J_{h_i,\phi_i}}\circ\eta_i \overset{\text{Example 6.8}}{\overset{\downarrow}{=}} i(\tilde\lambda_i). \quad (57)$$

For the second smoothness condition we have to show that there exists a smooth modification $\widetilde{\mathcal{E}}_{ij} : \tilde\lambda_j\circ\tilde\gamma_{ij} \Longrightarrow \tilde\gamma_{ij}'\circ\tilde\lambda_i$ such that

$$i(\widetilde{\mathcal{E}}_{ij})\bullet(\mathcal{L}_j\circ\mathcal{A}_{ij}) = (\mathcal{A}_{ij}'\circ\mathcal{L}_i)\bullet\mathcal{E}_{ij}. \quad (58)$$

Indeed, over double intersections we find by Remark 6.18 (b) a gauge-2-transformation

$$e_{ij} : (h_j,\phi_j)\bullet(g_{ij},\varphi_{ij}) \Longrightarrow (g_{ij}',\varphi_{ij}')\bullet(h_i,\phi_i),$$

from which we induce $\widetilde{\mathcal{E}}_{ij} := \mathcal{A}_{a_{ij}}$ via Section D. We have a diagram:

$$\begin{array}{ccccc}
\lambda_j\circ\gamma_{ij} & & \xRightarrow{\mathcal{L}_j\circ\mathcal{A}_{ij}} & & i(\tilde\lambda_j)\circ i(\tilde\gamma_{ij}) \\
 & \diagdown\!\!\diagdown & & \diagup\!\!\diagup & \\
 & \eta_j'^{-1}\circ\rho_{T_j'^{-1}\circ J\circ T_j}\circ\rho_{T_j^{-1}\circ T_i}\circ\eta_i & \xRightarrow{\mathrm{id}\circ\mathcal{A}_{\tilde\tau_j}\circ\mathcal{A}_{\tilde\sigma_{ij}}\circ\mathrm{id}} & \eta_j'^{-1}\circ\rho_{K_j}\circ\rho_{F_{ij}}\circ\eta_i & \\
{\scriptstyle\mathcal{E}_{ij}}\Big\Downarrow & {\scriptstyle\mathcal{A}_{d_i^{-1}}\circ\mathcal{A}_{d_j}}\Big\Downarrow & & \Big\Downarrow{\scriptstyle\mathrm{id}\circ\mathcal{A}_{a_{ij}}\circ\mathrm{id}} & \Big\Downarrow{\scriptstyle i(\widetilde{\mathcal{E}_{ij}})} \\
 & \eta_j'^{-1}\circ\rho_{T_j'^{-1}\circ T_i'}\circ\rho_{T_i'^{-1}\circ J\circ T_i}\circ\eta_i & \xRightarrow[\mathrm{id}\circ\mathcal{A}_{\tilde\sigma_{ij}'}\circ\mathcal{A}_{\tilde\tau_i}\circ\mathrm{id}]{} & \eta_j'^{-1}\circ\rho_{F_{ij}'}\circ\rho_{K_i}\circ\eta_i & \\
 & \diagup\!\!\diagup & & \diagdown\!\!\diagdown & \\
\gamma_{ij}'\circ\lambda_i & & \xRightarrow[\mathcal{A}_{ij}'\circ\mathcal{L}_i]{} & & i(\tilde\gamma_{ij}')\circ i(\tilde\lambda_i)
\end{array}$$

The subdiagrams at the bottom and at the top are the definitions of $\mathcal{A}_{ij}$, $\mathcal{A}'_{ij}$, and $\mathcal{L}_i$. The subdiagram on the left is the definition of $\mathcal{E}_{ij}$, and the subdiagram on the right is commutative due to Example 6.10. The subdiagram in the middle is commutative by Remark 6.18 (b). Hence, the whole diagram is commutative; this is Eq. (58). □

Proposition 6.19. *If $f : J_1 \Longrightarrow J_2$ is a 2-morphism in 2-$\mathcal{B}un_\Gamma^{\nabla ff}(M)$, then the modification $\mathcal{A}_f : \rho_{J_1} \Longrightarrow \rho_{J_2}$ of Proposition 6.9 is a 2-morphism of transport 2-functors.*

Remark 6.20. For the proof we extract and slightly reformulate the following results of [13, Propositions 5.4.6 & 5.4.9]. Suppose $J_1, J_2 : \mathcal{P} \longrightarrow \mathcal{P}'$ are 1-morphisms in 2-$\mathcal{B}un_\Gamma^{\nabla ff}(M)$ and $f : J_1 \Longrightarrow J_2$ is a 2-morphism. Suppose $U \subseteq M$ is an open set with Γ-connections (A, B) and (A', B'), 1-morphisms $T : \mathcal{I}_{A,B} \longrightarrow \mathcal{P}$ and $T' : \mathcal{I}_{A',B'} \longrightarrow \mathcal{P}'$, for $i = 1, 2$ gauge transformations (h_i, ϕ_i) with 2-morphism $\tilde{\tau}_i : T'^{-1} \circ J_i \circ T \Longrightarrow J_{h_i,\phi_i}$. In other words, we have for J_1 and J_2 the structure of Remark 6.18 (a). Then, there exists a unique gauge 2-transformation $a : (h_1, \phi_1) \Longrightarrow (h_2, \phi_2)$ such that the diagram

$$\begin{array}{ccc} T'^{-1}\circ J_1\circ T & \xRightarrow{\ \tilde\tau_1\ } & J_{h_1,\phi_1} \\ {\scriptstyle \mathrm{id}\circ f\circ\mathrm{id}}\Big\Downarrow & & \Big\Downarrow{\scriptstyle f_a} \\ T'^{-1}\circ J_2\circ T & \xRightarrow[\ \tilde\tau_2\ ]{} & J_{h_2,\phi_2} \end{array}$$

is commutative.

Proof of Proposition 6.19. Let $U \subseteq M$ be an open set over which we have the pseudonatural transformations τ and τ' of Eq. (53) for the two principal Γ-2-bundles $\mathcal{P}$ and $\mathcal{P}'$, respectively. We form the modification $\mathcal{F} : \lambda_1 \Longrightarrow \lambda_2$ by

$$\lambda_1 := \tau'^{-1} \circ \rho_{J_1} \circ \tau \xRightarrow{\mathrm{id}\circ\mathcal{A}_f\circ\mathrm{id}} \tau'^{-1} \circ \rho_{J_2} \circ \tau =: \lambda_2.$$

We let $\mathcal{L}_1$ and $\mathcal{L}_2$ be the modifications Eq. (57) associated to J_1 and J_2. The smoothness condition we have to check is that there exists a smooth modification $\widetilde{\mathcal{F}} : \tilde{\lambda}_1 \Longrightarrow \tilde{\lambda}_2$ such that $\mathcal{F} = \mathcal{L}_2^{-1} \bullet i(\widetilde{\mathcal{F}}) \bullet \mathcal{L}_1$. Let $a : (h_1, \phi_1) \Longrightarrow (h_2, \phi_2)$ be the gauge 2-transformation of Remark 6.20, and let $\widetilde{\mathcal{F}} := \mathcal{A}_a$ using Section D, which gives a smooth modification between $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$. We have a diagram:

$$\begin{array}{ccccc} \lambda_1 & & \xRightarrow{\qquad\qquad\mathcal{L}_1\qquad\qquad} & & i(\tilde\lambda_1) \\ & \searrow\!\!\!\!\searrow & & \nearrow\!\!\!\!\nearrow & \\ & \eta\circ\rho_{T'^{-1}}\circ\rho_{J_1}\circ\rho_T\circ\eta^{-1} & \xRightarrow{\mathrm{id}\circ\mathcal{A}_{\tilde\tau_1}\circ\mathrm{id}} & \eta\circ\rho_{K_1}\circ\eta^{-1} & \\ {\scriptstyle\mathcal{F}}\Big\Downarrow & {\scriptstyle \mathrm{id}\circ\mathrm{id}\circ\mathcal{A}_f\circ\mathrm{id}\circ\mathrm{id}}\Big\Downarrow & & \Big\Downarrow{\scriptstyle \mathrm{id}\circ\mathcal{A}_{f_a}\circ\mathrm{id}} & \Big\Downarrow{\scriptstyle i(\widetilde{\mathcal{F}})} \\ & \eta\circ\rho_{T'^{-1}}\circ\rho_{J_2}\circ\rho_T\circ\eta^{-1} & \xRightarrow[\mathrm{id}\circ\mathcal{A}_{\tilde\tau_2}\circ\mathrm{id}]{} & \eta\circ\rho_{K_2}\circ\eta^{-1} & \\ & \swarrow\!\!\!\!\swarrow & & \searrow\!\!\!\!\searrow & \\ \lambda_2 & & \xRightarrow[\qquad\qquad\mathcal{L}_2\qquad\qquad]{} & & i(\tilde\lambda_2) \end{array}$$

The subdiagram in the middle is induced from the commutative diagram of Remark 6.20, and the subdiagram on the right hand side commutes by Example 6.10. The other subdiagrams commute by definition. Hence, the whole diagram is commutative; this is what we had to show. □

Appendix A: 2-group connections and gauge transformations

We summarize the bicategory of Γ-connections following [10]. Let X be a smooth manifold and Γ be a Lie 2-group, given by a crossed module (G, H, t, α). A *Γ-connection on X* is a pair (A, B) of a 1-form $A \in \Omega^1(X, \mathfrak{g})$ and a 2-form $B \in \Omega^2(X, \mathfrak{h})$. The 2-form

$$\mathrm{fcurv}(A, B) := \mathrm{d}A + \frac{1}{2}[A \wedge A] - t_*(B) \in \Omega^2(X, \mathfrak{g})$$

is called the *fake-curvature*, and the 3-form

$$\mathrm{curv}(A, B) := \mathrm{d}B + \alpha_*(A \wedge B) \in \Omega^3(X, \mathfrak{h})$$

is called the *curvature*. A connection (A, B) is called *fake-flat*, if $\mathrm{fcurv}(A, B) = 0$, and it is called *flat*, if it is fake-flat and $\mathrm{curv}(A, B) = 0$. Let (A, B) and (A', B') be Γ-connections on X. A *gauge transformation*

$$(g, \varphi) : (A, B) \longrightarrow (A', B')$$

is a smooth map $g : X \longrightarrow G$ and a 1-form $\varphi \in \Omega^1(X, \mathfrak{h})$ such that:

$$A' + t_*(\varphi) = \mathrm{Ad}_g(A) - g^*\bar{\theta} \tag{59}$$

$$B' + \alpha_*(A' \wedge \varphi) + \mathrm{d}\varphi + \frac{1}{2}[\varphi \wedge \varphi] = (\alpha_g)_*(B). \tag{60}$$

Here, $\bar{\theta}$ is the right-invariant Maurer-Cartan form. The identity gauge transformation is given by $g = 1$ and $\varphi = 0$. The composition of gauge transformations

$$(A, B) \xrightarrow{(g_1, \varphi_1)} (A', B') \xrightarrow{g_2, \varphi_2} (A'', B'')$$

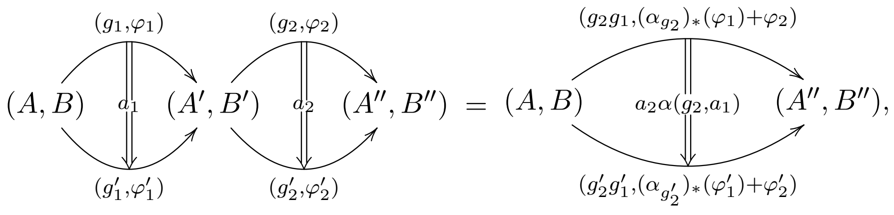
is given by the map $g_2 g_1 : X \longrightarrow G$ and the 1-form $\varphi_2 + (\alpha_{g_2})_*(\varphi_1)$. A *gauge 2-transformation* $a : (g_1, \varphi_1) \Longrightarrow (g_2, \varphi_2)$ is a smooth map $a : X \longrightarrow H$ such that

$$g_2 = (t \circ a) \cdot g_1 \quad \text{and} \quad \varphi_2 + (r_a^{-1} \circ \alpha_a)_*(A') = \mathrm{Ad}_a(\varphi_1) - a^*\bar{\theta}.$$

The vertical composition

$$(g, \varphi) \xRightarrow{a_1} (g', \varphi') \xRightarrow{a_2} (g'', \varphi'')$$

is given by $a_2 a_1$. The horizontal composition is



and the identity gauge 2-transformation is given by $a = 1$. Γ-connections on X, gauge transformations, and gauge 2-transformations form a strict bicategory $\mathcal{C}on_\Gamma(X)$. The restriction to fake-flat Γ-connections forms a full sub-bicategory $\mathcal{C}on_\Gamma^{ff}(X)$.

Appendix B: Path-ordered exponentials

For a 1-form $\omega \in \Omega^1(X,\mathfrak{g})$ with values in the Lie algebra of a Lie group G and a path $\gamma : [0,1] \longrightarrow X$ we denote by $poe_\omega(\gamma) \in G$ the path ordered exponential of ω along γ. That is, we let $g : [0,1] \longrightarrow G$ be the unique solution of the initial value problem

$$\dot{g}(\tau) = -\omega(\dot{\gamma}(\tau))g(\tau) \quad \text{with} \quad g(0) = 1,$$

and put $poe_\omega(\gamma) := g(1)$. We need the following well-known general properties of the path ordered exponential.

Lemma B.1. *Let $\omega \in \Omega^1(X,\mathfrak{g})$.*

(a) It depends only on the thin homotopy class of the path: if there is a fixed-ends homotopy between $\gamma,\gamma' : x \longrightarrow y$ whose rank is less than two, then $poe_\omega(\gamma) = poe_\omega(\gamma')$.

(b) It is compatible with path composition: if $\gamma : x \longrightarrow y$ and $\gamma' : y \longrightarrow z$ are composable paths, then $poe_\omega(\gamma' * \gamma) = poe_\omega(\gamma') \cdot poe_\omega(\gamma)$.

(c) It is natural under the pullback of differential forms: if $f : W \longrightarrow X$ is a smooth map, then $poe_{f^*\omega}(\gamma) = poe_\omega(f(\gamma))$.

(d) It is natural under Lie group homomorphisms: if $\varphi : G \longrightarrow G'$ is a Lie group homomorphism, then $\varphi(poe_\omega(\gamma)) = poe_{\varphi_*(\omega)}(\gamma)$.

(e) It is compatible with gauge transformations: if $g : X \longrightarrow G$ is a smooth map and $\omega' := \mathrm{Ad}_g^{-1}(\omega) + g^*\theta$, then $poe_\omega(\gamma) \cdot g(\gamma(0)) = g(\gamma(1)) \cdot poe_{\omega'}(\gamma)$.

The following propositions discuss special properties of path-ordered exponentials in the total space of principal 2-bundles and 1-morphisms between those; in combination with the notion of horizontality defined in Section 3.1.

Proposition B.2. *Let $\mathcal{P}$ be a principal Γ-2-bundle equipped with a connection Ω.*

(a) If β is a path in $\mathrm{Obj}(\mathcal{P})$ and γ a path in G with $\gamma(0) = 1$, then $poe_{\Omega^a}(R(\beta,\gamma)) = \gamma(1)^{-1} \cdot poe_{\Omega^a}(\beta)$.

(b) $poe_{\Omega^b}(\mathrm{id}_\beta) = 1$ for every path β in $\mathrm{Obj}(\mathcal{P})$.

(c) Let ρ be a horizontal path in $\mathrm{Mor}(\mathcal{P})$ such that $s(\rho)$ is horizontal, and let h be a path in H with $h(0) = 1$. Then, $poe_{\Omega^b}(R(\rho,(h,1))) = h(1)^{-1}$.

Proof. For (a) we consider $X := \mathrm{Obj}(\mathcal{P}) \times G$, so that (β,γ) is a path in X. It is easy to check using Eq. (1) that the map $g : X \longrightarrow G : (p,g') \longmapsto g'^{-1}$ is a gauge transformation between $R^*\Omega^a$ and $\mathrm{pr}_1^*\Omega^a$. Thus, by Lemmas B.1 (c) and B.1 (e) we have

$$poe_{\Omega^a}(R(\beta,\gamma)) = poe_{R^*\Omega^a}(\beta,g) \cdot g(\beta(0),\gamma(0)) = g(\beta(1),\gamma(1)) \cdot poe_{\mathrm{pr}_1^*\Omega^a}(\beta,g) = \gamma(1)^{-1} \cdot poe_{\Omega^a}(\beta)$$

For (b), we apply Lemma B.1 (c) to $\mathrm{id} : \mathrm{Obj}(\mathcal{P}) \longrightarrow \mathrm{Mor}(\mathcal{P})$ and use $\mathrm{id}^*\Omega^b = 0$. For (c) we note that Eq. (2) and the assumptions on ρ imply $R^*\Omega^b(\dot{\rho},\dot{h},0) = \eta^{-1}\dot{\eta}$. The corresponding initial value problem is then solved by η^{-1}; this shows the claim. □

Proposition B.3. *Suppose $J : \mathcal{P}_1 \longrightarrow \mathcal{P}_2$ is a 1-morphism in $2\text{-}\mathcal{B}un_\Gamma^\nabla(M)$. Let $\lambda : [0,1] \longrightarrow J$ be a horizontal path such that $\alpha_r(\lambda)$ is horizontal, and let $h : [0,1] \longrightarrow H$ be a path with $h(0) = 1$. Then, $poe_{\nu_0}(\lambda \cdot (h,1)) = h(1)^{-1}$.*

Proof. Let $\nu = (\nu_0,\nu_1)$ be the connective, connection-preserving pullback on J. Connectivity together with our assumptions on λ imply $\nu_0(\partial_t(\lambda \cdot (h,1))) = h^{-1}\dot{h}$. The corresponding initial value problem is then solved by η^{-1}; this shows the claim. □

Next we discuss an important application of the path-ordered exponential related to a gauge transformation $(g,\varphi):(A,B)\longrightarrow(A',B')$ between Γ-connections on X, see Section A. We note that (φ,A') is a 1-form on X with values in $\mathfrak{h}\ltimes\mathfrak{g}$, so that $poe_{\varphi,A'}(\gamma)\in H\ltimes G$ for any path γ in X. Since G acts on H the G-component of $poe_{\varphi,A'}(\gamma)$ is just $poe_{A'}(\gamma)$. The H-component, however, is an independent quantity; we denote it by $h_{g,\varphi}(\gamma)\in H$. In the following we study some of its properties.

Proposition B.4. *Let $(g,\varphi):(A,B)\longrightarrow(A',B')$ be a gauge transformation between Γ-connections on X.*

(a) If $\gamma:x\longrightarrow y$ and $\gamma':y\longrightarrow z$ are composable paths, then

$$h_{g,\varphi}(\gamma'\circ\gamma)=h_{g,\varphi}(\gamma')\cdot\alpha(poe_{A'}(\gamma'),h_{g,\varphi}(\gamma))$$

(b) For all paths $\gamma:x\longrightarrow y$, we have

$$poe_{A'}(\gamma)\cdot g(x)=t(h_{g,\varphi}(\gamma)^{-1})\cdot g(y)\cdot poe_{A}(\gamma).$$

(c) If $(g',\varphi'):(A',B')\longrightarrow(A'',B'')$ is a second gauge transformation, and $\gamma:x\longrightarrow y$ is a path, then

$$h_{(g',\varphi')\circ(g,\varphi)}(\gamma)=\alpha(g'(\gamma(y)),h_{g,\varphi}(\gamma))\cdot h_{g',\varphi'}(\gamma).$$

Proof. We have $poe_{\varphi,A'}(\gamma'\circ\gamma)=poe_{\varphi,A'}(\gamma')\cdot poe_{\varphi,A'}(\gamma)$ in $H\ltimes G$. Since the projection of $poe_{\varphi,A'}(\gamma)$ to G is $poe_{A'}(\gamma)$ we have (a). (b) is [10, Lemma 2.18]. (c) is the functoriality proved in [10, Section 2.3.4]. □

Gauge transformations can be produced from a fake-flat connection Ω on a principal Γ-2-bundle $\mathcal{P}$ in the following way. First we note that the pair $(\Omega^a,-\Omega^c)$ is a Γ-connection on $\mathrm{Obj}(\mathcal{P})$, with the sign chosen such that it is fake-flat in the sense of Section A. Consider the smooth manifold $X:=\mathrm{Mor}(\mathcal{P})\times G$ equipped with the maps $\chi_1,\chi_2:X\longrightarrow\mathrm{Obj}(\mathcal{P})$ defined by $\chi_1(\rho,g):=t(\rho)$ and $\chi_2(\rho,g):=R(s(\rho),g^{-1})$. Now we have the fake-flat Γ-connections $(A,B):=\chi_1^*(\Omega^a,-\Omega^c)$ and $(A',B'):=\chi_2^*(\Omega^a,-\Omega^c)$ over X. We define $g:=\mathrm{pr}_2:X\longrightarrow G$ and $\varphi:=(\alpha_g)_*(\mathrm{pr}_1^*\Omega^b)\in\Omega^1(X,\mathfrak{h})$.

Lemma B.5. *(g,φ) is a gauge transformation between (A,B) and (A',B').*

Proof. Identity Eq. (59) is proved by a direct calculation using only $t_*(\Omega^b)=\Delta(\Omega^a)$ and the transformation rule for Ω^a, [13, Eq. (5.1.1)]. Identity Eq. (59) is proved similarly using additionally the transformation rule for Ω^c (Eq. (3)) and the fake-flatness of Ω. □

Correspondingly, we have the quantity $h_{g,\varphi}(\rho,\gamma)\in H$ associated to any pair of paths ρ in $\mathrm{Mor}(\mathcal{P})$ and γ in G. The following two lemmas list its relevant properties.

Lemma B.6. *(a) $h_{g,\varphi}(\mathrm{id}_\beta,\gamma)=1$ for all paths β in $\mathrm{Obj}(\mathcal{P})$ and γ in G.*

(b) $h_{g,\varphi}(\rho_1\circ R(\rho_2,\gamma_1),\gamma_2\gamma_1)=\alpha(\gamma_2(1),h_{g,\varphi}(\rho_1,\gamma_1))\cdot h_{g,\varphi}(\rho_2,\gamma_2)$ for all paths ρ_1,ρ_2 in $\mathrm{Mor}(\mathcal{P})$ and γ_1,γ_2 in G such that $t(\rho_2)=R(s(\rho_1),\gamma_1^{-1})$.

Proof. For (a) we consider the map $i:\mathrm{Obj}(\mathcal{P})\times G\longrightarrow\mathrm{Mor}(\mathcal{P})\times G:(p,g)\longmapsto(\mathrm{id}_p,g)$, under which $i^*\varphi=0$ (because $\mathrm{id}^*\Omega^b=0$). From Lemma B.1 (c) we obtain

$$poe_{\varphi,A'}(\mathrm{id}_\beta,\gamma)=poe_{0,i^*A'}(\beta,\gamma)=(1,poe_{i^*A'}(\gamma));$$

this implies the claim. For (b) we consider $\tilde{X} := X\,{}_{\chi_1}\!\times_{\chi_2} X$, where $X = \mathrm{Mor}(\mathcal{P}) \times G$, so that $((\rho_2, \gamma_2), (\rho_1, \gamma_1))$ is a path in $\tilde{X}$. On $\tilde{X}$ we have the three Γ-connections

$$\begin{aligned}(A, B) &:= \mathrm{pr}_2^*\chi_1^*(\Omega^a, -\Omega^c)\\ (A', B') &:= \mathrm{pr}_2^*\chi_2^*(\Omega^a, -\Omega^c) = \mathrm{pr}_1^*\chi_1^*(\Omega^a, -\Omega^c)\\ (A'', B'') &:= \mathrm{pr}_1^*\chi_2^*(\Omega^a, -\Omega^c).\end{aligned}$$

and by Lemma B.5 two gauge transformations

$$\mathrm{pr}_2^*(g, \varphi) : (A, B) \longrightarrow (A', B') \quad \text{and} \quad \mathrm{pr}_1^*(g, \varphi) : (A', B') \longrightarrow (A'', B'').$$

We claim that the map $\mu : \tilde{X} \longrightarrow X$ defined by $\mu((\rho_2, g_2), (\rho_1, g_1)) := (\rho_1 \circ R(\rho_2, g_1), g_2 g_1)$ satisfies

$$\mu^*(g, \varphi) = \mathrm{pr}_1^*(g, \varphi) \circ \mathrm{pr}_2^*(g, \varphi),$$

where $\circ$ denotes the composition of gauge transformations. The only non-trivial part is to show the required identity for the $\mathfrak{h}$-valued differential forms, $\mu^*\varphi = \mathrm{pr}_1^*\varphi + (\alpha_{g\circ \mathrm{pr}_1})_*(\mathrm{pr}_2^*\varphi)$, which follows from the definition of φ and the identity $\Delta\Omega^b = 0$. By Proposition B.4 (c) and Lemma B.1 (c) we obtain

$$h_{g,\varphi}(\rho_1 \circ R(\rho_2, \gamma_1), \gamma_2\gamma_1) = h_{\mu^*(g,\varphi)}((\rho_2, \gamma_2), (\rho_1, \gamma_1)) = \alpha(\gamma_2(1), h_{g,\varphi}((\rho_1, \gamma_1)) \cdot h_{g,\varphi}(\rho_2, \gamma_2)),$$

this is the claim. □

Lemma B.7. $h_{g,\varphi}(\rho, \gamma) = \alpha(\gamma(1), h_{g,\varphi}(\rho, 1))$.

Proof. Put $\rho_2 = \mathrm{id}$ and $\gamma_1 = 1$ in Lemma B.6 (b) and then use Proposition B.8 (e). □

We can thus restrict ourselves to the case of constant paths $\gamma = 1$, and remain with a quantity $h_\Omega(\rho) := h_{g,\varphi}(\rho, 1) \in H$ associated to any path ρ in $\mathrm{Mor}(\mathcal{P})$. It has the following properties:

Proposition B.8. *Let $\mathcal{P}$ be a principal Γ-2-bundle over M with fake-flat connection Ω.*

(a) $h_\Omega(R(\rho, \gamma)) = \alpha(\gamma(1)^{-1}, h_\Omega(\rho))$.
(b) $h_\Omega(\rho_2)^{-1} = h_\Omega(\rho_2^{-1})$.
(c) $h_\Omega(\rho_1 \circ \rho_2) = h_\Omega(\rho_1) \cdot h_\Omega(\rho_2)$ *whenever ρ_1 and ρ_2 are pointwise composable.*
(d) $h_\Omega(\rho' * \rho) = h_\Omega(\rho') \cdot \alpha(poe_{\Omega^a}(s(\rho')), h_\Omega(\rho))$ *whenever ρ' and ρ are composable paths.*
(e) $h_\Omega(\rho) = poe_{\Omega^b}(\rho)$ *if $s(\rho)$ is horizontal.*
(f) $h_\Omega(R(\rho, (h, 1))) = h(1)^{-1}$ *if ρ and $s(\rho)$ are horizontal, and h is a path in H with $h(0) = 1$.*
(g) $h_\Omega(\rho) = 1$ *if ρ and $s(\rho)$ are horizontal.*

Proof. (a) follows in the same way by putting $\rho_1 = \mathrm{id}$ and $\gamma_2 = 1$ in Lemma B.6 (b), and then using Lemma B.7. (b) follows similarly with $\rho_1 = \rho_2^{-1}$ and $\gamma_1 = \gamma_2 = 1$, and (c) follows with $\gamma_1 = \gamma_2 = 1$. In (d), we have over X an identity $A' = \mathrm{Ad}_{\mathrm{pr}_2}(\mathrm{pr}_1^*\rho^*\Omega^a) - \mathrm{pr}_2^*\bar{\theta}$, i.e. the map $(\rho, g) \longmapsto g^{-1}$ is a gauge transformation between $s^*\Omega^a$ and A' in the sense of Lemma B.1 (e). Thus, Lemma B.1 (e) implies

$$poe_{\Omega^a}(s(\rho')) \cdot \gamma'(0)^{-1} = \gamma'(1)^{-1} \cdot poe_{A'}(\rho', \gamma')$$

and hence

$$h_{g,\varphi}(\rho' \circ \rho, 1) = h_{g,\varphi}(\rho', 1) \cdot \alpha(poe_{A'}(\rho', 1), h_{g,\varphi}(\rho, 1)) = h_{g,\varphi}(\rho', 1) \cdot \alpha(poe_{\Omega^a}(s(\rho')), h_{g,\varphi}(\rho, 1)).$$

(e) is proved by a direct calculation of $h_{g,\varphi}(\rho,1)$. Let (η,κ) be a path in $H\times G$ that solves the initial value problem for (φ,A'), i.e. $h_{g,\varphi}(\rho,1)=\eta(1)$. Employing the definitions of φ and A' the differential equation splits into two components

$$\begin{aligned}\dot\kappa(\tau)&=-\Omega^a(s_*(\dot\rho(\tau))\kappa(\tau)\\ \dot\eta(\tau)&=-\Omega^b(\dot\rho(\tau))\eta(\tau)-(\alpha_{\eta(\tau)})_*(\Omega^a(s_*(\dot\rho(\tau))))\end{aligned}$$

Since $s(\rho)$ is horizontal, we have $\kappa=1$, and we see that $\eta(1)=poe_{\Omega^b}(\rho)$. This shows the claim. (f) and (g) follow from (e) in combination with Proposition B.2 (c). $\square$

Another situation where a gauge transformation appears are 1-morphisms. Suppose $J:\mathcal{P}\longrightarrow\mathcal{P}'$ is a 1-morphism in $2\text{-}\mathcal{B}un_\Gamma^{\nabla ff}(M)$ between principal Γ-2-bundles with fake-flat connections Ω and Ω', respectively. Let $\nu=(\nu_0,\nu_1)$ be its connective, connection-preserving, and fake-flat Ω'-pullback. We consider the smooth manifold $Q:=J\times G$ equipped with the maps $\chi:Q\longrightarrow\mathrm{Obj}(\mathcal{P})$ and $\chi':Q\longrightarrow\mathrm{Obj}(\mathcal{P}')$ defined by $\chi(j,g):=\alpha_l(j)$ and $\chi'(j,g):=R(\alpha_r(j),g^{-1})$. Then we have the Γ-connections $(A,B):=\chi^*(\Omega^a,-\Omega^c)$ and $(A',B'):=\chi'^*(\Omega'^a,-\Omega'^c)$. We define the map $g:=\mathrm{pr}_G:Q\longrightarrow G$ and the 1-form $\varphi\in\Omega^1(Q,\mathfrak{h})$ by $\varphi:=(\alpha_g)_*(\mathrm{pr}_J^*\nu_0)$.

Lemma B.9. *(g,φ) is a gauge transformation between (A,B) and (A',B').*

Proof. Eq. (59) is a straightforward computation using Eq. (1) and the condition $t_*(\nu_0)=\alpha_l^*\Omega^a-\alpha_r^*\Omega'^a$ which is part of the relation $J_\nu^*\Omega'=\Omega$, see [13, Lemma 4.3.3]. For condition Eq. (60) we first compute using Eq. (1) that

$$\mathrm{d}\varphi+\frac{1}{2}[\varphi\wedge\varphi]+\alpha_*(A'\wedge\varphi)=(\alpha_g)_*(\mathrm{d}\nu_0)+\frac{1}{2}(\alpha_g)_*[\nu_0\wedge\nu_0]+(\alpha_g)_*\alpha_*(\alpha_r^*\Omega'^a\wedge\nu_0).$$

Using the fake-flatness of ν, this is equal to $-(\alpha_g)_*(\nu_1)$. Another part of the relation $J_\nu^*\Omega'=\Omega$ is $\nu_1=\alpha_l^*\Omega^c-\alpha_r^*\Omega'^c$; using this and Eq. (3) it is easy to show that $-(\alpha_g)_*(\nu_1)=(\alpha_g)_*(B)-B'$. This shows Eq. (60). $\square$

Thus, we have the quantity $h_{g,\varphi}(\lambda,\gamma)$ associated to any pair of paths λ in J and γ in G. Our first goal is to understand the dependence on γ.

Lemma B.10. $h_{g,\varphi}(\lambda,\gamma)=h_{g,\varphi}(\lambda\cdot\gamma^{-1},1)$.

Proof. We consider $\rho:Q\times G\longrightarrow Q$ defined by $\rho(j,g,g')=(j\cdot g',gg')$. It is easy to check that $\rho^*\varphi=\mathrm{pr}_Q^*\varphi$ and $\rho^*(\chi'^*\Omega'^a)=\mathrm{pr}_Q^*(\chi'^*\Omega'^a)$. Now, the definition of $h_{g,\varphi}$ and Lemma B.1 (c) give the claim. $\square$

By the lemma, it suffices to consider the quantity $h_\nu(\lambda)\in H$ associated to each path λ in J.

Proposition B.11. *Let $J:\mathcal{P}\longrightarrow\mathcal{P}'$ be a 1-morphism in $2\text{-}\mathcal{B}un_\Gamma^{\nabla ff}(M)$, and ν be its pullback. For every be a path λ in J such that $\alpha_r(\lambda)$ is horizontal, and every path γ in G we have $h_\nu(\lambda\cdot\gamma)=\alpha(\gamma(1)^{-1},poe_{\nu_0}(\lambda))$.*

Proof. Similar to the proof of Proposition B.8 (e). $\square$

Appendix C: Surface-ordered exponentials

If Γ is a Lie 2-group and (A, B) is a fake-flat Γ-connection on a smooth manifold X, then there exists a *surface-ordered exponential* $soe_{A,B}(\Sigma) \in H$ associated to any bigon $\Sigma : \gamma \Longrightarrow \gamma'$ in X. It is defined by a two-fold iteration of path-ordered exponentials in [10, Section 2.3.1]. We summarize the properties of the surface-ordered exponential in the following four propositions.

Proposition C.1. *Let (A, B) be a fake-flat Γ-connection, and $\Sigma : \gamma \Longrightarrow \gamma'$ be a bigon.*

(a) $soe_{A,B}(\Sigma)$ only depends on the thin homotopy class of Σ.

(b) It satisfies the target-source-matching condition $t(soe_{A,B}(\Sigma)) \cdot poe_A(\gamma) = poe_A(\gamma')$.

(c) If $\Sigma' : \gamma' \Longrightarrow \gamma''$ is vertically composable to a bigon $\Sigma' \bullet \Sigma : \gamma \Longrightarrow \gamma''$, then

$$soe_{A,B}(\Sigma' \bullet \Sigma) = soe_{A,B}(\Sigma') \cdot soe_{A,B}(\Sigma).$$

(d) If $\tilde{\Sigma} : \tilde{\gamma} \Longrightarrow \tilde{\gamma}'$ is horizontally composable to a bigon $\tilde{\Sigma} \circ \Sigma : \tilde{\gamma} \circ \gamma \Longrightarrow \tilde{\gamma}' \circ \gamma'$, then

$$soe_{A,B}(\tilde{\Sigma} \circ \Sigma) = soe_{A,B}(\tilde{\Sigma}) \cdot \alpha(poe_A(\tilde{\gamma}), soe_{A,B}(\Sigma)).$$

(e) If $f : X \longrightarrow Y$ is a smooth map, then $soe_{f^*(A,B)}(\Sigma) = soe_{A,B}(\Sigma \circ f)$.

(f) If $B = 0$, then $soe_{A,B}(\Sigma) = 1$.

Proof. (a) to (d) are a reformulation of [10, Proposition 2.17]. (e) follows from Lemma B.1 (c). Only for (f) we have to look into the details of the definition of the surface ordered exponential in [10, Section 2.3.1]. Since $B = 0$, we have $\mathcal{A}_\Sigma = 0$ for the 1-form $\mathcal{A}_\Sigma$ of Eq. (2.26) in that reference. Then, the function f_Σ vanishes, and so does the map $k_{A,0}$ which defines $soe_{A,0}(\Sigma)$. □

Remark C.2. Suppose $f : [0, 1]^2 \longrightarrow X$ is a smooth map, of which we can think of as a piece of surface in X. In order to compute the surface ordered exponential of f, we need the following terminology. A *bigon-parameterization* of f is a bigon $\Sigma : \gamma_r * \gamma_t \Longrightarrow \gamma_b * \gamma_l$ in X such that there exists a homotopy between f and Σ of rank less than three, which induces homotopies of rank less than two between the following pairs of paths: γ_t and the top edge $f(0, -)$, γ_b and $f(1, 0)$ the bottom edge, γ_r and the right edge $f(-, 0)$ and γ_r and the left edge $f(-, 1)$. It follows immediately that two bigon-parameterizations Σ and Σ' of f are thin homotopic. In particular, the surface-ordered exponential of f is well-defined. To see the existence of a bigon-parameterization, one can compose f with a standard bigon in $\mathbb{R}^2$, see [10, Eq. 2.5].

Let $\mathcal{P}$ be a principal Γ-2-bundle with fake-flat connection Ω. We recall that $(\Omega^a, -\Omega^c)$ is a fake-flat Γ-connection on $\mathrm{Obj}(\mathcal{P})$. Hence, we have a surface-ordered exponential

$$h_\Omega(\Sigma) := h_{\Omega^a, -\Omega^c}(\Sigma) \in H$$

associated to every bigon Σ in $\mathrm{Obj}(\mathcal{P})$. Next we study the surface-ordered exponential under gauge transformations. We start with the following.

Lemma C.3. *Let $(g, \varphi) : (A, B) \longrightarrow (A', B')$ be a gauge transformation between fake-flat Γ-connections, let $\Sigma : \gamma \Longrightarrow \gamma'$ be a bigon, and let $y := \gamma(1) = \gamma'(1)$. Then,*

$$soe_{A',B'}(\Sigma) \cdot h_{g,\varphi}(\gamma)^{-1} = h_{g,\varphi}(\gamma')^{-1} \cdot \alpha(g(y), soe_{A,B}(\Sigma)).$$

Proof. [10, Lemma 2.19]. □

Now recall from Lemma B.5 that every principal Γ-2-bundle $\mathcal{P}$ with fake-flat connection Ω induces a gauge transformation on the smooth manifold $X := \mathrm{Mor}(\mathcal{P}) \times G$. Lemma C.3 gives the following result.

Proposition C.4. *Let $\mathcal{P}$ be a principal Γ-2-bundle with fake-flat connection Ω. Let $\Psi : \rho \Longrightarrow \rho'$ be a bigon in* $\mathrm{Mor}(\mathcal{P})$*, and $\Theta : \gamma \Longrightarrow \gamma'$ be a bigon in G. Then,*

$$\alpha(\gamma(1)^{-1}, soe_\Omega(R(s(\Psi), \Theta^{-1}))) \cdot h_\Omega(\rho)^{-1} = h_\Omega(\rho')^{-1} \cdot soe_\Omega(t(\Psi)),$$

where Θ^{-1} denotes the point-wise inversion in G.

Corollary C.5. *If Σ is a bigon in* $\mathrm{Obj}(\mathcal{P})$ *and $\Theta : \gamma \Longrightarrow \gamma'$ is a bigon in G, then*

$$soe_\Omega(R(\Sigma, \Theta)) = \alpha(\gamma(1)^{-1}, soe_\Omega(\Sigma)).$$

Proof. We use Proposition C.4 with $\Psi = \mathrm{id}(\Sigma)$, and then Lemma B.6 (a). □

Finally, we recall from Lemma B.9 that every 1-morphism $J : \mathcal{P} \longrightarrow \mathcal{P}'$ in $2\text{-}\mathcal{B}un_\Gamma^{\nabla ff}(M)$ induces a gauge transformation on the smooth manifold $Q := J \times G$. Lemma C.3 gives the following result.

Proposition C.6. *Let $J : \mathcal{P} \longrightarrow \mathcal{P}'$ be a 1-morphism in $2\text{-}\mathcal{B}un_\Gamma^{\nabla ff}(M)$. Let Ω and Ω' denote the connections on $\mathcal{P}$ and $\mathcal{P}'$, respectively, and let ν denote the Ω'-pullback on J. Let $\Sigma : \lambda \Longrightarrow \lambda'$ be a bigon in J, let $\Theta : \gamma \Longrightarrow \gamma'$ be a bigon in G. Then,*

$$soe_{\Omega'}(R(\alpha_r(\Sigma), \Theta^{-1})) \cdot h_\nu(\lambda \cdot \gamma^{-1})^{-1} = h_\nu(\lambda' \cdot \gamma'^{-1})^{-1} \cdot \alpha(\gamma(1), soe_\Omega(\alpha_l(\Sigma))).$$

Appendix D: Smooth 2-functors on path 2-groupoids

The path 2-groupoid $\mathcal{P}_2(M)$ of a smooth manifold M is defined in the following way:

(a) Its objects are the points x of M.

(b) Its 1-morphisms are thin homotopy classes $[\gamma] : x \longrightarrow y$ of paths in M.

(c) Its 2-morphisms are thin homotopy classes $[\Sigma] : [\gamma] \Longrightarrow [\gamma']$ of bigons in M.

Using thin homotopy classes is one way to turn this structure into a strict bigroupoid, with the usual composition of paths, and the obvious vertical and horizontal composition of bigons sketched at the beginning of Section 4.3. A detailed definition is in [10, Section 2.1].

We recall the following constructions from [10, Section 2.3]:

(a) If (A, B) is a fake-flat Γ-connection on M, then we obtain a 2-functor

$$F_{A,B} : \mathcal{P}_2(M) \longrightarrow B\Gamma,$$

given by the following assignments:

$$x \longmapsto * \quad , \quad [\gamma] \longmapsto poe_A(\gamma) \quad \text{and} \quad [\Sigma] \longmapsto (soe_{A,B}(\Sigma), poe_A(\gamma)),$$

where $[\Sigma] : [\gamma] \Longrightarrow [\gamma']$. The well-definedness under thin homotopies was already mentioned in Lemma B.1 (a) and Proposition C.1 (a).

(b) If $(g, \varphi) : (A, B) \longrightarrow (A', B')$ is a gauge transformation between fake-flat Γ-connections, then we have a pseudonatural transformation

$$\rho_{g,\varphi} : F_{A,B} \longrightarrow F_{A',B'},$$

given by the assignments $x \longmapsto g(x)$ and $[\gamma] \longmapsto (h_{g,\varphi}(\gamma)^{-1}, g(y)poe_A(\gamma))$ for $[\gamma] : x \longrightarrow y$.

(c) If $a : (g_1, \varphi_1) \Longrightarrow (g_2, \varphi_2)$ is a a gauge 2-transformation, then we have a modification

$$\mathcal{A}_a : \rho_{g_1,\varphi_1} \Longrightarrow \rho_{g_2,\varphi_2},$$

given by the assignment $x \longmapsto (a(x), g(x))$.

These three constructions define a 2-functor

$$\mathcal{P} : \mathcal{C}on^{ff}_{\Gamma}(M) \longrightarrow \mathcal{F}un(\mathcal{P}_2(M), B\Gamma),$$

see [10, Section 2.3.4]. Besides of being a strict bigroupoid, the path 2-groupoid is naturally enriched in the category of diffeological spaces. Hence, there is a sub-bicategory

$$\mathcal{F}un^{\infty}(\mathcal{P}_2(M), B\Gamma) \subseteq \mathcal{F}un(\mathcal{P}_2(M), B\Gamma)$$

consisting of *smooth* 2-functors. The main result of [10] is that $\mathcal{P}$ induces an equivalence of bicategories, $\mathcal{C}on^{ff}_{\Gamma}(M) \cong \mathcal{F}un^{\infty}(\mathcal{P}_2(M), B\Gamma)$.

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