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Homological algebra in characteristic one

Alain Connes[abc] **and Caterina Consani**[d]

[a] *IHES, Bures sur Yvette, France*
[b] *Collège de France, France*
[c] *Department of Mathematics Ohio State University, Columbus OH, USA*
[d] *Department of Mathematics, The Johns Hopkins University, Baltimore MD, USA*

Abstract

This article develops several main results for a general theory of homological algebra in categories such as the category of idempotent semimodules. In the analogy with the development of homological algebra for abelian categories the present paper should be viewed as the analogue of the development of homological algebra for abelian groups. Our selected prototype, the category $\mathbb{B}$mod of semimodules over the Boolean semifield $\mathbb{B} := \{0,1\}$ is the replacement for the category of abelian groups. We show that the semi-additive category $\mathbb{B}$mod fulfills analogues of the axioms AB1 and AB2 for abelian categories. By introducing a precise comonad on $\mathbb{B}$mod we obtain the conceptually related Kleisli and Eilenberg-Moore categories. The latter category $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ is simply $\mathbb{B}$mod in the topos of sets endowed with an involution and as such it shares with $\mathbb{B}$mod most of its abstract categorical properties. The three main results of the paper are the following. First, when endowed with the natural ideal of null morphisms, the category $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ is a semiexact, homological category in the sense of M. Grandis. Second, there is a far reaching analogy between $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ and the category of operators in Hilbert spaces, and in particular results relating null kernel and injectivity for morphisms. The third fundamental result is that, even for finite objects of $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$, the resulting homological algebra is non-trivial and gives rise to a computable Ext functor. We determine explicitly this functor in the case provided by the diagonal morphism of the Boolean semiring into its square.

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Email addresses: alain@connes.org (Connes)
kc@math.jhu.edu (Consani)

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1. Introduction

This article advocates the existence of a meaningful theory of homological algebra in categories which share the same abstract properties of our studied prototype namely the category $\mathbb{B}$mod of semimodules over the Boolean semifield $\mathbb{B} := \{0, 1\}$, in which $1 + 1 = 1$ (*cf.* §2). With respect to the development of homological algebra for abelian groups as in [6] and its generalization for abelian categories [19], the main new difficulty in the development of homological algebra over $\mathbb{B}$ is that $\mathbb{B}$-semimodules are `not` cancellative and the general homology of semimodules as developed for example in [24–26] is not adapted for our purposes: see Remark 3.3.

The category $\mathbb{B}$mod is usually referred in books as the category of join semi-lattices with a least element, or idempotent semimodules and has a long history: we refer to [7] for an overview. In particular there is a well known equivalence between $\mathbb{B}$mod and the category of algebraic lattices. This equivalence together with the crucial notion of Galois connection are recalled in §2.3. This review is based on the duality results of [7] whose proofs are given, for the sake of completeness, in the simplest case of $\mathbb{B}$mod. Incidentally, the equivalence of $\mathbb{B}$mod with the category of algebraic lattices provides also the reader with a wealth of concrete and interesting examples even by restricting to the case of finite objects. In fact $\mathbb{B}$mod plays a universal role taking into account the transfer functor [18] which associates to an object of a category the lattice of its subobjects.

While $\mathbb{B}$mod is not an additive category it is semi-additive *i.e.* it has a zero object (both initial and final), all finite products and coproducts exist, and for any pair of objects M, N in $\mathbb{B}$mod the canonical morphism $M \amalg N \to M \times N$ from the coproduct to the product is an isomorphism. One sets $M \oplus N = M \amalg N \simeq M \times N$, and lets $p_j : M \oplus M \to M$ be the canonical projections and $s_j : M \to M \oplus M$ the canonical inclusions. In a semi-additive category $\mathscr{C}$ the set of the endomorphisms of an object in $\mathscr{C}$ is canonically endowed with the structure of semiring. Moreover, the analogues of the Axioms AB1 and AB2 for abelian categories ([19], §1.4) take the following form

AB1': $\mathscr{C}$ admits equalizers Equ and coequalizers Coequ.

Assuming AB1' we let, for $f \in \mathrm{Hom}_{\mathscr{C}}(L, M)$, $f^{(2)} := (f \circ p_1, f \circ p_2)$ and $\mathrm{Equ} f^{(2)}$ be the equalizer of $(f \circ p_1, f \circ p_2)$. It is the kernel pair of f. Similarly, with $f_{(2)} := (s_1 \circ f, s_2 \circ f)$ the coequalizer $\mathrm{Coequ} f_{(2)}$ is the cokernel pair of f. The analogue of the axiom AB2 is

AB2': For $f \in \mathrm{Hom}_{\mathscr{C}}(L, M)$, the natural morphism $\mathrm{Coim} f := \mathrm{Coequ}(\mathrm{Equ} f^{(2)}) \xrightarrow{f} \mathrm{Im}(f) := \mathrm{Equ}(\mathrm{Coequ} f_{(2)})$ is an isomorphism.

Then, for any morphism $f : L \to M$ in $\mathscr{C}$ one derives the sequence

$$\mathrm{Equ} f^{(2)} \overset{\iota_1}{\underset{\iota_2}{\rightrightarrows}} L \xrightarrow{f} M \overset{\gamma_1}{\underset{\gamma_2}{\rightrightarrows}} \mathrm{Coequ} f_{(2)} \tag{1}$$

In §3 we prove that the category $\mathbb{B}$mod fulfills the above Axioms $AB1'$ and $AB2'$. In this part a key role is played by a comonad $\perp$ on the category $\mathbb{B}$mod which encodes the endofunctor $M \longrightarrow M^2$ involved in the above construction of $f^{(2)}$ and $f_{(2)}$. The comonad $\perp$ is defined by

1. The endofunctor $\perp$: $\mathbb{B}\mathrm{mod} \longrightarrow \mathbb{B}\mathrm{mod}$, $\perp M = M^2$, $\perp f = (f, f)$.
2. The counit $\epsilon : \perp \to 1_{\mathbb{B}\mathrm{mod}}$, $\epsilon_M = p_1$, $p_1 : M^2 \to M$, $p_1(a, b) = a$.
3. The coproduct $\delta : \perp \to \perp \circ \perp$, $\delta_M = (M^2 \to (M^2)^2)$, $(x, y) \mapsto (x, y, y, x)$.

In Proposition 3.12 we determine the Kleisli and Eilenberg-Moore categories associated to the comonad $\perp$. The Kleisli category $\mathbb{B}\mathrm{mod}^2$ has the same objects as $\mathbb{B}$mod whereas the morphisms are pairs of morphisms in $\mathbb{B}$mod composing following the rule

$$(f,g)\circ(f',g') := (f\circ f' + g\circ g', f\circ g' + g\circ f'). \tag{2}$$

The category $\mathbb{B}\mathrm{mod}^2$ is the very natural enlargement of $\mathbb{B}$mod where pairs of morphisms are taken into account so that the notions of kernels and cokernels become similar to their incarnations in abelian categories. In fact to obtain the right notion of exact sequence not only one needs to use pairs of morphisms as in [24], but also to involve pairs of elements. Indeed, let M be an arbitrary $\mathbb{B}$-semimodule and $\psi : M \to \mathbb{B}$ the morphism $\psi(x) = 1 \iff x \neq 0$. Then the sequence

$$0 \underset{0}{\overset{0}{\rightrightarrows}} M \underset{0}{\overset{\psi}{\rightrightarrows}} \mathbb{B} \underset{0}{\overset{0}{\rightrightarrows}} 0$$

turns out to be exact for the cohomology defined in [24]. The same holds after refining the definition of the cohomology as the image of the equalizer in the coequalizer (see Remark 4.19).

This difficulty is overcome in §4 by implementing the following strict form of exactness

Definition 1.1. *We say that the sequence* $L \underset{\alpha_2}{\overset{\alpha_1}{\rightrightarrows}} M \underset{\beta_2}{\overset{\beta_1}{\rightrightarrows}} N$ *in* $\mathbb{B}\mathrm{mod}^2$ *is* strictly exact *at* M *if* $B(\alpha_1,\alpha_2)+\Delta = Z(\beta_1,\beta_2)$, *where* $\Delta \subset M\times M$ *is the diagonal and one sets*

$$B(\alpha_1,\alpha_2) := \{(\alpha_1(x)+\alpha_2(y), \alpha_2(x)+\alpha_1(y)) \mid x,y\in L\} \tag{3}$$

$$Z(\beta_1,\beta_2) := \{(u,v)\in M^2 \mid \beta_1(u)+\beta_2(v) = \beta_2(u)+\beta_1(v)\}. \tag{4}$$

In §4.3, Proposition 4.12, we show that a sequence $0 \underset{0}{\overset{0}{\rightrightarrows}} L \underset{g}{\overset{f}{\rightrightarrows}} M \underset{0}{\overset{0}{\rightrightarrows}} 0$ in $\mathbb{B}\mathrm{mod}^2$ is strictly exact if and only if $\phi = (f,g)$ is an isomorphism in $\mathbb{B}\mathrm{mod}^2$ (*i.e.* is invertible). Moreover, we find that the group $\mathrm{Aut}_{\mathbb{B}\mathrm{mod}^2}(M)$ of automorphisms of an object in $\mathbb{B}\mathrm{mod}^2$ is the semi-direct product of the 2-group of decompositions $M = M_1\times M_2$ of M as a product, by the group of automorphisms of the $\mathbb{B}$-semimodule M. In §§4.2, 4.4, we compare strict exactness with the categorical notions of monomorphisms and epimorphisms in $\mathbb{B}\mathrm{mod}^2$. The conclusion is that while being an epimorphism is equivalent to the strict exactness of $L \underset{g}{\overset{f}{\rightrightarrows}} M \underset{0}{\overset{0}{\rightrightarrows}} 0$, being a monomorphism is a more restrictive notion than claiming the strict exactness of $0 \underset{0}{\overset{0}{\rightrightarrows}} L \underset{g}{\overset{f}{\rightrightarrows}} M$ (*cf.* Example 4.9 in §4.2).

Example 4.14 displays a critical defect of $\mathbb{B}\mathrm{mod}^2$ by showing that some specific morphism in the category does not admit the epi-mono factorization "monomorphism $\circ$ epimorphism". Additionally, while a subsemimodule $N\subset M$ of a $\mathbb{B}$-semimodule M defines a natural covariant functor

$$M/N : \mathbb{B}\mathrm{mod}^2 \longrightarrow \mathbb{B}\mathrm{mod}, \quad M/N(X) := \{(f,g)\in \mathrm{Hom}_{\mathbb{B}\mathrm{mod}^2}(M,X) \mid f(x)=g(x)\,,\ \forall x\in N\}$$

this functor is in general not representable in $\mathbb{B}\mathrm{mod}^2$. The general theory of [3] suggests that these defects are eliminated by enlarging $\mathbb{B}\mathrm{mod}^2$ to the Eilenberg-Moore category of the comonad $\perp$. This extension turns out to be exactly the category $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ of $\mathbb{B}[s]$-semimodules which was first introduced in the thesis of S. Gaubert [15] (see also [1]). We refer to [27] for an updated account of this approach.

Definition 1.2. *We let* $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ *be the category of* $\mathbb{B}$*-semimodules endowed with an involution* σ*. The morphisms in* $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ *are the morphisms of* $\mathbb{B}$*-semimodules commuting with* σ*, i.e. equivariant for the action of* $\mathbb{Z}/2\mathbb{Z}$*.*

It should be noted from the beginning that the category $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ is simply the category $\mathbb{B}\mathrm{mod}$ in the topos of sets endowed with an involution and as such it shares with $\mathbb{B}\mathrm{mod}$ most of its abstract categorical properties. Moreover $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ is the same as the category of $\mathbb{B}[s]$-semimodules where $\mathbb{B}[s]$ is the semiring which is the symmetrization of $\mathbb{B}$ introduced in [15], p 71 (see also [1], definition 2.18). One has $\mathbb{B}[s] = \mathbb{B} \oplus s\mathbb{B}$, $s^2 = 1$ and the element $p = 1 + s$ fulfills $p^2 = p$ and plays the role of "null" element since $N = \{0, p\}$ is an ideal in $\mathbb{B}[s]$.

A main finding of this paper is that the Eilenberg-Moore category $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ falls within the framework of the homological categories developed by M. Grandis in [18], where the theory of lattices of subobjects in an ambient category is tightly linked to the existence and coherence of fundamental constructions in homological algebra. The astonishing outcome is that even in the simplest case of the category $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$, the resulting homological algebra is highly non-trivial and gives rise to a computable Ext functor.

In §6 we prove the fundamental result stating that the pair $(\mathbb{B}\mathrm{mod}^{\mathfrak{s}}, \mathcal{N})$, where $\mathcal{N}$ is the ideal (closed as in §1.3.2 of [18]) of the null morphisms f in $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ (*i.e.* $\sigma \circ f = f$) is a semiexact homological category.

Theorem 1.3. *The pair given by the category* $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ *and the null morphisms:* $\mathcal{N} \subset Hom_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(L, M)$ *forms a homological category in the sense of [17, 18].*

It turns out that $\mathcal{N} = \mathcal{N}(\mathcal{O})$ *i.e.* $\mathcal{N}$ is generated by the class $\mathcal{O}$ of null objects in $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ *i.e.* the objects N such that $\mathrm{Id}_N \in \mathcal{N}$ (equivalently the N's for which the involution σ_N is the identity). This class of objects is stable under retracts. The pair $(\mathbb{B}\mathrm{mod}^{\mathfrak{s}}, \mathcal{N})$ is a semiexact category and as such is provided with kernels and cokernels with respect to the ideal $\mathcal{N}$.

Definition 1.4. *For* $h \in Hom_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(L, M)$ *one sets* $\mathrm{Ker}(h) := h^{-1}(M^\sigma)$ *endowed with the induced involution. One also defines* $\mathrm{Coker}(h) := M/\!\sim$ *where*

$$b \sim b' \iff f(b) = f(b') \ \& \ \forall X, \forall f \in Hom_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(M, X): \ \mathrm{Range}(h) \subset \mathrm{Ker}(f). \qquad (5)$$

The homological structure of $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ allows for the definition of subquotients and their induced morphisms, for the construction of the connecting morphism and for the introduction of the homology sequence associated to a short exact sequence of chain complexes over $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$.

By applying the functor $y_{\mathbb{B}}(-) = \mathrm{Hom}_{\mathbb{B}\mathrm{mod}^2}(\mathbb{B}, -)$, the category $\mathbb{B}\mathrm{mod}^2$ becomes a full subcategory of $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ and the new objects in this latter category provide both a factorization of any morphism as "monomorphism $\circ$ epimorphism" and a representation of the functors M/N. The notion of cokernel in $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ gives an interpretation of the coequalizer $M \overset{\gamma_1}{\underset{\gamma_2}{\rightrightarrows}} \mathrm{Coequ} f_{(2)}$ of the sequence (1) as a cokernel. The notion of strict exactness continues to make sense in $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$. A sequence $L \overset{f}{\to} M \overset{g}{\to} N$ in $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ is *strictly exact* at M when $\mathrm{Range}(f) + M^\sigma = \mathrm{Ker}(g)$. In $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$, not all subobjects $E \subset F$ of an object F, with $F^\sigma \subset E$, qualify as kernels of morphisms (see example 6.4). This fact suggests to introduce the following new notion of exactness which turns out to be the same as the notion of exactness given in [18] for semiexact categories.

Definition 1.5. (i) *For* $f \in Hom_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(L, M)$*, we let the* normal image $\overline{\mathrm{Im}}(f) \subset M$ *be the kernel of its cokernel.*

(ii) A sequence of $\mathbb{B}\text{mod}^{\mathfrak{s}}$*:* $L \xrightarrow{f} M \xrightarrow{g} N$ *is* exact *at* M *when* $\overline{\text{Im}}(f) = Ker(g)$*.*

(iii) A subobject $E \subset F$ *of an object in* $\mathbb{B}\text{mod}^{\mathfrak{s}}$ *is* normal *if* E *is the kernel of some morphism in* $\mathbb{B}\text{mod}^{\mathfrak{s}}$*.*

Note that strict exactness implies exactness in $\mathbb{B}\text{mod}^{\mathfrak{s}}$, and the definition of the normal image of a morphism as the kernel of its cokernel corresponds to the standard definition of the image of a morphism in an abelian category. By applying these concepts we find

Proposition 1.6. *Let* $f \in Hom_{\mathbb{B}\text{mod}^{\mathfrak{s}}}(L, M)$*. The sequence*

$$0 \to \text{Ker}(f) \to L \xrightarrow{f} M \to \text{Coker}(f) \to 0.$$

is exact in $\mathbb{B}\text{mod}^{\mathfrak{s}}$*.*

The slight difference between the notions of strict exactness and exactness in $\mathbb{B}\text{mod}^{\mathfrak{s}}$ is due to the fact that not all subobjects $E \subset F$, with $F^\sigma \subset E$, are normal. For this reason it is crucial to determine the "closure" of a subobject in $\mathbb{B}\text{mod}^{\mathfrak{s}}$ *i.e.* the intersection of all kernels containing this subobject or equivalently find the normal image $\overline{\text{Im}}(E)$ of the inclusion $E \subset F$. This part is developed in Proposition 6.16. We report here the output of this computation. Let $p(x) := x + \sigma(x)$ be the projection on null elements.

Proposition 1.7. *Let* $E \subset F$ *be a subobject in* $\mathbb{B}\text{mod}^{\mathfrak{s}}$*, with* $F^\sigma \subset E$*. Then for* $\xi \in F$ *one has* $\xi \in \overline{\text{Im}}(E)$ *(the normal image of the inclusion) if and only if there exists a finite sequence* $a_0, a'_0, a_1, a'_1, \ldots, a_n, a'_n$ *of elements of* E *such that* $\xi = a_0 + \xi$*,* $p(a_0) = p(a'_0)$*,* $a'_0 + \xi = a_1 + \xi$*,* $p(a_1) = p(a'_1), \ldots, p(a_n) = \xi + \sigma(\xi)$*.*

A corresponding explicit description of the cokernel of a morphism in $\mathbb{B}\text{mod}^{\mathfrak{s}}$ is given in Proposition 6.19 and states as follows

Proposition 1.8. *Let* $E \subset F$ *be a subobject in* $\mathbb{B}\text{mod}^{\mathfrak{s}}$ *containing* F^σ*. Then the cokernel of the inclusion* $E \subset F$ *is the quotient of* $F^\sigma \cup E^c$ *(*$E^c := F \setminus E$*), by the smallest equivalence relation such that*

$$\xi \in E^c,\ u, v \in E,\ p(u) = p(v) \implies \xi + u \sim \xi + v. \tag{6}$$

The cokernel map is the quotient map on E^c *and the projection* p *on* E*.*

In §7 we develop a second fundamental topic of this paper namely the analogy between the category $\mathbb{B}\text{mod}^{\mathfrak{s}}$ and the category of linear operators in a Hilbert space. This parallel relies on a duality property holding in $\mathbb{B}\text{mod}^{\mathfrak{s}}$ which we shall now review. Let $\mathfrak{s} : \mathbb{B}\text{mod} \to \mathbb{B}\text{mod}^{\mathfrak{s}}$ be the functor acting on objects as $M \mapsto (M^2, \sigma_M)$, where $\sigma_M(x, y) = (y, x)$ and on morphisms $f : M \to N$ as $\mathfrak{s}(f)(x, y) = (f(x), f(y))$. One defines the orthogonal object of a subobject E of an object F in $\mathbb{B}\text{mod}^{\mathfrak{s}}$ by implementing the following natural pairing connecting F and $F^* := \text{Hom}_{\mathbb{B}\text{mod}^{\mathfrak{s}}}(F, \mathfrak{s}\mathbb{B})$:

$$\langle x, y\rangle_\sigma := y(x)\,,\ \ \forall x \in F, y \in F^*, \quad E^\perp := \{y \mid \langle x, y\rangle_\sigma \in (\mathfrak{s}\mathbb{B})^\sigma,\ \forall x \in E\}.$$

Then the following fact holds

Proposition 1.9. *(i) Let* $E \subset F$ *be a subobject of the object* F *of* $\mathbb{B}\text{mod}^{\mathfrak{s}}$*. Then the least normal subobject* $\overline{\text{Im}}(E)$ *containing* E *satisfies:* $\overline{\text{Im}}(E) = (E^\perp)^\perp$*.*

(ii) Let $\phi \in Hom_{\mathbb{B}\text{mod}^{\mathfrak{s}}}(E, F)$*. Then the dual of the cokernel of* ϕ *is canonically isomorphic to the kernel of* $\phi^* \in Hom_{\mathbb{B}\text{mod}^{\mathfrak{s}}}(F^*, E^*)$ *i.e.* $\text{Coker}(\phi)^* = \text{Ker}(\phi^*)$*.*

In the paper we construct a large class of objects X of $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ such that for any morphism $\phi \in \mathrm{Hom}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(X, Y)$ the nullity of $\mathrm{Ker}(\phi)$ is equivalent to the injectivity of ϕ. More precisely we obtain the following

Theorem 1.10. *Let M be a finite object of $\mathbb{B}\mathrm{mod}$ whose dual M^* is generated by its minimal non-zero elements. Then, for any object N of $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ a morphism $\phi \in Hom_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(\mathfrak{s}M, N)$ is a monomorphism if and only if its kernel is null.*

In §8, we finally apply to the semiexact, homological category $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ the general theory developed in [18]. The key notion is the analogue of the "kernel-cokernel pairs" in additive categories [5], which is introduced in §1.3.5 of [18]. In our context such short doubly exact sequences are given by a pair of maps

$$L \overset{m}{\rightarrowtail} M \overset{p}{\twoheadrightarrow} N : \quad m = \ker(p) \ \&\ p = \mathrm{cok}(m)$$

where $L = \mathrm{Ker}(p)$ is a normal subobject of M and $N = \mathrm{Coker}(m)$ is a normal quotient of M. Notice that the definition of a short doubly exact sequence as above is more restrictive than requiring the exactness of $0 \to L \overset{m}{\to} M \overset{p}{\to} N \to 0$ and the two notions become equivalent if one requires the maps m, p to be exact (Proposition 6.7).

Connected sequences of functors and satellites exist for semiexact categories and their limit-construction is given as pointwise Kan extensions. For the definition of a cohomology theory, we are mainly interested in the construction of the right satellite of the left exact functor defined by the internal Hom functor: $\underline{\mathrm{Hom}}(M, -)$. In an abelian category this functor is known to be left exact and one obtains the Ext functors as right derived functors. In our context, the right satellite is constructed as a left Kan extension (*cf.* [21] p. 240). The great advantage to work within a semiexact, homological category $\mathscr{C}$ (such as $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$) is that one can construct a homology sequence for any short doubly exact sequence of chain complexes over $\mathscr{C}$. This homology sequence is always of order two, meaning that the composition of two consecutive morphisms is null, and a very simple condition on the middle complex gives a partial exactness property for the homology sequence, sufficient for studying the universality of chain homology. The main obstacle for a straightforward application of the results of [18] is that the categories we work with do not satisfy the modularity requirement which allows one to bypass the role of the difference to relate for instance the injectivity of a morphism with the nullity of its kernel. This drawback inputs a further obstruction to a direct proof of the main condition (a) of Theorem 4.2.2 of [18], which entails, if satisfied, to compute satellite functors using semi-resolutions. The key result of §8 is a sleek solution of this problem stating that for the right satellite of $\underline{\mathrm{Hom}}(M, -)$ one has the following result

Theorem 1.11. *Let $\iota : I' \overset{i'}{\rightarrowtail} I \overset{i''}{\twoheadrightarrow} I''$ be a short doubly exact sequence in $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ with the middle term I being both injective and projective. Then the functor $F := \underline{\mathrm{Hom}}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(I'', -)$ satisfies the condition $a)$ of [18] with respect to an arbitrary morphism of short doubly exact sequences $c \to \iota$, provided this holds for all endomorphisms of I' in $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$.*

In the final §9 we study the cokernel of the diagonal which enters in the construction of the Čech version of sheaf cohomology, as the dual of the differential of the Koszul resolution $d : \wedge^2 \to \wedge$. We apply Theorem 1.11 to determine explicitly the Ext functor associated to this key example provided by the cokernel of the diagonal $\Delta : \mathbb{B} \to \mathbb{B} \times \mathbb{B}$. This cokernel is represented by a computable object Q of $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$, which is described in details in §9.1. It fits in the following short doubly exact sequence

$$\alpha : K = \mathrm{Ker}(\mathrm{cok}(\mathfrak{s}\Delta)) \overset{\subset}{\rightarrowtail} \mathbb{B}^2 \times \mathbb{B}^2 \overset{\mathrm{cok}(\mathfrak{s}\Delta)}{\twoheadrightarrow} Q \tag{7}$$

and the main point is to detect the obstruction to lift back from Q to $\mathbb{B}^2 \times \mathbb{B}^2$. Thus we consider the representable functor $F := \underline{\mathrm{Hom}}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(Q, -)$, viewed as a covariant endofunctor in $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ (using the natural internal Hom) and we compute its right derived functor. Lemma 9.3 shows that the short doubly exact sequence (7), gives rise, after applying the functor F, to a non-null cokernel for the map $F(\mathrm{cok}(\mathfrak{s}\Delta))$. This yields naturally an element of the satellite functor $SF(\mathrm{Ker}(\mathrm{cok}(\mathfrak{s}\Delta)))$. In Lemma 9.5 we compute $\mathrm{Coker}(F(\mathrm{cok}(\mathfrak{s}\Delta)))$: its graph is displayed in Figure 14. In §9.4 we investigate the correspondence between endomorphisms of $K := \mathrm{Ker}(\mathrm{cok}(\mathfrak{s}\Delta))$ and of Q as provided by the two maps from the endomorphisms $\mathrm{End}(\alpha)$ of the short doubly exact sequence α of (7)

$$\mathrm{End}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(K) \overset{res}{\leftarrow} \mathrm{End}(\alpha) \overset{quot}{\rightarrow} \mathrm{End}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(Q) \tag{8}$$

We show that this correspondence is multivalued. The remaining issue then is to prove that after applying the functor F it becomes single valued so that one obtains functoriality. This result is achieved in §9.5 (Theorems 9.12, 9.15) and provides one with the following third fundamental result of this paper

Theorem 1.12. *The short doubly exact sequence* $\alpha : \mathrm{Ker}(\mathrm{cok}(\mathfrak{s}\Delta)) \overset{\subset}{\rightarrowtail} \mathbb{B}^2 \times \mathbb{B}^2 \overset{\mathrm{cok}(\mathfrak{s}\Delta)}{\twoheadrightarrow} Q$ *satisfies condition a) of [18] with respect to the functor* $F = \underline{\mathrm{Hom}}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(Q, -)$. *The right satellite functor* SF *is non-null and* $SF(K) = \mathrm{Coker}(F(\mathrm{cok}(\mathfrak{s}\Delta)))$.

This result shows that the Ext functor is non-trivial already in the case of finite objects of $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$. For this reason the present paper provides a strong motivation for a systematic development of homological algebra for categories of sheaves of semimodules over semirings of characteristic one over a topos.

2. $\mathbb{B}$-semimodules and algebraic lattices

A $\mathbb{B}$-semimodule M is a commutative monoid written additively such that $x + x = x \ \forall x \in M$. The idempotent addition is often denoted by $x \vee y$ instead of $x + y$. The action of $\mathbb{B}$ on M is set to be: $0 \cdot x = 0$ and $1 \cdot x = x$, $\forall x \in M$. In §2.1 we recall why $\mathbb{B}$-semimodules correspond to join-semilattices with a least element. In §2.2 we review basic material on duality [7], and §2.3 surveys the classical equivalence with algebraic lattices. In §2.4 we implement Galois connections [7] to give a canonical decomposition of morphisms in $\mathbb{B}\mathrm{mod}$. This construction will play an important role in the analogy with operators developed in §7. Finally, in §2.5 we discuss for future application in Theorem 7.10, the condition on a $\mathbb{B}$-semimodule E that ensures that its dual E^* is generated by its minimal non-zero elements.

2.1 $\mathbb{B}$-semimodules as partially ordered sets. The traditional terminology for $\mathbb{B}$-semimodules is of join-semilattices with a least element (written 0), where "upper-semilattice" is also used in place of join-semilattice. The join of two elements is denoted $x \vee y$ and the morphisms $f : S \to T$ in the category satisfy $f(x \vee y) = f(x) \vee f(y)$ and $f(0) = 0$. This category is the same as that of $\mathbb{B}$-semimodules as shown in the following proposition which is well known and whose proof is straightforward.

Proposition 2.1. (i) *Let M be a $\mathbb{B}$-semimodule, then the condition*

$$x \leq y \iff x \vee y = y \tag{9}$$

defines a partial order on M such that any two elements of M have a join, i.e. there is a smallest element $z = x \vee y$ among the elements larger than x and y, and moreover 0 is the smallest element.

Conversely:

(ii) *Let $(E, \leq)$ be a partially ordered set with a smallest element in which any two elements $x, y \in E$ have a join $x \vee y$. Then E endowed with the addition $(x, y) \mapsto x \vee y$ is a $\mathbb{B}$-semimodule.*

(iii) *The morphisms of $\mathbb{B}$-semimodules correspond to increasing maps of semi-lattices preserving the sup and the minimal element and (i) and (ii) define functors which are inverse of each other.*

2.2 Duality in $\mathbb{B}\text{mod}$. This section recalls in the simple case of the semiring $\mathbb{B}$ the duality results of [7]. We give complete proofs for convenience. The functor $\mathrm{Hom}_{\mathbb{B}}$ with one of the two entries fixed, determines an endofunctor in $\mathbb{B}\text{mod}$. The simplest case to consider is the duality determined by the contravariant endofunctor

$$M \longrightarrow M^* := \mathrm{Hom}_{\mathbb{B}}(M, \mathbb{B}). \tag{10}$$

The terminology "ideal" is often used in the context of partially ordered sets to denote a hereditary subsemimodule. We prefer to adopt the latter to avoid confusion with the theory of semirings.

Definition 2.2. (i) *A subset $H \subset E$ of a partially ordered set E is hereditary if*

$$y \in H \,\&\, x \leq y \implies x \in H.$$

(ii) *A subset $H \subset M$ of a $\mathbb{B}$-semimodule M is hereditary if it is so for the partial order (9) on M.*

Next proposition is easily deduced from [7] (*cf.* §4.2 and Corollaries 37 and 40). In particular (iii) implies that $\mathbb{B}$ is injective in the category of $\mathbb{B}$-semimodules

Proposition 2.3. (i) *The map $M^* \to \{H \subset M\}$, $\phi \mapsto \phi^{-1}(0)$ is a bijection between M^* and the set of non-empty hereditary subsemimodules $H \subset M$.*

(ii) *A subset $H \subset M$ is an hereditary subsemimodule if and only if it is a filtering union of intervals $I_x := \{y \in M \mid y \leq x\}$.*

(iii) *Let $N \subset M$ be a subsemimodule, then the restriction map $M^* \to N^*$ is surjective.*

(iv) *The pairing between M and M^* is separating.*

(v) *Let E, F be arbitrary objects in $\mathbb{B}\text{mod}$ and $f \in Hom_{\mathbb{B}}(E, F)$, then*

$$f(\xi) = f(\eta) \iff \langle \xi, u \rangle = \langle \eta, u \rangle\,, \quad \forall u \in \mathrm{Range}(f^*). \tag{11}$$

Proof. (i) For $\phi \in M^*$ the subset $\phi^{-1}(0)$ is a hereditary subsemimodule of M. Conversely, if $H \subset M$ is a non-empty hereditary subsemimodule of M the equality $\phi(x) = 0 \iff x \in H$ defines a map $\phi : M \to \mathbb{B}$ satisfying $\phi(x \vee y) = \phi(x) \vee \phi(y)$. In fact, either $x, y \in H$ in which case $x \vee y \in H$ and the two sides are 0, or say $x \notin H$ and then $x \vee y \notin H$ since H is hereditary and then both sides are equal to 1.

(*ii*) Let $H \subset M$. If H is a hereditary subsemimodule one has $H = \cup_{x \in H} I_x$ and the inclusion $I_x \cup I_y \subset I_{x \vee y}$ shows that the union is filtering. Conversely, any interval I_x is a hereditary subsemimodule, any union of hereditary subsets is hereditary and any filtering union of subsemimodules is a subsemimodule.

(*iii*) Let $N \subset M$ be a subsemimodule. Let $\phi \in N^*$. Let

$$H \subset M, \quad H = \cup_{x \in N | \phi(x)=0} I_x = \{y \in M \mid \exists x \in N,\, y \leq x \,\&\, \phi(x) = 0\}.$$

Then $H \subset M$ is a filtering union of intervals and hence a hereditary subsemimodule, thus the equality $\psi(x) = 0 \iff x \in H$ defines a map $\psi : M \to \mathbb{B}$ and $\psi \in M^*$. Moreover one has $N \cap H = \phi^{-1}(0)$ and thus the restriction of ψ to N is equal to ϕ.

(*iv*) Let $x, y \in M$, such that $\phi(x) = \phi(y)\ \forall \phi \in M^*$, then $x \in I_y$ and $y \in I_x$ so $x = y$.

(*v*) The pairing of F with F^* is separating by (*iv*), and this shows the implication $\Leftarrow$. The implication $\Rightarrow$ is straightforward. $\square$

Note that by construction the map $\psi : M \to \mathbb{B}$ defined in the proof of (*iii*) is the largest extension of $\phi : N \to \mathbb{B}$ to M since for any extension ψ' one has $\psi'(x) = 0$ for any $x \in H$ for H as in (*iii*). Applying (*iii*) to $N = 0$ one thus obtains the largest element of M^*, as the map $\alpha : M \to \mathbb{B}$ such that $\alpha(x) = 1$, $\forall x \neq 0$.

Next proposition shows that $\mathbb{B}$ is a cogenerator in the category of $\mathbb{B}$-semimodules.

Proposition 2.4. (*i*) *The product $\mathbb{B}^X$ (resp. the coproduct $\mathbb{B}^{(X)}$) of any number of copies of $\mathbb{B}$ is an injective (resp. projective) object in the category of $\mathbb{B}$-semimodules.*

(*ii*) *Let M be a $\mathbb{B}$-semimodule then M is isomorphic to a subsemimodule of a product $\mathbb{B}^X$ and there exists a surjective map of $\mathbb{B}$-semimodules of the form $\mathbb{B}^{(X)} \to M$.*

Proof. (*i*) A morphism $\phi : N \to \mathbb{B}^X$ can be equivalently described by a family of morphisms $\phi_x \in \mathrm{Hom}_{\mathbb{B}}(N, \mathbb{B}) = N^*$. Since when $N \subset M$ each ϕ_x admits an extension to M, one obtains that the naturally associated restriction map $\mathrm{Hom}_{\mathbb{B}}(M, \mathbb{B}^X) \to \mathrm{Hom}_{\mathbb{B}}(N, \mathbb{B}^X)$ is surjective. Similarly, a morphism $\phi : \mathbb{B}^{(X)} \to M$ is equivalent to a family of morphisms $\phi_x \in \mathrm{Hom}_{\mathbb{B}}(\mathbb{B}, M) = M$ and one derives in this way that $\mathbb{B}^{(X)}$ is projective.

(*ii*) Let $\iota : M \to (M^*)^*$ be defined by $\iota(x)(\phi) := \phi(x)$. One has

$$\iota(x \vee y)(\phi) = \phi(x \vee y) = \phi(x) \vee \phi(y) = \iota(x)(\phi) \vee \iota(y)(\phi).$$

Then ι defines a morphism $M \to \mathbb{B}^{M^*}$. This morphism is injective since for $x \in M$ the element $\phi_x \in M^*$ defined by $\phi_x^{-1}(0) = I_x = \{y \mid y \leq x\}$ satisfies the rule $\phi_x(y) = \phi_x(x) \iff y \leq x$. Thus if $x \neq y$ one has either that $\phi_x(y) \neq \phi_x(x)$ or that $\phi_y(y) \neq \phi_y(x)$. The existence of a surjective map $\mathbb{B}^{(X)} \to M$ is clear by taking $X = M$ and using the addition. $\square$

Remark 2.5. It might seem contradictory that the lattice $\mathcal{L}$ of subspaces $E \subset V$ of a finite dimensional vector space V over a field K can be embedded as a subsemimodule of a product $\mathbb{B}^X$. To see why this construction is meaningful one first considers the orthogonal $E^\perp \subset V^*$ and this step replaces the lattice operation $E_1 \vee E_2$ by the dual operation *i.e.* the intersection: $(E_1 \vee E_2)^\perp = E_1^\perp \cap E_2^\perp$. In this way $\mathcal{L}$ embeds as a sub-lattice of the lattice of subsets of the set V^* endowed with the operation of intersection. Finally, by using the map from a subset to its complement, this lattice is the same as the product $\mathbb{B}^X$, for $X = V^*$.

We end this subsection with the simple form of the Separation Theorem of [7] in the case of $\mathbb{B}$-semimodules. It gives the following analogue of the Hahn-Banach Theorem. Since the proof of [7] takes a simple form in our context we include it here below.

Lemma 2.6. *Let N be a $\mathbb{B}$-semimodule and $E \subset N$ be a subsemimodule: we denote $\iota : E \to N$ the inclusion. Then*

$$x \in E \iff \phi(x) = \psi(x)\,, \ \ \forall \phi, \psi \in Hom_{\mathbb{B}}(N, \mathbb{B}) \text{ s.t. } \phi \circ \iota = \psi \circ \iota. \tag{12}$$

Proof. Let $\xi \in N$, $\xi \notin E$. It is enough to show that there are two morphisms $\phi, \psi \in \mathrm{Hom}_{\mathbb{B}}(N, \mathbb{B})$ which agree on E but take different values at ξ. Let $\phi = \phi_\xi$ so that $\phi^{-1}(0) = \{\eta \mid \eta \leq \xi\}$. We let

$$F = \{\alpha \in N \mid \exists \eta \in E,\ \alpha \leq \eta \leq \xi\}.$$

One has $\alpha, \alpha' \in F \Rightarrow \alpha \vee \alpha' \in F$ since for $\eta, \eta' \in E$, $\alpha \leq \eta \leq \xi$, $\alpha' \leq \eta' \leq \xi$, $\eta \vee \eta'$ is an element of E, and one has $\alpha \vee \alpha' \leq \eta \vee \eta' \leq \xi$. Moreover by construction one has $\alpha \in F\ \&\ \beta \leq \alpha \Rightarrow \beta \in F$. It follows that the equality $\psi(\alpha) = 0 \iff \alpha \in F$ defines an element $\psi \in \mathrm{Hom}_{\mathbb{B}}(N, \mathbb{B})$. Moreover $\xi \notin F$ since $\xi \notin E$, thus $\psi(\xi) = 1$. One has $\phi(\xi) = 0 \neq \psi(\xi) = 1$. Since $F \cap E = [0, \xi] \cap E$ one has $\phi(\eta) = \psi(\eta)$, $\forall \eta \in E$. □

Lemma 2.6 allows one to determine the epimorphisms in $\mathbb{B}$mod and one gets the following corollary (part (i) is true for every variety in the sense of universal algebra).

Proposition 2.7. *Let $f \in Hom_{\mathbb{B}}(M, N)$ be a morphism in $\mathbb{B}$mod.*

(i) f is a monomorphism if and only if the underlying map is injective.

(ii) f is an epimorphism if and only if the underlying map is surjective.

Proof. (i) This follows from the identification $M = \mathrm{Hom}_{\mathbb{B}}(\mathbb{B}, M)$ which associates to any $x \in M$ the unique morphism $\mathbb{B} \to M$ sending 1 to x.

(ii) This follows from Lemma 2.6. □

2.3 Algebraic lattices.

In this subsection we describe the well-known equivalence between the category $\mathbb{B}$mod and the category of algebraic lattices. The notion of algebraic lattice is recalled in the following

Definition 2.8. (i) *A complete lattice $\mathcal{L}$ is a partially ordered set in which all subsets have both a supremum (join) and an infimum (meet).*

(ii) *An element $x \in \mathcal{L}$ of a complete lattice is* compact *(one writes $x \in K(\mathcal{L})$) if the following rule applies*

$$x \leq \vee_A y \implies \exists F \subset A,\ F \text{ finite},\ \text{s.t. } x \leq \vee_F y.$$

(iii) *A lattice $\mathcal{L}$ is* algebraic *iff it is complete and every element x is the supremum of compact elements below x.*

Examples of algebraic lattices are:

- Subgroups(G) for a group G, where the compact elements are the finitely generated subgroups
- Subspaces(V) for a vector space V, where the compact elements are the finite dimensional subspaces
- Id(R) *i.e.* ideals for a ring R, where the compact elements are the finitely generated ideals

A morphism of algebraic (complete) lattices is a compactness-preserving complete join homomorphism *i.e.*

$$f : S \to T \text{ s.t. } f(\vee_A y) = \vee_A f(y)\,, \ \ \forall A \subset S, \ \ f(K(S)) \subset K(T). \tag{13}$$

To understand this notion we show

Lemma 2.9. *Let $f : S \to T$ be a morphism of algebraic lattices. Then f is injective if and only if its restriction $f : K(S) \to K(T)$ is injective.*

Proof. We assume that the restriction to $f : K(S) \to K(T)$ is injective. Let then $s, s' \in S$ such that $f(s) = f(s')$. Let $A = \{x \in K(S) \mid x \le s\}$, $B = \{y \in K(S) \mid y \le s'\}$. One has $s = \vee_A x$, $s' = \vee_B y$, and $A, B \subset K(S)$. One has $\vee_A f(x) = \vee_B f(y)$ and the compactness of $f(y)$ for $y \in B$ shows that there exists $x \in A$ with $f(x) \ge f(y)$. This means $f(x) + f(y) = f(x)$ and hence, as $x + y, x \in K(S)$, that $x + y = x$, *i.e.* $y \le x$. This proves that A and B are cofinal and thus $s = s'$. $\square$

The notion of compactness of elements is best understood in terms of the small category $\mathcal{C}(\mathcal{L})$ whose objects are the elements of $\mathcal{L}$ and there is a single morphism from x to y iff $x \le y$, while otherwise $\mathrm{Hom}(x, y) = \emptyset$. In these terms, an element $x \in \mathcal{L}$ is compact iff it is *finitely presentable* in the sense that the functor $\mathrm{Hom}(x, -) : \mathcal{C}(\mathcal{L}) \longrightarrow \mathfrak{Sets}$ preserves filtered colimits. In this language and for any ordered set $\mathcal{L}$ one has the equivalence

$$\mathcal{L} \text{ is a complete algebraic lattice } \iff \mathcal{C}(\mathcal{L}) \text{ is locally finitely presentable}$$

By definition a locally finitely presentable category has all small colimits and any object is a filtered colimit of the canonical diagram of finitely presentable objects mapping into it. There is a classical equivalence of categories between join-semilattices with a least element, *i.e.* the category $\mathbb{B}$mod and the category $\mathcal{A}$ of complete algebraic lattices with compactness-preserving complete join-homomorphisms. This equivalence is a special case of the Gabriel-Ulmer duality [14].

To a $\mathbb{B}$-semimodule M one associates the algebraic complete lattice $\mathrm{Id}(M)$ of hereditary subsemimodules of Definition 2.2. The join operation in the lattice is the hereditary subsemimodule generated by the union. The join of the intervals $I_x := \{z \in M \mid z \le x\}$ and I_y is the interval $I_{x \vee y}$. Proposition 2.3 then shows that one gets an algebraic lattice and that the intervals are the most general compact elements of $\mathrm{Id}(M)$. To a morphism $f : M \to N$ in $\mathbb{B}$mod one associates the morphism $\mathrm{Id}(f) : \mathrm{Id}(M) \to \mathrm{Id}(N)$ obtained by mapping an hereditary subsemimodule $I \subset M$ to the hereditary subsemimodule generated by $f(I) \subset N$. This defines a compactness-preserving complete join-homomorphism but since in general $f(I \cap J) \neq f(I) \cap f(J)$, such homomorphism does not respect the intersections. By applying Proposition 2.3 one has a canonical identification of $\mathrm{Id}(M)$ with $M^* := \mathrm{Hom}_{\mathbb{B}}(M, \mathbb{B})$ but this identification is not compatible with the join since

$$(\phi \vee \psi)^{-1}(0) = \phi^{-1}(0) \cap \psi^{-1}(0)\,, \ \ \forall \phi, \ \psi \in M^*.$$

Moreover the functor $\mathrm{Id} : \mathbb{B}\mathrm{mod} \longrightarrow \mathcal{A}$ is covariant while the endofunctor $M \mapsto M^*$ on $\mathbb{B}$mod is contravariant. The precise link between these two functors is derived from the following notion of Galois connection: we refer to [12] Chapter VII [18] 1.1.3, and [16] to read more details.

Definition 2.10. *A covariant Galois connection $f \dashv g$ between two partially ordered sets E, F consists of two monotone functions: $f : E \to F$ and $g : F \to E$, such that*

$$f(a) \le b \iff a \le g(b). \tag{14}$$

Note that in fact one does not have to assume that the maps f and g are monotone since this property follows from (14). Indeed, we show that $x \leq y \Rightarrow f(x) \leq f(y)$. It follows from (14) that $f(x) \leq f(y) \iff x \leq g(f(y))$ and moreover using $f(y) \leq f(y)$ one derives $y \leq g(f(y))$. Thus if $x \leq y$ one gets $x \leq g(f(y))$ and hence $f(x) \leq f(y)$.

For a Galois connection $f \dashv g$, f is called the left adjoint of g and g is called the right adjoint of f. This terminology fits with that of adjoint functors by considering the category $\mathcal{C}(E)$. The fact that f is monotone means that it defines a covariant functor $\mathcal{C}(f) : \mathcal{C}(E) \to \mathcal{C}(F)$. The relation $f \dashv g$ means exactly that $\mathcal{C}(f) \dashv \mathcal{C}(g)$ as a pair of adjoint functors. There is a natural composition law for Galois connections: given $f \dashv g$ with $f : E \to F$ and $h \dashv k$ with $h : F \to G$, the pair $(h \circ f, g \circ k)$ is a Galois connection $h \circ f \dashv g \circ k$, with $h \circ f : E \to G$. This is a particular case of a general fact, true for general adjunctions. Moreover a Galois connection $f \dashv g$ with $f : E \to F$ automatically fulfills the equalities: $f = fgf$ and $g = gfg$ since $f(a) \leq f(a) \Rightarrow a \leq (g \circ f)(a) \Rightarrow f(a) \leq (f \circ g \circ f)(a)$, while $g(b) \leq g(b) \Rightarrow (f \circ g)(b) \leq b$ and taking $b = f(a)$ one gets $(f \circ g \circ f)(a) \leq f(a)$ and thus $(f \circ g \circ f)(a) = f(a)$. This is a particular case of the classical triangular identities for adjunctions.

Any homomorphism between complete lattices which preserves all joins is the left adjoint of some Galois connection (this is a special case of the adjoint functor theorem). This fact holds in particular for $\mathrm{Id}(f) : \mathrm{Id}(M) \to \mathrm{Id}(N)$ with the above notations (*i.e.* with $f \in \mathrm{Hom}_{\mathbb{B}}(M, N)$) and one derives the following conclusion

Proposition 2.11. *Let M, N be objects of $\mathbb{B}\mathrm{mod}$ and $f \in Hom_{\mathbb{B}}(M, N)$. Then, under the canonical identifications of partially ordered sets $Id(M) \simeq (M^*)^{\mathrm{op}}$ and $Id(N) \simeq (N^*)^{\mathrm{op}}$, the following pair defines a monotone Galois connection*

$$Id(f) \dashv f^*.$$

Proof. The map $f^* : N^* \to M^*$ is defined as composition with f, *i.e.* $\psi \in N^* \mapsto \psi \circ f \in M^*$. One has for $\phi \in M^*$, $\psi \in N^*$

$$\mathrm{Id}(f)(\phi^{-1}(0)) \subset \psi^{-1}(0) \iff f(\phi^{-1}(0)) \subset \psi^{-1}(0).$$

In turns this is equivalent to $\phi(x) = 0 \Rightarrow \psi(f(x)) = 0$ *i.e.* to $\phi \geq f^*\psi$. Since the identification $\mathrm{Id}(M) \simeq (M^*)^{\mathrm{op}}$ is order reversing one gets the required equivalence of Definition 2.10. ☐

The next statement is a corollary of the theorem asserting the equivalence of $\mathbb{B}\mathrm{mod}$ with the category $\mathcal{A}$

Corollary 2.12. *Let M be a $\mathbb{B}$-semimodule, and $M^* = Hom_{\mathbb{B}}(M, \mathbb{B})$. Then $(M^*)^{\mathrm{op}}$ is a complete algebraic lattice and the map which to $x \in M$ associates $\phi_x \in M^*$, with $\phi_x^{-1}(0) = [0, x]$ is an order preserving bijection of M with the compact elements of $(M^*)^{\mathrm{op}}$.*

Proof. Since the partially ordered set $(M^*)^{\mathrm{op}}$ is isomorphic to $\mathrm{Id}(M)$ it is a complete algebraic lattice. The compact elements of $\mathrm{Id}(M)$ are the intervals $[0, x]$ and the corresponding elements of $M^* = \mathrm{Hom}_{\mathbb{B}}(M, \mathbb{B})$ are the ϕ_x. ☐

Next, we compare the complete algebraic lattice $\mathrm{Id}(M)$ with the bidual $(M^*)^*$. In general the latter is strictly larger than $\mathrm{Id}(M)$ and there is a simple notion of complete join homomorphism between complete lattices, *i.e.* those morphisms which preserve arbitrary $\vee$. It is natural to compare them with normal positive maps in the von Neumann algebra context. Thus we let

$$N^*_{\mathrm{norm}} := \{\phi \in \mathrm{Hom}_{\mathbb{B}}(N, \mathbb{B}) \mid \phi(\vee x_\alpha) = \vee(\phi(x_\alpha))\}.$$

Proposition 2.13. (*i*) *The canonical map* $\epsilon : M \to (M^*)^*$ *given by evaluation* $\epsilon(x)(\psi) = \psi(x)$ *extends to an isomorphism* $\tilde{\epsilon} : Id(M) \simeq (M^*)^*_{\rm norm}$.

(*ii*) *The* $\mathbb{B}$*-semimodule* M *is the subsemimodule of* $(M^*)^*_{\rm norm}$ *of the compact elements of this complete algebraic lattice.*

Proof. (*i*) The $\mathbb{B}$-semimodule $N = M^*$ is a complete lattice. An element $\phi \in N^*$ is characterized by the hereditary subsemimodule $J = \phi^{-1}(\{0\})$ and $\phi \in N^*_{\rm norm}$ iff J is closed under arbitrary $\vee$. This fact holds iff $J = [0, n]$ for some element $n \in N$. One thus obtains a canonical order preserving bijection $(M^*)^*_{\rm norm} \simeq (M^*)^{\rm op} = \mathrm{Id}(M)$ which associates to $\phi \in (M^*)^*_{\rm norm}$ the unique $\psi \in M^*$ such that $\phi(\alpha) = 0 \iff \alpha \leq \psi$. Let $x \in M$ and $\phi = \epsilon_x$. One has $\phi \in (M^*)^*_{\rm norm}$ and the associated $\psi \in M^*$ is such that $\alpha(x) = 0 \iff \alpha \leq \psi$ and thus $\psi = \phi_x$ with $\phi_x^{-1}(0) = [0, x]$ as in Corollary 2.12. This shows that the canonical order preserving bijection $(M^*)^*_{\rm norm} \simeq (M^*)^{\rm op} = \mathrm{Id}(M)$ is the extension by completion of the map $\epsilon : M \to (M^*)^*$ given by evaluation.

(*ii*) follows from (*i*) and the identification of M as the subsemimodule of compact elements in $\mathrm{Id}(M)$. □

Remark 2.14. The additive structure of a semifield F of characteristic one is that of a lattice as a partially ordered set. Moreover, the morphisms of such semifields preserve not only the join $\vee$ but also the meet $\wedge$. Indeed, while for the natural partial order on F given by $x \leq y \iff x + y = y$ the sum gives the $\vee$, the map $x \mapsto x^{-1}$ is an order reversing isomorphism of $F^* = F \setminus \{0\}$ onto itself since one has

$$x \leq y \iff x/y + 1 = 1 \iff 1/y + 1/x = 1/x \iff 1/y \leq 1/x.$$

In fact, one obtains in this way an equivalence of categories between the category of ℓ-groups (*i.e.* lattice ordered abelian groups) and the category of idempotent semifields.

2.4 Decomposition of morphisms in $\mathbb{B}$**mod.** The construction of the Galois connection reviewed in §2.3 suggests that one may associate to a subset $E \subset F$ of an object in $\mathbb{B}$mod a projection $q_E : F \to E$ defined by the formula of [7] Remark 14

$$q_E(\xi) := \wedge\{a \in E \mid a \geq \xi\}. \tag{15}$$

Here E is assumed to be a lattice for the partial order induced by F, and the $\wedge$ is taken with respect to this induced order. For simplicity we restrict to the case of finite objects and obtain the following decomposition of morphisms which plays an important role in §7.

Proposition 2.15. *Let* $f \in Hom_{\mathbb{B}}(F, G)$ *be a morphism of finite objects in* $\mathbb{B}$mod. *Let* $S = \{z \in F \mid f(y) \leq f(z) \Rightarrow y \leq z\}$. *Then*

1. *S is a lattice for the partial order induced by F.*
2. *Formula (15) defines a surjective morphism $q_S \in Hom_{\mathbb{B}}(F, S)$, for S endowed with the operation $\vee$ induced by its order, and the lowest element 0_S.*
3. *The restriction of f to S defines an injective morphism $f|_S \in Hom_{\mathbb{B}}(S, G)$.*
4. *One has $f = f|_S \circ q_S$.*
5. *The inclusion $\iota : S \to F$ preserves the $\wedge$: $\iota(x \wedge_S y) = \iota(x) \wedge \iota(y)$, $\forall x, y \in S$.*

Proof. Since F and G are finite they are lattices and thus there exists a unique monotone map $g : G \to F$ such that $f \dashv g$ is a Galois connection. By construction one has, for $x \in F$ and $y \in G$,

$$f(x) \leq y \iff x \leq g(y). \tag{16}$$

It follows from the general theory on Galois connections that $g(x \wedge y) = g(x) \wedge g(y)$ for all $x, y \in G$ and that $c(x) := g(f(x))$ is a closure operator, *i.e.* c fulfills (using the equality $f = fgf$) the following facts

1. $x \leq y \Rightarrow c(x) \leq c(y)$.
2. $c(c(x)) = c(x)\,, \;\; \forall x \in F$.
3. $x \leq c(x)\,, \;\; \forall x \in F$.

Since (16) with $y = f(z)$ means that $f(x) \leq f(z) \iff x \leq c(z)$, one has $S = \{z \in F \mid f(y) \leq f(z) \iff y \leq z\} = \{z \in F \mid c(z) = z\}$. Moreover, using the above relations one derives

$$c(x) \leq y \iff x \leq y\,, \;\; \forall x \in F, \;\; y \in S. \tag{17}$$

This shows that $c \dashv \iota$ where $\iota : S \to F$ is the inclusion of S endowed with the induced partial order in F. Thus c is the left adjoint of ι, it only depends upon S and is defined by

$$\iota^{-1}([x, \infty)) = [c(x), \infty) \subset S\,, \;\; \forall x \in F.$$

One has $[x \vee y, \infty) = [x, \infty) \cap [y, \infty)$ and thus in the partially ordered set S one gets

$$[c(x \vee y), \infty) = [c(x), \infty) \cap [c(y), \infty).$$

This formula shows that any pair $a, b \in S$ has a least upper bound $a \vee_S b = c(a \vee b) \in S$ and that the map $c : F \to S$ fulfills $c(x \vee y) = c(x) \vee_S c(y)$, $\forall x, y \in F$. Thus since S is finite it is a lattice and moreover $O_S := c(0)$ is the smallest element of S since $\iota^{-1}([0, \infty)) = [c(0), \infty)$. Hence S endowed with the operation $\vee_S$ and the zero element $0_S \in S$ is an object of $\mathbb{B}$mod and $c : F \to S$ is a morphism in $\mathbb{B}$mod. Since $c \circ \iota = \mathrm{Id}$ the morphism c is surjective. By construction and using (15) one has $c = q_S$. Let $h = f|_S$ be the restriction of f to S. We prove that $h \in \mathrm{Hom}_{\mathbb{B}}(S, G)$. First, one has $h(0_S) = f(c(0)) = f(0) = 0$ since $c = gf$ and $fgf = f$. Next, let $a, b \in S$ then

$$h(a \vee_S b) = h(c(a \vee b)) = fgf(a \vee b) = f(a \vee b) = f(a) \vee f(b) = h(a) \vee h(b).$$

Moreover the map $h : S \to G$ is injective since for $x \in S$ one has $x = c(x) = g(f(x))$.

We show 4). One has $q_S = c$ and $c = gf$ so 4) follows from $f = fgf$. We have thus proven that $S \subset F$ fulfills the first 4 conditions. The last one can be derived from the adjunction $c \dashv \iota$. $\square$

The subset $S \subset F$ is called the *support* of $f : F \to G$ and denoted $\mathrm{Support}(f)$. To stress the relation between the operator $q_S : F \to S$ of Proposition 2.15 and an orthogonal projection we state the following

Lemma 2.16. *With the notations of Proposition 2.15, let $\hat{S} \subset F^* = Hom_{\mathbb{B}}(F, \mathbb{B})$ be the sub-semimodule of F^* given by the ϕ_s, $s \in S$. Then one has $\hat{S} = \mathrm{Range}(f^*)$ and*

$$q_S(\xi) = q_S(\eta) \iff \langle \xi, \phi_s \rangle = \langle \eta, \phi_s \rangle\,, \;\; \forall s \in S \iff \langle \xi, u \rangle = \langle \eta, u \rangle\,, \;\; \forall u \in \hat{S} \tag{18}$$

Proof. We show first that $\hat{S} = \mathrm{Range}(f^*)$. Since $c = gf$ and $g = gfg$, S is the range of g and moreover the maps ϕ_s, $s \in S$, fulfill $\phi_{g(y)} = f^*(\phi_y)$ since

$$\phi_{g(y)}(x) = 0 \iff f(x) \leq y \iff f^*(\phi_y)(x) = 0.$$

We now prove (18). By construction $\phi_s(\zeta) = 0 \iff \zeta \leq s$, thus the equality in the middle of (18) means: $\xi \leq s \iff \eta \leq s$ for any $s \in S$. One has $q_S = c$ and thus by (17): $\xi \leq s \iff \eta \leq s$ $\forall s \in S$ if and only if $q_S(\xi) = q_S(\eta)$. □

Remark 2.17. Let F be an object of $\mathbb{B}\mathrm{mod}$, $\sigma \in \mathrm{Aut}_{\mathbb{B}\mathrm{mod}}(F)$ an involution (*i.e.* $\sigma^2 = \mathrm{Id}$) and $E = F^\sigma \subset F$ the fixed subset by σ. Then formula (15) applied to the subset $E \subset F$ defines the projection $p : F \to E$, $p(x) := x + \sigma(x)$. The equality $q_E = p$ shows that p only depends upon the subsemimodule F^σ.

2.5 The radical of an object of $\mathbb{B}\mathrm{mod}$. In this subsection we discuss for future use (*cf.* Theorem 7.10) the condition on an object E of $\mathbb{B}\mathrm{mod}$ equivalent to state that the dual E^* is generated by its minimal non-zero elements. To make clear the analogy with the notion of the radical we shall use in the following the terminology “ideal" in place of “hereditary subsemimodule" as in Definition 2.2. For an object E of $\mathbb{B}\mathrm{mod}$ the correspondence between ideals $J \subset E$ and elements $\phi \in E^*$ given by: $J = \phi^{-1}(\{0\})$ & $\phi(x) = 0 \Leftrightarrow x \in J$, for $J \subset E$ hereditary subsemimodule (*cf.* Proposition 2.3) fulfills the rule: $J \subset J' \Leftrightarrow \phi \geq \phi'$. Thus minimal non-zero elements of E^* correspond to maximal ideals of E, where E itself is not counted as an ideal by convention. We can thus reformulate, when E is finite, the condition: "E^* is generated by its minimal non-zero elements", in terms of ideals of E as follows

Every ideal $J \subset E$ is the intersection of maximal ideals containing J.

Indeed, the above statement means exactly that any element of E^* is a supremum of minimal non-zero elements. This condition makes sense in general, without any finiteness hypothesis and in fact it also suggests to define, for any object E of $\mathbb{B}\mathrm{mod}$, the following congruence relation on E.

Definition 2.18. *The radical* $\mathrm{Rad}(E)$ *of an object E of* $\mathbb{B}\mathrm{mod}$ *is defined as the following congruence*

$$x \sim_{\mathrm{Rad}(E)} y \iff (\forall \text{ maximal ideal } J \subset E,\ x \in J \iff y \in J).$$

An equivalent formulation of this definition can be given in terms of the quotients E/J, for maximal ideals $J \subset E$. Here, the quotient E/J is defined as the $\mathbb{B}$-semimodule of equivalence classes for the relation: $u \sim v \iff \exists i, j \in J \mid u + i = v + j$. One checks that this is an equivalence relation compatible with $+$ and that $u \sim v$ holds iff $f(u) = f(v)$, for any morphism f with $J \subset f^{-1}(\{0\})$. The following lemmas describe properties of $\mathrm{Rad}(E)$.

Lemma 2.19. *The radical* $\mathrm{Rad}(E)$ *is the same as the relation stating that $x, y \in E$ have the same image in E/J, for all maximal ideals $J \subset E$.*

Proof. To prove the statement one shows that $E/J = \mathbb{B}$ for any maximal ideal $J \subset E$. Let $J \subset E$ be a maximal ideal, then any proper ideal in $F = E/J$ is reduced to $\{0\}$ since its inverse image by the quotient map is an ideal of E containing J. It follows that $F = \mathbb{B}$ since all intervals $[0, \xi] \neq F$ are reduced to $\{0\}$. Moreover the class of an element $u \in E$ is $0 \in E/J = \mathbb{B}$ iff $u + i = j \in J$ for some $i \in J$ and this implies $u \in J$. Thus the image of u in E/J is entirely determined as 0 if $u \in J$ and 1 if $u \notin J$. This provides the equivalence with Definition 2.18. □

Lemma 2.20. *Let E be an object of* $\mathbb{B}$mod. *An ideal $J \subset E$ is maximal if and only if the quotient $E/J \simeq \mathbb{B}$.*

Proof. The proof of the previous lemma shows that if J is maximal then the quotient $E/J \simeq \mathbb{B}$. Conversely, assume that the quotient E/J is $\mathbb{B}$. Let $J' \supset J$ be an ideal containing J with $J' \neq E$. Let $u \sim_J v$ be the equivalence relation defined as: $u \sim_J v \iff \exists i, j \in J \mid u + i = v + j$. One has $u \sim_J v \Rightarrow u \sim_{J'} v$ and thus one gets an induced surjective map $s : \mathbb{B} = E/J \to E/J'$. The class of any $x \notin J'$ in E/J' is non zero since $u \sim_{J'} 0 \Rightarrow u \in J'$. Thus E/J' contains two distinct elements, s is also injective and $J = J'$. $\square$

Lemma 2.21. *Let E be an object of* $\mathbb{B}$mod. *Then the congruence* $\mathrm{Rad}(E)$ *is trivial (i.e. $x \sim_{\mathrm{Rad}(E)} y \Rightarrow x = y$) if and only if every principal ideal $J \subset E$ is the intersection of maximal ideals containing J.*

Proof. Assume that every principal ideal $J \subset E$ is the intersection of maximal ideals containing J. Apply this for $x \in E$ to the ideal $J_x = [0, x] \subset E$. Since this ideal uniquely determines x (as its largest element) and since for any maximal ideal $K \subset E$ one has $J_x \subset K \iff x \in K$, it follows that the congruence $\mathrm{Rad}(E)$ is trivial *i.e.* that $x \sim y \Rightarrow x = y$. Conversely, assume that the congruence $\mathrm{Rad}(E)$ is trivial. Let $J_x = [0, x] \subset E$ be a principal ideal. Let $J \subset E$ be the intersection of all maximal ideals containing J_x. One has $J_x \subset J$. Let $y \in J$: we show that $x + y \sim_{\mathrm{Rad}(E)} x$. For any maximal ideal K one has: $x \in K \Rightarrow x + y \in K \Rightarrow x \in K$. Thus since the congruence $\mathrm{Rad}(E)$ is trivial one gets $x + y = x$, *i.e.* $y \leq x$ so that $y \in J_x$. $\square$

Let E be an object of $\mathbb{B}$mod. We let $\kappa : E^* \to \mathrm{Id}(E)$ be the bijection which associates to $\phi \in E^*$ the ideal $\phi^{-1}(\{0\}) = J$.

Lemma 2.22. *Let E be an object of* $\mathbb{B}$mod *and* $\mathrm{Max}(E)$ *the set of maximal ideals of E.*

(i) The map κ induces a bijection of the set of minimal non-zero elements $M \subset E^*$ with $\mathrm{Max}(E) \subset Id(E)$.

(ii) For $\phi \in E^*$, the following conditions are equivalent

1. *ϕ belongs to the complete subsemimodule of E^* generated by the minimal elements.*
2. *$J = \kappa(\phi)$ is an intersection of maximal ideals.*

Proof. The equivalence follows from the fact that the bijection $\kappa : E^* \to \mathrm{Id}(E)$ transforms the operation $\vee$ on ϕ into the intersection of the ideals J. $\square$

We now replace the condition that the dual E^* is generated by its minimal non-zero elements by the weaker condition given by the triviality of the congruence $\mathrm{Rad}(E)$. In view of Lemma 2.21 the triviality of $\mathrm{Rad}(E)$ is analogous to the statement of the vanishing of the radical in ordinary algebra.

Proposition 2.23. *Let E be a finite object of* $\mathbb{B}$mod. *Then*

(i) The dual of the quotient $E/\mathrm{Rad}(E)$ *is the subsemimodule $S \subset E^*$ generated by the minimal non-zero elements.*

(ii) The quotient $E/\mathrm{Rad}(E)$ *has a trivial radical congruence.*

(iii) For $\phi \in E^*$, $J = \kappa(\phi)$ is an intersection of maximal ideals if and only if $x \sim_{\mathrm{Rad}(E)} y \Rightarrow \phi(x) = \phi(y)$.

Proof. (i) Let S be the subsemimodule of E^* generated by the minimal non-zero elements $M \subset E^*$ and $j : S \to E^*$ the inclusion. The morphism $j^* : E = (E^*)^* \to S^*$ is surjective. Moreover $j^*(x) = j^*(y) \iff x \sim_{\mathrm{Rad}(E)} y$ since M generates S. Thus, one has $E/\mathrm{Rad}(E) = S^*$ and $(E/\mathrm{Rad}(E))^* = (S^*)^* = S$.

(ii) Since $(E/\mathrm{Rad}(E))^* = S$ is generated by its minimal non-zero elements, $E/\mathrm{Rad}(E)$ has a trivial radical congruence.

(iii) This follows from (i) and Lemma 2.22. □

Proposition 2.24. *Let E, F be finite objects of $\mathbb{B}$mod and $f \in Hom_{\mathbb{B}}(E, F)$. Then one has $x \sim_{\mathrm{Rad}(E)} y \Rightarrow f(x) \sim_{\mathrm{Rad}(F)} f(y)$ if and only if for every maximal ideal J of F, the ideal $f^{-1}(J)$ is an intersection of maximal ideals.*

Proof. Let us first assume that $x \sim_{\mathrm{Rad}(E)} y \Rightarrow f(x) \sim_{\mathrm{Rad}(F)} f(y)$. Let J be a maximal ideal in F and $\phi \in F^*$, $\phi^{-1}(\{0\}) = J$. Then by Proposition 2.23 one has $x \sim_{\mathrm{Rad}(F)} y \Rightarrow \phi(x) = \phi(y)$, and thus $u \sim_{\mathrm{Rad}(E)} v \Rightarrow \phi(f(u)) = \phi(f(v))$. Thus the ideal $f^{-1}(J)$ is an intersection of maximal ideals. Conversely, if this fact holds for any maximal ideal $J \subset F$, then if $x \sim_{\mathrm{Rad}(E)} y$ one has $x \in f^{-1}(J) \iff y \in f^{-1}(J)$ or equivalently $f(x) \in J \iff f(y) \in J$, *i.e.* $f(x) \sim_{\mathrm{Rad}(F)} f(y)$. □

Next we provide the simplest example of a morphism which does not fulfill the condition of Proposition 2.24. Example 2.25 shows also that in general a morphism f in $\mathbb{B}$mod does not induce a morphism on the quotients by the radical congruence.

Example 2.25. Let $N = \{0, m, n\}$ with $0 < m < n$, and $M = \{0, m, x, n\}$. The idempotent addition is defined as follows: $n \vee m = n$, $x \vee m = n$, $x \vee n = n$. Let $f \in \mathrm{Hom}_{\mathbb{B}}(N, M)$ be the natural inclusion. One sees that in N, $\{0, m\}$ is the only maximal ideal and thus $m \sim_{\mathrm{Rad}(N)} 0$. On the other hand, in M the maximal ideal $\{0, x\}$ contains 0 but not m so that $m \nsim_{\mathrm{Rad}(M)} 0$.

3. The category $\mathbb{B}$mod of $\mathbb{B}$-semimodules and the comonad $\perp$

The category $\mathbb{B}$mod is a symmetric, closed monoidal category for the tensor product of $\mathbb{B}$-semimodules. The object $\{0\}$ is both initial and final. Thus for any pair of objects M, N in $\mathbb{B}$mod there is a natural morphism $\gamma_{M,N} : M \amalg N \to M \times N$ from the coproduct of M and N to their product. Indeed, by construction of $M \amalg N$, a morphism $f : M \amalg N \to P$ is a pair of morphisms $M \to P$, $N \to P$, then for $P = M \times N$ one takes the morphisms $(\mathrm{Id}, 0) : M \to P$ and $(0, \mathrm{Id}) : N \to P$. We recall the proof of the following well known Lemma ([21])

Lemma 3.1. *In the category of $\mathbb{B}$-semimodules the morphisms $\gamma_{M,N}$ are isomorphisms.*

Proof. Let M and N be two $\mathbb{B}$-semimodules. By definition $M \amalg N$ is the initial object for pairs of morphisms $\alpha : M \to X$, $\beta : N \to X$, where X is any $\mathbb{B}$-semimodule. Let $P = M \times N$ endowed with the operation

$$(x, y) + (x', y') = (x + x', y + y'), \ \ \forall x, x' \in M, \forall y, y' \in N.$$

P is a $\mathbb{B}$-semimodule with zero element $(0, 0)$. Moreover one has canonical morphisms $s : M \to P$, $s(x) = (x, 0)$ and $t : N \to P$, $t(y) = (0, y)$. Given a pair of morphisms $\alpha : M \to X$ and $\beta : N \to X$, there exists a unique morphism $\rho : M \times N \to X$, $\rho(x, y) = \alpha(x) + \beta(y)$ such that $\rho \circ s = \alpha$ and $\rho \circ t = \beta$. This proves that $(M \times N, s, t)$ is also an initial object for pairs of morphisms α,β as above. Given a pair of morphisms $\phi : X \to M$ and $\psi : X \to N$, the map $a \mapsto (\phi(a),\psi(a))$ defines a morphism $\sigma : X \to P$ uniquely characterized by $\phi = p_1\circ\sigma$ and $\psi = p_2\circ\sigma$, where the p_j are the projections. This shows that $(M\times N, p_1, p_2)$ is the product in the category of $\mathbb{B}$-semimodules. The morphism $\gamma_{M,N} : M \amalg N \to M\times N$ is defined by implementing the pair of morphisms $\alpha = (\mathrm{Id}, 0) : M \to P$ and $\beta = (0,\mathrm{Id}) : N \to P$. The corresponding morphism ρ is the identity map and this shows that the coproduct $M \amalg N$ is isomorphic to the product by means of the morphism $\gamma_{M,N}$. $\square$

In view of Lemma 3.1 we shall denote by $M\oplus N$ the $\mathbb{B}$-semimodule $M\amalg N \simeq M\times N$. Given two morphisms $\alpha,\beta : L \rightrightarrows M$ of $\mathbb{B}$-semimodules, their equalizer is given by the subsemimodule $\mathrm{Equ}(\alpha,\beta) = \{x\in L \mid \alpha(x) = \beta(x)\}$. We now describe the coequalizer.

3.1 Coequalizer of two morphisms of $\mathbb{B}$-semimodules. We give the explicit construction of the coequalizer in the category of $\mathbb{B}$-semimodules. By definition, a congruence on a $\mathbb{B}$mod M is a subsemimodule of $M\times M$ characterized as the graph $\mathcal{C}$ of an equivalence relation. Thus $\mathcal{C}\subset M\times M$ fulfills the following conditions

1. $(f_i, g_i)\in\mathcal{C}, i = 1,2 \implies (f_1+f_2, g_1+g_2)\in\mathcal{C}$
2. $(f,f)\in\mathcal{C}$, $\forall f\in M$
3. $(f,g)\in\mathcal{C} \implies (g,f)\in\mathcal{C}$
4. $(f,g)\in\mathcal{C},\ (g,h)\in\mathcal{C} \implies (f,h)\in\mathcal{C}$.

These conditions ensure that the quotient $M/\mathcal{C}$ is a $\mathbb{B}$-semimodule. We use this notion to construct the coequalizer of two morphisms $\alpha,\beta : L\rightrightarrows M$ of $\mathbb{B}$-semimodules. We let $\mathcal{C}$ be the intersection of all subsemimodules of $M\times M$ which fulfill the above four conditions and also contain $\{(\alpha(x),\beta(x)) \mid x\in L\}$. Let $E = M/\mathcal{C}$ and $\rho : M\to E$ be the quotient map. By construction one has $\rho\circ\alpha = \rho\circ\beta$.

Lemma 3.2. *(i) The pair (E,ρ) is the coequalizer of the morphisms $\alpha,\beta : L\rightrightarrows M$:* $L\overset{\alpha}{\underset{\beta}{\rightrightarrows}}M \xrightarrow{\rho} E$.

(ii) The coequalizer of the morphisms $\alpha,\beta : L\rightrightarrows M$ is the quotient of M by the equivalence relation

$$x\sim x' \iff h(x) = h(x')\,,\ \ \forall X,\, h : M\to X \mid h\circ\alpha = h\circ\beta \tag{19}$$

Proof. (i) We have seen that $\rho\circ\alpha = \rho\circ\beta$. To test the universality, we let $\phi : M\to X$ be a morphism of $\mathbb{B}$-semimodules such that $\phi\circ\alpha = \phi\circ\beta$. Let then $\mathcal{C}_\phi := \{(x,y)\in M\times M \mid \phi(x) = \phi(y)\}$. One gets $\mathcal{C}\subset\mathcal{C}_\phi$ since by construction $\mathcal{C}_\phi$ fulfills the above four conditions and also contains $\{(\alpha(x),\beta(x)) \mid x\in S\}$. Thus one sees that $\phi(x)$ only depends on the image $\rho(x)\in E$ and moreover the map factors through ρ as required by the universal property of the coequalizer.

(ii) This follows from (i) since any $h : M\to X \mid h\circ\alpha = h\circ\beta$ factors through ρ. $\square$

Remark 3.3. In some contexts (see eg [26] Definition 3.1) the following equivalence relation on M is introduced in the presence of two morphisms $\alpha,\beta : L\rightrightarrows M$ of $\mathbb{B}$-semimodules

$$x,y\in M \quad x\sim y \iff \exists u,v\in L \ \text{ s.t. } \ x+\alpha(u)+\beta(v) = y+\alpha(v)+\beta(u).$$

This is an additive congruence which coequalizes α and β, however next example shows that in general this equivalence relation is not the coequalizer of α,β. Let $\alpha = \beta = \mathrm{Id} : M\to M$. Then the coequalizer as in Lemma 3.2 is the identity map $\mathrm{Id} : M\to M$ whereas the above equivalence relation reads as: $x\sim y \iff \exists a\in M$ s.t. $x+a = y+a$. In characteristic 1 (*i.e.* for idempotent structures) this gives $x \sim 0\ \forall x \in M$. Thus the above congruence does not provide in general the coequalizer of two maps.

3.2 Kernel and coimage of morphisms of $\mathbb{B}$-semimodules. The category $\mathbb{B}$mod is a particular example for the general category RSmd of semimodules over a unital semiring R considered in [18] §1.6.2. In $\mathbb{B}$mod, the naive notion of kernel given, for a morphism $f \in \mathrm{Hom}_{\mathbb{B}}(M,N)$, by $f^{-1}(\{0\})$ (*i.e.* the equalizer of f and the 0-morphism) is not adequate. In fact, using this notion one can find plenty of examples of morphisms with trivial kernels but failing to be monomorphisms. In fact one has in general

Lemma 3.4. *Let M be an object of* $\mathbb{B}$mod *and $f \in Hom_{\mathbb{B}}(M,\mathbb{B})$ be defined by $f(x) = 1$, $\forall x \neq 0$. Then $f^{-1}(\{0\}) = \{0\}$ and f is not a monomorphism unless $M = \mathbb{B}$.*

Proof. It is enough to show that f is additive and this follows from $x + y = 0 \Rightarrow x = y = 0$. ☐

In order to deal with this problem of the inadequacy of the naive notion of kernel, we use the classical technique of replacing kernels with kernel pairs and cokernels with cokernel pairs.

For a morphism $f \in \mathrm{Hom}_{\mathbb{B}}(M,N)$, our goal is to construct an exact sequence which replaces, in this context, the sequence

$$0 \to \mathrm{Ker} f \to M \xrightarrow{f} N \to \mathrm{Coker} f \to 0 \tag{20}$$

holding in an abelian category. To this end, we introduce the extended notion of morphism obtained by considering pairs (f,g), with $f,g \in \mathrm{Hom}_{\mathbb{B}}(M,N)$ which compose as follows:

$$(f,g) \circ (f',g') := (f \circ f' + g \circ g', f \circ g' + g \circ f'). \tag{21}$$

This law makes sense since addition of morphisms makes sense in $\mathbb{B}$mod. This set-up determines the new category $\mathbb{B}\mathrm{mod}^2$ that will be considered in details in §4. Any ordinary morphism $f \in \mathrm{Hom}_{\mathbb{B}}(M,N)$ is therefore seen as the pair $(f,0)$ of $\mathbb{B}\mathrm{mod}^2$. Moreover, for pairs of morphisms one introduces the following definition

Definition 3.5. *Let $f,g \in Hom_{\mathbb{B}}(M,N)$ then*

1. $\mathrm{Equ}(f,g)$ *is the equalizer of* (f,g)*:* $\mathrm{Equ}(f,g) \to M \underset{g}{\overset{f}{\rightrightarrows}} N$.
2. $\mathrm{Coequ}(f,g)$ *is the coequalizer of* (f,g)*:* $M \underset{g}{\overset{f}{\rightrightarrows}} N \to \mathrm{Coequ}(f,g)$.

Diagonal pairs such as (f,f) play a special role since $\forall f \in \mathrm{Hom}_{\mathbb{B}}(M,N)$ one has: $\mathrm{Equ}(f,f) = (M, \mathrm{Id}_M)$, and $\mathrm{Coequ}(f,f) = (N, \mathrm{Id}_N)$. Thus the result is independent of the choice of f and the outcome is similar to what happens for the morphism $0 \in \mathrm{Hom}_{\mathbb{B}}(M,N)$. It would however be too naive to simplify by such pairs and the outcome would correspond to the over-simplification described in Remark 3.3. To properly define the replacement for $\mathrm{Ker} f$ of (20) we introduce the pair

$$f^{(2)} := (f \circ p_1, f \circ p_2), \quad f \circ p_j \in \mathrm{Hom}_{\mathbb{B}}(M^2, N),\ j = 1,2$$

where the $p_j : M^2 \to M$ are the two canonical projections. By definition $\mathrm{Equ} f^{(2)}$ is the equalizer of $(f \circ p_1, f \circ p_2)$: $\mathrm{Equ} f^{(2)} \to M^2 \underset{f \circ p_2}{\overset{f \circ p_1}{\rightrightarrows}} N$, hence it is, by construction, a subobject of M^2 (By

Proposition 2.7 any monomorphism in $\mathbb{B}$mod is an injective map). This determines two maps $\mathrm{Equ}f^{(2)} \underset{\iota_2}{\overset{\iota_1}{\rightrightarrows}} M$ by composition with the p_j's. Thus one obtains:

$$\mathrm{Equ}f^{(2)} \underset{\iota_2}{\overset{\iota_1}{\rightrightarrows}} M \xrightarrow{f} N \tag{22}$$

where by construction $f \circ \iota_1 = f \circ \iota_2$. Next, recall that in an abelian category one defines the coimage $\mathrm{Coim}f$ of a morphism f as the cokernel of its kernel. In our context we use the kernel pair.

Definition 3.6. *Let $f \in Hom_{\mathbb{B}}(M, N)$ then*

1. $\mathrm{Ker}_p\, f = \mathrm{Equ}f^{(2)} \subset M^2$ *(as in (22)).*
2. $\mathrm{Coim}f = \mathrm{Coequ}(\iota_1, \iota_2) = M/\sim$ *(as in (19)).*

It follows from Lemma 3.2 that $\mathrm{Coim}f$ is the quotient of M by the equivalence relation fulfilling the 4 conditions of that lemma and containing the pairs $(\iota_1(x, y), \iota_2(x, y))$. These pairs are characterized by the equality $f(x) = f(y)$ and thus one derives

Lemma 3.7. *Let $f \in Hom_{\mathbb{B}}(M, N)$, then $\mathrm{Coim}f$ is the quotient of M by the congruence*

$$x \sim y \iff f(x) = f(y).$$

As a corollary one obtains that $\mathrm{Coim}f$ is isomorphic to the naive notion of image of f, *i.e.* as the subsemimodule of N given by

$$\mathrm{Range}\ (f) := \{f(x) \mid x \in M\} \subset N. \tag{23}$$

3.3 Cokernel and image of morphisms of $\mathbb{B}$-semimodules. Next, we investigate the cokernel and the image of a morphism in $\mathbb{B}$mod. Likewise for the kernel, we associate to $f \in \mathrm{Hom}_{\mathbb{B}}(M, N)$ the pair

$$f_{(2)} := (s_1 \circ f, s_2 \circ f), \quad s_j \circ f \in \mathrm{Hom}_{\mathbb{B}}(M, N \oplus N), j = 1, 2$$

where the $s_j : N \to N \oplus N$ are the two canonical inclusions. By definition, $\mathrm{Coequ}f_{(2)}$, which plays the role of the cokernel, is the coequalizer of $(s_1 \circ f, s_2 \circ f)$ and is hence a quotient of $N \oplus N$. This provides two maps $\gamma_j : N \to \mathrm{Coequ}f_{(2)}$ by composition with s_j. Thus one obtains:

$$M \xrightarrow{f} N \underset{\gamma_2}{\overset{\gamma_1}{\rightrightarrows}} \mathrm{Coequ}f_{(2)} \tag{24}$$

where by construction $\gamma_1 \circ f = \gamma_2 \circ f$ and the coequalizer is the universal solution of this equation. This equation only involves the naive notion of image of f *i.e.* the subsemimodule of N given by (23) since the equality $\alpha_1 \circ f = \alpha_2 \circ f$ is equivalent to $\alpha_1 \circ \iota = \alpha_2 \circ \iota$, where $\iota : \mathrm{Range}\ (f) \to N$ is the inclusion.

As in universal algebra we define the cokernel pair $\mathrm{Coker}_p\, f$ of the morphism $f \in \mathrm{Hom}_{\mathbb{B}}(M, N)$ as the $\mathbb{B}$-semimodule $\mathrm{Coequ}f_{(2)}$ together with the morphism $(\gamma_1, \gamma_2) : N \oplus N \to \mathrm{Coequ}f_{(2)}$, and the categorical notion of image is defined as follows

Definition 3.8. *Let $f \in Hom_{\mathbb{B}}(M, N)$ then*

1. $\mathrm{Coker}_p\, f = (\mathrm{Coequ}f_{(2)}) = (N \oplus N)/\sim$ *(as in (24)).*
2. $\mathrm{Im}(f) = \mathrm{Equ}(\gamma_1, \gamma_2) \subset N$*.*

Thus we derive the sequence (analogue of (20))

$$\mathrm{Ker}_p\, f \underset{\iota_2}{\overset{\iota_1}{\rightrightarrows}} M \xrightarrow{f} N \underset{\gamma_2}{\overset{\gamma_1}{\rightrightarrows}} \mathrm{Coker}_p\, f.$$

Next, we shall compare the naive image of a morphism as defined in (23) with the categorical notion given in Definition 3.8. Both notions depend only upon the subsemimodule Range $(f) \subset N$ and by construction one has Range $(f) \subset \mathrm{Im}(f)$.

Proposition 3.9. *Let N be a $\mathbb{B}$-semimodule, $E \subset N$ a subsemimodule and let $\iota : E \to N$ the inclusion. Then*

(*i*) $\mathrm{Im}(\iota) = \tilde{E}$, *where $\tilde{E} \subset N$ is defined using arbitrary $\mathbb{B}$-semimodules as follows*

$$x \in \tilde{E} \iff \phi(x) = \psi(x)\,,\ \ \forall X,\ \phi, \psi \in Hom_{\mathbb{B}}(N, X) \text{ s.t. } \phi \circ \iota = \psi \circ \iota. \tag{25}$$

(*ii*) *One has $\tilde{E} = E$.*

Proof. (*i*) Note that formula (25) defines the kernel (*i.e.* the equalizer) of the cokernel pair of the inclusion $\iota : E \to N$. Indeed, one lets $s_j : N \to N \oplus N$ be the canonical inclusions $(\mathrm{Id}, 0)$ and $(0, \mathrm{Id})$ and $\rho : N \oplus N \to N$ the coequalizer of $(s_1 \circ \iota, s_2 \circ \iota)$. Then one has in general

$$x \in \tilde{E} \iff \rho \circ s_1(x) = \rho \circ s_2(x). \tag{26}$$

(*ii*) follows from Lemma 2.6. ☐

As a bi-product, one derives for $\mathbb{B}$mod the key property AB2 holding for abelian categories

Proposition 3.10. *For a morphism $f \in Hom_{\mathbb{B}}(M, N)$, the natural map from* Coim$f$ *to* $\mathrm{Im}(f)$ *is an isomorphism.*

Proof. By Lemma 3.7 the coimage of f is the quotient of M by the congruence $f(x) = f(y)$. Thus the coimage of f is isomorphic to the subsemimodule $E =$ Range $(f) \subset N$. The image of f as in Definition 3.8 is $\tilde{E}$ which is equal to E by Proposition 3.9. ☐

3.4 The comonad $\perp$ and its Eilenberg-Moore and Kleisli categories.

In the above discussion of kernels and cokernels of morphisms of $\mathbb{B}$mod we made repeated use of the operation which replaces an object M of $\mathbb{B}$mod by its square M^2. This operation is an endofunctor and leads one to consider the following comonad (or cotriple).

Proposition 3.11. *The following rules define a comonad:*

1. *The endofunctor* $\perp\colon \mathbb{B}\mathrm{mod} \longrightarrow \mathbb{B}\mathrm{mod}$, $\perp M = M^2$, $(\perp f) := (f, f)$.
2. *The counit* $\epsilon : \perp \to 1_{\mathbb{B}\mathrm{mod}}$, $\epsilon_M = p_1$, $p_1 : M^2 \to M$, $p_1(x, y) = x$.
3. *The coproduct* $\delta : \perp \to \perp \circ \perp$, $\delta_M = (M^2 \to (M^2)^2)$, $(x, y) \mapsto (x, y, y, x)$.

Proof. We first check the co-associativity, *i.e.* the commutativity of the following diagram:

$$\begin{tikzcd}
\perp M \arrow[r, "\delta_M"] \arrow[d, "\delta_M"'] & \perp(\perp M) \arrow[d, "\delta_{\perp M}"] \\
\perp(\perp M) \arrow[r, "\perp(\delta_M)"] & \perp\perp\perp M
\end{tikzcd}$$

One has $\delta_{\perp M}\delta_M((x,y)) = \delta_{\perp M}(((x,y),(y,x))) = (((x,y),(y,x)),((y,x),(x,y)))$ and $\perp(\delta_M)\delta_M((x,y)) = (\delta_M((x,y)),\delta_M((y,x))) = ((x,y,y,x),(y,x,x,y))$ so one gets the required equality. We now check the defining property of the counit, *i.e.* the commutativity

$$\begin{tikzcd} \perp M \arrow[r, "\delta_M"] \arrow[d, "\delta_M"'] \arrow[dr, "\mathrm{Id}"] & \perp(\perp M) \arrow[d, "\epsilon_{\perp M}"] \\ \perp(\perp M) \arrow[r, "\perp(\epsilon_M)"] & \perp M \end{tikzcd}$$

One has $\epsilon_{\perp M}\delta_M((x,y)) = \epsilon_{\perp M}(((x,y),(y,x))) = (x,y)$ and similarly $\perp(\epsilon_M)\delta_M((x,y)) = (\epsilon_M((x,y)),\epsilon_M((y,x))) = (x,y)$ so one gets the required equality. □

A comonad gives rise to two categories, its Kleisli category and its Eilenberg-Moore category ([3] §3.2) which we now determine for the comonad $\perp$.

Proposition 3.12. (*i*) *The Kleisli category of the comonad* $\perp$ *is the category* $\mathbb{B}\mathrm{mod}^2$ *whose objects are* $\mathbb{B}$*-semimodules and morphisms are pairs of morphisms in* $\mathbb{B}\mathrm{mod}$ *with composition assigned by the formula*

$$(f,g)\circ(f',g') := (f\circ f' + g\circ g', f\circ g' + g\circ f'). \tag{27}$$

(*ii*) *The Eilenberg-Moore category of the comonad* $\perp$ *is the category* $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ *of* $\mathbb{B}$*-semimodules endowed with an involution* σ.

Proof. (*i*) By construction of the Kleisli category $\mathcal{K}_\perp$ its objects are the objects of $\mathbb{B}\mathrm{mod}$. The morphisms $M \to N$ in $\mathcal{K}_\perp$ are given by $\mathrm{Hom}_{\mathbb{B}}(\perp M, N)$, *i.e.* , since $M^2 = M \oplus M$, by pairs (f,g) of morphisms $f,g \in \mathrm{Hom}_{\mathbb{B}}(M,N)$. The composition $(f',g')\circ(f,g)$ is given by:

$$\perp M \stackrel{\delta_M}{\to} \perp\perp M \stackrel{\perp((f,g))}{\to} \perp N \stackrel{(f',g')}{\to} P.$$

One has $\delta_M((x,y)) = (x,y,y,x)$ and $\perp(f,g)(\delta_M((x,y))) = (f(x)+g(y), f(y)+g(x)) \in \perp N$. By applying (f',g') one obtains

$$(f',g')\circ \perp(f,g)(\delta_M((x,y))) = f'(f(x)+g(y)) + g'(f(y)+g(x))$$

which coincides with $(f'\circ f + g'\circ g)(x) + (f'\circ g + g'\circ f)(y)$.

(*ii*) By construction the Eilenberg-Moore category of the comonad $\perp$ is the category of coalgebras for this comonad. A coalgebra in this context is given by an object M of $\mathbb{B}\mathrm{mod}$ and a morphism $\alpha : M \to \perp M$ that makes the following diagrams commutative

$$\begin{tikzcd} & M \arrow[dl, "\alpha"'] \arrow[dr, "\mathrm{Id}_M"] & \\ \perp M \arrow[rr, "\epsilon_M"] & & M \end{tikzcd} \tag{28}$$

$$\begin{tikzcd} M \arrow[r, "\alpha"] \arrow[d, "\alpha"'] & \perp M \arrow[d, "\perp(\alpha)"] \\ \perp M \arrow[r, "\delta_M"] & \perp\perp M \end{tikzcd} \tag{29}$$

The commutativity of the diagram (28) means that $\alpha(x) = (x, \sigma(x))$ for some morphism $\sigma \in \mathrm{End}_{\mathbb{B}}(M)$. In the diagram (29) one gets $\delta_M(\alpha(x)) = (x, \sigma(x), \sigma(x), x)$ while $\perp \alpha(\alpha(x)) = (x, \sigma(x), \sigma(x), \sigma^2(x))$. Thus the commutativity of the diagram (29) means that $\sigma^2 = \mathrm{Id}$. □

Proposition 3.12 gives the conceptual meaning of the categories $\mathbb{B}\mathrm{mod}^2$ and $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ which are studied in §§4,5. A number of their properties are corollaries of general properties of Kleisli and Eilenberg-Moore categories and for instance Lemma 5.2 (ii) which identifies $\mathbb{B}\mathrm{mod}^2$ as a full subcategory of $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ is a special case of Proposition 13.2.11 of [2].

There is a natural notion of a projective object in a category $\mathcal{C}$ endowed with a comonad $\perp$: see *e.g.* [28] Definition 8.6.5.

Definition 3.13. *An object P of $\mathcal{C}$ is $\perp$-projective if the counit map $\epsilon_P : \perp P \to P$ has a section, i.e. there is a map $s : P \to \perp P$ such that $\epsilon_P \circ s = Id_P$.*

In our setup we have

Lemma 3.14. *Any object P of $\mathbb{B}\mathrm{mod}$ is $\perp$-projective for the monad $\perp = I \circ \mathfrak{s}$.*

Proof. Let $\iota_P : P \to \perp P$ be given by $\iota_P(x) = (x, 0)$. Then one has $\epsilon_P \circ \iota_P = \mathrm{Id}$. In fact the section ι_P can be characterized as the smallest one in the sense that for any other section $s : P \to \perp P$ one has $s + \iota_P = s$. □

4. The Kleisli category $\mathbb{B}\mathrm{mod}^2$

In this section we study the Kleisli category $\mathbb{B}\mathrm{mod}^2$ of Proposition 3.12 whose introduction can be justified independently as follows. The lack of the additive inverse for morphisms of the category $\mathbb{B}\mathrm{mod}$ leads one to consider formal pairs (f, g) of morphisms in $\mathbb{B}\mathrm{mod}$ as a substitute for $f - g$. More precisely one introduces (see Proposition 3.12 (i))

Definition 4.1. *We denote by $\mathbb{B}\mathrm{mod}^2$ the category whose objects are $\mathbb{B}$-semimodules and morphisms are pairs of morphisms in $\mathbb{B}\mathrm{mod}$ with composition assigned by the formula*

$$(f, g) \circ (f', g') := (f \circ f' + g \circ g', f \circ g' + g \circ f'). \tag{30}$$

By construction $\mathbb{B}\mathrm{mod}^2$ is enriched over the category $\mathbb{B}\mathrm{mod}$. In this section (*cf.*§4.1) we study a provisional notion of strict exactness of sequences in $\mathbb{B}\mathrm{mod}^2$. This definition will be refined later on, in §6. In §§4.2, 4.3, 4.4, we investigate the link of strict exactness respectively with monomorphisms, isomorphisms and epimorphisms. Finally in §4.5 we consider the issue of defining quotients in $\mathbb{B}\mathrm{mod}^2$.

4.1 Strictly exact sequences of $\mathbb{B}$-semimodules.

The category $\mathbb{B}\mathrm{mod}$ embeds as a subcategory of $\mathbb{B}\mathrm{mod}^2$ by applying the functor $\kappa : \mathbb{B}\mathrm{mod} \longrightarrow \mathbb{B}\mathrm{mod}^2$ which is defined as the identity on objects while on morphisms one sets

$$\kappa(f) := (f, 0) \in \mathrm{Hom}_{\mathbb{B}\mathrm{mod}^2}(M, N), \ \ \forall f \in \mathrm{Hom}_{\mathbb{B}}(M, N). \tag{31}$$

Definition 3.6 gives the kernel in the case of pairs of the form $(f, 0)$ thus the next step is to extend the construction of the kernel to an arbitrary pair (f, g) of morphisms $M \to N$ in

$\mathbb{B}$mod. The main idea is to re-interpret a congruence relation involving differences of the form $f(x)-g(x)=f(y)-g(y)$ as follows

$$f(x)-g(x)=f(y)-g(y) \iff f(x)+g(y)=f(y)+g(x).$$

This means that in the product $M\times M$ one looks for all pairs (x,y) such that $f(x)+g(y)=f(y)+g(x)$.

Proposition 4.2. *Let $f,g\in Hom_{\mathbb{B}}(M,N)$.*

(i) $Z(f,g):=\{(x,y)\in M\times M \mid f(x)+g(y)=f(y)+g(x)\}$ is a subsemimodule of $M\times M$ that we call the algebraic kernel of the pair (f,g).

(ii) Let $\iota_j: Z(f,g)\to M$ be the restrictions of the canonical projections $p_j: M^2\to M$, $j=1,2$. The sequence

$$Z(f,g) \underset{\iota_2}{\overset{\iota_1}{\rightrightarrows}} M \underset{g}{\overset{f}{\rightrightarrows}} N \tag{32}$$

satisfies the relation: $f\circ\iota_1+g\circ\iota_2=f\circ\iota_2+g\circ\iota_1$.

(iii) Let $\alpha,\beta\in Hom_{\mathbb{B}}(L,M)$ be such that $f\circ\alpha+g\circ\beta=f\circ\beta+g\circ\alpha$. Then the range of the map $(\alpha,\beta):L\to M^2$ is included in $Z(f,g)\subset M^2$.

(iv) $Z(f,0)=\mathrm{Ker}_p\, f$ (cf. Definition 3.6).

Proof. (i) The pairs $(x,y)\in M\times M$ such that $f(x)+g(y)=f(y)+g(x)$ form a subsemimodule since

$$(x,y),(x',y')\in Z \ \Rightarrow\ f(x+x')+g(y+y')=f(y+y')+g(x+x').$$

(ii) By definition $Z(f,g)$ is a subobject of M^2, thus one defines two maps $Z(f,g) \underset{\iota_2}{\overset{\iota_1}{\rightrightarrows}} M$ by composing with the p_j's. From this one derives the sequence (32). By definition of $Z(f,g)$ one also has $f\circ\iota_1+g\circ\iota_2=f\circ\iota_2+g\circ\iota_1$.

(iii) For any $x\in L$ one has $(\alpha(x),\beta(x))\in Z(f,g)$.

(iv) Both sides of the equality are defined as the subsemimodule of M^2 given by the equation $f(x)=f(y)$. □

The statement (iii) of the above proposition means that when the composition $(f,g)\circ(\alpha,\beta)$ of two successive pairs is equivalent to 0, *i.e.* given by a diagonal pair, one derives a factorization through the kernel $Z(f,g)$. Thus, by requiring that this factorization is onto $Z(f,g)$, one gets a first hint for the notion of strict exactness. One has

Proposition 4.3. *Let $f,g\in Hom_{\mathbb{B}}(L,M)$.*

(i) $B(f,g)=\{(f(x)+g(y),f(y)+g(x))\mid x,y\in L\}$ is a subsemimodule of $M\times M$.

(ii) Let $\alpha,\beta\in Hom_{\mathbb{B}}(M,N)$, then one has

$$B(f,g)\subset Z(\alpha,\beta) \iff \alpha\circ f+\beta\circ g=\beta\circ f+\alpha\circ g \tag{33}$$

(iii) Let $\phi=(f,g)\in Hom_{\mathbb{B}\mathrm{mod}^2}(L,M)$, $\psi=(h,k)\in Hom_{\mathbb{B}\mathrm{mod}^2}(M,N)$, then one has

$$Z(f,g)=Z(\phi)\subset Z(\psi\circ\phi),\quad B(h,k)=B(\psi)\supset B(\psi\circ\phi). \tag{34}$$

Proof. (i) is straightforward.

(ii) $B(f,g)\subset Z(\alpha,\beta)$ if and only if for any $x,y\in L$ one has

$$\alpha(f(x)+g(y))+\beta(f(y)+g(x))=\beta(f(x)+g(y))+\alpha(f(y)+g(x))$$

or equivalently

$$(\alpha \circ f + \beta \circ g)(x) + (\beta \circ f + \alpha \circ g)(y) = (\alpha \circ f + \beta \circ g)(y) + (\beta \circ f + \alpha \circ g)(x)$$

This equality clearly holds if $\alpha \circ f + \beta \circ g = \beta \circ f + \alpha \circ g$. Conversely, taking $y = 0$, one gets $\alpha \circ f + \beta \circ g = \beta \circ f + \alpha \circ g$.

(iii) By Proposition 4.2, $Z(\psi \circ \phi)$ is given as the set of pairs $(x, y) \in L \times L$ such that

$$(h \circ f + k \circ g)(x) + (h \circ g + k \circ f)(y) = (h \circ f + k \circ g)(y) + (h \circ g + k \circ f)(x).$$

Then the first inclusion in (34) follows by rewriting the above equality as

$$h(f(x) + g(y)) + k(g(x) + f(y)) = h(f(y) + g(x)) + k(g(y) + f(x)).$$

By (i), $B(\psi \circ \phi)$ is given as the set of pairs

$$((h \circ f + k \circ g)(x) + (h \circ g + k \circ f)(y), (h \circ f + k \circ g)(y) + (h \circ g + k \circ f)(x)).$$

The second inclusion in (34) then follows by rewriting such pairs as

$$(h(X) + k(Y), h(Y) + k(X)), \quad X = f(x) + g(y),\ Y = f(y) + g(x). \qquad \square$$

It is clear that in general one has $\Delta \subset Z(\alpha, \beta)$, where $\Delta \subset M \times M$ is the diagonal, *i.e.* $\Delta = \{(x, x) \mid x \in M\}$. Keeping in mind this fact, we introduce the notion of strictly exact sequence as follows

Definition 4.4. *The sequence* $L \underset{\alpha_2}{\overset{\alpha_1}{\rightrightarrows}} M \underset{\beta_2}{\overset{\beta_1}{\rightrightarrows}} N$ *in* $\mathbb{B}\mathrm{mod}^2$ *is strictly exact at* M *if* $B(\alpha_1, \alpha_2) + \Delta = Z(\beta_1, \beta_2)$.

In the following we shall test this new notion in several cases. As a first case, we assume that $N = 0$ so that both arrows $\beta_j = 0$. Then, $Z(\beta_1, \beta_2) = M \times M$ and one needs to find out the relation between the coequalizer of the α_j and the strict exactness of the sequence at M. This is provided by the following

Proposition 4.5. *Consider the sequence* $L \underset{\alpha_2}{\overset{\alpha_1}{\rightrightarrows}} M \underset{0}{\overset{0}{\rightrightarrows}} 0$ *in* $\mathbb{B}\mathrm{mod}^2$.

(i) *The following three conditions are equivalent*

a) The sequence is strictly exact at M,

b) $\{\alpha_1(x) + \alpha_2(y) \mid x, y \in L,\ \alpha_2(x) + \alpha_1(y) = 0\} = M$

c) $B(\alpha_1, \alpha_2) = M \times M$.

(ii) *If* $\alpha_2 = 0$ *then the sequence is strictly exact at* M *if and only if* α_1 *is surjective.*

(iii) *If the* α_j*'s have a non-trivial coequalizer then the sequence is not strictly exact at* M.

Proof. (i) One has $Z(0, 0) = M \times M$. By symmetry, one has $B(\alpha_1, \alpha_2) + \Delta = M \times M$ if and only if $M \times \{0\} \subset B(\alpha_1, \alpha_2) + \Delta$. But an equality $a + b = 0$ for $a, b \in M$ implies $a = 0$ and $b = 0$ since $a = a + a + b = a + b = 0$. Thus for $x, y \in L$ and $z \in M$ such that $(\alpha_1(x) + \alpha_2(y) + z, \alpha_2(x) + \alpha_1(y) + z) = (t, 0)$, one has $z = 0$ and $\alpha_2(x) + \alpha_1(y) = 0$.

(ii) If $\alpha_2 = 0$, (i) states that the strict exactness at M holds if and only if $\{\alpha_1(x) \mid x, y \in L,\ \alpha_1(y) = 0\} = M$. This requirement is nothing but the surjectivity of α_1.

(iii) Consider a non-trivial morphism $\phi : M \to N$ such that $\phi \circ \alpha_1 = \phi \circ \alpha_2$. Let $t \in M$ with $\phi(t) \neq 0$. Then the pair $(t, 0)$ cannot belong to $B(\alpha_1, \alpha_2) + \Delta$ since $\phi(\alpha_1(x) + \alpha_2(y) + z) = \phi(\alpha_2(x) + \alpha_1(y) + z)\ \forall x, y \in L,\ z \in M$. $\square$

Next we test the notion of strict exactness of $0 \underset{0}{\overset{0}{\rightrightarrows}} M \underset{\beta_2}{\overset{\beta_1}{\rightrightarrows}} N$.

Proposition 4.6. *Consider the sequence* $0 \underset{0}{\overset{0}{\rightrightarrows}} M \underset{\beta_2}{\overset{\beta_1}{\rightrightarrows}} N$ *in* $\mathbb{B}\mathrm{mod}^2$.

(i) *The sequence is strictly exact at* M *if and only if*

$$\beta_1(x) + \beta_2(y) = \beta_2(x) + \beta_1(y) \;\Leftrightarrow\; x = y$$

(ii) *If* $\beta_2 = 0$ *then the sequence is strictly exact at* M *if and only if* β_1 *is injective.*

Proof. (i) One has $B(0,0) = 0$ and thus the strict exactness holds if and only if $Z(\beta_1, \beta_2) = \Delta$.

(ii) One has $\beta_1(x) = \beta_1(y) \;\Leftrightarrow\; x = y$ if and only if β_1 is injective. ☐

Corollary 4.7. *The sequence* $0 \underset{0}{\overset{0}{\rightrightarrows}} M \underset{0}{\overset{\beta}{\rightrightarrows}} N \underset{0}{\overset{0}{\rightrightarrows}} 0$ *in* $\mathbb{B}\mathrm{mod}^2$ *is strictly exact if and only if* β *is an isomorphism in* $\mathbb{B}\mathrm{mod}$.

Proof. This follows from Propositions 4.5 and 4.6. ☐

4.2 Monomorphisms and strict exact sequences.

In this subsection we investigate the meaning of a strictly exact sequence as in Proposition 4.6.

Proposition 4.8. *Let* $0 \underset{0}{\overset{0}{\rightrightarrows}} M \underset{g}{\overset{f}{\rightrightarrows}} N$ *be a sequence in* $\mathbb{B}\mathrm{mod}^2$ *strictly exact at* M*, then:*

(i) *The pair* (f, g) *embeds* M *as a subsemimodule of* N^2.

(ii) *The map* $f + g$ *embeds* M *as a subsemimodule of* N.

Proof. By Proposition 4.6 one has

$$f(x) + g(y) = g(x) + f(y) \iff x = y \tag{35}$$

(i) Assume that for elements $x, y \in M$ one has $(f(x), g(x)) = (f(y), g(y))$. Then $f(x)+g(y) = g(x) + f(y)$ thus by hypothesis $x = y$.

(ii) Assume that for elements $x, y \in M$ one has $f(x) + g(x) = f(y) + g(y)$. Then one has $f(x)+g(x+y) = f(x+y)+g(x)$ since adding $g(y)$ or $f(y)$ to $f(x)+g(x) = f(y)+g(y)$ does not change the result. Then by (35) one gets $x = x + y$ and similarly $y = x + y$ so that $x = y$. ☐

Thus, Proposition 4.8 (i) shows that to understand the strict exactness of $0 \underset{0}{\overset{0}{\rightrightarrows}} M \underset{g}{\overset{f}{\rightrightarrows}} N$ at M we can reduce to the case where $M \subset N \times N$ is a subsemimodule of $N \times N$, while $f = p_1$ and $g = p_2$ are the two projections restricted to this subsemimodule. The condition of strict exactness on the subsemimodule $M \subset N \times N$ reads as the implication for $x, y \in M \subset N \times N$

$$x + \sigma(y) = \sigma(x) + y \Rightarrow y = x \tag{36}$$

where $\sigma : N \times N \to N \times N$ is the involution $\sigma(a, b) := (b, a)$. One might guess at first that (36) implies the injectivity of the map $\iota : M \times M \to N \times N$ given by

$$\iota(x, y) := x + \sigma(y) \tag{37}$$

But this fails as shown by the following

Example 4.9. Let $X = \{1,2\}$ and $N = 2^X$. Consider the subsemimodule $M \subset N \times N$ defined as

$$M := \{(\emptyset,\emptyset), (\emptyset,\{1\}), (\{1\},\{1,2\})\}. \tag{38}$$

Then the sequence $0 \underset{0}{\overset{0}{\rightrightarrows}} M \underset{p_2}{\overset{p_1}{\rightrightarrows}} N$ is strictly exact at M, but the map $\iota : M \times M \to N \times N$ given by (37) is not injective.

Proof. The structure of $\mathbb{B}$-semimodule of M is the same as that of the totally ordered set $0 < 1 < 2$. For $x, y \in M$ the sum $x + \sigma(y)$ cannot be symmetric if x or y is $(\emptyset,\emptyset)$ and $x \neq y$. Moreover the sum of the two non-zero elements is $(\emptyset,\{1\}) \vee (\{1\},\{1,2\}) = (\{1\},\{1,2\})$ which is not symmetric. Thus M fulfills the condition of strict exactness at M of the sequence $0 \underset{0}{\overset{0}{\rightrightarrows}} M \underset{p_2}{\overset{p_1}{\rightrightarrows}} N$. We now consider the map $\iota : M \times M \to N \times N$ as in (37). We claim that its range has 7 elements, precisely:

$$\iota(M \times M) = \{(\emptyset,\emptyset), (\emptyset,\{1\}), (\{1\},\{1,2\}), (\{1\},\emptyset), (\{1,2\},\{1\}), (\{1\},\{1\}), (\{1,2\},\{1,2\})\}$$

and the element $(\{1\},\{1,2\})$ is obtained twice since

$$(\{1\},\{1,2\}) \vee \sigma((\emptyset,\{1\})) = (\{1\},\{1,2\}) = (\{1\},\{1,2\}) \vee \sigma((\emptyset,\emptyset)).$$

Note that by applying σ on both sides one gets an equality of the form $\iota(x,y) = \iota(x',y)$ with $x \neq x'$. ∎

Example 4.9 was the simplest case to consider. To understand the various choices of $M \subset N \times N$ which fulfill the strict exactness of Proposition 4.8, we consider the next case where $N = 2^X$, $|X| = 3$. In this case we shall display all the maximal choices of M up to permutations and symmetry. The condition of strict exactness (36) is preserved by any automorphism α of $N \times N$ which commutes with the symmetry σ and we simplify further using such automorphisms. Since $N = 2^X$ one has $N \times N = 2^{X \cup Y}$ where we represent X and Y as the first (in blue) and second line (in red) of the rectangle, and the elements of $N \times N$ as subsets of the rectangle. When the cardinality of M is $|M| = 4$ the number of maximal cases up to symmetries is two. They are displayed in Figure 1 where each line gives a choice of $M \subset N \times N$.

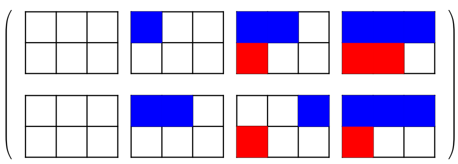

Figure 1: Reduction to two cases $|M| = 4$.

Notice that these cases are not isomorphic since in the first line the semimodule M is totally ordered. Similarly for $|M| = 5$ the number of maximal cases up to symmetries is two.

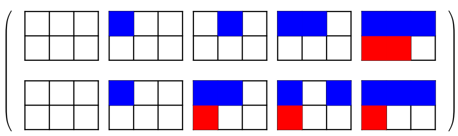

Figure 2: Reduction to two cases $|M| = 5$.

For $|M| = 6$ the number of maximal cases up to symmetries is one.

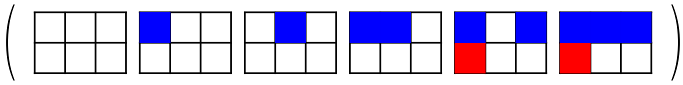

Figure 3: Reduction to one case $|M| = 6$.

as well as the case $|M| = 8$

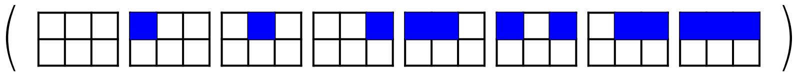

Figure 4: Reduction to one case $|M| = 8$.

Example 4.9 shows that strict exactness of a sequence of the form $0 \underset{0}{\overset{0}{\rightrightarrows}} M \underset{g}{\overset{f}{\rightrightarrows}} N$ does not imply that the morphism $\phi = (f, g)$ is a monomorphism. More precisely one has

Proposition 4.10. *(i) The morphism $\phi : M \underset{g}{\overset{f}{\rightrightarrows}} N$ is a monomorphism in the category $\mathbb{B}\mathrm{mod}^2$ if and only if the map $M^2 \to N^2$, $(x, y) \mapsto (f(x) + g(y), g(x) + f(y))$ is injective.*

(ii) If $\phi = (f, g)$ is a monomorphism, the sequence $0 \underset{0}{\overset{0}{\rightrightarrows}} M \underset{g}{\overset{f}{\rightrightarrows}} N$ is strictly exact at M.

Proof. (i) If the map $(x, y) \mapsto (f(x)+g(y), g(x)+f(y))$ fails to be injective, consider $(x, y) \in M^2$, $(x', y') \in M^2$ such that $(x, y) \neq (x', y')$ and $f(x)+g(y) = f(x')+g(y')$, $g(x)+f(y) = g(x')+f(y')$. Let $\psi \in \mathrm{Hom}_{\mathbb{B}\mathrm{mod}^2}(\mathbb{B}, M)$ be given by $\mathbb{B} \underset{y}{\overset{x}{\rightrightarrows}} M$, where x stands for the unique morphism in $\mathbb{B}\mathrm{mod}$ such that $1 \mapsto x$. One uses similar notations for y and ψ'. The above equality then means that $\phi \circ \psi = \phi \circ \psi'$, where $\phi = (f, g) \in \mathrm{Hom}_{\mathbb{B}\mathrm{mod}^2}(M, N)$. Thus ϕ fails to be a monomorphism in $\mathbb{B}\mathrm{mod}^2$. Conversely, if the map $(x, y) \mapsto (f(x) + g(y), g(x) + f(y))$ is injective then the composition law in $\mathbb{B}\mathrm{mod}^2$ shows that $\phi = (f, g)$ is a monomorphism. Indeed, the equality $(f \circ \alpha + g \circ \beta, f \circ \beta + g \circ \alpha) = (f \circ \alpha' + g \circ \beta', f \circ \beta' + g \circ \alpha')$ implies, by restriction to diagonal pairs (a, a) that $\alpha(a) = \alpha'(a)$ and $\beta(a) = \beta'(a)$, $\forall a$.

(ii) When $\phi = (f, g)$ is a monomorphism, the above sequence is strictly exact since the equality $f(x)+g(y) = g(x)+f(y)$ implies that (x, y) and (y, x) have the same image by the map $(x, y) \mapsto (f(x) + g(y), g(x) + f(y)) : M^2 \to N^2$. $\square$

4.3 Isomorphisms and strict exact sequences. To understand how restrictive the notion of strictly exact sequence is we investigate the exact sequences in $\mathbb{B}\mathrm{mod}^2$ of the form

$$0 \underset{0}{\overset{0}{\rightrightarrows}} M \underset{g}{\overset{f}{\rightrightarrows}} N \underset{0}{\overset{0}{\rightrightarrows}} 0. \tag{39}$$

Proposition 4.11. *Let the sequence (39) be strictly exact in $\mathbb{B}\mathrm{mod}^2$. Then there exists a unique decomposition $N = N_1 \times N_2$ and a unique isomorphism of $\mathbb{B}$-semimodules $\alpha : M \to N$ such that*

$$f = (p_{N_1} \circ \alpha, 0), \quad g = (0, p_{N_2} \circ \alpha), \quad p_{N_j} : N_1 \times N_2 \to N_j \tag{40}$$

Proof. By the strict exactness of the sequence at M and Proposition 4.8, we can reduce to the case where $M \subset N \times N$, $f = p_1$ and $g = p_2$. By applying Proposition 4.5 the strict exactness of the sequence at N means that

$$\{p_1(x) + p_2(y) \mid x, y \in M,\ p_2(x) + p_1(y) = 0\} = N \tag{41}$$

One then obtains the following two hereditary subsemimodules $J_k \subset M$

$$J_1 := \{x \in M \mid p_2(x) = 0\} = M \cap (N \times 0), \quad J_2 := \{x \in M \mid p_1(x) = 0\} = M \cap (0 \times N).$$

The addition gives an injection $(N \times 0) \times (0 \times N) \to N \times N$ and hence by restriction an injection $j : J_1 \times J_2 \to M$. Let $J = j(J_1 \times J_2) = J_1 + J_2 \subset M$. The map $p : J \to N$, $p(x) := p_1(x) + p_2(x)$ is surjective by (41). By Proposition 4.8 (ii) the strict exactness at M shows that the map $\tilde{p} : M \to N$, $\tilde{p}(x) := p_1(x) + p_2(x)$ is injective. Since its restriction $p : J \to N$ to $J \subset M$ is surjective this shows that $M = J$ and that p is bijective and is an isomorphism. To the decomposition of J as $J_1 \times J_2$ corresponds a decomposition $N = N_1 \times N_2$ where $N_k = p(J_k)$. This shows the existence of the decomposition (40). We finally prove its uniqueness. Assuming (40), one has $N_1 = f(M)$, $N_2 = g(M)$. This determines the decomposition $N = N_1 \times N_2$ uniquely. Moreover both $p_{N_1} \circ \alpha$ and $p_{N_2} \circ \alpha$ are uniquely determined by f and g and thus α is unique. $\square$

Proposition 4.11 suggests that a morphism $M \overset{f}{\underset{g}{\rightrightarrows}} N$ in the category $\mathbb{B}\mathrm{mod}^2$ is an isomorphism iff the sequence $0 \overset{0}{\underset{0}{\rightrightarrows}} M \overset{f}{\underset{g}{\rightrightarrows}} N \overset{0}{\underset{0}{\rightrightarrows}} 0$ is strictly exact. Let M be a $\mathbb{B}$-semimodule, and consider two decompositions $M = M_1 \times M_2$, $M = M'_1 \times M'_2$ of M as a product. For $x \in M$ we denote by $x = x_1 + x_2$ (resp. $x = x_{1'} + x_{2'}$) its unique decomposition with $x_j \in M_j$ (resp. $x_{j'} \in M'_j$). Since M_j is a hereditary subsemimodule one has, with $x_j = (x_j)_{1'} + (x_j)_{2'}$ that $(x_j)_{k'} \in M_j \cap M'_k$, and that any element $x \in M$ is uniquely decomposed as a sum $x = \sum x_{jk'}$ where $x_{jk'} \in M_j \cap M'_k$. It follows that the projections p_j and p'_k associated to the two decompositions commute pairwise and that the composition in the category $\mathbb{B}\mathrm{mod}^2$ of the morphisms (p_1, p_2) and (p'_1, p'_2) is given by the pair corresponding to the decomposition

$$M = \big((M_1 \cap M'_1) \oplus (M_2 \cap M'_2)\big) \times \big((M_1 \cap M'_2) \oplus (M_2 \cap M'_1)\big).$$

It follows that such pairs (p_1, p_2) form a subgroup $\mathrm{Aut}^{(1)}_{\mathbb{B}\mathrm{mod}^2}(M) \subset \mathrm{Aut}_{\mathbb{B}\mathrm{mod}^2}(M)$ of the group of automorphisms $\mathrm{Aut}_{\mathbb{B}\mathrm{mod}^2}(M)$. By construction this subgroup is abelian and every element is of order two.

Proposition 4.12. (i) *The sequence* $0 \overset{0}{\underset{0}{\rightrightarrows}} M \overset{f}{\underset{g}{\rightrightarrows}} N \overset{0}{\underset{0}{\rightrightarrows}} 0$ *is strictly exact if and only if the pair* $\phi = (f, g)$ *is an isomorphism in* $\mathbb{B}\mathrm{mod}^2$.

(ii) *The group of automorphisms* $\mathrm{Aut}_{\mathbb{B}\mathrm{mod}^2}(M)$ *is the semi-direct product of the 2-group* $\mathrm{Aut}^{(1)}_{\mathbb{B}\mathrm{mod}^2}(M)$ *of decompositions* $M = M_1 \times M_2$ *by the group of automorphisms of the* $\mathbb{B}$-*semimodule* M.

Proof. (i) Assume first that (40) holds, with $\alpha : M \to N$ an isomorphism in the category of $\mathbb{B}$-semimodules. The composition law $(u, v) \circ (\alpha, 0) = (u \circ \alpha, v \circ \alpha)$ shows that to prove that the morphism $\phi = (f, g)$ is an isomorphism in $\mathbb{B}\mathrm{mod}^2$ one can assume that $N = M$ and $\alpha = \mathrm{Id}$. Thus we can assume that $f = (p_{M_1}, 0)$, $g = (0, p_{M_2})$, $p_{M_j} : M_1 \times M_2 \to M_j$. One then has, since both

f and g are idempotent and with vanishing product: $(f,g)\circ(f,g) = (f\circ f + g\circ g, f\circ g + g\circ f) = (\mathrm{Id}, 0)$. This shows that strict exactness implies isomorphism.

Conversely, if $\phi = (f,g)$ is an isomorphism in $\mathbb{B}\mathrm{mod}^2$ it admits a left inverse ψ such that $\psi\circ\phi = (\mathrm{Id}, 0)$ and Proposition 4.3 (iii) shows that $Z(f,g) \subset Z(\psi\circ\phi) = Z((\mathrm{Id},0)) = \Delta$. Similarly, ϕ admits a right inverse ψ such that $\phi\circ\psi = (\mathrm{Id},0)$ and Proposition 4.3 (iii) shows that $B(f,g) = B(\phi) \supset B(\phi\circ\psi) = B((\mathrm{Id},0)) = N\times N$.

(ii) The map $\alpha\mapsto(\alpha,0) = \rho(\alpha)$ defines an injective group homomorphism $\rho : \mathrm{Aut}_{\mathbb{B}\mathrm{mod}}(M) \to \mathrm{Aut}_{\mathbb{B}\mathrm{mod}^2}(M)$. Moreover by (i) together with Proposition 4.11, we derive that any element of $\mathrm{Aut}_{\mathbb{B}\mathrm{mod}^2}(M)$ is uniquely a product $\beta\circ\rho(\alpha)$, with $\beta\in\mathrm{Aut}^{(1)}_{\mathbb{B}\mathrm{mod}^2}(M)$ and $\alpha\in\mathrm{Aut}_{\mathbb{B}\mathrm{mod}}(M)$. Thus one obtains the equality: $\mathrm{Aut}_{\mathbb{B}\mathrm{mod}^2}(M) = \mathrm{Aut}^{(1)}_{\mathbb{B}\mathrm{mod}^2}(M)\rtimes\mathrm{Aut}_{\mathbb{B}\mathrm{mod}}(M)$, using the natural action of $\mathrm{Aut}_{\mathbb{B}\mathrm{mod}}(M)$ by conjugation on $\mathrm{Aut}^{(1)}_{\mathbb{B}\mathrm{mod}^2}(M)$. ∎

4.4 Epimorphisms and strict exact sequences.

There is a direct relation between strict exactness and the categorical notion of epimorphism in $\mathbb{B}\mathrm{mod}^2$, it is given by the following

Proposition 4.13. *Let $\phi = (f,g)\in Hom_{\mathbb{B}\mathrm{mod}^2}(M,N)$. The following conditions are equivalent*

1. *$B(f,g) = N\times N$.*
2. *The sequence $M \underset{g}{\overset{f}{\rightrightarrows}} N \underset{0}{\overset{0}{\rightrightarrows}} 0$ is strictly exact at N.*
3. *The morphism ϕ is an epimorphism in the category $\mathbb{B}\mathrm{mod}^2$.*

Proof. By Proposition 4.5 the strict exactness of the sequence at N is equivalent to $B(f,g) = N\times N$ and the addition of Δ is irrelevant in that case. By Lemma 2.6 the equality $B(f,g) = N\times N$ holds iff for any $\mathbb{B}$-semimodule X and any morphisms $\psi,\psi'\in\mathrm{Hom}_{\mathbb{B}}(N\times N, X)$ one has

$$\psi = \psi' \iff \psi|_{B(f,g)} = \psi'|_{B(f,g)}.$$

Let ψ_j (resp. ψ'_j) be the components of ψ so that $\psi((a,b)) = \psi_1(a) + \psi_2(b)$ (resp. $\psi'((a,b)) = \psi'_1(a) + \psi'_2(b)$). One then derives

$$\psi((f(x)+g(y), f(y)+g(x)) = \psi_1(f(x)+g(y)) + \psi_2(g(x)+f(y)),\ \forall x,y\in N.$$

Thus (taking $y=0$ or $x=0$) the condition $\psi|_{B(f,g)} = \psi'|_{B(f,g)}$ is equivalent to

$$\psi_1\circ f + \psi_2\circ g = \psi'_1\circ f + \psi'_2\circ g,\quad \psi_1\circ g + \psi_2\circ f = \psi'_1\circ g + \psi'_2\circ f$$

which means exactly, using composition in $\mathbb{B}\mathrm{mod}^2$, that $\psi\circ\phi = \psi'\circ\phi$, where ψ is viewed as the element $(\psi_1,\psi_2)\in\mathrm{Hom}_{\mathbb{B}\mathrm{mod}^2}(N,X)$ and similarly for ψ'. Thus strict exactness holds iff the morphism $\phi = (f,g)$ is an epimorphism in the category $\mathbb{B}\mathrm{mod}^2$. ∎

By using the inclusion $f(N)\subset M$ and Proposition 2.7 every morphism f in $\mathbb{B}\mathrm{mod}$ admits a factorization "monomorphism ∘ epimorphism". This fact no longer holds in the category $\mathbb{B}\mathrm{mod}^2$ since the size of the image of a monomorphism $M \underset{g}{\overset{f}{\rightrightarrows}} N$, when viewed as the subsemimodule $E = \{(f(x),g(x)) \mid x\in M\}\subset N\times N$, is limited by the condition of injectivity of the map $(x,y)\mapsto(f(x)+g(y), g(x)+f(y)) : M^2\to N^2$. More precisely, this condition implies the injectivity of the map $E\times E\to N\times N$, $(\xi,\eta)\mapsto\xi+\sigma(\eta)$. The failure of this condition ought to imply the impossibility of a factorization "monomorphism ∘ epimorphism" but one needs to be careful since the above definition of E uses the diagonal.

To test our guess, we reconsider the simplest Example 4.9.

Example 4.14. With the notations of Example 4.9, the morphism $\phi : M \underset{p_2}{\overset{p_1}{\rightrightarrows}} N$ does not admit a factorization "monomorphism $\circ$ epimorphism" in the category $\mathbb{B}\text{mod}^2$.

Proof. Assume that $\phi = \psi \circ \eta$ in $\mathbb{B}\text{mod}^2$. By (34) one has $Z(\eta) \subset Z(\psi \circ \eta) = Z(\phi)$ and $Z(\phi) = \Delta$ since the sequence $0 \underset{0}{\overset{0}{\rightrightarrows}} M \underset{p_2}{\overset{p_1}{\rightrightarrows}} N$ is strictly exact. Thus $Z(\eta) = \Delta$ for any choice of η, and if moreover η is an epimorphism then Proposition 4.13 shows, together with Proposition 4.12, that it is an isomorphism. The $\mathbb{B}$-semimodule M does not admit a non-trivial decomposition as a product, and thus the only possible choices for η are $M \underset{0}{\overset{\rho}{\rightrightarrows}} M$ or $M \underset{\rho}{\overset{0}{\rightrightarrows}} M$, where ρ is an isomorphism. This reduces the possible factorizations "mono $\circ$ epi" to ϕ itself and since this map fails to be a monomorphism, there is no factorization "$\phi =$ mono $\circ$ epi" in the category $\mathbb{B}\text{mod}^2$. $\square$

This example points out a serious issue which is in fact independent of the definition of exactness and is formulated simply in terms of the category $\mathbb{B}\text{mod}^2$. This defect is resolved by the extension of the category $\mathbb{B}\text{mod}^2$ performed below in §5.1.

Remark 4.15. For M a $\mathbb{B}$-semimodule, let $M^* := \text{Hom}_{\mathbb{B}}(M, \mathbb{B})$, and for $f \in \text{Hom}_{\mathbb{B}}(M, N)$ let $f^* : N^* \to M^*$ be given by $f^*(\phi) := \phi \circ f$. This defines a contravariant endofunctor of $\mathbb{B}\text{mod}$ and also of $\mathbb{B}\text{mod}^2$ using $(f, g)^* := (f^*, g^*)$ and the compatibility of the composition law (27) with $f \mapsto f^*$. One shows

(*i*) If the sequence $N^* \underset{g^*}{\overset{f^*}{\rightrightarrows}} M^* \underset{0}{\overset{0}{\rightrightarrows}} 0$ is strictly exact so is the sequence $0 \underset{0}{\overset{0}{\rightrightarrows}} M \underset{g}{\overset{f}{\rightrightarrows}} N$.

(*ii*) If the sequence $M \underset{g}{\overset{f}{\rightrightarrows}} N \underset{0}{\overset{0}{\rightrightarrows}} 0$ is strictly exact so is the dual sequence $0 \underset{0}{\overset{0}{\rightrightarrows}} N^* \underset{g^*}{\overset{f^*}{\rightrightarrows}} M^*$.

Example 4.9 shows that the converse of both statements fails.

4.5 Quotients in $\mathbb{B}\text{mod}^2$.

The issue of the existence of quotients in the category $\mathbb{B}\text{mod}^2$ arises naturally because one would like to define the cohomology of the sequence

$$M \underset{\alpha_2}{\overset{\alpha_1}{\rightrightarrows}} N \underset{\beta_2}{\overset{\beta_1}{\rightrightarrows}} P \tag{42}$$

at N as the quotient $H_N := Z(\beta_1, \beta_2)/(\Delta + B(\alpha_1, \alpha_2))$. The subsemimodule $\Delta + B(\alpha_1, \alpha_2) \subset Z(\beta_1, \beta_2)$ is not hereditary in general and the quotient is ill defined. Thus we proceed as follows by using functors and the Yoneda embedding.

Proposition 4.16. *Let M be a $\mathbb{B}$-semimodule and $N \subset M$ a subsemimodule.*

(*i*) *The following equality defines a covariant additive functor $F = M/N : \mathbb{B}\text{mod}^2 \longrightarrow \mathbb{B}\text{mod}$*

$$M/N(X) := \{(f, g) \in Hom_{\mathbb{B}\text{mod}^2}(M, X) \mid f(x) = g(x)\,, \ \forall x \in N\}. \tag{43}$$

(*ii*) *The functor $F = M/N$ is a subfunctor of the representable functor $y_M := Hom_{\mathbb{B}\text{mod}^2}(M, -)$.*

(*iii*) *The map $(f, g) \mapsto (g, f)$ induces an involution σ on $M/N(X)$ and σ is the identity for any X if and only if $N = M$.*

Proof. (*i*) The pairs (f, g) fulfilling $f(x) = g(x), \forall x \in N$ form a $\mathbb{B}$-subsemimodule of the $\mathbb{B}$-semimodule $\text{Hom}_{\mathbb{B}\text{mod}^2}(M, X)$. Let $X \underset{\beta}{\overset{\alpha}{\rightrightarrows}} Y$ be a morphism in $\text{Hom}_{\mathbb{B}\text{mod}^2}(X, Y)$. The composition $(\alpha, \beta) \circ (f, g)$ as in (27) defines a map $F(\alpha, \beta) : M/N(X) \to M/N(Y)$ since one has

$$(\alpha \circ f + \beta \circ g)(x) = (\alpha \circ g + \beta \circ f)(x)\,, \ \forall x \in N.$$

The map $F(\alpha,\beta)$ is additive by construction, *i.e.* it defines a morphism of $\mathbb{B}$-semimodules. Moreover it depends additively upon the pair (α,β).

(ii) By construction the functor $F = M/N$ is a subfunctor of the Yoneda functor y_M which associates to any $\mathbb{B}$-semimodule X the $\mathbb{B}$-semimodule $\mathrm{Hom}_{\mathbb{B}\mathrm{mod}^2}(M,X)$.

(iii) Lemma 2.6 states that $N \neq M$ iff there exists a $\mathbb{B}$-semimodule E and two distinct morphisms $f,g \in \mathrm{Hom}_{\mathbb{B}}(M,E)$ which agree on N. $\square$

Proposition 4.16 gives a meaning to the quotient M/N as a covariant additive functor $F = M/N : \mathbb{B}\mathrm{mod}^2 \longrightarrow \mathbb{B}\mathrm{mod}$. By the Yoneda Lemma the opposite of the category $\mathbb{B}\mathrm{mod}^2$ embeds fully and faithfully in the category of covariant additive functors $\mathbb{B}\mathrm{mod}^2 \longrightarrow \mathbb{B}\mathrm{mod}$. The inclusion of $F = M/N$ as a subfunctor of the representable functor $y_M : \mathbb{B}\mathrm{mod}^2 \longrightarrow \mathbb{B}\mathrm{mod}$ corresponds to the "quotient map" $M \to M/N$. The following example shows that the functor $M/N : \mathbb{B}\mathrm{mod}^2 \longrightarrow \mathbb{B}\mathrm{mod}$ is not representable in general.

Example 4.17. We let $M = \{0,1,2\}$ with the operation of max, and take $N = \{0,2\} \subset M$. The functor F associates, to a $\mathbb{B}$-semimodule X, two morphisms $u,v \in \mathrm{Hom}_{\mathbb{B}}(M,X)$ which agree on N. With $a = u(1)$, $b = v(1)$, $c = u(2) = v(2)$ this corresponds to the subset of X^3 formed of triples (a,b,c) such that $a \leq c$ and $b \leq c$. Let then Q be the $\mathbb{B}$-semimodule

$$Q := \{0,\alpha,\beta,\alpha\vee\beta,\gamma\}, \quad x\vee\gamma = \gamma\,, \ \ \forall x \in Q$$

One has a natural identification $F(X) \simeq \mathrm{Hom}_{\mathbb{B}}(Q,X)$ by sending $\alpha \mapsto a$, $\beta \mapsto b$, $\gamma \mapsto c$. Assume that the functor $M/N : \mathbb{B}\mathrm{mod}^2 \longrightarrow \mathbb{B}\mathrm{mod}$ is represented by a $\mathbb{B}$-semimodule Z. Then $F(X) \simeq \mathrm{Hom}_{\mathbb{B}\mathrm{mod}^2}(Z,X) = \mathrm{Hom}_{\mathbb{B}}(Z,X)^2$ and one gets a contradiction for $X = \mathbb{B}$ since $F(\mathbb{B})$ has five elements.

The above example suggests that given a sub-$\mathbb{B}$-semimodule $N \subset M$, one can define an analogue of the above semimodule Q as follows.

Proposition 4.18. *Let $N \subset M$ be a subsemimodule. On $E = M \times M$ define the following relation*

$$(x,y) = z \sim z' = (x',y') \iff f(x)+g(y) = f(x')+g(y')\,, \ \ \forall X, \ (f,g) \in F(X) \tag{44}$$

where $F(X)$ is defined in (43).

(i) The quotient $Q = E/\sim$ is the cokernel pair $\mathrm{Coker}_p(\iota)$ *of the inclusion $\iota : N \to M$.*

(ii) The canonical maps $\gamma_j : M \to \mathrm{Coker}_p(\iota)$ of the sequence

$$N \xrightarrow{\iota} M \underset{\gamma_2}{\overset{\gamma_1}{\rightrightarrows}} \mathrm{Coker}_p(\iota) \tag{45}$$

are injective and the intersection of the $\gamma_j(M)$ is $\gamma_1(N) = \gamma_2(N)$.

(iii) One has canonical isomorphisms of endofunctors of $\mathbb{B}\mathrm{mod}$

$$r : F\circ\kappa(X) \to Hom_{\mathbb{B}}(Q,X), \quad \gamma : Hom_{\mathbb{B}}(Q,X) \to F\circ\kappa(X),$$

where $\kappa : \mathbb{B}\mathrm{mod} \longrightarrow \mathbb{B}\mathrm{mod}^2$ is defined in (31).

Proof. By construction (44) defines an equivalence relation that is compatible with the addition, *i.e.* $z_j \sim z'_j (j=1,2) \implies z_1+z_2 \sim z'_1+z'_2$. Thus the quotient $Q = E/\sim$ is a well defined $\mathbb{B}$-semimodule.

(*i*) By Definition 3.8, $\mathrm{Coker}_p(\iota)$ is the coequalizer $\mathrm{Coequ}(\iota_{(2)})$ of the two maps $s_j \circ \iota$ where the $s_j : M \to M \oplus M$ are the canonical inclusions from M to $M \oplus M$. A morphism $\rho \in \mathrm{Hom}_{\mathbb{B}}(M \oplus M, X)$ is given by a pair of morphisms $f, g \in \mathrm{Hom}_{\mathbb{B}}(M, X)$ satisfying the following rule

$$\rho \circ s_1 \circ \iota = \rho \circ s_2 \circ \iota \iff (f, g) \in F(X).$$

Thus the quotient $Q = E/\!\sim$ coincides with the construction of the coequalizer of Lemma 3.2.

(*ii*) Let $x \in N$, then $f(x) = g(x)$ for $(f, g) \in F(X)$ and one gets $(x, 0) \sim (0, x)$. Note also that the map $\gamma_1 : M \to Q$, $\gamma_1(x) = (x, 0)$ is injective since the pair $(\mathrm{Id}_M, \mathrm{Id}_M)$ belongs to $F(M)$. The same applies to $\gamma_2 : M \to Q$, $\gamma_2(x) = (0, x)$. By construction any element of Q is a sum of two elements of $\gamma_j(M)$. Let $z \in \gamma_1(M) \cap \gamma_2(M)$, $z = \gamma_1(x) = \gamma_2(y)$. Since the pair $(\mathrm{Id}_M, \mathrm{Id}_M)$ belongs to $F(M)$ one gets $x = y$. By Lemma 2.6 one gets $x \in N$ which shows that the intersection of the $\gamma_j(M)$ is $\gamma_1(N) = \gamma_2(N)$.

(*iii*) We compare $\mathrm{Hom}_{\mathbb{B}}(Q, X)$ and $F \circ \kappa(X) = F(X)$. First let $(f, g) \in F(X)$, then the map $E = M \times M \to X$, $(x, y) \mapsto f(x) + g(y)$ is compatible with the equivalence relation $\sim$ and thus induces a map $r(f, g) : Q \to X$. One has $r(f, g) \circ \gamma_1 = f$ and $r(f, g) \circ \gamma_2 = g$. Conversely, let $\rho \in \mathrm{Hom}_{\mathbb{B}}(Q, X)$, then let $f = \rho \circ \gamma_1$, $g = \rho \circ \gamma_2$. The equivalence $(x, 0) \sim (0, x)$ for $x \in N$ shows that $f(x) = g(x)$ for $x \in N$, *i.e.* $(f, g) \in F(X)$. Denote $\gamma(\rho) := (\rho \circ \gamma_1, \rho \circ \gamma_2)$. Consider the maps $r : F(X) \to \mathrm{Hom}_{\mathbb{B}}(Q, X)$ and $\gamma : \mathrm{Hom}_{\mathbb{B}}(Q, X) \to F(X)$. For $\rho \in \mathrm{Hom}_{\mathbb{B}}(Q, X)$ one has $\rho = r(\gamma(\rho))$ since any $z \in Q$ is of the form $z = (x, y) = \gamma_1(x) + \gamma_2(y)$ while $\rho(z) = f(x) + g(y)$, where $f = \rho \circ \gamma_1$, $g = \rho \circ \gamma_2$. Let similarly $(f, g) \in F(X)$ then $r(f, g) \circ \gamma_1 = f$ and $r(f, g) \circ \gamma_2 = g$, so that $(f, g) = \gamma(r(f, g))$. Thus the maps r and γ are inverse of each other and give a natural identification $F(X) \simeq \mathrm{Hom}_{\mathbb{B}}(Q, X)$. $\square$

Remark 4.19. Consider the sequence (42). As in Definition 3.5, the equalizer of the β_j, $\iota : \mathrm{Equ}(\beta_1, \beta_2) \to N$ is a subsemimodule of N and the coequalizer of the α_j is a quotient of N, *i.e.* $\gamma : N \to \mathrm{Coequ}(\alpha_1, \alpha_2)$. One can thus consider the composition $\gamma \circ \iota : \mathrm{Equ}(\beta_1, \beta_2) \to \mathrm{Coequ}(\alpha_1, \alpha_2)$ which is a morphism of $\mathbb{B}$-semimodules, and define the *weak cohomology* H_N^{weak} at N of the sequence (42) as:

$$H_N^{\mathrm{weak}} := \mathrm{Range}(\gamma \circ \iota), \quad \gamma \circ \iota : \mathrm{Equ}(\beta_1, \beta_2) \to \mathrm{Coequ}(\alpha_1, \alpha_2). \tag{46}$$

If the sequence (42) is strictly exact at N then $H_N^{\mathrm{weak}} = 0$. Indeed, if $H_N^{\mathrm{weak}} \neq 0$ there exists a $\mathbb{B}$-semimodule E and $\phi \in \mathrm{Hom}_{\mathbb{B}}(N, E)$ such that $\phi \circ \alpha_1 = \phi \circ \alpha_2$, while the restriction of ϕ to $\mathrm{Equ}(\beta_1, \beta_2)$ is non-zero, *i.e.* there exists $t \in N$, $\beta_1(t) = \beta_2(t)$ and $\phi(t) \neq 0$. One then has $(t, 0) \in Z(\beta_1, \beta_2)$ while $(t, 0) \notin B(\alpha_1, \alpha_2) + \Delta$. Indeed otherwise let $x, y \in M$, $z \in N$ such that $(t, 0) = (\alpha_1(x) + \alpha_2(y) + z, \alpha_2(x) + \alpha_1(y) + z)$. By applying ϕ one gets that $(\phi(t), 0)$ is diagonal which contradicts $\phi(t) \neq 0$.

The converse does not hold since for ψ the maximal element of M^* the sequence: $0 \underset{0}{\overset{0}{\rightrightarrows}} M \underset{0}{\overset{\psi}{\rightrightarrows}} \mathbb{B} \underset{0}{\overset{0}{\rightrightarrows}} 0$ fulfills $H_M^{\mathrm{weak}} = 0$ and $H_{\mathbb{B}}^{\mathrm{weak}} = 0$ but by Corollary 4.7 this sequence is strictly exact only when $M = \mathbb{B}$.

5. The Eilenberg-Moore category $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ of the comonad $\perp$

In this Section (*cf.* §§5.1,5.2) we take up the problem of the representability of the functor associated to quotients of objects of $\mathbb{B}\mathrm{mod}$ and we provide a solution by considering the natural extension of $\mathbb{B}\mathrm{mod}^2$ to the Eilenberg-Moore category $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ of Proposition 3.12 which is simply the category $\mathbb{B}$mod in the topos of "sets endowed with an involution" (and as such shares with $\mathbb{B}$mod most of its abstract categorical properties).

Definition 5.1. *Let $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ be the category of $\mathbb{B}$-semimodules endowed with an involution σ. The morphisms in $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ are the morphisms of $\mathbb{B}$-semimodules commuting with σ, i.e. equivariant for the action of $\mathbb{Z}/2\mathbb{Z}$.*

In §3.4 we provided the conceptual meaning of the construction of $\mathbb{B}\mathrm{mod}^2$ and $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ from the category $\mathbb{B}$mod. In §5.1 we provide another direct construction of $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ based on the need to represent the functor associated to quotients. In §5.2 we prove the required representability. In §5.3 we analyze the monad T on $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ corresponding to the adjunction and show its simplifying role. The notion of strict exact sequence extends naturally from $\mathbb{B}\mathrm{mod}^2$ to $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ and we end the section with a table comparing these two categories.

5.1 Extending the category $\mathbb{B}\mathrm{mod}^2$.

For any $\mathbb{B}$-semimodule M one gets the covariant functor $y_M(-) : \mathbb{B}\mathrm{mod}^2 \longrightarrow \mathbb{B}\mathrm{mod}$, $y_M(N) = \mathrm{Hom}_{\mathbb{B}\mathrm{mod}^2}(M, N)$. For $M = \mathbb{B}$ the functor $y_{\mathbb{B}}(-)$ lands in a finer category than $\mathbb{B}$mod since one can use the group $S := \mathrm{Aut}_{\mathbb{B}\mathrm{mod}^2}(\mathbb{B})$ to act on the right on morphisms. The only non-trivial element of S is $\sigma = (0, \mathrm{Id})$ and it has order two. One has $(f, g) \circ (0, \mathrm{Id}) = (g, f)$, thus this action exchanges the two copies of N in $y_{\mathbb{B}}(N) = \mathrm{Hom}_{\mathbb{B}\mathrm{mod}^2}(\mathbb{B}, N) = N \times N$. One thus obtains a functor $y_{\mathbb{B}}(-) : \mathbb{B}\mathrm{mod}^2 \longrightarrow \mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ which embeds the category $\mathbb{B}\mathrm{mod}^2$ as a full subcategory of $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$.

Lemma 5.2. *(i) The functor $y_{\mathbb{B}}(-) : \mathbb{B}\mathrm{mod}^2 \longrightarrow \mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ associates to a $\mathbb{B}$-semimodule N the square $y_{\mathbb{B}}(N) = Hom_{\mathbb{B}\mathrm{mod}^2}(\mathbb{B}, N) = N \times N$ endowed with the involution which exchanges the two copies of N and to a morphism $\phi = (f, g) \in Hom_{\mathbb{B}\mathrm{mod}^2}(N, N')$ the following map $y_{\mathbb{B}}(\phi) : y_{\mathbb{B}}(N) \to y_{\mathbb{B}}(N')$*

$$N \times N \ni (x, y) \mapsto (f(x) + g(y), f(y) + g(x)) \in N' \times N'. \tag{47}$$

(ii) The functor $y_{\mathbb{B}}(-) : \mathbb{B}\mathrm{mod}^2 \longrightarrow \mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ is fully faithful.

(iii) For any morphism $\phi = (f, g) \in Hom_{\mathbb{B}\mathrm{mod}^2}(N, N')$ one has

$$B(\phi) = \mathrm{Range}(y_{\mathbb{B}}(\phi)), \quad Z(\phi) = y_{\mathbb{B}}(\phi)^{-1}(\Delta) \tag{48}$$

where Δ is the diagonal in $N' \times N'$.

Proof. (i) An element of $y_{\mathbb{B}}(N) = \mathrm{Hom}_{\mathbb{B}\mathrm{mod}^2}(\mathbb{B}, N)$ is given by a pair of morphisms of $\mathbb{B}$-semimodules from $\mathbb{B}$ to N and hence characterized by a pair of elements of N. The composition of morphisms in $\mathbb{B}\mathrm{mod}^2$ can be seen easily to give (47).

(ii) Let $\phi = (f, g), \phi' = (f', g')$ be elements of $\mathrm{Hom}_{\mathbb{B}\mathrm{mod}^2}(N, N')$ such that $y_{\mathbb{B}}(\phi) = y_{\mathbb{B}}(\psi)$. One then gets, taking $y = 0$ in (47) $(f(x), g(x)) = (f'(x), g'(x))$, $\forall x \in N$. This proves that $\phi = \phi'$. We prove that $y_{\mathbb{B}}(-)$ is full *i.e.* that for any morphism $\rho : y_{\mathbb{B}}(N) \to y_{\mathbb{B}}(N')$ commuting with σ there exists a morphism $\phi \in \mathrm{Hom}_{\mathbb{B}\mathrm{mod}^2}(N, N')$ such that $\rho = y_{\mathbb{B}}(\phi)$. The restriction of ρ to $N \times \{0\} \subset N \times N = y_{\mathbb{B}}(N)$ is given by a pair of morphisms $f, g \in \mathrm{Hom}_{\mathbb{B}}(N, N')$. Since ρ commutes with σ one has $\rho((x, y)) = \rho((x, 0)) + \rho(\sigma((y, 0))) = (f(x), g(x)) + \sigma((f(y), g(y))$ which shows by (47) that $\rho = y_{\mathbb{B}}(\phi)$ for $\phi = (f, g)$.

(iii) Follows from (47) and Propositions 4.2 and 4.3. ∎

Thus the notion of strict exact sequence in $\mathbb{B}\mathrm{mod}^2$ can be re-interpreted, through the functor $y_{\mathbb{B}}(-) : \mathbb{B}\mathrm{mod}^2 \longrightarrow \mathbb{B}\mathrm{mod}^{\mathfrak{s}}$, in terms of the notions of range and of inverse image in the category $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ but involving the additional structure given by the diagonal in the square which replaces the zero element for abelian groups. This implies that rather than asking the composition of two consecutive maps in a complex to be zero, one requires instead $\mathrm{Range}(y_{\mathbb{B}}(\phi \circ \psi)) \subset \Delta$ or equivalently $\mathrm{Range}(y_{\mathbb{B}}(\psi)) \subset y_{\mathbb{B}}(\phi)^{-1}(\Delta)$, *i.e.* $B(\psi) \subset Z(\phi)$. Moreover Proposition 4.10 can be now re-interpreted stating that a morphism $\phi \in \mathrm{Hom}_{\mathbb{B}\mathrm{mod}^2}(N, N')$ is a monomorphism iff $y_{\mathbb{B}}(\phi)$ is injective, *i.e.* a monomorphism in $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$. Similarly, Proposition 4.13 now states that a morphism $\phi \in \mathrm{Hom}_{\mathbb{B}\mathrm{mod}^2}(N, N')$ is an epimorphism in $\mathbb{B}\mathrm{mod}^2$ iff $y_{\mathbb{B}}(\phi)$ is surjective, *i.e.* an epimorphism in $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$.

Next lemma states that the forgetful functor $I : \mathbb{B}\mathrm{mod}^{\mathfrak{s}} \longrightarrow \mathbb{B}\mathrm{mod}$ is left adjoint to the squaring functor $\mathfrak{s} = y_{\mathbb{B}} \circ \kappa : \mathbb{B}\mathrm{mod} \to \mathbb{B}\mathrm{mod}^{\mathfrak{s}}$, $\mathfrak{s}(M) = (M^2, \sigma)$, which is obtained by composition of $y_{\mathbb{B}}$ with the functor $\kappa : \mathbb{B}\mathrm{mod} \longrightarrow \mathbb{B}\mathrm{mod}^2$ of (31).

Lemma 5.3. *Let $I : \mathbb{B}\mathrm{mod}^{\mathfrak{s}} \longrightarrow \mathbb{B}\mathrm{mod}$ be the forgetful functor. The composition with the first projection gives, for any object M of $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ and any object N of $\mathbb{B}\mathrm{mod}$, a canonical isomorphism*

$$\pi : Hom_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(M, \mathfrak{s}(N)) \to Hom_{\mathbb{B}}(I(M), N). \tag{49}$$

which is natural in M and N.

Proof. Let $\phi \in \mathrm{Hom}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(M, \mathfrak{s}(N))$, then the composition $p_1 \circ \phi$ with the first projection fulfills $p_1 \circ \phi \in \mathrm{Hom}_{\mathbb{B}}(I(M), N)$. Let $\psi \in \mathrm{Hom}_{\mathbb{B}}(I(M), N)$, define $\tilde{\psi} \in \mathrm{Hom}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(M, \mathfrak{s}(N))$ by $\tilde{\psi}(z) = (\psi(z), \psi(\sigma(z)))$ for $z \in M$. It commutes with σ by construction and $p_1 \circ \tilde{\psi} = \psi$. Moreover, for $\phi \in \mathrm{Hom}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(M, \mathfrak{s}(N))$ one has for any $z \in M$, $\phi(z) = (p_1 \circ \phi(z), p_2 \circ \phi(z)) = (p_1 \circ \phi(z), p_1 \circ \phi(\sigma(z)))$, so that $\phi = \widetilde{p_1 \circ \phi}$. By construction the isomorphism π is natural in M. The naturality in N follows from $g \circ p_1 = p_1 \circ \mathfrak{s}(g)$, $\forall g \in \mathrm{Hom}_{\mathbb{B}}(N, N')$. $\square$

To the adjunction $I \dashv \mathfrak{s}$ corresponds the comonad $\perp$ for the category $\mathbb{B}\mathrm{mod}$ of Proposition 3.11. Indeed, one has $\perp = I \circ \mathfrak{s}$ which proves 1 (of Proposition 3.11). The counit of the adjunction is the first projection and this gives 2. Using the unit η of the adjunction, one gets the coproduct δ as $I \circ \eta \circ \mathfrak{s} : I \circ \mathfrak{s} \longrightarrow I \circ \mathfrak{s} \circ I \circ \mathfrak{s}$ and since $\sigma((x, y)) = (y, x)$ one obtains 3. The adjunction[1] $I \dashv \mathfrak{s}$ is comonadic and this corresponds to the construction of $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ in §3.4.

5.2 Representability in $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$. Next, for a subsemimodule $N \subset M$ of a $\mathbb{B}$-semimodule M we shall show that the covariant functor $F = M/N : \mathbb{B}\mathrm{mod}^2 \longrightarrow \mathbb{B}\mathrm{mod}$ (*cf.* (43))

$$F(X) = \{(f, g) \mid f, g \in \mathrm{Hom}_{\mathbb{B}}(M, X),\ f|_N = g|_N\}. \tag{50}$$

extends to a representable functor $\mathrm{Hom}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(Q, -) : \mathbb{B}\mathrm{mod}^{\mathfrak{s}} \longrightarrow \mathbb{B}\mathrm{mod}$. Thus we look for an object Q of $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ such that, for any $\mathbb{B}$-semimodule X, one has a natural isomorphism,

$$F(X) \simeq \mathrm{Hom}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(Q, y_{\mathbb{B}}(X)) \tag{51}$$

We first reconsider Example 4.17 and take the category $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ into account.

[1] This remark was pointed to us by M. Grandis

Example 5.4. Let $M = \{0, 1, 2\}$, $N = \{0, 2\} \subset M$ and Q as in Example 4.17. A morphism $\phi = (f, g) \in \mathrm{Hom}_{\mathbb{B}\mathrm{mod}^2}(X, X')$ maps (a, b, c) to $(f(a) + g(b), f(b) + g(a), f(c) + g(c))$ and this can be re-interpreted in the following form

$$\mathrm{Hom}_{\mathbb{B}}(Q, X) \ni \tau \mapsto f \circ \tau + g \circ \tau \circ \sigma$$

where $\sigma \in \mathrm{Aut}_{\mathbb{B}\mathrm{mod}}(Q)$ is the involution which interchanges α and β and fixes γ. This suggests to view Q as an object of $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ and to compare $F(X)$ with $\mathrm{Hom}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(Q, y_{\mathbb{B}}(X))$. Let $\rho \in \mathrm{Hom}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(Q, y_{\mathbb{B}}(X))$, then with $\rho(\alpha) = (a, b) \in X \times X$, one has $\rho(\beta) = \rho(\sigma(\alpha)) = \sigma((a, b)) = (b, a) \in X \times X$, while $\rho(\gamma)$ is fixed by σ and hence diagonal, *i.e.* of the form $\rho(\gamma) = (c, c)$ with $a \vee c = c$, $b \vee c = c$. Thus one obtains: $F(X) \simeq \mathrm{Hom}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(Q, y_{\mathbb{B}}(X))$, and the composition $\tau \mapsto f \circ \tau + g \circ \tau \circ \sigma$ corresponds to (47). This shows the representability of the functor F in the category $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$.

The above example suggests that the quotient functor M/N ought to be representable in the category $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$. We take the notations of Proposition 4.18.

Proposition 5.5. *(i) The involution $(x, y) \mapsto (y, x)$ on $E = M \times M$ induces an involution σ on $Q \simeq \mathrm{Coker}_p(\iota)$ which turns $\mathrm{Coker}_p(\iota)$ into an object of $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$.*

(ii) The functor $F : \mathbb{B}\mathrm{mod}^2 \longrightarrow \mathbb{B}\mathrm{mod}$, associated by (50) to the quotient "M/N" is represented in the category $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ by the object $\mathrm{Coker}_p(\iota : N \to M) = (Q = E/\sim, \sigma)$.

Proof. (i) The congruence $\sim$ on E is compatible with the involution $\sigma((x, y)) = (y, x)$, *i.e.* $z \sim z' \Rightarrow \sigma(z) \sim \sigma(z')$ since $(f, g) \in F(X) \iff (g, f) \in F(X)$. Thus one obtains an involution, still denoted by σ, on the $\mathbb{B}$-semimodule Q and this turns it into an object of the category $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$.

(ii) Proposition 4.18 shows that one has $F(X) \simeq \mathrm{Hom}_{\mathbb{B}}(Q, X)$, with $Q = I((Q, \sigma))$. Thus by Lemma 5.3 one deduces a canonical isomorphism $F(X) \simeq \mathrm{Hom}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}((Q, \sigma), y_{\mathbb{B}}(X))$. It remains to show that this identification is compatible with the functoriality. To this end, one needs to keep track on the description of both sides after composing with a morphism $y_{\mathbb{B}}(\phi) : y_{\mathbb{B}}(X) \to y_{\mathbb{B}}(X')$ where by (47) and with $\phi = (\alpha, \beta)$, the map $y_{\mathbb{B}}(\phi)$ is given by $(x, y) \mapsto (\alpha(x) + \beta(y), \alpha(y) + \beta(x))$. Let then $(f, g) \in F(X)$, then the corresponding pair $(f', g') \in F(X')$ is given by $f' = \alpha \circ f + \beta \circ g$, $g' = \alpha \circ g + \beta \circ f$. Now, to $(f, g) \in F(X)$ corresponds the morphism $r(f, g) : Q \to X$ with $r(f, g)((x, y)) = f(x) + g(y)$. Moreover the associated morphism $\rho(f, g)$ in $\mathrm{Hom}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}((Q, \sigma), y_{\mathbb{B}}(X))$ is given by $\rho(f, g)((x, y)) = (f(x) + g(y), f(y) + g(x))$. Thus one has

$$y_{\mathbb{B}}(\phi)(\rho(f, g)((x, y))) = (\alpha(f(x) + g(y)) + \beta(f(y) + g(x)), \alpha(f(y) + g(x)) + \beta(f(x) + g(y))).$$

One checks that this is the same as

$$\rho(f', g')((x, y)) = ((\alpha \circ f + \beta \circ g)(x) + (\alpha \circ g + \beta \circ f)(y), (\alpha \circ f + \beta \circ g)(y) + (\alpha \circ g + \beta \circ f)(x))$$

This proves that the two functors are the same. □

5.3 The monad $T = \mathfrak{s} \circ I$ on the category $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ and strict exactness. We consider the monad associated to the adjunction $I \dashv \mathfrak{s}$ displayed in Lemma 5.3. We determine the counit and the unit of this adjunction. The counit is a natural transformation $I \circ \mathfrak{s} \longrightarrow \mathrm{Id}_{\mathbb{B}\mathrm{mod}}$ which associates to an object M of $\mathbb{B}\mathrm{mod}$ a morphism $\epsilon_M \in \mathrm{Hom}_{\mathbb{B}}(I \circ \mathfrak{s}(M), M)$. By Lemma 5.3, ϵ_M corresponds to the identity $\mathrm{Id} \in \mathrm{Hom}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(\mathfrak{s}(M), \mathfrak{s}(M))$. In fact the proof of that lemma shows that ϵ_M is the first projection $p_1 : M \times M \to M$.

Proposition 5.6. *The monad associated to the adjunction $I \dashv \mathfrak{s}$ is described by*

1. *The endofunctor $T : \mathbb{B}\mathrm{mod}^{\mathfrak{s}} \longrightarrow \mathbb{B}\mathrm{mod}^{\mathfrak{s}}$, $T(M,\sigma) = (M^2, \sigma_M)$, $\sigma_M(x,y) = (y,x)$, and for $f \in Hom_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(M,N)$, $T(f) = (f,f)$.*
2. *The unit $\eta : 1_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}} \to T$, $\eta_{(M,\sigma)} = ((M,\sigma) \to (M^2,\sigma_M),\ a \mapsto (a,\sigma(a)))$.*
3. *The product $\mu : T^2 \to T$, $\mu_{(M,\sigma)} = (((M^2)^2, \sigma_{M^2}) \to (M^2,\sigma_M),\ ((x,y),(x',y')) \mapsto (x,x'))$.*

Proof. From the general construction of a monad associated to an adjunction, the endofunctor is of the form $G \circ F$, where $F \dashv G$. The adjunction of Lemma 5.3 is $I \dashv \mathfrak{s}$ and thus $T = \mathfrak{s} \circ I$ which shows 1. The unit η of this adjunction is a natural transformation $\mathrm{Id}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}} \longrightarrow \mathfrak{s} \circ I$. It associates to an object (M,σ) of $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ a morphism $\eta_M \in \mathrm{Hom}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(M, \mathfrak{s} \circ I(M))$. This morphism corresponds to the identity $\mathrm{Id} \in \mathrm{Hom}_{\mathbb{B}}(I(M), I(M))$ and the proof of Lemma 5.3 shows that it is given, for every object (M,σ) of $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$, by the morphism $M \ni a \mapsto (a,\sigma(a)) \in T(M) = (M^2,\sigma_M)$. The product μ is a natural transformation $T \circ T \longrightarrow T$ and for an adjunction $F \dashv G$ is given by $G\epsilon F$ where ϵ is the counit of the adjunction. The counit of the adjunction $I \dashv \mathfrak{s}$ is given by the first projection $p_1 : M \times M \to M$. Thus $\mu = \mathfrak{s} p_1 I$ as a natural transformation $T \circ T(M) = \mathfrak{s} \circ I \circ \mathfrak{s} \circ I(M) \to T(M) = \mathfrak{s} \circ I(M)$. To obtain it one applies the functor $\mathfrak{s}$ to the morphism $\epsilon_{I(M)} \in \mathrm{Hom}_{\mathbb{B}}(I \circ \mathfrak{s}(I(M)), I(M))$. Given a morphism $f \in \mathrm{Hom}_{\mathbb{B}}(N,N')$ the morphism $\mathfrak{s}(f) \in \mathrm{Hom}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(\mathfrak{s}(N), \mathfrak{s}(N'))$ acts diagonally as $(z,z') \mapsto (f(z), f(z'))$. This gives the description 3. of the product. $\square$

To clarify the fact that the cokernel of a morphism of $\mathbb{B}\mathrm{mod}$ should be viewed as an object of $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$, we continue with the investigation of the monad $T = \mathfrak{s} \circ I$ defined in Proposition 5.6. There is a classical natural notion of an injective object in a category $\mathcal{C}$ endowed with a monad T, it is given by the following

Definition 5.7. *An object I of $\mathcal{C}$ is T-injective if the unit map $\eta_I : I \to T(I)$ has a retraction, i.e. there is a map $f : T(I) \to I$ such that $f \circ \eta_I = Id_I$.*

In our setup we derive

Lemma 5.8. *Any object of $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ is T-injective for the monad $T = \mathfrak{s} \circ I$.*

Proof. Let (M,σ) be an object of $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$. Let $r : T(M) \to (M,\sigma)$ be defined by $r((x,y)) = x + \sigma(y)$, $\forall x,y \in M$. One has $r(\sigma_M(x,y)) = r((y,x)) = y + \sigma(x) = \sigma(x + \sigma(y)) = \sigma(r((x,y)))$. Thus r is a morphism in $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$. Moreover one also has $r(\eta_M(a)) = r((a,\sigma(a))) = a + a = a$, $\forall a \in M$. This shows that r is a retraction of η_M. $\square$

Lemma 5.8 will be used below in the proof of Lemma 5.11.

In the following part, we shall recast our previous main constructions and results in terms of the category $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ and its monad structure. For example, Lemma 5.2 shows that the notion of strict exactness in the category $\mathbb{B}\mathrm{mod}^2$ as provided in Definition 4.4, corresponds to the following definition in $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$.

Definition 5.9. *A sequence $L \xrightarrow{f} M \xrightarrow{g} N$ in $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ is strictly exact at M if*

$$\mathrm{Range}(f) + M^\sigma = g^{-1}(N^\sigma). \tag{52}$$

Notice that this definition implies the weaker condition $\mathrm{Range}(g \circ f) \subset N^\sigma$. The heuristic behind is that the fixed points M^σ of an object M of $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ play the role of the zero element. The kernel and cokernel of a morphism in $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ are defined as follows

Definition 5.10. *For* $h \in Hom_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(L,M)$ *one sets* $\mathrm{Ker}(h) := h^{-1}(M^\sigma)$ *endowed with the induced involution. One also sets* $\mathrm{Coker}(h) := M/\!\sim$ *where for* $a, a' \in M$

$$a \sim a' \iff f(a) = f(a') \quad \forall X, \forall f \in Hom_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(M,X) \text{ s.t. } \mathrm{Range}(h) \subset \mathrm{Ker}(f). \tag{53}$$

The above formula defines an equivalence relation which is compatible with the addition and the involution and thus the quotient $\mathrm{Coker}(h) := M/\!\sim$ is an object of $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$. We shall show below in Proposition 6.1 that the above notions of kernel and cokernel are the natural ones in the context of homological categories of [17]. What remains to be seen is whether this notion of cokernel is compatible with the notion given earlier on, in Definition 3.8 *i.e.* as $\mathrm{Coker}_p\, h = \mathrm{Coequ}(s_1 \circ h, s_2 \circ h)$. The issue is to control the morphisms $f \in \mathrm{Hom}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(y_{\mathbb{B}}(M), X)$ coming from objects X of $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ which are not in the range of $y_{\mathbb{B}}$. This is indeed possible thanks to Lemma 5.8 and one obtains

Lemma 5.11. *Let* $h \in Hom_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(L,M)$*, then the equivalence relation* $\sim$ *of* (53) *defining its cokernel is the same as the following, for* $a, a' \in M$

$$a \sim a' \iff f(a) = f(a') \quad \forall X, \forall f \in Hom_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(M,T(X)) \text{ s.t. } \mathrm{Range}(h) \subset \mathrm{Ker}(f). \tag{54}$$

Proof. Let X be an object of $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$. We show that if $g(a) = g(a')\ \forall g \in \mathrm{Hom}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(M,T(X))$ s.t. $\mathrm{Range}(h) \subset \mathrm{Ker}(g)$, one has $f(a) = f(a')\ \forall f \in \mathrm{Hom}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(M,X)$ s.t. $\mathrm{Range}(h) \subset \mathrm{Ker}(f)$. Let η_X and r be as in Lemma 5.8. Let $f \in \mathrm{Hom}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(M,X)$ s.t. $\mathrm{Range}(h) \subset \mathrm{Ker}(f)$ and $g = \eta_X \circ f$. One has $g \in \mathrm{Hom}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(M,T(X))$ s.t. $\mathrm{Range}(h) \subset \mathrm{Ker}(g)$, since $\mathrm{Ker}(f) \subset \mathrm{Ker}(\eta_X \circ f)$. Thus one has by hypothesis $g(a) = g(a')$. Since $r \circ g = r \circ \eta_X \circ f = f$, one concludes that $f(a) = f(a')$. $\square$

Proposition 5.12. *Let $f : M \to N$ be a morphism in $\mathbb{B}$mod.*

(i) $\mathrm{Ker}_p f = (\mathrm{Equ} f^{(2)}, \iota_1, \iota_2)$ *as in* (22)*, endowed with its canonical involution is isomorphic as an object of* $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ *to* $Ker(\mathfrak{s} f)$ *as in Definition 5.10.*

(ii) $\mathrm{Coker}_p f = \mathrm{Coequ}(s_1 \circ f, s_2 \circ f)$ *as in Definition 3.8, endowed with its canonical involution, is isomorphic as an object of* $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ *to* $\mathrm{Coker}(\mathfrak{s} f)$ *as in* (53).

Proof. (i) By definition $\mathrm{Equ} f^{(2)}$ is the equalizer of $(f \circ p_1, f \circ p_2)$: $\mathrm{Equ} f^{(2)} \to M^2 \underset{f\circ p_2}{\overset{f\circ p_1}{\rightrightarrows}} N$, and coincides as a subobject of M^2 with $(\mathfrak{s} f)^{-1}(\Delta)$ where $\Delta = (N^2)^\sigma$ is the diagonal.

(ii) By Lemma 5.11, and using $T = \mathfrak{s} \circ I$, $\mathrm{Coker}(\mathfrak{s} f)$ of (53) is the quotient of $\mathfrak{s} N = N^2$ by the equivalence relation, where X varies among objects of $\mathbb{B}$mod, and $a, a' \in \mathfrak{s} N$

$$a \sim a' \iff \phi(a) = \phi(a') \quad \forall X, \forall \phi \in \mathrm{Hom}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(\mathfrak{s}(N), \mathfrak{s}(X)) \text{ s.t. } \mathrm{Range}(\mathfrak{s} f) \subset \mathrm{Ker}(\phi). \tag{55}$$

By Lemma 5.2 one has a natural isomorphism $\mathrm{Hom}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(\mathfrak{s}(N), \mathfrak{s}(X)) \simeq \mathrm{Hom}_{\mathbb{B}\mathrm{mod}^2}(N, X)$ defined by the map which associates to a morphism $\phi \in \mathrm{Hom}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(\mathfrak{s}(N), \mathfrak{s}(X))$ given by the matrix $\begin{pmatrix} u & v \\ v & u \end{pmatrix}$ the morphism in $\mathbb{B}\mathrm{mod}^2$ given by $N \underset{v}{\overset{u}{\rightrightarrows}} X$. The condition $\mathrm{Range}(\mathfrak{s} f) \subset \mathrm{Ker}(\phi)$ is equivalent to $u \circ f = v \circ f$, since it means that $\phi \circ \mathfrak{s} f$ is null (*i.e.* fixed by the involution). This condition means exactly that the pair (u, v) defines a morphism $N^2 \to X$ which coequalizes the $s_j \circ f$. Thus the equivalence relation (55) in $\mathfrak{s} N = N^2$ takes the form

$$(x, y) \sim (x', y') \iff u(x) + v(y) = u(x') + v(y')\,, \;\; \forall u, v \in \mathrm{Hom}_{\mathbb{B}}(N, X) \text{ s.t. } u \circ f = v \circ f.$$

This is exactly the same as the equivalence relation which defines the coequalizer of the $s_j \circ f$. □

When $\sigma_M = \mathrm{Id}$, the strict exactness in $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ of the sequence $L \xrightarrow{f} M \xrightarrow{g} N$ is automatic. This shows that the strict exactness of $0 \to L \xrightarrow{f} M$ in $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ does not imply that f is a monomorphism, and similarly that the strict exactness of $M \xrightarrow{f} N \to 0$ in $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ does not imply that f is an epimorphism. The application of the monad T improves these statements considerably.

Proposition 5.13. *Let* $f \in Hom_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(L, M)$*, then*

1. *f is a monomorphism $\iff$ f is injective $\iff$ $0 \to TL \xrightarrow{Tf} TM$ is strictly exact.*
2. *f is an epimorphism $\iff$ f is surjective $\iff$ $TL \xrightarrow{Tf} TM \to 0$ is strictly exact.*

Proof. 1. f is a monomorphism if the underlying map of $\mathbb{B}$-semimodules is injective. Conversely, for $a \in L$ there exists a unique morphism $\xi_a : (\mathbb{B} \times \mathbb{B}, \sigma_{\mathbb{B}}) \to (L, \sigma)$ such that $\xi_a((1, 0)) = a$. Thus if for some $a \neq b \in L$ one has $f(a) = f(b)$, f cannot be a monomorphism. Hence f is a monomorphism in $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ $\Leftrightarrow$ $I(f)$ is a monomorphism in $\mathbb{B}$mod $\Leftrightarrow$ f is injective. Moreover, the strict exactness of $0 \to TL \xrightarrow{Tf} TM$ is equivalent to the injectivity of f. Indeed one has $T(f) = y_{\mathbb{B}}((I(f), 0))$ since $T = \mathfrak{s} \circ I = y_{\mathbb{B}} \circ \kappa \circ I$ and the equivalence follows from (ii) of Proposition 4.6 since the faithful functor $y_{\mathbb{B}}$ preserves strict exactness.

2. f is an epimorphism if the underlying map of $\mathbb{B}$-semimodules is surjective. Conversely, if the underlying map of $\mathbb{B}$-semimodules is not surjective there exists by Lemma 2.6 a $\mathbb{B}$-semimodule X and a pair $h \neq k$ of morphisms of $\mathbb{B}$-semimodules, $h, k \in \mathrm{Hom}_{\mathbb{B}}(M, X)$, such that $h \circ f = k \circ f$. Then the corresponding morphisms $\tilde h(x) = (h(x), h(\sigma(x))$, $\tilde k(x) = (k(x), k(\sigma(x))$ fulfill

$$\tilde h,\, \tilde k \in \mathrm{Hom}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(M, (X \times X, \sigma_X)), \quad \tilde h \circ f = \tilde k \circ f, \quad \tilde h \neq \tilde k$$

where the equality $h(\sigma(f(x))) = k(\sigma(f(x)))$ follows from $\sigma(f(x)) = f(\sigma(x))$. This shows that f is an epimorphism in $\mathbb{B}\mathrm{mod}^{\mathfrak{s}} \Leftrightarrow I(f)$ is an epimorphism in $\mathbb{B}\mathrm{mod} \Leftrightarrow f$ is surjective. Finally, the strict exactness of $TL \stackrel{Tf}{\to} TM \to 0$ is equivalent to the surjectivity of f by (ii) of Proposition 4.5. ☐

In the following table we compare the interpretation of several definitions in the categories $\mathbb{B}\mathrm{mod}^2$ and $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$.

Category $\mathbb{B}\mathrm{mod}^2$	**Category** $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$
$\phi = (u, v) \in \mathrm{Hom}_{\mathbb{B}\mathrm{mod}^2}(M, N)$	$h \in \mathrm{Hom}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(M, N)$
$Z(\phi) = Z(u, v)$	$\mathrm{Ker}(h) = h^{-1}(N^\sigma)$
$B(\phi) = \{(u(x) + v(y), u(y) + v(x))\}$	$\mathrm{Range}(h) = h(M)$
Diagonal $\Delta \subset M \times M$	Fixed points M^σ
strict exactness at M of $L \underset{\alpha_2}{\overset{\alpha_1}{\rightrightarrows}} M \underset{\beta_2}{\overset{\beta_1}{\rightrightarrows}} N$ $B(\alpha_1, \alpha_2) + \Delta = Z(\beta_1, \beta_2)$	strict exactness at M of $L \xrightarrow{f} M \xrightarrow{g} N$ $\mathrm{Range}(f) + M^\sigma = \mathrm{Ker}(g)$
$\beta_1 \circ \alpha_1 + \beta_2 \circ \alpha_2 = \beta_2 \circ \alpha_1 + \beta_1 \circ \alpha_2$	$\mathrm{Range}(f) \subset \mathrm{Ker}(g)$

6. The category $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ and the null morphisms

By construction, $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ is the category $\mathbb{B}\mathrm{mod}$ in the topos of sets endowed with an involution. The category $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ contains, as a subcategory stable under retract, the category $\mathbb{B}\mathrm{mod}$ of those objects of $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$, called "null objects", whose involution is the identity. The morphisms which factor through a null object are called the null morphisms. They form an ideal [13, 20] in the category $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$. In §6.1 we show that one obtains in this way a homological category in the sense of [18]. In §6.2 we prove two results which control the least normal subobject containing a given subobject: *cf.* Proposition 6.16 and Proposition 6.17. Finally, in §6.3 (Proposition 6.19) we provide an explicit description of the cokernel of morphisms in $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$.

6.1 $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ **as a semiexact homological category.** The general notion of semiexact category has been developed by Grandis in [17,18]. A category $\mathcal{E}$ is called **semiexact** if it fulfills the following conditions (ex0) & (ex1).

(ex 0) There is a **closed, 2-sided ideal** $\mathcal{N}$ of morphisms in $\mathcal{E}$ s.t. $\mathcal{N} = N(\mathcal{O})$ for some class of objects $\mathcal{O}$ where $N(\mathcal{O}) := \{f \in \mathrm{Mor}(\mathcal{E}) \mid f$ factors through some object in $\mathcal{O}\}$.

(ex 1) All morphisms in $\mathcal{E}$ have kernels and cokernels **with respect to** $\mathcal{N}$**:** *i.e.* kernels and cokernels of morphisms in $\mathcal{E}$ fulfill the universal property w.r.t $\mathcal{N}$.

We view $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ as the category of $\mathbb{B}[s]$-semimodules where $\mathbb{B}[s]$ is the semiring generated over $\mathbb{B}$ by s, $s^2 = 1$ ([15], p 71 and [1], Definition 2.18). One has $\mathbb{B}[s] = \{0, 1, s, p\}$ where $p = 1 + s$, $p^2 = p$. We use the notion of null morphisms associated to the ideal $N = \{0, p\} \subset \mathbb{B}[s]$. We shall show that the category $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ is semiexact.

Proposition 6.1. *The pair given by the category* $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ *and the null morphisms:* $\mathcal{N} \subset$ $Hom_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(L, M)$

$$f \in \mathcal{N} \iff f(x) = \sigma(f(x))\,, \quad \forall x \in L$$

forms a semiexact category in the sense of [17, 18]. The corresponding notions of kernel and cokernel are the same as in Definition 5.10.

Proof. First we show that $\mathcal{N}$ is a closed ideal in $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$. For $f \in \mathcal{N}$ one has $g \circ f \in \mathcal{N}$, $\forall g \in \mathrm{Hom}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(M, N)$ since $g(M^\sigma) \subset N^\sigma$. One has also $f \circ h \in \mathcal{N}$, $\forall h \in \mathrm{Hom}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(E, L)$. The closedness means that any null morphism factors through a null identity. This is the case here since any $f \in \mathcal{N} \subset \mathrm{Hom}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(L, M)$ factors through M^σ which is a null object (the identity morphism belongs to $\mathcal{N}$). In fact any $f \in \mathcal{N} \subset \mathrm{Hom}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(L, M)$ also factors through L^σ using the projection $p : L \to L^\sigma$, $p(x) := x + \sigma(x)$. Next we prove that every morphism has a kernel and cokernel with respect to $\mathcal{N}$. In general the kernel $\ker(f) : \mathrm{Ker}(f) \to L$ is characterized by

$$f \circ \ker(f) \in \mathcal{N}, \quad f \circ g \in \mathcal{N} \iff \exists!\, h \text{ s.t. } g = \ker(f) \circ h.$$

For $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ the condition $f \circ g \in \mathcal{N}$ means that the range of g is contained in $f^{-1}(M^\sigma)$ and this gives the required unique factorization $g = \ker(f) \circ h$, where $\ker(f) : f^{-1}(M^\sigma) \to L$ is the inclusion. Thus one gets the agreement with Definition 5.10.

The cokernel $\mathrm{cok}(f) : M \to \mathrm{Coker}(f)$ is characterized in turn by

$$\mathrm{cok}(f) \circ f \in \mathcal{N}, \quad g \circ f \in \mathcal{N} \iff \exists!\, h \text{ s.t. } g = h \circ \mathrm{cok}(f).$$

We have defined in (53) the cokernel as the quotient $\mathrm{Coker}(f) := M/\!\sim$ where for $b, b' \in M$

$$b \sim b' \iff g(b) = g(b') \quad \forall X, \forall g \in \mathrm{Hom}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(M, X) \text{ s.t. } g \circ f \in \mathcal{N}. \tag{56}$$

Let $\mathrm{cok}(f) : M \to \mathrm{Coker}(f)$ be the quotient map. One has $\mathrm{cok}(f) \circ f \in \mathcal{N}$, since for $b = f(a)$ one has $b \sim \sigma(b)$ as $g(b) = g(\sigma(b))$ for any g such that $g \circ f \in \mathcal{N}$. Indeed

$$g(b) = (g \circ f)(a) = \sigma((g \circ f)(a)) = g(\sigma(f(a)) = g(\sigma(b)).$$

Moreover, by construction, any g s.t. $g \circ f \in \mathcal{N}$ factors uniquely as $g = h \circ \mathrm{cok}(f)$. We have thus shown that every morphism has a kernel and cokernel with respect to $\mathcal{N}$ and that they agree with Definition 5.10. $\square$

Definition 6.2. *(i) Let $f \in Hom_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(L, M)$, then the normal image $\overline{\mathrm{Im}}(f) \subset M$ is the kernel of the cokernel* $\mathrm{cok}(f)$.

(ii) We say that a sequence of $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$*: $L \xrightarrow{f} M \xrightarrow{g} N$ is* exact *at M if $\overline{\mathrm{Im}}(f) = Ker(g)$.*

Notice that this definition of the normal image as the kernel of the cokernel corresponds to the definition of the image in an abelian category. By definition the cokernel of f is the quotient of M by the relation (53). Thus the kernel of the cokernel is given by the elements of M whose image in the cokernel is fixed under σ. Thus one has

$$b \in \overline{\mathrm{Im}}(f) \iff g(b) = \sigma(g(b)), \ \forall X, \ g \in \mathrm{Hom}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(M, X) \text{ s.t. } \mathrm{Range}(f) \subset \mathrm{Ker}(g). \tag{57}$$

Proposition 6.3. *Let $f \in Hom_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(L, M)$.*

(i) The sequence $0 \to L \xrightarrow{f} M$ is exact at L if and only if it is strictly exact at L.

(ii) $\mathrm{Range}(f) + M^\sigma \subset \overline{\mathrm{Im}}(f)$.

(iii) $\xi + \sigma(\xi) \in \mathrm{Range}(f) + \sigma(\xi)$, $\forall \xi \in \overline{\mathrm{Im}}(f)$.

(iv) Strict exactness (in the sense of Definition 5.9) implies exactness.

(v) Proposition 5.13 continues to hold if one replaces strict exactness by exactness.

(vi) The following sequence is exact in $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$

$$0 \to \mathrm{Ker}(f) \to L \xrightarrow{f} M \to \mathrm{Coker}(f) \to 0. \tag{58}$$

Proof. (i) Strict exactness of $0 \to L \xrightarrow{f} M$ means that $L^\sigma = \mathrm{Ker}(f)$, while exactness of $0 \to L \xrightarrow{f} M$ means that $\overline{\mathrm{Im}}(0) = \mathrm{Ker}(f)$. Thus it is enough to show that for any object F of $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ one has $\overline{\mathrm{Im}}(0) = F^\sigma$. One has $F^\sigma \subset \overline{\mathrm{Im}}(0)$. Moreover the kernel of the identity map $\mathrm{Id} : F \to F$ is F^σ and thus $\overline{\mathrm{Im}}(0) \subset F^\sigma$.

(ii) This follows from (57).

(iii) Let $E = \mathrm{Range}(f)$ and $\zeta = \xi + \sigma(\xi)$. Assume $\zeta = \xi + \sigma(\xi) \notin \xi + E$. We show that $\xi \notin \overline{\mathrm{Im}}(f)$. The result will then follow since $\overline{\mathrm{Im}}(f)$ is σ-invariant. Let $P = \{\alpha \in M \mid \exists \eta \in \xi + E, \ \alpha \le \eta \le \zeta\}$. By construction P is hereditary and is a sub-$\mathbb{B}$-semimodule since for $\alpha \le \eta \le \zeta$, $\alpha' \le \eta' \le \zeta$ one has $\alpha + \alpha' \le \eta + \eta' \le \zeta$, $\ \eta + \eta' \in \xi + E$. Let $\omega \in \mathrm{Hom}_{\mathbb{B}}(M, \mathbb{B})$ be such that $\omega(\alpha) = 0 \Leftrightarrow \alpha \in P$. One has $P \cap E = [0, \zeta] \cap E$, since for $\alpha \in E$, $\alpha \le \zeta$, $\eta = \xi + \alpha$ one has $\eta \in \xi + E$, $\alpha \le \eta \le \zeta$. Thus one gets $\omega(\sigma(\alpha)) = \omega(\alpha)$ for any $\alpha \in E$. Since $\xi \in P$ (with $\eta = \xi$) one has $\omega(\xi) = 0$. One has by hypothesis that $\zeta = \xi + \sigma(\xi) \notin \xi + E$. Thus $\omega(\zeta) = 1$ and it follows that $\omega(\sigma(\xi)) = 1$. Let then $h \in \mathrm{Hom}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(M, \mathfrak{s}\mathbb{B})$ be given by $h(\alpha) = (\omega(\alpha), \omega(\sigma(\alpha))) \ \forall \alpha \in M$. Since $\omega(\sigma(\alpha)) = \omega(\alpha)$ for any $\alpha \in E$, one has $h(E) \subset (\mathfrak{s}\mathbb{B})^\sigma$ while $h(\xi) = (0, 1) \notin (\mathfrak{s}\mathbb{B})^\sigma$.

(iv) Strict exactness means $\mathrm{Range}(f) + M^\sigma = \mathrm{Ker}(g)$ and by (ii) $\mathrm{Range}(f) + M^\sigma \subset \overline{\mathrm{Im}}(f)$ with $\overline{\mathrm{Im}}(f) \subset \mathrm{Ker}(g)$, thus it follows that $\overline{\mathrm{Im}}(f) = \mathrm{Ker}(g)$.

(v) By (i) statement 1. of Proposition 5.13 continues to hold if one replaces strict exactness by exactness. To see that this also holds for statement 2. it is enough to show that exactness of $TL \xrightarrow{Tf} TM \to 0$ implies that f is surjective. But if $\overline{\mathrm{Im}}(Tf) = TM$ one has $\xi + \sigma(\xi) \in \mathrm{Range}(Tf) + \sigma(\xi)$ for any $\xi \in TM$, using (iii), and taking $\xi = (x, 0)$ for $x \in M$ gives $x \in \mathrm{Range}(f)$ and the required surjectivity of f.

(vi) We first show exactness at $\mathrm{Ker}(f)$. The normal image $\overline{\mathrm{Im}}(0) = \mathrm{Ker}(f)^\sigma$. Since the inclusion $\iota : \mathrm{Ker}(f) \to L$ is injective and compatible with σ, one has $z \in \mathrm{Ker}(\iota) = \iota^{-1}(L^\sigma) \Leftrightarrow z \in \mathrm{Ker}(f)^\sigma$. Since $\overline{\mathrm{Im}}(0) = \mathrm{Ker}(f)^\sigma$ this gives exactness at $\mathrm{Ker}(f)$. We now show exactness at L. By (ii) $\mathrm{Range}(\iota) \subset \overline{\mathrm{Im}}(\iota)$. By construction $\mathrm{Range}(\iota) = \mathrm{Ker}(f)$. The other inclusion $\overline{\mathrm{Im}}(\iota) \subset \mathrm{Ker}(f)$ follows from $\mathrm{Range}(f \circ \iota) \subset M^\sigma$. Exactness at M follows from the definition

$\overline{\mathrm{Im}}(f) = \mathrm{Ker}(\mathrm{Coker}(f))$. Finally concerning exactness at $\mathrm{Coker}(f)$, note that the kernel of the last map (to 0) is $\mathrm{Coker}(f)$ while the quotient map $\eta : M \to \mathrm{Coker}(f)$ is surjective by construction and hence $\mathrm{Coker}(f) \subset \mathrm{Range}(\eta) \subset \overline{\mathrm{Im}}(\eta)$ (using (ii)) so that the equality $\mathrm{Coker}(f) = \overline{\mathrm{Im}}(\eta)$ is proved. $\square$

In [17] a sequence $L \xrightarrow{f} M \xrightarrow{g} N$ of morphisms in a semiexact category is called of order two if $g \circ f \in \mathcal{N}$. The sequence is said to be exact if $\mathrm{Ker}(\mathrm{cok}(f)) = \mathrm{Ker}(g)$. This definition agrees with Definition 6.2. The notion of normal image introduced in [17] is the same as $\overline{\mathrm{Im}}(f) = \mathrm{Ker}(\mathrm{cok}(f))$. The notion of normal coimage again as in [17] is $\mathrm{Coim}(g) = \mathrm{Coker}(\mathrm{ker}(g))$ and the condition $\mathrm{Coker}(f) = \mathrm{Coim}(g)$ is equivalent to exactness because one has in general the equalities $\mathrm{cok}(\mathrm{ker}(\mathrm{cok}(f))) = \mathrm{cok}(f)$ and $\mathrm{ker}(\mathrm{cok}(\mathrm{ker}(f))) = \mathrm{ker}(f)$. The notion of an exact morphism is defined in [17] based on a diagram such as

$$\begin{tikzcd} \mathrm{Ker}(f) \arrow[r, "\mathrm{ker}(f)"] & L \arrow[r, "f"] \arrow[d, "\mathrm{cok}(\mathrm{ker}(f))"'] & M \arrow[r, "\mathrm{cok}(f)"] & \mathrm{Coker}(f) \\ & \mathrm{Coim}(f) \arrow[r, "\tilde{f}"] & \overline{\mathrm{Im}}(f) \arrow[u, "\mathrm{ker}(\mathrm{cok}(f))"'] & \end{tikzcd} \tag{59}$$

where for f to be exact one requires that the map $\tilde{f}$ is an isomorphism.

When $M = 0$, one has $\mathrm{ker}(f) = \mathrm{Id}_L$ and the cokernel $\mathrm{cok}(\mathrm{ker}(f))$ is the quotient of L by the equivalence relation (56) for $f = \mathrm{Id}_L$. One checks that this equivalence relation is given by $b \sim b' \iff p(b) = p(b')$ where $p(x) = x + \sigma(x)$ gives the projection $p : L \to L^\sigma$. Thus one has

$$\mathrm{Coim}(L \xrightarrow{0} M) = L^\sigma, \quad \mathrm{cok}(\mathrm{ker}(0)) = L \xrightarrow{p} L^\sigma \tag{60}$$

and this shows that the zero map $L \xrightarrow{0} M$ is never exact unless $L = 0$.

The following example shows that $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ is not generalized exact in the sense of [17] §1.3.6 (*i.e.* a semiexact category in which every morphism is exact).

Example 6.4. Let $M = \mathbb{B}^2$. It has 4 elements: $M = \{0, \ell, m, \ell \vee m = n\}$. Let $\iota : N = \{0, m, n\} \to M = \{0, \ell, m, n\}$ be the inclusion. We consider the morphism $f = \mathfrak{s}\iota : \mathfrak{s}N = N^2 \to \mathfrak{s}M = M^2$. The normal image $\overline{\mathrm{Im}} f = \mathrm{Ker}(\mathrm{cok}(f))$ is given by the elements of $\mathfrak{s}M = M^2$ which are fixed points of the involution in the quotient $\mathrm{Coker}(f)$ of $\mathfrak{s}M$ by the equivalence relation

$$(x, y) \sim (x', y') \iff h(x) + g(y) = h(x') + g(y') \tag{61}$$

$$\forall X, \ \forall (h, g) \in \mathrm{Hom}_{\mathbb{B}}(M, X) \text{ s.t. } h|_N = g|_N.$$

One has $(\ell, m) \in M^2 \setminus (N^2 + \Delta)$ and $(\ell, m) \in \overline{\mathrm{Im}} f$, *i.e.* $(\ell, m) \sim (m, \ell)$ that is

$$h, g \in \mathrm{Hom}_{\mathbb{B}}(M, X), \quad h|_N = g|_N \Rightarrow h(\ell) + g(m) = h(m) + g(\ell).$$

Indeed, since $\ell \vee m = n \in N$ one has $h(\ell) + h(m) = g(m) + g(\ell)$, but since $h(m) = g(m)$ this gives $h(\ell) + g(m) = h(m) + g(\ell)$. The quotient $\mathrm{Coker}(f)$ of M^2 by the equivalence relation (61) is the object of $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ obtained by adjoining to N (endowed with $\sigma = \mathrm{Id}$) three elements $\alpha, \sigma(\alpha), \alpha + \sigma(\alpha)$ with addition given by $\alpha + z = n$, $\forall z \neq 0$, $z \in N$. The quotient map $\eta : M^2 \to M^2/\!\sim$ is such that

$$\eta((u, v)) = u + v \,, \ \ \forall u, v \in N, \quad \eta((\ell, 0)) = \alpha.$$

In the above example, since f is injective, its kernel is a null object and the cokernel of its kernel is the identity map (using (53) and the identity map $\mathrm{Id} \in \mathrm{Hom}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(\mathfrak{s}N, \mathfrak{s}N)$ to show that the equivalence relation is trivial). Thus $\tilde{f}$ is the inclusion of N^2 in $\overline{\mathrm{Im}}(T\iota)$ and is not an isomorphism. The above example also shows that $f = \mathfrak{s}\iota$ is not a normal monomorphism in the sense of [17]. In fact kernels and cokernels establish an anti-isomorphism between the ordered sets of normal subobjects (kernels of morphisms in $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$) and normal quotients as one uses in the context of abelian categories. Normal subobjects and normal quotients form anti-isomorphic lattices. Thus normal subobjects give the relevant notion of subobject. It follows from [18] 1.3.3. that in a semiexact category, every morphism has a normal factorization through its normal coimage and its normal image as in (59). In [18] one denotes a normal monomorphism by the symbol $L \overset{f}{\rightarrowtail} M$ while for a normal epimorphism one uses $L \overset{f}{\twoheadrightarrow} M$. The next Lemma holds in any semiexact category and we include its proof for completeness.

Lemma 6.5. *Let $L \xrightarrow{f} N$ be a morphism with $f = k \circ h$, where $h : L \twoheadrightarrow M$ is a normal epimorphism and $k : M \rightarrowtail N$ is a normal monomorphism. Then f is exact.*

Proof. Since k is injective $k^{-1}(N^\sigma) = M^\sigma$ and the kernel of f is equal to the kernel of h. By hypothesis h is the cokernel of its kernel and thus $h = \mathrm{cok}(\ker(f))$ and $M = \mathrm{Coim}(f)$. Since h is surjective $h(L^\sigma) = M^\sigma$ and the condition $\phi \circ f$ null is equivalent to $\phi \circ k$ null. Thus the cokernel of f is the same as the cokernel of k and the factorization (59) takes the form

$$\begin{tikzcd} \mathrm{Ker}(h) = \mathrm{Ker}(f) \arrow[r] & L \arrow[r, "f"] \arrow[d, "h=\mathrm{cok}(\ker(f))"'] & N \arrow[r] & \mathrm{Coker}(f) = \mathrm{Coker}(k) \\ & M \arrow[r, "\mathrm{Id}_M"] & M \arrow[u, "k=\ker(\mathrm{cok}(f))"'] & \end{tikzcd} \tag{62}$$

which shows that f is exact. ∎

Exact sequences $L \xrightarrow{f} M \xrightarrow{g} N$ are defined by $\mathrm{Ker}(\mathrm{cok}(f)) = \mathrm{Ker}(g)$. In [18] 1.3.5. the following notion of short exact sequence[2] is introduced.

Definition 6.6. *A short doubly exact sequence is given by a pair of maps*

$$L \overset{m}{\rightarrowtail} M \overset{p}{\twoheadrightarrow} N, \quad m = \ker(p) \ \& \ p = \mathrm{cok}(m).$$

This means that L is a normal subobject of M and N is a normal quotient of M. As pointed out in [18] §1.5.2, one has

Proposition 6.7. *let $f : L \to M$ and $g : M \to N$ be morphisms in $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$. The following conditions are equivalent*

1. *The sequence $L \overset{f}{\rightarrowtail} M \overset{g}{\twoheadrightarrow} N$ is short doubly exact.*
2. *The sequence $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ is exact and the morphisms f and g are exact.*

Direct and inverse images of normal subobjects can be organized by a transfer functor with values in the category of lattices and Galois connections. Exact functors are introduced in [18] 1.7.

Next, we test in our context the axiom *ex*2 of [18] 1.3.6. *i.e.* the stability under composition of the normal monos and normal epis.

[2] Such short exact sequences are the analogues of the kernel-cokernel pairs in additive categories: [5]

Lemma 6.8. *Let $L \subset M$ be a normal subobject of the object M of $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$. There exists an injective object F of the category $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ and a morphism $\phi : M \to F$ such that $L = Ker(\phi)$.*

Proof. By hypothesis there exists $f \in \mathrm{Hom}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(M, X)$ such that $L = \mathrm{Ker}(f)$. Let η_X and r be as in Lemma 5.8. One has $L = \mathrm{Ker}(\eta_X \circ f)$ since $\eta_X^{-1}(\Delta) = X^\sigma$ by construction. Let E be an injective object of $\mathbb{B}\mathrm{mod}$ and $\iota : I(X) \to E$ an embedding as in Proposition 2.4. Then $g = \mathfrak{s}(\iota) : TX \to \mathfrak{s}E = F$ is a morphism in $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ such that $g^{-1}(F^\sigma) = T(X)^\sigma$. Thus one has $L = \mathrm{Ker}(g \circ \eta_X \circ f) = \mathrm{Ker}\phi$, with $\phi = g \circ \eta_C \circ f$. By Lemma 7.17, $\mathfrak{s}E = F$ is an injective object of $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ since the underlying $\mathbb{B}$-semimodule is E^2. □

Lemma 6.9. *The normal monomorphisms in the category $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ are stable under composition.*

Proof. Let $L \subset M$ be a normal subobject of an object M of $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ and similarly let $M \subset N$ be a normal subobject of an object N of $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$. By Lemma 6.8 there exists an injective object X of the category $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ and a morphism $\phi : M \to X$ such that $L = \mathrm{Ker}(\phi)$. Since X is injective we can extend ϕ to a morphism $\psi \in \mathrm{Hom}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(N, X)$. Let $f \in \mathrm{Hom}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(N, Y)$ such that $M = \mathrm{Ker}(f)$. We let $\rho = (\psi, f) \in \mathrm{Hom}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(N, X \times Y)$. The inverse image by ρ of $(X \times Y)^\sigma = X^\sigma \times Y^\sigma$ is contained in $M = \mathrm{Ker}(f)$ and coincides with the inverse image of X^σ by the restriction of ψ to M, *i.e.* with $L = \mathrm{Ker}(\phi)$. Thus $L = \mathrm{Ker}(\rho)$. □

Lemma 6.10. *In the category $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ the normal epis are stable under composition.*

Proof. Let $\alpha : L \to M$ and $\beta : M \to N$ be normal epis. One needs to show that

$$\beta \circ \alpha(a) = \beta \circ \alpha(b) \iff f(a) = f(b)\,,\ \ \forall f : L \to Z,\ \mathrm{Ker}(f) \supset \mathrm{Ker}(\beta \circ \alpha). \tag{63}$$

We first show, using the fact that α is a normal epi, that one has the equivalence

$$\mathrm{Ker}(f) \supset \mathrm{Ker}(\beta \circ \alpha) \iff \exists \psi : M \to Z \mid f = \psi \circ \alpha,\ \mathrm{Ker}(\psi) \supset \mathrm{Ker}(\beta).$$

Indeed, the implication $\Leftarrow$ is immediate. Conversely, if $\mathrm{Ker}(f) \supset \mathrm{Ker}(\beta \circ \alpha)$ one has $\mathrm{Ker}(f) \supset \mathrm{Ker}(\alpha)$ and since α is normal one can factorize $f = \psi \circ \alpha$. Moreover, for any element u of $\mathrm{Ker}\beta$ one can find using the surjectivity of α an element $v \in L$ such that $\alpha(v) = u$. One has $v \in \mathrm{Ker}(\beta \circ \alpha)$ since $\alpha(v) \in \mathrm{Ker}(\beta)$ and thus since $\mathrm{Ker}(f) \supset \mathrm{Ker}(\beta \circ \alpha)$ one has $v \in \mathrm{Ker}(f)$ and $f(v) \in Z^\sigma$ which shows that $\psi(u) \in Z^\sigma$ and $\mathrm{Ker}(\psi) \supset \mathrm{Ker}(\beta)$. Thus one gets the required equivalence. Next, since β is a normal epi one has

$$\beta \circ \alpha(a) = \beta \circ \alpha(b) \iff \psi(\alpha(a)) = \psi(\alpha(b))\,,\ \ \forall \psi : M \to Z \mid \mathrm{Ker}(\psi) \supset \mathrm{Ker}(\beta).$$

Thus one obtains (63) and that $\beta \circ \alpha$ is a normal epi. □

The axiom *ex*3 of [18] 1.3.6, (sub-quotient axiom, or homology axiom) which if satisfied, defines a homological category is stated as follows:

*ex*3: Given a normal mono $M \overset{m}{\rightarrowtail} N$ and a normal epi $N \overset{q}{\twoheadrightarrow} Q$ with $m \geq \mathrm{Ker}(q)$, the morphism $q \circ m$ is exact.

In [18] the basic example of a homological category is the category $\mathfrak{Sets}_2$ of pairs of sets (X, X_0) with $X_0 \subset X$ and maps $f : X \to Y$ such that $f(X_0) \subset Y_0$. The null maps are those with $f(X) \subset Y_0$. This category is shown to be semiexact and homological. A morphism of $\mathfrak{Sets}_2$ is exact iff f is injective and $Y_0 \subset f(X)$. This example suggests that for $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ a necessary condition for $f : L \to M$ to be exact should be that $M^\sigma \subset f(L)$. Indeed

Lemma 6.11. *In the category* $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ *if* $f : L \to M$ *is exact then* $M^\sigma \subset f(L)$.

Proof. In the factorization (59), the left vertical arrow is surjective and the right vertical arrow is a kernel so that $\overline{\mathrm{Im}}(f) = \mathrm{Ker}(\mathrm{cok}(f))$ automatically contains M^σ. Thus if the horizontal arrow is an isomorphism one gets $M^\sigma \subset f(L)$. $\square$

Now, given m and q as in *ex*3 we have $\mathrm{Ker}(q) = q^{-1}(Q^\sigma)$ and since $N \xrightarrow{q} Q$ is a normal epi, it is surjective and thus the map q restricted to $\mathrm{Ker}(q)$ surjects onto Q^σ. The condition $m \geq \mathrm{Ker}(q)$ means that the normal subobject $M \overset{m}{\rightarrowtail} N$ contains $\mathrm{Ker}(q)$ and it follows that the map $q \circ m$ surjects onto Q^σ, *i.e.* it fulfills the necessary condition of Lemma 6.11. In fact, since $M \overset{m}{\rightarrowtail} N$ is a normal subobject let $f : N \to Z$ be a morphism of $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ such that $M = \mathrm{Ker}(f)$. One has $\mathrm{Ker}(q) \subset \mathrm{Ker}(f)$ and since $N \xrightarrow{q} Q$ is a normal epi it follows that $q = \mathrm{cok}(\mathrm{Ker}(q))$ which according to (53) means

$$q(b) = q(b') \iff h(b) = h(b')\ \forall X, \forall h \in \mathrm{Hom}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(N, X) \text{ s.t. } \mathrm{Ker}(q) \subset \mathrm{Ker}(h).$$

This shows, taking $h = f$, that $f(b)$ only depends upon $q(b)$ and that there exists a morphism $\phi \in \mathrm{Hom}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(Q, Z)$ such that $f = \phi \circ q$. Thus one has $\mathrm{Ker}(f) = q^{-1}(\mathrm{Ker}(\phi))$ and the normal subobject $S = \mathrm{Ker}(\phi) \subset Q$ is the natural candidate to make the following diagram commutative

$$\begin{tikzcd} M \arrow[r, tail, "m"] \arrow[d, two heads, "q'"'] & N \arrow[d, two heads, "q"] \\ S \arrow[r, tail, "k"] & Q \end{tikzcd} \tag{64}$$

Since $S = \mathrm{Ker}(\phi)$ its inclusion in Q is a normal mono $S \overset{k}{\rightarrowtail} Q$. The map $q \circ m$ has range in S and one needs to show that the induced map $q' : M \to S$ is a normal epi. Since q is a normal epi it is surjective and thus $q' : q^{-1}(\mathrm{Ker}(\phi)) \to \mathrm{Ker}(\phi)$ is also surjective. To show that q' is a normal epi one needs to prove that it is the cokernel of its kernel. Its kernel is $\mathrm{Ker}(q) \subset \mathrm{Ker}(f)$. The equivalence relation defining the cokernel of $\mathrm{Ker}(q) \subset \mathrm{Ker}(f)$ is, for $b, b' \in M = \mathrm{Ker}(f)$

$$b \sim_1 b' \iff g(b) = g(b')\ \forall X, \forall g \in \mathrm{Hom}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(\mathrm{Ker}(f), X) \text{ s.t. } \mathrm{Ker}(q) \subset \mathrm{Ker}(g).$$

We need to show that this equivalence relation is the same as that defined by $q(b) = q(b')$ which in turns is

$$b \sim_2 b' \iff h(b) = h(b')\ \forall X, \forall h \in \mathrm{Hom}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(N, X) \text{ s.t. } \mathrm{Ker}(q) \subset \mathrm{Ker}(h).$$

Now, any such h gives a g by restriction to $\mathrm{Ker}(f)$ and thus for for $b, b' \in M = \mathrm{Ker}(f)$, one has $b \sim_1 b' \Rightarrow b \sim_2 b'$. Conversely we need to show that if $g(b) \neq g(b')$ for some $g \in \mathrm{Hom}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(\mathrm{Ker}(f), X)$ s.t. $\mathrm{Ker}(q) \subset \mathrm{Ker}(g)$, then the same holds for some h. Let η_X and r be as in Lemma 5.8. One has $\mathrm{Ker}(g) = \mathrm{Ker}(\eta_X \circ g)$ since $\eta_X^{-1}(\Delta) = X^\sigma$ by construction. Let E be an injective object of $\mathbb{B}\mathrm{mod}$ and $\iota : I(X) \to E$ an embedding as in Proposition 2.4. Then $u = \mathfrak{s}(\iota) : T(X) \to \mathfrak{s}(E) = F$ is a morphism of $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ such that $u^{-1}(F^\sigma) = T(X)^\sigma$. Thus replacing g by $g' = u \circ \eta_X \circ g$ one still has $g'(b) \neq g'(b')$, $\mathrm{Ker}(q) \subset \mathrm{Ker}(g')$ while now the object $\mathfrak{s}(E) = F$ is injective in the category $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$. Thus one can extend $g' \in \mathrm{Hom}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(\mathrm{Ker}(f), F)$ to $h \in \mathrm{Hom}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(N, F)$ and one has $\mathrm{Ker}(q) \subset \mathrm{Ker}(h)$ since $\mathrm{Ker}(q) \subset \mathrm{Ker}(g')$. Moreover since $b, b' \in M = \mathrm{Ker}(f)$ one has $h(b) \neq h(b')$ since $g'(b) \neq g'(b')$. Thus we have shown that the equivalence relation defining the cokernel of $\mathrm{Ker}(q) \subset \mathrm{Ker}(f)$ is the same as $q(b) = q(b')$. This shows that the restriction q' of q to $\mathrm{Ker}(f)$ is a normal epimorphism and hence that the vertical arrow q' of the diagram (64) is a normal epimorphism. Thus $q \circ m$ factorizes as $k \circ q'$ and is exact by Lemma 6.5. We can therefore state our main result.

Theorem 6.12. *The category* $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ *is homological.*

Proof. By Proposition 6.1 the category $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ is semiexact. It follows from Lemma 6.9 and Lemma 6.10 that $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ satisfies $ex2$. We show that it also satisfies $ex3$. Given a normal mono $M \overset{m}{\rightarrowtail} N$ and a normal epi $N \overset{q}{\twoheadrightarrow} Q$ with $m \geq \mathrm{Ker}(q)$, the above construction determines a factorization (64), and one has $q \circ m = k \circ q'$ where k is a normal mono and q' a normal epi. Thus Lemma 6.5 applies and shows that $q \circ m = k \circ q'$ is exact. Thus the category $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ is homological. $\square$

Next, we investigate how morphisms in $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ act on normal subobjects.

For any object E of $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ we let $\mathrm{Nsb}(E)$ be the lattice of normal subobjects of E. For $N, M \in \mathrm{Nsb}(E)$ one has $N \cap M \in \mathrm{Nsb}(E)$ since the intersection of two kernels $\mathrm{Ker}(f) \cap \mathrm{Ker}(g)$ is the kernel of the map (f, g) to the product of the codomains. Thus the lattice operation $\wedge$ on $\mathrm{Nsb}(E)$ coincides with the intersection. The operation $\vee$ of the lattice is more delicate. By construction $E_1 \vee E_2$ contains $E_1 + E_2$ but next example shows the existence of two normal subsemimodules $E_j \in \mathrm{Nsb}(E)$ whose sum $E_1 + E_2$ is not normal.

Example 6.13. Let $M = \{0, m, \ell, n\}$ be as in Example 6.4 and take $E = \mathfrak{s}M$. One easily checks that the following subsemimodules are normal subsemimodules $E_j \in \mathrm{Nsb}(E)$

$$E_1 = \{0, m\} \times \{0, m\} + \Delta, \quad E_2 = \{0, n\} \times \{0, n\} + \Delta.$$

Example 6.4 shows that the smallest element of $\mathrm{Nsb}(E)$ which contains $E_1 + E_2$ is $\overline{\mathrm{Im}}(T\iota) = M^2 \setminus \{(\ell, 0), (0, \ell)\}$ and this normal subsemimodule $E_1 \vee E_2$ is strictly larger than $E_1 + E_2$ since it contains $(\ell, m) \notin E_1 + E_2$.

We recall that the modular condition for a lattice states that: $(E \vee F) \wedge G = E \vee (F \wedge G)$, for $E \subset G$. Figure 5 provides the graph of the lattice of normal subsemimodules of $\mathfrak{s}N = N^2$ for $N = \{0, m, n\}$..

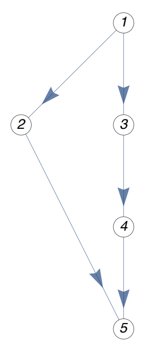

Figure 5: The lattice of normal subsemimodules of $\mathfrak{s}N = N^2$

The figure shows clearly that this lattice is not modular since taking $E = e(3) \subset G = e(4)$ and $F = e(2)$ one gets $(E \vee F) \wedge G = G$ while $E \vee (F \wedge G) = E$.

Following [18] 1.5.6., a morphism in a semiexact category determines direct and inverse images of normal subobjects. In our setup this gives

Proposition 6.14. *Let $f \in Hom_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(E, F)$.*

(i) For $m \in \mathrm{Nsb}(E)$, the direct image $f_*(m)$ is equal to $\overline{\mathrm{Im}}(f \circ m)$.

(ii) For $M \in \mathrm{Nsb}(F)$, the inverse image $f^*(M)$ is equal to $f^{-1}(M)$.

Proof. (i) This is by definition *cf.* (1.69) of [18] 1.5.6.

(ii) It is enough, using equation (1.70) of [18] 1.5.6., to show that for $M = \mathrm{Ker}(\mathrm{cok}(n))$ one has $\mathrm{Ker}(\mathrm{cok}(n) \circ f) = f^{-1}(M)$: this is immediate. □

6.2 Kernels and normal subobjects in $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$. In this subsection we show that for $E \subset F$ a subobject in $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$, the normal image $\overline{\mathrm{Im}}(E)$ of the inclusion is determined in a local manner. More precisely, let $\xi \in F$ and assume that $\xi + \sigma(\xi) \in E$ (this is possible since adding F^σ to E does not change $\overline{\mathrm{Im}}(E)$). We look for a morphism $L \in \mathrm{Hom}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(F, \mathfrak{s}\mathbb{B})$ such that $L(E) \subset (\mathfrak{s}\mathbb{B})^\sigma$ and $L(\xi) \notin (\mathfrak{s}\mathbb{B})^\sigma$. The composition $\phi = p_j \circ L$ with the first or second projection is an element of $\mathrm{Hom}_{\mathbb{B}}(F, \mathbb{B})$ such that

$$\phi(\sigma(x)) = \phi(x)\,, \ \ \forall x \in E, \quad \phi(\xi) = 0, \quad \phi(\sigma(\xi)) = 1. \tag{65}$$

Conversely, the existence of such an element of $\mathrm{Hom}_{\mathbb{B}}(F, \mathbb{B})$ suffices to reconstruct $L = (\phi, \phi \circ \sigma)$. Note now that the restriction ρ of ϕ to E^σ uniquely determines ϕ on $E + \mathbb{B}\xi \subset F$ by means of the equality

$$\phi(x + \epsilon\xi) = \rho(x + \sigma(x))\,, \ \ \forall x \in E, \ \epsilon \in \mathbb{B}.$$

The relative position of ξ with respect to E determines a triple $(\mathcal{R}, \mathcal{S}, \alpha)$ where

1. $\mathcal{R} \subset E \times E$ is a subsemimodule (in $\mathbb{B}\mathrm{mod}$) of $E \times E$ described by an equivalence relation.
2. $\mathcal{S} \subset E$ is an additive subset.
3. $\alpha = \sigma(\alpha) \in \mathcal{S}$.

Indeed, one lets $\mathcal{R} = \{(x, y) \mid x + \xi = y + \xi\}$. This is a subsemimodule (in $\mathbb{B}\mathrm{mod}$) of $E \times E$. $\mathcal{S} = \{z \mid z + \xi = z\}$ is an additive subset of E, and $\alpha = \xi + \sigma(\xi)$ belongs to $\mathcal{S}$, since $\alpha + \xi = \alpha$. The existence of ϕ verifying (65) is determined by the triple $(\mathcal{R}, \mathcal{S}, \alpha)$ as follows, where $p : E \to E^\sigma$ is the projection $p(x) = x + \sigma(x)$.

Lemma 6.15. *The existence of ϕ verifying (65) is equivalent to the existence of $\rho \in Hom_{\mathbb{B}}(E^\sigma, \mathbb{B})$ such that*

$$\rho(p(x)) = \rho(p(y))\,, \ \ \forall (x, y) \in \mathcal{R}, \quad \rho(\alpha) = 1.$$

Proof. The conditions are necessary as follows from

$$\rho(p(x)) = \phi(x) = \phi(x + \xi) = \phi(y + \xi) = \phi(y) = \rho(p(y))$$

and from

$$s + \xi = s \Rightarrow p(s) + \alpha = p(s), \quad \rho(\alpha) = 1 \Rightarrow \rho(p(s)) = 1.$$

Conversely, define ϕ on $E + \mathbb{B}\xi \subset F$ by the equality

$$\phi(x + \epsilon\xi) = \rho(p(x))\,, \ \ \forall x \in E, \ \epsilon \in \mathbb{B}.$$

If $x + \epsilon\xi = x' + \epsilon'\xi$ then $x + \xi = x' + \xi$ since one can add ξ to both terms. Thus $(x, x') \in \mathcal{R}$ and by hypothesis $\rho(p(x)) = \rho(p(x'))$. This shows that ϕ is well defined on $E + \mathbb{B}\xi \subset F$ and

$\phi \in \mathrm{Hom}_{\mathbb{B}}(E+\mathbb{B}\xi,\mathbb{B})$. Moreover $\phi(\xi)=0$. By Proposition 2.3 (*iii*) one can extend ϕ to an element $\tilde\phi \in \mathrm{Hom}_{\mathbb{B}}(F,\mathbb{B})$. Since $\phi(\alpha)=1$ by hypothesis one has

$$\tilde\phi(\sigma(\xi))=\tilde\phi(\xi+\sigma(\xi))=\phi(\alpha)=1.$$

Thus $\tilde\phi$ fulfills (65). □

Note that any pair (a,b) in the projection $p(\mathcal{R})$ of the equivalence relation $\mathcal{R}$ fulfills $\alpha+a=\alpha+b$ since $a=x+\sigma(x)$, $b=y+\sigma(y)$ with $x+\xi=y+\xi$. Thus $p(\mathcal{R})\subset\mathcal{R}_\alpha$. To $p(\mathcal{R})$ corresponds a canonical element $\psi\in\mathrm{Hom}_{\mathbb{B}}(E^\sigma,\mathbb{B})$, namely the largest element compatible with the relation. This is obtained as

$$\psi=\vee\{\rho\in\mathrm{Hom}_{\mathbb{B}}(E^\sigma,\mathbb{B})\mid \rho(p(x))=\rho(p(y))\,,\ \forall(x,y)\in\mathcal{R}\}.$$

Thus the existence of ρ as in Lemma 6.15 is equivalent to the condition $\psi(\alpha)=1$.

Proposition 6.16. *Let $E\subset F$ be a subobject in $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ containing F^σ. Then for $\xi\in F$ one has $\xi\in\overline{\mathrm{Im}}(E)$ (the normal image of the inclusion) if and only if there exists a finite sequence $a_0,a_0',a_1,a_1',\dots,a_n,a_n'$ of elements of E such that*

$$\xi=a_0+\xi,\ p(a_0)=p(a_0'),\ a_0'+\xi=a_1+\xi,\ p(a_1)=p(a_1'),\dots,p(a_n)=\xi+\sigma(\xi).$$

Proof. Let us assume that there exists a finite sequence $a_0,a_0',a_1,a_1',\dots,a_n,a_n'$ of elements of E as in the statement. Let $f\in\mathrm{Hom}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(F,C)$ be such that $E\subset\mathrm{Ker}(f)$ and let us show that $\xi\in\mathrm{Ker}(f)$. One has, using the equality $f(a)=f(p(a))$ for all $a\in E$ which implies $f(a_j)=f(a_j')$ for all j

$$f(\xi)=f(a_0+\xi)=f(a_0')+f(\xi)=f(a_0'+\xi)=f(a_1+\xi)=\ldots=f(a_n+\xi)=f(p(a_n)+\xi)$$

and since $p(a_n)+\xi=\xi+\sigma(\xi)\in F^\sigma$ one has $f(\xi)\in C^\sigma$, *i.e.* $\xi\in\mathrm{Ker}(f)$. Thus the existence of the sequence ensures that $\xi\in\overline{\mathrm{Im}}(E)$. We now assume that no such sequence exists and we define a subsemimodule $Z\subset E^\sigma$ as the equivalence class $\mathcal{U}(0)$ of 0 for the equivalence relation $\mathcal{U}$ on E^σ generated by the relation $p(\mathcal{R})$ where $\mathcal{R}=\{(x,y)\mid x+\xi=y+\xi\}\subset E\times E$ was defined above. The non-existence of the sequence means that $\alpha\notin\mathcal{U}(0)$ where $\alpha=\xi+\sigma(\xi)$. Let then $\rho\in\mathrm{Hom}_{\mathbb{B}}(E^\sigma,\mathbb{B})$ be defined by $u\in\rho^{-1}(0)\Leftrightarrow\exists v\in\mathcal{U}(0)\mid u\le v$. This condition defines an hereditary subsemimodule, thus ρ is well defined. Since $\mathcal{U}(0)$ is saturated for the relation $p(\mathcal{R})$, ρ fulfills the first condition of Lemma 6.15 and it remains to show that $\rho(\alpha)=1$. Note that the subset $V\subset E^\sigma$ determined as $V=\{v\in E^\sigma\mid v+\alpha=\alpha\}$ contains 0 and is such that if $(v,v')\in p(\mathcal{R})$ then $v\in V\iff v'\in V$. Indeed for $v=p(x)$, $v'=p(x')$ and $x+\xi=x'+\xi$ one has $p(x)+\alpha=p(x')+\alpha$. It follows that $\mathcal{U}(0)\subset V$. Now if $\alpha\in\rho^{-1}(0)$ one has $\exists v\in\mathcal{U}(0)\mid\alpha\le v$ and since $v\in V$ one has $v\le\alpha$ and thus $v=\alpha$ which is a contradiction since $\alpha\notin\mathcal{U}(0)$. Thus one has $\rho(\alpha)=1$ and by Lemma 6.15 there exists $L\in\mathrm{Hom}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(F,\mathfrak{s}\mathbb{B})$ such that $E\subset\mathrm{Ker}(L)$ but $\xi\notin\mathrm{Ker}(L)$. □

Proposition 6.16 provides a natural filtration of $\overline{\mathrm{Im}}(E)$ as: $\overline{\mathrm{Im}}(E)=\cup_n\overline{\mathrm{Im}}^{(n)}(E)$. Thus, we set

$$\overline{\mathrm{Im}}^{(1)}(E):=\{\xi\in F\mid\exists a\in E,\ a+\xi=\xi,\ p(a)=\xi+\sigma(\xi)\}.$$

One has $E\subset\overline{\mathrm{Im}}^{(1)}(E)$ since for $\xi\in E$ one can take $a=\xi$. Also $\overline{\mathrm{Im}}^{(1)}(E)$ is a subobject since it is invariant under σ and with a,a' associated to ξ,ξ' the sum $a+a'$ is associated to $\xi+\xi'$. The next level is $\overline{\mathrm{Im}}^{(2)}(E)$ which is defined by

$$\{\xi\in F\mid\exists a_0,a_0',a_1\in E,\ \xi=a_0+\xi,\ p(a_0)=p(a_0'),\ a_0'+\xi=a_1+\xi,\ p(a_1)=\xi+\sigma(\xi)\}.$$

Taking $a_0 = a'_0 = 0$ one sees that $\overline{\mathrm{Im}}^{(1)}(E) \subset \overline{\mathrm{Im}}^{(2)}(E)$.

Next, we show with an example that this inclusion is strict in general.

One lets $F = \mathfrak{s}N$, $N = \{0, m, n\}$, $m < n$, and considers the inclusion $E \subset F$ with

$$E = \{(0,0), (0,m), (m,0), (m,m), (m,n), (n,m), (n,n)\} \subset F.$$

For $\xi = (0, n)$ the relation $p(\mathcal{R})$ on $F^\sigma = N$ is in fact given by the same symmetric set

$$p(\mathcal{R}) = \{(0,0), (0,m), (m,0), (m,m), (m,n), (n,m), (n,n)\} \subset N \times N$$

and one sees that it is not an equivalence relation since it contains $(0, m)$ and (m, n) but not $(0, n)$. One has $\xi + \sigma(\xi) = n$ when viewed as an element of $F^\sigma = N$ and the composition $p(\mathcal{R}) \circ p(\mathcal{R})$ is required to get n in the equivalence class of 0. This shows that $\xi \in \overline{\mathrm{Im}}^{(2)}(E)$ but $\xi \notin \overline{\mathrm{Im}}^{(1)}(E)$.

Proposition 6.17. *(i) Let $\phi \in Hom_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(L, M)$ and $\xi \in M$, $\xi \neq \sigma(\xi)$ be an indecomposable element, i.e. such that the subset $\{0, \xi\} \subset M$ is a hereditary sub $\mathbb{B}$-semimodule. Then $\xi \in \phi(L) \Leftrightarrow \xi \in \overline{\mathrm{Im}}(\phi)$.*

(ii) Assume that M is generated by its indecomposable elements, then one has

$$\overline{\mathrm{Im}}(\phi) = M \iff \phi(L) + M^\sigma = M. \tag{66}$$

(iii) Let $\phi \in Hom_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(L, M)$ and $\xi \in M$, be minimal among non-null elements, i.e. such that ξ is the only non-null element in $[0, \xi] \subset M$. Then $\xi \in \phi(L) \iff \xi \in \overline{\mathrm{Im}}(\phi)$.

Proof. (i) This follows from Proposition 6.16 but we provide here a direct proof. One has to show that if $\xi \notin \phi(L)$ there exists a morphism $\psi \in \mathrm{Hom}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(M, X)$ such that $\phi(L) \subset \mathrm{Ker}(\psi)$ while $\xi \notin \mathrm{Ker}(\psi)$. Let $f \in \mathrm{Hom}_{\mathbb{B}}(M, \mathbb{B})$ be defined by $f(b) = 0 \iff b \in \{0, \xi\}$. One has $f \in \mathrm{Hom}_{\mathbb{B}}(M, \mathbb{B})$ since the subset $\{0, \xi\} \subset M$ is an hereditary subsemimodule by hypothesis. Take $X = \mathfrak{s}\mathbb{B}$ and define ψ as

$$\psi(b) = (f(b), f(\sigma(b))), \quad \forall b \in M.$$

Since $\xi \neq \sigma(\xi)$ one has $f(\sigma(\xi)) = 1$ and thus $\xi \notin \mathrm{Ker}(\psi)$. Since $\xi \notin \phi(L)$ one has $f(u) = 1$, $\forall u \in \phi(L)$, $u \neq 0$ and since $\phi(L)$ is globally invariant under σ one gets that $\phi(L) \subset \mathrm{Ker}(\psi)$.

(ii) We show the implication $\Rightarrow$. By (i) if $\xi \in M$ is indecomposable, one has either $\xi \in M^\sigma$ or $\xi \in \phi(L)$. Since M is generated by its indecomposable elements one gets $\phi(L) + M^\sigma = M$.

(iii) Let $f \in \mathrm{Hom}_{\mathbb{B}}(M, \mathbb{B})$ be defined by $f(b) = 0 \iff b \in [0, \xi]$, one has $f \in \mathrm{Hom}_{\mathbb{B}}(M, \mathbb{B})$. Define ψ as above by $\psi(b) = (f(b), f(\sigma(b)))$, $\forall b \in M$. Then as above $\xi \notin \mathrm{Ker}(\psi)$. Moreover if $\xi \notin \phi(L)$ one has for $a \in L$ that $\phi(a) \neq \xi$ and $\phi(\sigma(a)) \neq \xi$, thus either $f(\phi(a)) = 1 = f(\phi(\sigma(a)))$, or $\phi(a) < \xi$ (or $\phi(\sigma(a)) < \xi$) in which case $\phi(a)$ is null and hence $f(\phi(a)) = 0 = f(\phi(\sigma(a)))$. This shows that $\phi(L) \subset \mathrm{Ker}(\psi)$, and hence $\xi \notin \overline{\mathrm{Im}}(\phi)$. ∎

6.3 Cokernels in $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$.

Proposition 6.16 gives a good control on the normal image and we now discuss the cokernel. The first guess for the cokernel of a normal subsemimodule $E \subset F$ is the disjoint union of F^σ with the complement of E in F, while the map $\tilde{p} : F \to (F^\sigma \cup E^c)$ is defined as

$$\tilde{p}(x) = p(x) = x + \sigma(x), \quad \forall x \in E, \quad \tilde{p}(x) = x, \quad \forall x \in E^c.$$

In §9 we shall provide several examples where this rule defines the cokernel. For this to hold one needs to show that the operation in $F^\sigma \cup E^c$ given by

$$(\xi,\eta) \mapsto \{\tilde{p}(u+v) \mid \tilde{p}(u)=\xi,\ \tilde{p}(v)=\eta\} \tag{67}$$

is single valued. This is clear when $\xi,\eta \in E^c$, since then the fibers have one element. It also holds for $\xi,\eta \in F^\sigma$ since then one has $u,v \in E$ and by linearity of p one has $u+v \in E$, $p(u+v)=p(u)+p(v)=\xi+\eta$. One can thus assume that $\xi \in E^c$ and $\eta \in E$. One then has to show that

$$\#\{\tilde{p}(u+\xi) \mid u \in E,\ p(u)=\eta\} = 1. \tag{68}$$

When all of the $u+\xi$ which appear are in E, the uniqueness follows from the linearity of p. Thus the interesting case to study is when $\xi \in E^c$, $u+\xi \in E^c$.

The following example provides a case where uniqueness fails.

Example 6.18. Take $F = \mathfrak{s}M$ with $M = \{0, m, \ell, n\}$ and consider the subsemimodule

$$E_1 = \{(0,0),(0,m),(m,0),(m,m),(\ell,\ell),(\ell,n),(n,\ell),(n,n)\}.$$

Take $\xi = (0,\ell)$ and $\eta = (m,m)$. The fiber $\{u \in E_1,\ p(u)=\eta\}$ consists of the three elements $(0,m),(m,0),(m,m)$. The elements $u+\xi$ with u in the fiber are all in E_1^c and are the three distinct elements $(0,n),(m,n),(m,\ell)$. It follows that these three elements have the same image in the cokernel.

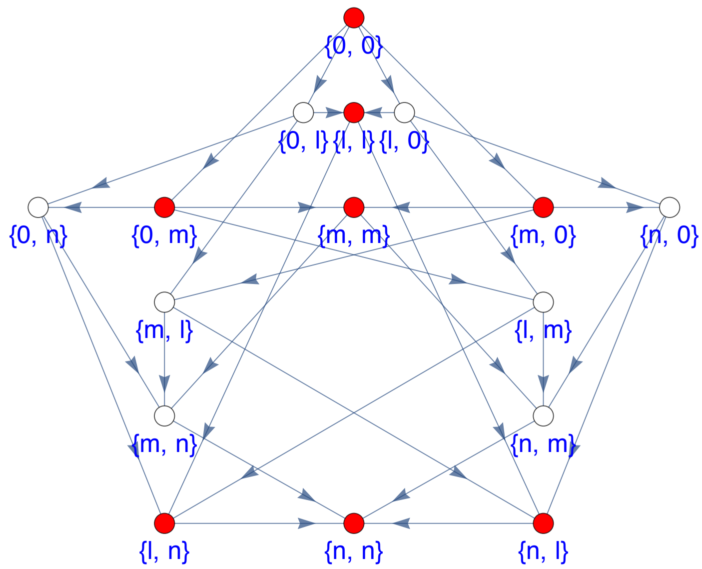

Figure 6: The subsemimodule $E_1 \subset F = \mathfrak{s}M$

The next proposition shows that in general, to obtain the cokernel, it suffices to divide E^c by the equivalence relation generated by (69) which is the analog in characteristic 1 of the operation of quotient by the subsemimodule.

Proposition 6.19. *Let $E \subset F$ be a subobject in $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ containing F^σ. Then the cokernel of the inclusion $E \subset F$ is the quotient of $F^\sigma \cup E^c$ by the smallest equivalence relation such that*

$$\xi \in E^c,\ u,v \in E,\ p(u)=p(v) \Rightarrow \xi+u \sim \xi+v. \tag{69}$$

The cokernel map is the quotient map on E^c and the projection p on E.

Proof. By construction the cokernel of the inclusion $E \subset F$ is the quotient of F by the equivalence relation

$$\alpha \sim_{\rm cok} \beta \iff f(\alpha) = f(\beta)\,, \ \ \forall f \text{ s.t. } E \subset \mathrm{Ker}(f).$$

Let $f \in \mathrm{Hom}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(F, X)$ be such that $E \subset \mathrm{Ker}(f)$. Then for $u \in E$ one has $f(u) \in X^\sigma$ and hence $f(u) = f(p(u))$. This shows that $u \sim_{\rm cok} p(u)$, $\forall u \in E$ and hence that the cokernel is the quotient of $F^\sigma \cup E^c$ by the restriction to $F^\sigma \cup E^c$ of the equivalence relation $\sim_{\rm cok}$. This equivalence relation fulfills (69) and we need to show that it coincides with the equivalence relation $\sim$ generated by (69). The latter equivalence relation is given by $\xi \sim \eta \iff \exists (\xi_j, u_j, v_j)$, $\xi_j \in F$, $u_j, v_j \in E$

$$\xi_1 + u_1 = \xi,\ \xi_n + v_n = \eta,\ \ p(u_j) = p(v_j),\ \xi_j + v_j = \xi_{j+1} + u_{j+1}\ \forall j, 1 \leq j \leq n.$$

This equivalence relation is compatible with the addition, *i.e.* if $\xi \sim \eta$ and $\xi' \sim \eta'$ one has $\xi + \xi' \sim \eta + \eta'$ as one gets by adding term by term the sequences (ξ_j, u_j, v_j), (ξ'_j, u'_j, v'_j). Moreover $\xi \sim \eta \Rightarrow p(\xi) = p(\eta)$. Now as above define a map $\tilde p : F \to Z := (F^\sigma \cup E^c)/\!\sim$, by

$$\tilde p(x) = p(x)\,,\ \ \forall x \in E,\ \ \tilde p(x) = x/\sim\,,\ \ \forall x \in E^c.$$

We have shown that $x \sim y \Rightarrow x \sim_{\rm cok} y$ and to show the converse it is enough to prove that the addition in F descends to Z, while $\tilde p : F \to Z$ is additive. This follows from the compatibility

$$\xi \sim \eta\ \&\ \xi' \sim \eta' \Rightarrow \xi + \xi' \sim \eta + \eta'.$$

The commutation with σ is automatic. □

In the statement of Proposition 6.19 we did not assume that E is normal as a subsemimodule of F. It is important to relate Proposition 6.16 with Proposition 6.19. We check that the existence of a finite sequence $a_0, a'_0, a_1, a'_1, \ldots, a_n, a'_n$ of elements of E such that

$$\xi = a_0 + \xi,\ p(a_0) = p(a'_0),\ a'_0 + \xi = a_1 + \xi,\ p(a_1) = p(a'_1), \ldots, p(a_n) = \xi + \sigma(\xi)$$

implies that $\tilde p(\xi) = \sigma(\tilde p(\xi))$ with the above notations. Indeed, to show that $\xi \sim p(\xi)$ we can take all $\xi_j = \xi$, $u_1 = a_0$, $v_1 = a'_0$, $u_j = a_{j-1}$, $v_j = a'_{j-1}$. The equalities $p(a_j) = p(a'_j)$ and $a'_{j-1} + \xi = a_j + \xi$ mean $p(u_j) = p(v_j)$, $\xi_j + v_j = \xi_{j+1} + u_{j+1}$ which gives the equivalence of ξ with the last term $a_n + \xi \sim p(a_n) + \xi = \xi + \sigma(\xi)$. This shows that $\xi \in \overline{\mathrm{Im}}$, as required.

As a corollary of Proposition 6.19 we obtain

Proposition 6.20. *Let $E \subset F$ be a subobject in $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ containing F^σ. Let $\mathrm{Coker}(\iota)$ be the cokernel of the inclusion $\iota : E \to F$. Then there are canonical isomorphisms of $\mathbb{B}$-semimodules*

$$F^\sigma \overset{\gamma}{\simeq} (\mathrm{Coker}(\iota))^\sigma \overset{\mathrm{cok}(Id_F)}{\simeq} F^\sigma \tag{70}$$

whose composition is the identity on F^σ.

Proof. $\mathrm{cok}(\iota) : F \to \mathrm{Coker}(\iota)$ restricts to a map of $\mathbb{B}$-semimodules $\gamma : F^\sigma \to (\mathrm{Coker}(\iota))^\sigma$. This map is surjective because $\mathrm{cok}(\iota) : F \to \mathrm{Coker}(\iota)$ is surjective by construction, and the projection p on the σ-fixed points commutes with morphisms in $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$. To show that γ is injective it is enough to check that the composition $\mathrm{cok}(\mathrm{Id}_F) \circ \gamma$ is the identity on F^σ. One has by Proposition 6.19, $\mathrm{Coker}(\mathrm{Id}_F) = F^\sigma$ and $\mathrm{cok}(\mathrm{Id}_F)$ is the projection $p : F \to F^\sigma$, thus one gets the required result. It follows that the two maps of (70) give an isomorphism and its inverse. □

7. Analogy with operators

The category $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ is homological but not modular. As exlained in [18] 2.3.5, the modularity property is used as a replacement of the standard argument in diagram chasing for additive categories, *e.g.* $f(a) = f(b) \Rightarrow a - b \in \mathrm{Ker}(f)$. In §7.1 we develop the analogy of the category $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ with the category of operators in Hilbert spaces and in §7.2 we introduce a substitute for the modularity property by showing in Theorem 7.10 that for a large class of objects E of $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ the nullity of the kernel of a morphism with domain E is equivalent to the injectivity of the restriction of the morphism to each fiber of the projection on the null elements. Finally, in §7.3 we discuss injective and projective objects of $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ and analyze the simplest example of a non-trivial short doubly exact sequence of finite objects.

7.1 Duality in $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$. In this subsection we define a duality in the category $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$. One first notes that there is an internal Hom functor $\underline{\mathrm{Hom}}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(L, M)$ obtained by involving the symmetry $\sigma(f) := \sigma \circ f = f \circ \sigma$ on the $\mathbb{B}$-semimodule $\mathrm{Hom}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(L, M)$. To dualize we use the object $\mathfrak{s}\mathbb{B}$ of $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ and we define the dual $E^* := \underline{\mathrm{Hom}}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(E, \mathfrak{s}\mathbb{B})$. By Lemma 5.3 one has a canonical isomorphism $I(E^*) \simeq I(E)^*$ of the dual E^* in the above sense with the dual of $I(E)$ in the sense of §2.3 *i.e.* $I(E)^* = \mathrm{Hom}_{\mathbb{B}}(I(E), \mathbb{B})$. It is given explicitly by the formula

$$E^* \ni \phi \mapsto \psi = p_1 \circ \phi \in I(E)^*, \quad p_1((x, y)) := x\,, \ \ \forall (x, y) \in \mathfrak{s}\mathbb{B}. \tag{71}$$

Conversely, given $\psi \in I(E)^*$ the associated $\phi \in E^*$ is obtained by symmetrization *i.e.* as $\phi(x) = (\psi(x), \psi(\sigma(x)))$. The additional structure given by the involution corresponds to the involution $\psi \mapsto \psi \circ \sigma$, for $\psi \in \mathrm{Hom}_{\mathbb{B}}(I(E), \mathbb{B})$. In particular, the results of §2.3 apply and allow one to reconstruct E from E^* by biduality as follows. One defines the normal dual as

$$E^*_{\mathrm{norm}} := \{\phi \in \mathrm{Hom}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(E, \mathfrak{s}\mathbb{B}) \mid \phi(\vee x_\alpha) = \vee(\phi(x_\alpha)\}.$$

Note that $\mathrm{Id}(I(E))$ involves hereditary subsemimodules of E which are not in general σ-invariant. The map $J \mapsto \sigma(J)$ endows $\mathrm{Id}(I(E))$ with an involution which coincides with the involution of the dual E^* under the identifications $\mathrm{Id}(I(E)) \simeq I(E)^* \simeq E^*$. The isomorphism $\tilde{\epsilon} : \mathrm{Id}(I(E)) \simeq (I(E)^*)^*_{\mathrm{norm}}$ of Proposition 2.13 is compatible with the involutions and one obtains

Proposition 7.1. *The object E of $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ is the subobject of $(E^*)^*_{\mathrm{norm}}$ given by the compact elements of this latter complete algebraic lattice.*

Proof. The evaluation map $\phi \mapsto \phi(x) \in \mathfrak{s}\mathbb{B}$ determines an embedding $\rho : E \to (E^*)^*_{\mathrm{norm}}$. This embedding is compatible with the embedding $\epsilon : I(E) \subset (I(E)^*)^*_{\mathrm{norm}}$ of Proposition 2.13 using the isomorphisms of (71)

$$I((E^*)^*_{\mathrm{norm}}) \simeq (I(E^*))^*_{\mathrm{norm}} \simeq (I(E)^*)^*_{\mathrm{norm}}$$

and by applying the equality $(p_1 \circ \phi)(x) = p_1(\phi(x))$. $\square$

One defines the notion of the orthogonal of a subobject by using the natural pairing between E and its dual $E^* := \underline{\mathrm{Hom}}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(E, \mathfrak{s}\mathbb{B})$, *i.e.*

$$\langle x, y\rangle_\sigma := y(x) \in \mathfrak{s}\mathbb{B}\,, \ \ \forall x \in E, y \in E^*, \quad F^\perp := \{y \mid \langle x, y\rangle_\sigma \text{ null}\,, \ \ \forall x \in F\}.$$

We recall that the term "null" means fixed under σ, so that $\langle x, y\rangle_\sigma$ null $\Leftrightarrow \langle x, y\rangle_\sigma \in (\mathfrak{s}\mathbb{B})^\sigma$.

Lemma 7.2. *Let $F \subset E$ be a subobject of an object E of $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$. Then the least normal subobject containing F is $(F^\perp)^\perp$.*

Proof. Note first that $(F^\perp)^\perp$ is normal since it is a kernel by construction. Moreover it contains F. Let $F \subset \mathrm{Ker}(f)$ with $f \in \mathrm{Hom}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(E, X)$. Using the injection $\eta_X : X \to T(X)$ one can replace X by $T(X)$ and using an embedding $I(X) \to \mathbb{B}^Y$ (Proposition 2.4) one can replace X by $\mathfrak{s}(\mathbb{B}^Y) = (\mathfrak{s}\mathbb{B})^Y$. Then one sees that $f(z)$ is null iff $\forall x \in Y$ the component $f_x(z) \in \mathfrak{s}\mathbb{B}$ is null. Since $F \subset \mathrm{Ker}(f)$ one has $F \subset \mathrm{Ker}(f_x)$ and thus $f_x \in F^\perp$ $\forall x$. Hence, for $z \in (F^\perp)^\perp$ the component $f_x(z) \in \mathfrak{s}\mathbb{B}$ is null $\forall x \in Y$ and one gets $(F^\perp)^\perp \subset \mathrm{Ker}(f)$. □

Proposition 7.1 shows that an object of $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ is uniquely determined by its dual. The following result (to be compared with Proposition 6.19) thus gives an efficient way to determine the cokernel of any morphism $\phi \in \mathrm{Hom}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(E, F)$.

Proposition 7.3. *Let $\phi \in Hom_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(E, F)$. The dual of the cokernel of ϕ is canonically isomorphic to the kernel of $\phi^* \in Hom_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(F^*, E^*)$:* $\mathrm{Coker}(\phi)^* = \mathrm{Ker}(\phi^*)$.

Proof. By construction $\mathrm{Coker}(\phi)$ is the quotient of F by the equivalence relation

$$\alpha \sim_{\mathrm{cok}} \beta \iff f(\alpha) = f(\beta)\,,\ \ \forall f \text{ s.t. } \ \phi(E) \subset \mathrm{Ker}(f).$$

Since the map $\mathrm{cok} : F \to \mathrm{Coker}(\phi)$ is surjective, the map $\mathrm{cok}^* : \mathrm{Coker}(\phi)^* \to F^*$ is injective and it remains to show that its range is the kernel of ϕ^*. The map $\theta = \mathrm{cok} \circ \phi$ is null, *i.e.* $\sigma \circ \theta = \theta \circ \sigma = \theta$, thus the same holds for $\theta^* = \phi^* \circ \mathrm{cok}^*$. This shows that $\mathrm{cok}^*(\mathrm{Coker}(\phi)^*) \subset \mathrm{Ker}(\phi^*)$. Conversely, let $f \in \mathrm{Ker}(\phi^*) \subset F^*$. One has $f \in \mathrm{Hom}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(F, \mathfrak{s}\mathbb{B})$ and since $f \in \mathrm{Ker}(\phi^*)$ then $f \circ \phi$ is null and $\phi(E) \subset \mathrm{Ker}(f)$. Thus $\alpha \sim_{\mathrm{cok}} \beta \Rightarrow f(\alpha) = f(\beta)$ and f induces a morphism $g : \mathrm{Coker}(\phi) \to \mathfrak{s}\mathbb{B}$ such that $g \circ \mathrm{cok} = f$. This means $\mathrm{cok}^*(g) = f$ and hence $f \in \mathrm{cok}^*(\mathrm{Coker}(\phi)^*)$. Thus we get $\mathrm{cok}^*(\mathrm{Coker}(\phi)^*) = \mathrm{Ker}(\phi^*)$. □

Remark 7.4.
(i) Proposition 7.3 implies that the cokernel of the inclusion $j : E \to F$ of a subobject is the same as the cokernel of the identity $\mathrm{Id}_F : F \to F$ iff the kernel of $j^* : F^* \to E^*$ is null. One has $\mathrm{Ker}(j^*) = \{\phi \in F^* \mid \phi \circ j \text{ null}\} = E^\perp$. Thus $\mathrm{Ker}(j^*) = (F^*)^\sigma \iff E^\perp = (F^*)^\sigma$ and the above statement follows directly from Lemma 7.2.

(ii) When the object E of $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ is finite, Proposition 7.1 simplifies and gives a canonical isomorphism of biduality $E \simeq (E^*)^*$.

(iii) The subtlety arising from the existence of non-normal subobjects in the category $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ is analogous to the existence of non-closed subspaces of a Hilbert space in the category of Hilbert spaces and linear operators. In the latter category the range of a morphism $T : \mathcal{H}_1 \to \mathcal{H}_2$ is in general not closed and it is natural to define the cokernel as

$$\mathrm{Coker}(T) := \mathcal{H}_2/\sim, \ \ \alpha \sim \beta \iff f(\alpha) = f(\beta)\,,\ \ \forall f : \mathcal{H}_2 \to \mathcal{H}_3 \text{ s.t. } f \circ T = 0.$$

Since f is continuous, $f = 0$ on the *closure* of the range of T and thus $\mathrm{Coker}(T) = \mathcal{H}_2/(\overline{\mathrm{Range}(T)})$. The equality $\mathrm{Coker}(T)^* = \mathrm{Ker}(T^*)$ holds for any morphism T.

(iv) In the category of operators in Hilbert spaces one introduces ([4]) the notion of *strict morphism* f. These are morphisms with closed range, and they are characterized by the existence of a quasi-inverse g such that $f = fgf$ and $g = gfg$. The morphism g is obtained as the composition of the orthogonal projection on the closed range of f with the inverse of $f_{|\mathrm{Ker}(f)^\perp}$. Thus fg is the orthogonal projection on the closed range of f and gf is the orthogonal projection on the support of f, *i.e.* $\mathrm{Ker}(f)^\perp$, hence $f = fgf$ and $g = gfg$. However in general it is not true that the composition of strict morphisms is strict. Indeed, a selfadjoint idempotent is strict, but in general the product of two selfadjoint idempotents is not strict since the operator valued angle can have arbitrary spectrum.

In the Hilbert space context any operator admits a canonical decomposition as the product of its restriction to its support, which is the orthogonal of its kernel, with the orthogonal projection on its support. We give an analogous statement for morphisms in $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$.

Lemma 7.5. *Let $f \in Hom_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(E,F)$, then $Ker(f) = \mathrm{Range}(f^*)^\perp$.*

Proof. One has $\mathrm{Ker}(f) = \{x \mid f(x) = \sigma(f(x))\}$, moreover since the duality between F and F^* is separating one has $f(x) = \sigma(f(x)) \iff \phi(f(x)) = \phi(\sigma(f(x)))\ \forall \phi \in F^*$. Thus

$$x \in \mathrm{Ker}(f) \iff \phi \circ f(x) \text{ null}\,,\ \ \forall \phi \in F^* \iff x \in \mathrm{Range}(f^*)^\perp.$$

□

Next, we apply the canonical decomposition of Proposition 2.15 to obtain the following analogue in the category $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ of the above decomposition of operators in Hilbert space.

Proposition 7.6. *Let $f \in Hom_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(E,F)$ be a morphism of finite objects in $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$. Then*

(i) The involution σ on E restricts to the support $S = \mathrm{Support}(I(f))$ of $I(f) \in Hom_{\mathbb{B}}(I(E), I(F))$ and the morphisms q_S and $f|_S$ are morphisms in $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$.

(ii) The canonical factorization $f = f|_S \circ q_S$ holds in $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$.

(iii) The kernel of f is the orthogonal of its support: $Ker(f) = \hat{S}^\perp$, where $\hat{S}$ is a subsemimodule of E^* using the canonical isomorphism $E^{\mathrm{op}} \simeq E^*$.

Proof. (i) The definition of the support as $S = \{z \in E \mid f(y) \le f(z) \Rightarrow y \le z\}$ shows that $z \in S \iff \sigma(z) \in S$. Similarly the definition of q_S shows that it commutes with σ and since by hypothesis f commutes with σ one gets (i).

(ii) Follows from Proposition 2.15.

(iii) By Lemma 2.16 one has $\hat{S} = \mathrm{Range}(f^*)$, thus the result follows from Lemma 7.5. □

7.2 Nullity of kernel and injectivity. As a general tool to prove injectivity of a morphism $f \in \mathrm{Hom}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(E,F)$ one has

Theorem 7.7. *Let E be an object of $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ and $x, y \in E$ such that $x+\sigma(y)$ is not a null element (i.e. is not fixed by σ). Then for any morphism $f \in Hom_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(E,F)$ with null kernel, one has $f(x) \neq f(y)$.*

Proof. If $f(x) = f(y)$ one has $f(x) + \sigma(f(y))$ null and hence $x + \sigma(y) \in \mathrm{Ker}(f)$, thus we get

$$f(x) = f(y) \Rightarrow x + \sigma(y) \in \mathrm{Ker}(f). \tag{72}$$

When $\mathrm{Ker}(f)$ is null we obtain the required statement. □

Remark 7.8. (i) The relation "$x\mathcal{R}y \iff x+\sigma(y) \in E^\sigma$", between elements x, y of an object E of $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ is symmetric and reflexive but in general not transitive. The simplest example of a non transitive relation is for $E = \mathfrak{s}\mathbb{B}$. Here one sees that $(1,0)\mathcal{R}(1,1)$, $(0,1)\mathcal{R}(1,1)$ while $(1,0)\mathcal{R}(0,1)$ does not hold.

(ii) In fact, we show that for any object E of $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$, and $x, y \in E$, if $p(x) = p(y)$ then x and y belong to the same class of the equivalence relation $\sim_{\mathcal{R}}$ generated by the relation $\mathcal{R}$. Indeed one has

$$x + \sigma(x) = y + \sigma(y) \Rightarrow x + \sigma(x + y) = (x + y) + \sigma(x).$$

This shows that $x + \sigma(x + y) \in E^{\sigma}$, hence $x\mathcal{R}(x + y)$. Thus $x \sim_{\mathcal{R}} x + y$. Similarly $y \sim_{\mathcal{R}} x + y$ so that $x \sim_{\mathcal{R}} y$ by transitivity of $\sim_{\mathcal{R}}$.

The next theorem shows that, by implementing a technical hypothesis on an object E of $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$, the kernel of a morphism $f : E \to F$ in $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ plays a role similar to the kernel of a linear map inasmuch as its nullity is equivalent to injectivity. Since on the null objects the nullity of the kernel is automatic, the statement of injectivity is relative to the projection $p : E \to E^{\sigma}$, $p(x) = x + \sigma(x)$.

Definition 7.9. *A morphism $f \in Hom_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(E, F)$ is said to be σ-*injective *if the restriction of f to each fiber of p is injective.*

Notice that a morphism $f \in \mathrm{Hom}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(E, F)$ is injective if and only if it is σ-injective and its restriction to the null elements is injective.

Theorem 7.10. *Let E be a finite object of $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ whose dual E^* is generated by its minimal non-zero elements. Then for $f \in Hom_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(E, F)$ the following statement holds*

$$f \textit{ is } \sigma\textit{-injective} \iff Ker(f) \textit{ is null.}$$

Proof. First assume that the restriction of f to the fibers of p is injective, then since x and $p(x)$ are on the same fiber of p, the equality $f(x) = \sigma(f(x))$ implies $f(x) = f(p(x))$ and hence $x = p(x)$ which shows that the kernel of f is null. Conversely, assume that $\mathrm{Ker}(f)$ is null. By Proposition 7.6 (ii) one has $\mathrm{Ker}(f) = \mathrm{Range}(f^*)^{\perp}$ and hence by Lemma 7.2 the least normal subobject of E^* containing $\mathrm{Range}(f^*)$ is $(\mathrm{Range}(f^*)^{\perp})^{\perp} = (\mathrm{Ker}(f))^{\perp} = E^*$ since $\mathrm{Ker}(f)$ is null. Note that we use here the finiteness of E to apply Lemma 7.2 to E^*, using the identification of E with the dual of E^*. Thus the normal image of the inclusion $\mathrm{Range}(f^*) \subset E^*$ is E^* and Proposition 6.17 (ii) shows that $\mathrm{Range}(f^*) + (E^*)^{\sigma} = E^*$. Let now $\xi, \eta \in E$ be such that $p(\xi) = p(\eta)$ and $f(\xi) = f(\eta)$. We prove that $\xi = \eta$. It is enough to show that one has $L(\xi) = L(\eta)$ for any $L \in E^*$ since the duality is separating (both in $\mathbb{B}\mathrm{mod}$ and in $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$). Since $\mathrm{Range}(f^*) + (E^*)^{\sigma} = E^*$ it is enough to show that $L(\xi) = L(\eta)$ when $L \in \mathrm{Range}(f^*)$ and when $L = L \circ \sigma$. In the latter case the equality follows from $p(\xi) = p(\eta)$. For $L \in \mathrm{Range}(f^*)$ the equality follows from $f(\xi) = f(\eta)$. ☐

Remark 7.11. It is not true in general that for $f \in \mathrm{Hom}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(E, F)$ and $\mathrm{Ker}(f)$ null, one has $f^{-1}(\{0\}) = \{0\}$. However the nullity of $\mathrm{Ker}(f)$ implies $f^{-1}(\{0\}) \subset E^{\sigma}$, and that the fiber of p above any $a \in f^{-1}(\{0\})$ is reduced to a. Thus the restriction of f to these fibers is injective.

The condition that the dual E^* is generated by its minimal non-zero elements can be weakened still ensuring the conclusion of Theorem 7.10

Corollary 7.12. *Let E be a finite object of $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ such that for any fiber F of p the dual of the $\mathbb{B}$-semimodule $\{0\} \cup F$ is generated by its minimal non-zero elements. Then for $f \in Hom_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(E, F)$ the following statement holds*

$$f \textit{ is } \sigma\textit{-injective} \iff Ker(f) \textit{ is null.}$$

Proof. This follows from Theorem 7.10 applied to the restriction of f to the object $\{0\} \cup F$ of $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$. The kernel of this restriction is null if $\mathrm{Ker}(f)$ is null. ☐

The hypothesis of Corollary 7.12 holds for the object $E = \mathrm{Coker}(F(\alpha''))$ described in Figure 14 while the dual E^* is not generated by its minimal non-zero elements (see Remark 9.14).

Next, we show using Theorem 7.10, that the condition that the dual M^* of a $\mathbb{B}$-semimodule M is generated by its minimal non-zero elements suffices to derive the implication

$$0 \underset{0}{\overset{0}{\rightrightarrows}} M \underset{g}{\overset{f}{\rightrightarrows}} N \text{ exact at } M \Rightarrow (f,g) \text{ is a monomorphism in } \mathbb{B}\mathrm{mod}^2.$$

This statement was discussed at length in §4.2 where several counterexamples have been given none of which though fulfills the above condition. Moreover, the second statement of next Corollary 7.13 exhibits a large class of objects X in $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ which fulfill the analogue of the fundamental property holding for morphisms in an abelian category *i.e.*

$$f \in \mathrm{Hom}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(X,Y) \text{ monomorphism} \iff \mathrm{Ker}(f) \text{ null.} \tag{73}$$

Corollary 7.13. *Let M be a finite object of $\mathbb{B}\mathrm{mod}$ whose dual M^* is generated by its minimal non-zero elements.*

(i) The sequence $0 \underset{0}{\overset{0}{\rightrightarrows}} M \underset{g}{\overset{f}{\rightrightarrows}} N$ is strictly exact at M if and only if (f,g) is a monomorphism in $\mathbb{B}\mathrm{mod}^2$.

(ii) $\phi \in Hom_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(\mathfrak{s}M, X)$ is a monomorphism if and only if its kernel is null.

Proof. (i) By Proposition 4.10, we need to show that, assuming strict exactness, the map $\iota : M^2 \to N^2$, $\iota(x,y) = (f(x)+g(y), g(x)+f(y))$ of (37) is injective. Proposition 4.8 (ii) shows that its restriction to the null elements (*i.e.* to the diagonal) is injective. Moreover by hypothesis the kernel of ι (viewed as a morphism in $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$) is null and the dual of $\mathfrak{s}M = M^2$ is generated by its minimal elements. Thus Theorem 7.10 shows that ι is σ-injective and hence injective since its restriction to the null elements is injective.

(ii) The map ϕ is a monomorphism iff it is injective. If it is injective its kernel is null since $\phi \circ \sigma = \sigma \circ \phi$. Conversely, assume that the kernel of ϕ is null. Let $\eta_X : X \to TX$ be the unit, $\eta_X(b) = (b, \sigma(b))$. The kernel of $\eta_X \circ \phi$ is null. By Lemma 5.2, the morphism $\eta_X \circ \phi \in \mathrm{Hom}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(\mathfrak{s}M, TX)$ derives from a morphism (f,g) in $\mathbb{B}\mathrm{mod}^2$. Moreover the normal image of the morphism $0 : 0 \to \mathfrak{s}M$ in $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ is the diagonal $\Delta \subset \mathfrak{s}M$ and thus exactness in $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ at $\mathfrak{s}M$ is equivalent to strict exactness. Then the required injectivity follows from (i). $\square$

Next, we investigate the meaning of the technical hypothesis that the dual E^* of an object E of $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ is generated by its minimal elements. The next example shows a class of cases where it fails.

Example 7.14. Let X be an arbitrary $\mathbb{B}$-semimodule. We define an object $E = X \vee X$ of $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ as follows. The underlying $\mathbb{B}$-semimodule is $X \cup_0 X' \cup \{t\}$, where X' is a second copy of X and the zero elements are identified, while the additional element t is the largest element in E. The addition restricts to the addition in each copy of X and is otherwise defined by $x + x' = t$ whenever $x \in X$, $x' \in X'$ are non-zero. The element t is fixed by the involution σ which interchanges the two copies of X. The elements $0, t$ are the only null elements. One has $\mathbb{B} \vee \mathbb{B} = \mathfrak{s}\mathbb{B}$ and moreover the maximal element μ of $X^* = \mathrm{Hom}_{\mathbb{B}}(X, \mathbb{B})$, *i.e.* the element such that $\mu^{-1}(\{0\}) = \{0\}$, defines a morphism $\mu \vee \mu : X \vee X \to \mathbb{B} \vee \mathbb{B} = \mathfrak{s}\mathbb{B}$ in $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$. By construction the kernel of $\mu \vee \mu$ is null but as soon as X has more than one element $\mu \vee \mu$ fails to be σ-injective.

To understand what happens in Example 7.14, we assume for simplicity that X is finite. The support S of $f = \mu \vee \mu$ is by definition $S = \{z \in E \mid f(y) \leq f(z) \Rightarrow y \leq z\}$ and this selects the subset $S = \{0, u, u', t\} \subset E = X \vee X$ where $u := \sum_X j$ is the largest element of X (which exists since X is assumed to be finite). When viewed as a subset of $E^* \simeq E^{\rm op}$, the sub-object $\hat{S}$ contains the null elements and we seek to understand why its normal closure is E^*. This follows from Proposition 6.16 since for any $\xi \in X^{\rm op} \subset E^{\rm op}$, non null, one has with $a = u^{\rm op}$: $\xi + a = \xi$ (since a is minimal) and $a + \sigma(a) = \xi + \sigma(\xi)$. Thus the condition of Proposition 6.16 holds with $n = 0$ and $a_0 = a$. When X is not finite, $u := \sum_X j$ no longer exists in general as an element of X but it still makes sense as the ideal $J = X \subset E = X \cup_0 X' \cup \{t\}$, so that the associated element $\phi_u \in E^*$ is meaningful.

Remark 7.15. Let $\mu \vee \mu : X \vee X \to \mathbb{B} \vee \mathbb{B} = \mathfrak{s}\mathbb{B}$ be as in Example 7.14. Then the sequence

$$0 \to X \vee X \stackrel{\mu \vee \mu}{\to} \mathbb{B} \vee \mathbb{B} \to 0 \tag{74}$$

is strictly exact since the kernel of $\mu \vee \mu$ is null and its range is $\mathbb{B} \vee \mathbb{B}$. This result is in sharp contrast with Proposition 4.12 which forbids for any object of $\mathbb{B}\mathrm{mod}^2$ not isomorphic to $\mathbb{B} \vee \mathbb{B}$ to be on the left hand side of (74).

We can now refine Theorem 7.10.

Theorem 7.16. *Let E be a finite object of $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$, $f \in Hom_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(E, F)$ a morphism with null kernel and $x, y \in E$ such that $p(x) = p(y)$ and $f(x) = f(y)$. Then $x \sim_{\mathrm{Rad}(E)} y$.*

Proof. As in the proof of Theorem 7.10 one gets that the normal image of the inclusion $\mathrm{Range}(f^*) \subset E^*$ is E^*. Let then $\xi \in E^*$ be a minimal element, then we show that $\xi(x) = \xi(y)$. If $\sigma(\xi) = \xi$ this follows from $p(x) = p(y)$. If $\sigma(\xi) \neq \xi$, Proposition 6.17 states that $\xi \in \mathrm{Range}(f^*)$, *i.e.* $\xi = L \circ f$ for some $L \in F^*$, but then $\xi(x) = L(f(x)) = L(f(y)) = \xi(y)$. ☐

7.3 Injective and Projective objects in $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$.

Proposition 2.4 states that the product $\mathbb{B}^X$ of any number of copies of $\mathbb{B}$ is an injective object in the category of $\mathbb{B}$-semimodules and that any $\mathbb{B}$-semimodule is isomorphic to a subsemimodule of a product $\mathbb{B}^X$. Next lemma relates the study of injective/projective objects in $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ to the corresponding one in $\mathbb{B}\mathrm{mod}$ by applying the (forgetful) functor $I : \mathbb{B}\mathrm{mod}^{\mathfrak{s}} \longrightarrow \mathbb{B}\mathrm{mod}$.

Lemma 7.17.
(i) An object E of $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ is injective if and only if the underlying $\mathbb{B}$-semimodule $I(E)$ is injective in $\mathbb{B}\mathrm{mod}$.

(ii) An object E of $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ is projective if and only if the underlying $\mathbb{B}$-semimodule $I(E)$ is projective in $\mathbb{B}\mathrm{mod}$.

(iii) For the finite *objects in $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ the properties of being injective/projective are equivalent and they mean that an object is a retract of a finite product $(\mathfrak{s}\mathbb{B})^n$.*

Proof. (i) Assume that $I(E)$ is injective in $\mathbb{B}\mathrm{mod}$, consider an inclusion $L \subset M$ in $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ and let $f : L \to E$ be a morphism in $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$. Since $I(E)$ is injective, let $g : M \to E$ be a morphism in $\mathbb{B}\mathrm{mod}$ extending f. Then $h = g + \sigma \circ g \circ \sigma$ agrees with f on L and commutes with σ so that it defines an extension of f to a morphism $M \to E$ in $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$. Thus E is injective in $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$. Conversely, let E be an object of $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$, then by Proposition 2.4 there exists an embedding $\iota : I(E) \subset \mathbb{B}^X$, and the map $x \mapsto u(x) = (\iota(x), \iota(\sigma(x))) \in \mathfrak{s}\mathbb{B}^X$ gives an embedding in $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ of E as a subobject of $\mathfrak{s}\mathbb{B}^X$. If the object E of $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ is injective there exists a retraction of u, *i.e.* a morphism of $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$, $v : \mathfrak{s}\mathbb{B}^X \to E$ such that $v \circ u = \mathrm{Id}$. It follows that the same equality holds in $\mathbb{B}\mathrm{mod}$ and since $I(\mathfrak{s}\mathbb{B}^X)$ is injective in $\mathbb{B}\mathrm{mod}$, so is $I(E)$.

(ii) Assume that $I(E)$ is projective in $\mathbb{B}\mathrm{mod}$, and let $M \to N$ be an epimorphism in $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ and $f : E \to N$ a morphism in $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$. Since $I(E)$ is projective, let $g : E \to M$ be a morphism in $\mathbb{B}\mathrm{mod}$ lifting f. Then as for (i) $h = g + \sigma \circ g \circ \sigma$ determines a lift of f in $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$. For the converse one uses Proposition 2.4 and the existence of a surjective morphism $k : \mathbb{B}^{(X)} \to I(E)$ which extends to a surjective morphism of $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ from $\mathfrak{s}\mathbb{B}^{(X)}$ to E.

(iii) By Proposition 2.4 for finite object of $\mathbb{B}\mathrm{mod}$ the properties of being injective/projective are equivalent and they mean that an object is a retract of a finite product $\mathbb{B}^n$. Thus by (i) and (ii) one obtains the required statement. □

7.3.1 Normal subobjects of injective objects in $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$. We now investigate the condition on an object A of $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ that A embeds as a *normal* subobject of an injective object of $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$.

Lemma 7.18. *Let E be an object of $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ such that $E^\sigma = \mathbb{B}$. Then if E is projective one has $E = \mathbb{B}$ or $E = \mathfrak{s}\mathbb{B}$.*

Proof. Let X be the complement of $E^\sigma = \mathbb{B}$ in E, and $Y = X/\sigma$ the orbit space of the action of σ on X and $\iota : Y \to X$ an arbitrary section of the canonical surjection $X \to Y$. Let $\phi : (\mathfrak{s}\mathbb{B})^{(Y)} \to E$ be the morphism in $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ which is given on the copy of $\mathfrak{s}\mathbb{B}$ corresponding to $y \in Y$ by

$$\phi((1,0)_y) := \iota(y), \quad \phi((0,1)_y) := \sigma(\iota(y))$$

The morphism $\phi : (\mathfrak{s}\mathbb{B})^{(Y)} \to E$ is surjective by construction. Let $p : (\mathfrak{s}\mathbb{B})^{(Y)} \to \mathbb{B}^{(Y)}$ be the projection on the fixed points of the involution, we identify $\mathbb{B}^{(Y)}$ with the boolean $\mathbb{B}$-semimodule of finite subsets of Y. Let $\psi : E \to (\mathfrak{s}\mathbb{B})^{(Y)}$ be a morphism in $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ such that $\phi \circ \psi = \mathrm{Id}_E$. Since $E^\sigma = \mathbb{B}$ contains only one non-zero element τ one has $p(\psi(\xi)) = \psi(\tau)$ for any non-zero $\xi \in E$. Let $S \subset Y$ be the finite subset of Y corresponding to $\psi(\tau)$. Let us show that the existence of ψ implies the following property of E:

$$\xi + \eta = \tau\,, \quad \forall \xi, \eta \in E \setminus \{0\}, \xi \neq \eta \tag{75}$$

To prove this we can assume that $\xi, \eta \in E \setminus \{0, \tau\}$. For $\xi \in E \setminus \{0, \tau\}$ the components $\psi(\xi)_y$ are 0 for $y \notin S$ and are either $(1,0)$, in which case we write $y \in S_\xi$, or $(0,1)$ for $y \in S$ since they are non-zero and if one component is $(1,1)$ one would get $\phi(\psi(\xi)) = \tau$. Thus to each $\xi \in E \setminus \{0, \tau\}$ corresponds a partition of S as a disjoint union of S_ξ and its complement. For two distinct elements $\xi, \eta \in E \setminus \{0, \tau\}$ these partitions are different and thus there exists $y \in S$ such that the components $\psi(\xi)_y$ and $\psi(\eta)_y$ are different and this gives $\phi \circ \psi(\xi + \eta) = \tau$ and thus (75). We have shown that if E is projective it fulfills (75), and hence that the underlying $\mathbb{B}$-semimodule $I(E)$ is obtained from the set X by adjoining 0 and the element τ while the addition is idempotent and uniquely specified by (75). Let then $\rho : \mathbb{B}^{(X)} \to I(E)$ the surjective morphism in $\mathbb{B}\mathrm{mod}$ which associates to any finite subset $F \subset X$ the sum $\sum_F \xi$. If E is projective so is $I(E)$ by Lemma 7.17, and thus there exists a section $\delta \in \mathrm{Hom}_{\mathbb{B}}(I(E), \mathbb{B}^{(X)})$, $\rho \circ \delta = \mathrm{Id}_E$. For any $x \in X$ the element $\delta(x)$ is given by the subset $F = \{x\}$ and since the range of δ must be a subsemimodule of $\mathbb{B}^{(X)}$ one gets that X has at most two elements. This shows that one is in one of the two cases $E = \mathbb{B}$ or $E = \mathfrak{s}\mathbb{B}$. □

Proposition 7.19. *Let E be a finite object of $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ such that $E^\sigma = \mathbb{B}$. Then if E is isomorphic to a normal subobject of a finite injective object of $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ one has $E = \mathbb{B}$ or $E = \mathfrak{s}\mathbb{B}$.*

Proof. Consider an embedding $E \subset I$ of E as a normal subobject of a finite injective object of $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$. The equality $E^\sigma = I^\sigma$ shows that $I^\sigma = \mathbb{B}$. Since I is finite and injective it is also projective by Lemma 7.17. Thus by Lemma 7.18 one has $I = \mathbb{B}$ or $I = \mathfrak{s}\mathbb{B}$ and hence $E = \mathbb{B}$ or $E = \mathfrak{s}\mathbb{B}$. $\square$

As a corollary of Proposition 7.19 we get that among the examples 7.14 only the trivial ones give a normal subobject of a finite injective object.

7.3.2 An example: a kernel-cokernel sequence in $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$*.* We investigate a specific example of an object of $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ which is not projective. Let S be the object of $\mathbb{B}\mathrm{mod}$ with three generators a, b, c such that $a + b = b + c$. It follows that $a + b = b + c = a + b + c$. Thus in the Boolean object $\mathbb{B}^3$ of $\mathbb{B}\mathrm{mod}$ freely generated by a, b, c (Figure 7) one identifies $a + b = b + c = a + b + c$.

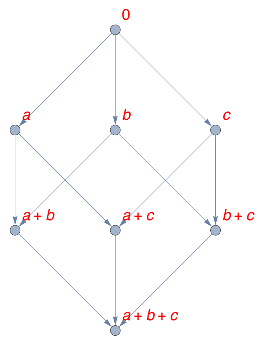

Figure 7: Boolean object of $\mathbb{B}\mathrm{mod}$ freely generated by a, b, c.

The object $\mathbb{B}^3$ of $\mathbb{B}\mathrm{mod}$ becomes an object of $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ when endowed with the involution $\sigma(a) = c$, $\sigma(b) = b$. Let S (see Figure 8) be the quotient of $\mathbb{B}^3$ by the relation $a + b = b + c$ and endowed with the induced involution. Let $\phi : \mathbb{B}^3 \to S$ the quotient map. Consider the functor $H := \underline{\mathrm{Hom}}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(S, -)$ viewed as a covariant endofunctor on $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$, using the natural internal Hom.

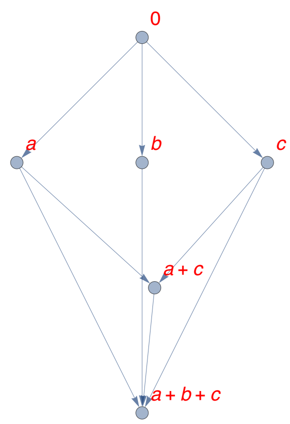

Figure 8: The object S of $\mathbb{B}\mathrm{mod}$.

Proposition 7.20. *(i) The kernel of $\phi : \mathbb{B}^3 \to S$ is the subobject $J \subset \mathbb{B}^3$ which is the complement of a, c in $\mathbb{B}^3$.*

(ii) The sequence $s: \; J \rightarrowtail \mathbb{B}^3 \xrightarrow{\phi} S$ is a short doubly exact sequence and is non split in $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$.

(iii) $Id_S, \sigma_S \in H(S) = \underline{\mathrm{Hom}}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(S, S)$ are the only elements which do not belong to the range of $H(\phi) : H(\mathbb{B}^3) \to H(S)$.

(iv) The cokernel of the morphism $H(\phi) : H(\mathbb{B}^3) \to H(S)$ is given by the projection p on the range of $H(\phi)$ and by the identity on the complement of the range.

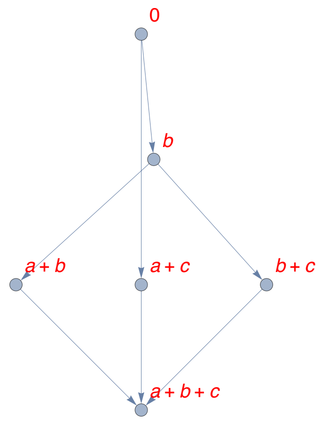

Figure 9: Kernel J of the map ϕ from $\mathbb{B}^3$ to S.

Proof. (i) The map $\phi : \mathbb{B}^3 \to S$ is defined as

$$0 \to 0, a \to a, b \to b, c \to c, a+b \to a+b+c, a+c \to a+c, b+c \to a+b+c, a+b+c \to a+b+c.$$

and the subsemimodule $S^\sigma \subset S$ of null elements is $S^\sigma = \{0, b, a+c, a+b+c\}$. It follows that the only elements of $\mathbb{B}^3$ which do not belong to the kernel of ϕ are a, c.

(ii) We show that the cokernel of the inclusion $J \subset \mathbb{B}^3$ is given by the map ϕ. By Proposition 6.19, this cokernel is the quotient of $(\mathbb{B}^3)^\sigma \cup J^c$ by the smallest equivalence relation fulfilling the rule

$$\xi \in J^c, \;\; u, v \in J, \;\; p(u) = p(v) \Rightarrow \xi + u \sim \xi + v.$$

$\mathrm{cok}(\phi)$ agrees with p on J and with the quotient map on J^c. For $\xi \in J^c = \{a, c\}$ one has $\xi + u \in J$ for any non-zero element $u \in J$. This shows that the equivalence relation is trivial, thus $\mathrm{Coker}(\phi) = (\mathbb{B}^3)^\sigma \cup J^c$. Since the kernel of $\phi : \mathbb{B}^3 \to S$ is the subobject $J \subset \mathbb{B}^3$, the map ϕ factors through $\mathrm{Coker}(\phi)$ and hence agrees with it. The proof of (iii) below shows that s is not split.

(iii) We first show that the element $\mathrm{Id}_S \in H(S) = \underline{\mathrm{Hom}}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(S, S)$ does not belong to the range of the morphism $H(\phi) : H(\mathbb{B}^3) \to H(S)$. For any object M of $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$, an element $f \in H(M)$ is given by a pair (α, β) of elements of M such that $\beta \in M^\sigma$ and $\alpha + \beta \in M^\sigma$. One lets $\alpha = f(a)$ and $\beta = f(b)$. For the element $\mathrm{Id}_S \in H(S) = \underline{\mathrm{Hom}}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(S, S)$ one has $\alpha = a$ and $\beta = b$. These two elements lift uniquely to elements of $\mathbb{B}^3$ but the lifts $\alpha' = a, \beta' = b$ no longer satisfy $\alpha' + \beta' \in (\mathbb{B}^3)^\sigma$. In the same way one sees that the element $\sigma_S \in H(S) = \underline{\mathrm{Hom}}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(S, S)$ does not belong to the range of the morphism $H(\phi) : H(\mathbb{B}^3) \to H(S)$. There are 6 endomorphisms of

S which are not null, they correspond to the values of (α, β) reported here below

$$\begin{pmatrix} a & b \\ a & a+c \\ a & a+b+c \\ c & b \\ c & a+c \\ c & a+b+c \end{pmatrix}$$

Then, Id_S which corresponds to (a,b), and σ_S which correponds to (c,b) are both not liftable. But the other 4 endomorphisms are liftable since the sum $\alpha+\beta$ is null already in $\mathbb{B}^3$.

(iv) Let $\xi = \mathrm{Id}_S \in H(S) = \underline{\mathrm{Hom}}_{\mathbb{B}\mathrm{mod}^s}(S,S)$. We determine the interval $[0,\xi] \subset H(S)$. One has $f \in [0,\xi]$ iff $\alpha \le a$ and $\beta \le b$. For $\alpha = a$ and $\beta = 0$ this does not fulfill $\alpha + \beta \in S^\sigma$. Thus the only non-trivial element $f \in [0,\xi]$ corresponds to $\alpha = 0$ and $\beta = b$. This element is null, and thus Proposition 6.17 (iii) applies to show that $\xi \notin \overline{\mathrm{Im}}(H(\phi))$. This shows that the cokernel of the morphism $H(\phi) : H(\mathbb{B}^3) \to H(S)$ is non null. The result then follows from Proposition 6.19. □

Next, we consider the endomorphisms of the short exact sequence $s: \ J \rightarrowtail \mathbb{B}^3 \overset{\phi}{\twoheadrightarrow} S$, *i.e.* the endomorphisms $f \in \mathrm{End}_{\mathbb{B}\mathrm{mod}^s}(\mathbb{B}^3)$ such that $f(J) \subset J$. They define a correspondence between $\mathrm{End}_{\mathbb{B}\mathrm{mod}^s}(J)$ and $\mathrm{End}_{\mathbb{B}\mathrm{mod}^s}(S)$ displayed in the following diagram

$$\mathrm{End}_{\mathbb{B}\mathrm{mod}^s}(J) \overset{res}{\leftarrow} \mathrm{End}(s) \overset{quot}{\to} \mathrm{End}_{\mathbb{B}\mathrm{mod}^s}(S). \tag{76}$$

The left arrow is given by restriction of f to $J \subset \mathbb{B}^3$ and the right one is defined by the induced morphism on the cokernel.

Proposition 7.21. *(i) $f \in \mathrm{End}_{\mathbb{B}\mathrm{mod}^s}(\mathbb{B}^3)$ is uniquely specified by the pair $(f(a), f(b)) \in \mathbb{B}^3 \times (\mathbb{B}^3)^\sigma$.*

(ii) Among the 32 endomorphisms $f \in \mathrm{End}_{\mathbb{B}\mathrm{mod}^s}(\mathbb{B}^3)$ only two do not fulfill $f(J) \subset J$, they correspond to the pairs $(a,0)$ and $(c,0)$.

(iii) For any $h \in \mathrm{End}_{\mathbb{B}\mathrm{mod}^s}(S)$ such that $h \notin \overline{\mathrm{Im}}(H(\phi))$ the restriction of h to S^σ is an automorphism.

(iv) Let $v \in \mathrm{End}_{\mathbb{B}\mathrm{mod}^s}(J)$ admit more than one extension to $\mathbb{B}^3$. Then for any of these extensions w the restriction to S^σ of the induced morphisms $w'' \in \mathrm{End}_{\mathbb{B}\mathrm{mod}^s}(S)$ fails to be surjective and $w'' \in \overline{\mathrm{Im}}(H(\phi))$.

Proof. (i) The commutation of f with the involution σ of $\mathbb{B}^3$ entails that $f(c) = \sigma(f(a))$ and that $f(b) \in (\mathbb{B}^3)^\sigma$. Conversely, any pair $(f(a), f(b)) \in \mathbb{B}^3 \times (\mathbb{B}^3)^\sigma$ uniquely defines an $f \in \mathrm{End}_{\mathbb{B}\mathrm{mod}^s}(\mathbb{B}^3)$.

(ii) Let $f \in \mathrm{End}_{\mathbb{B}\mathrm{mod}^s}(\mathbb{B}^3)$. One has $f(b) \in (\mathbb{B}^3)^\sigma \subset J$. Thus if $f(a) = 0$ then $f(\mathbb{B}^3) \subset (\mathbb{B}^3)^\sigma \subset J$. Also, if $f(a) \in J$ then $f(\mathbb{B}^3) \subset J$. Thus, by Proposition 7.20 (i), we just need to consider the cases $f(a) = a$ and $f(a) = c$. Assume $f(a) = a$. Then if $f(b) \neq 0$ one gets $f(J) \subset J$ since $f(c) = c$ and for any subset $Y \subset \{a,b,c\}$ not reduced to a or c one has $\sum_Y f(y) \in J$. If $f(b) = 0$ one has $f(a+b) = a$ and this contradicts $f(J) \subset J$.

(iii) By Proposition 7.20 (iii), the only such h are Id_S and σ_S.

(iv) The restriction map of (76) is surjective since $\mathbb{B}^3$ is injective. By (ii) there are 30 endomorphisms of the short doubly exact sequence s and when we take their restriction to J one obtains the following 4 non-trivial fibers with three elements each (where we list the corresponding pairs $(f(a), f(b)) \in \mathbb{B}^3 \times (\mathbb{B}^3)^\sigma$):

$$\begin{pmatrix} a & a+c \\ c & a+c \\ a+c & a+c \end{pmatrix}, \begin{pmatrix} a+b & a+c \\ b+c & a+c \\ a+b+c & a+c \end{pmatrix}, \begin{pmatrix} a & a+b+c \\ c & a+b+c \\ a+c & a+b+c \end{pmatrix}, \begin{pmatrix} a+b & a+b+c \\ b+c & a+b+c \\ a+b+c & a+b+c \end{pmatrix}$$

In all these cases the restriction of f to $(\mathbb{B}^3)^\sigma$ fails to be surjective. This restriction is the same as the restriction to S^σ of the induced morphisms w'' on S by Proposition 6.20. Thus by *(iii)* one gets $w'' \in \overline{\mathrm{Im}}(H(\phi))$. There are 22 elements in $\mathrm{End}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(J)$ and since the restriction map (76) is surjective one concludes that all other fibers are reduced to a single element. $\square$

8. Homological algebra in a homological category

The conceptual definition of sheaf cohomology as derived functors of the functor of global sections suggests to study, in the category $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$, the right derived functors Ext^n of the functor $F := \mathrm{Hom}(L, -)$ for a fixed object L. In this section we apply the construction of chapter 4 of [18] of satellite functors as Kan extensions. After recalling in §8.1 the framework of homological algebra from [18], we give a straightforward adaptation of the construction to left exact functors and right satellite in §8.2. In §8.3 we explain the key condition (a) of [18] which allows one to compute the satellite functor SF. The latter is defined, on an object X as a colimit indexed by a comma category $J = \mathrm{Sh}(\mathcal{E}) \downarrow_{P'} X$ built from short doubly exact sequences. This indexing category does not in general admit a final object but under good circumstances it admits a weakly final object coming from a semi-resolution i of X. Condition (a) of [18], Theorem 4.2.2, is the requirement that such weakly final object of the indexing category J becomes final after applying the functor $\alpha \longrightarrow \mathrm{Coker}(F(\alpha''))$ which defines the colimit. In other words applying this functor should erase the ambiguity created by the weak finality of the semi-resolution. Our main result in this development is explained in §8.4, it is Theorem 8.10 which, under the hypothesis that the middle term of the semi-resolution is both injective and projective, reduces the proof of condition (a) to endomorphisms of the weakly final short doubly exact sequence. This result is applied in Theorem 8.11 to show that the satellite functor of the hom functor is not null for a specific finite object of $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$.

8.1 Short doubly exact sequences of chain complexes. We first recall that in the case of an abelian category the functor $\mathrm{Hom}(A, -)$ (for fixed A) is left exact and hence admits right derived functors $R^nT = \mathrm{Ext}^n(A, -)$ which are computed, by using an injective resolution I^-, as the cohomology of the complex

$$\cdots \to \mathrm{Hom}(A, I^j) \to \mathrm{Hom}(A, I^{j+1}) \to \mathrm{Hom}(A, I^{j+2}) \to \cdots$$

We start this section by comparing two notions of “left exactness" in this setup. The first notion we take up is the purely categorical notion

Definition 8.1. *A functor between finitely complete categories is called left exact (or flat) if it preserves finite limits.*

Let $F \dashv G$ be a pair of functors with F left adjoint to G. Then F preserves all colimits and G preserves all limits. For the category $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ and a fixed object A we consider the internal Hom functor $\underline{\mathrm{Hom}}(A,-)$ as a covariant functor $G:\mathbb{B}\mathrm{mod}^{\mathfrak{s}}\longrightarrow\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$, then one has the adjunction which follows from the construction of the tensor product as representing bilinear maps

$$\mathrm{Hom}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(X,\underline{\mathrm{Hom}}(A,Y))\simeq\mathrm{Hom}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(X\otimes A,Y) \tag{77}$$

where $\otimes$ denotes the tensor product over $\mathbb{B}$ endowed with the involution, *i.e.* the tensor product of the two involutions. This suffices to show that G is left exact in the categorical sense of Definition 8.1. We shall now review some results taken from Chapter III, 3.3 of [18] concerning the homology sequence associated to a short doubly exact sequence of chain complexes. In a homological category $\mathcal{E}$ one defines unbounded chain complexes as sequences

$$\cdots\to A_{n+1}\xrightarrow{\partial_{n+1}}A_n\xrightarrow{\partial_n}A_{n-1}\to\cdots$$

indexed by $n\in\mathbb{Z}$ and satisfying the condition that $\partial_n\circ\partial_{n+1}$ is always null. A morphism of complexes is a sequence of morphisms $f_n:A_n\to B_n$ such that $\partial_n\circ f_n=f_{n-1}\circ\partial_n$. One defines the null morphisms of complexes as those for which all the components f_n are null. This yields the category $\mathrm{Ch}_\bullet(\mathcal{E})$ of chain complexes over $\mathcal{E}$ and the subcategory $\mathrm{Ch}_+(\mathcal{E})$ of positive chain complexes (a positive chain complex is completed in negative indices by null objects). These categories are also homological.

One of the key results of [18] is the following statement (homology sequence) *cf.* 3.3.5.

Theorem 8.2. *Let $\mathcal{E}$ be a homological category and consider a short doubly exact sequence of chain complexes*

$$U\overset{m}{\rightarrowtail}V\overset{p}{\twoheadrightarrow}W,\quad m=\ker(p),\ p=\mathrm{cok}(m).$$

a) There is a homology sequence of order two, natural for morphisms of short doubly exact sequences

$$\cdots\longrightarrow H_n(V)\xrightarrow{H_n(p)}H_n(W)\xrightarrow{\partial_n}H_{n-1}(U)\xrightarrow{H_{n-1}(m)}H_{n-1}(V)\longrightarrow\cdots \tag{78}$$

where ∂_n is induced by the differential ∂_n^V of the complex V.

b) If the differential ∂_n^V of the central complex V is an exact morphism, so is the differential ∂_n of the homology sequence; moreover, the sequence itself is exact in the domain of ∂_n (i.e. $H_n(W)$) and in its codomain (i.e. $H_{n-1}(U)$).

c) If the following conditions hold for every $n\geq 0$, the homology sequence is exact

$$(B_nV\vee U_n)\wedge Z_nV=B_nV\vee(U_n\wedge Z_nV) \tag{79}$$

$$\partial^*\partial_*(U_n)=U_n\vee Z_nV,\quad \partial_*\partial^*(U_{n-1})=U_{n-1}\wedge B_{n-1}V. \tag{80}$$

The central exactness stated in (*b*) is the key to prove the universality of chain homology for non-exact categories (cf. [18] Section 4.5). The exact couples are treated in 3.5 of *op.cit.*

8.2 The right satellite of a left exact functor. In this subsection we transpose the treatment of the left satellite of a right exact functor explained in [18] Section 4.1, to the construction of the right satellite of a left exact functor such as the internal Hom functor $\underline{\mathrm{Hom}}(A,-)$. The right satellite is constructed as a left Kan extension in the sense of [21] p. 240. One considers a homological category $\mathcal{E}$ and the category $\mathrm{Sh}(\mathcal{E})$ of short doubly exact sequences of $\mathcal{E}$ *i.e.* of sequences of the form

$$A'\overset{a'}{\rightarrowtail}A\overset{a''}{\twoheadrightarrow}A''\ \text{s.t.}\ a'=\ker(a'')\ \&\ a''=\mathrm{cok}(a').$$

The morphisms between two such sequences of $\mathcal{E}$ are the same as the morphisms for the corresponding 2-truncated chain complexes (*i.e.* one has morphisms $A' \to B'$ etc. commuting with the a', b' etc). Then, one considers a left exact functor $F : \mathcal{E} \longrightarrow \mathcal{B}$ where $\mathcal{B}$ is also a homological category and cocomplete. One seeks to construct a long sequence of derived functors F^n where $F^0 = F$ and where each short doubly exact sequence of $\mathcal{E}$ gives rise to a long sequence of the form

$$\cdots \to F^n A' \overset{F^n a'}{\to} F^n A \overset{F^n a''}{\to} F^n A'' \overset{d^n}{\to} F^{n+1} A' \overset{F^{n+1} a'}{\to} F^{n+1} A \overset{F^{n+1} a''}{\to} F^{n+1} A'' \to \cdots$$

In order to construct the first right satellite $SF = F^1$ one associates to a short doubly exact sequence of $\mathcal{E}$ the object of $\mathcal{B}$ given by $\mathrm{Coker}(F(a''))$. In fact, one should define SF in such a way that all these objects map to $SF(A')$. Thus $SF(A')$ is defined as a colimit of the $\mathrm{Coker}(F(a''))$'s. One fixes an object X of $\mathcal{E}$ and considers the comma category $\mathrm{Sh}(\mathcal{E}) \downarrow_{P'} X$, where $P' : \mathrm{Sh}(\mathcal{E}) \longrightarrow \mathcal{E}$ is the functor of projection on A'. Thus, an object of this comma category is of the form

$$X \overset{x}{\leftarrow} A' \overset{a'}{\rightarrowtail} A \overset{a''}{\twoheadrightarrow} A''. \tag{81}$$

We introduce the following notion to handle the preservation of null objects by satellite functors.

Definition 8.3. *An $\mathcal{N}$-retraction ρ of an homological category $\mathcal{E}$ is provided by an endofunctor $\rho : \mathcal{E} \longrightarrow \mathcal{E}$ which projects on the subcategory of null objects, and by a natural transformation p from the identity functor to ρ.*

In fact the retraction exists in general for any homological category $\mathcal{E}$ and it is defined by the functor

$$\rho : \mathcal{E} \longrightarrow \mathcal{E}, \quad \rho(X) := \mathrm{Coker}(\mathrm{Id}_X), \quad p_X := \mathrm{cok}(\mathrm{Id}_X) : X \to \mathrm{Coker}(\mathrm{Id}_X).$$

For the category $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ the endofunctor ρ associates to an object N of $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ the fixed points N^σ under σ. The natural transformation p is defined by $p(x) = x + \sigma(x)$.

Lemma 8.4. *Let $\mathrm{Sh}_{\mathrm{small}}$ be a small subcategory of the category $\mathrm{Sh}(\mathcal{E})$ of short doubly exact sequences in the homological category $\mathcal{E}$. Let $F : \mathcal{E} \longrightarrow \mathcal{B}$ be a covariant functor where $\mathcal{B}$ is a cocomplete homological category. The following colimit exists for any object X of $\mathcal{E}$ and defines a covariant functor*

$$SF : \mathcal{E} \longrightarrow \mathcal{B}, \quad SF(X) := \varinjlim_{I} \mathrm{Coker}(F(a'')), \quad I = \mathrm{Sh}_{\mathrm{small}}(\mathcal{E}) \downarrow_{P'} X. \tag{82}$$

Moreover, assuming that F sends null objects to null objects, one obtains for any object of $\mathrm{Sh}_{\mathrm{small}}$, an order two sequence

$$FA' \overset{Fa'}{\to} FA \overset{Fa''}{\to} FA'' \overset{d}{\to} SFA' \overset{SFa'}{\to} SFA \overset{SFa''}{\to} SFA''. \tag{83}$$

Proof. The smallness of $\mathrm{Sh}_{\mathrm{small}}$ ensures that the comma category $\mathrm{Sh}_{\mathrm{small}}(\mathcal{E}) \downarrow_{P'} X$ is small. Since $\mathcal{B}$ is cocomplete, the colimit $SF(X) := \varinjlim_I \mathrm{Coker}(F(a''))$ makes sense. Let $f : X \to Y$ be a morphism in $\mathcal{E}$. The following assignment defines a functor $I \longrightarrow J = \mathrm{Sh}_{\mathrm{small}}(\mathcal{E}) \downarrow_{P'} Y$

$$\left(X \overset{x}{\leftarrow} A' \overset{a'}{\rightarrowtail} A \overset{a''}{\twoheadrightarrow} A'' \right) \mapsto \left(Y \overset{f \circ x}{\leftarrow} A' \overset{a'}{\rightarrowtail} A \overset{a''}{\twoheadrightarrow} A'' \right).$$

This yields a natural morphism $SF(f) \in \mathrm{Hom}_{\mathcal{B}}(SFX, SFY)$, where $SF(X) := \varinjlim_I \mathrm{Coker}(F(a''))$ and $SF(Y) := \varinjlim_J \mathrm{Coker}(F(a''))$. Thus $SF : \mathcal{E} \longrightarrow \mathcal{B}$ is a covariant functor. To obtain the map

d of (83) one uses the object i of $\mathrm{Sh}_{\mathrm{small}}(\mathcal{E}) \downarrow_{P'} A'$ given by $i:\ A' \overset{\mathrm{Id}}{\leftarrow} A' \overset{a'}{\rightarrowtail} A \overset{a''}{\twoheadrightarrow} A''$ and the natural morphism $\phi_i : \mathrm{Coker}(F(a'')) \to SF(A')$ given by the index i and the construction of the colimit. After composition with $\mathrm{cok}(F(a'')) : A'' \to \mathrm{Coker}(F(a''))$, one obtains the morphism $FA'' \overset{d}{\to} SFA'$, $d = \phi_i \circ \mathrm{cok}(F(a''))$. One has to show that $d \circ F(a'')$ is null and this follows from $\mathrm{cok}(F(a'') \circ F(a'')$ null. Next, we consider $SFa' \circ d$. By the functoriality of SF one gets the map $\phi_j \circ \mathrm{cok}(F(a''))$, where the object j of $\mathrm{Sh}_{\mathrm{small}}(\mathcal{E}) \downarrow_{P'} A$ is given by composition with i as

$$i = \left(A' \overset{\mathrm{Id}}{\leftarrow} A' \overset{a'}{\rightarrowtail} A \overset{a''}{\twoheadrightarrow} A''\right) \mapsto j = \left(A \overset{a'}{\leftarrow} A' \overset{a'}{\rightarrowtail} A \overset{a''}{\twoheadrightarrow} A''\right).$$

The following commutative diagram defines a morphism in the comma category $\mathrm{Sh}_{\mathrm{small}}(\mathcal{E}) \downarrow_{P'} A$ from the object j to the object j' represented by the lower horizontal line

$$\begin{tikzcd}
& A' \arrow[r, tail, "a'"] \arrow[dl, "a'"'] \arrow[d, "a'"] & A \arrow[r, two heads, "a''"] \arrow[d, "\mathrm{Id}"'] & A'' \arrow[d] \\
A & A \arrow[l, "\mathrm{Id}"'] \arrow[r, tail, "\mathrm{Id}"] & A \arrow[r, two heads, "\mathrm{cok}(\mathrm{Id})"] & \mathrm{Coker}(\mathrm{Id})
\end{tikzcd} \tag{84}$$

By construction $\mathrm{Coker}(\mathrm{Id})$ is a null object and it follows that $SFa' \circ d$ is null, since in the colimit one has $\phi_j \circ \mathrm{cok}(F(a'')) \sim \phi_{j'} \circ \mathrm{cok}(\mathrm{Id})$ which factorizes through the null object $\mathrm{Coker}(\mathrm{Id})$. It remains to show that the composition $SFa'' \circ SFa'$ is null. This will follow if we show that the image by SF of a null object N is a null object. In order to prove this statement we use an $\mathcal{N}$-retraction ρ of $\mathcal{E}$

$$\begin{tikzcd}
& A' \arrow[r, tail, "a'"] \arrow[dl, "x"'] \arrow[d, "p_{A'}"] & A \arrow[r, two heads, "a''"] \arrow[d, "p_A"'] & A'' \arrow[d, "p_{A''}"] \\
N & \rho(A') \arrow[l, "\rho(x)"'] \arrow[r, tail, "\rho(a')"] & \rho(A) \arrow[r, two heads, "\rho(a'')"] & \rho(A'')
\end{tikzcd} \tag{85}$$

The diagram (85) is commutative when N is a null object and this shows that $SF(N)$ is a colimit of null objects and hence a null object. □

The notion of normally injective object in a homological category is introduced in [18] 4.2.1. An object I is said to be normally injective if for any normal monomorphism $m : A \to B$, every morphism $f : A \to I$ extends to a morphism $g : B \to I$ such that $f = g \circ m$.

Lemma 8.5. *Let I be a normally injective object of $\mathcal{E}$ then, with the notations of Lemma 8.4, $SF(I)$ is a null object of $\mathcal{B}$.*

Proof. Let i be an object of the comma category $\mathrm{Sh}_{\mathrm{small}}(\mathcal{E}) \downarrow_{P'} I$ as in the upper horizontal line of the diagram

$$\begin{tikzcd}
I & A' \arrow[l, "x"'] \arrow[r, tail, "a'"] \arrow[d, "a'"] & A \arrow[r, two heads, "a''"] \arrow[d, "\mathrm{Id}"'] & A'' \arrow[d] \\
& A \arrow[ul, "\tilde{x}"] \arrow[r, tail, "\mathrm{Id}"] & A \arrow[r, two heads, "\mathrm{cok}(\mathrm{Id})"] & \mathrm{Coker}(\mathrm{Id})
\end{tikzcd} \tag{86}$$

Since I is normally injective the morphism $x : A' \to I$ extends to a morphism $\tilde{x} : A \to I$ such that the diagram (86) is commutative. By construction of the colimit it follows that i gives a null object in $SF(I)$. Indeed, the cokernel $\mathrm{Coker}(\mathrm{Id})$ is a null object and hence both $F(\mathrm{Coker}(\mathrm{Id}))$ and $\mathrm{Coker}(F(\mathrm{cok}(\mathrm{Id})))$ are null. □

8.3 Condition (a). Consider two morphisms of short doubly exact sequences in $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ of the form

$$\begin{tikzcd} C' \arrow[r, tail, "c'"] \arrow[d, "v"] & C \arrow[r, two heads, "c''"] \arrow[d, shift right, "w_1"'] \arrow[d, shift left, "w_2"] & C'' \arrow[d, shift right, "w_1''"'] \arrow[d, shift left, "w_2''"] \\ I' \arrow[r, tail, "\alpha'"] & I \arrow[r, two heads, "\alpha''"] & I'' \end{tikzcd} \tag{87}$$

so that both morphisms coincide on $v : C' \to I'$. Thus one starts with two extensions $w_j : C \to I$ of the morphism $v : C' \to I'$ using the fact that I is assumed to be injective. Given a covariant functor $F : \mathbb{B}\mathrm{mod}^{\mathfrak{s}} \longrightarrow \mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ one wishes to show the equality

$$F(w_1'') = F(w_2'') : \mathrm{Coker}(F(c'')) \to \mathrm{Coker}(F(\alpha'')). \tag{88}$$

In order to understand the inherent difficulty to obtain this condition (labelled condition (a) in [18], Theorem 4.2.2) in relation with the functoriality, we first recall the proof that such condition holds for the functor $F := \mathrm{Hom}(Q, -)$ in abelian categories. In this classical setup

$$\mathrm{Coker}(F(c'')) = \mathrm{Hom}(Q, C'')/(c'' \circ \mathrm{Hom}(Q, C)), \ \mathrm{Coker}(F(\alpha'')) = \mathrm{Hom}(Q, I'')/(\alpha'' \circ \mathrm{Hom}(Q, I)).$$

In the abelian context, one can take $w = w_1 - w_2$ and this induces $w'' = w_1'' - w_2''$ in $\mathrm{Hom}(C'', I'')$. Since $w_1 \circ c' = w_2 \circ c'$, one has $w \circ c' = 0$ and this determines (diagram chasing) a lift $\tilde{w} \in \mathrm{Hom}(C'', I)$ such that $\alpha'' \circ \tilde{w} = w''$. It follows that $F(w'') : \mathrm{Coker}(F(c'')) \to \mathrm{Coker}(F(\alpha''))$ is zero since for any $\phi \in \mathrm{Hom}(Q, C'')$ the composition $w'' \circ \phi$ lifts to $\tilde{w} \circ \phi \in \mathrm{Hom}(Q, I)$ and thus vanishes in the quotient $\mathrm{Coker}(F(\alpha'')) = \mathrm{Hom}(Q, I'')/(\alpha'' \circ \mathrm{Hom}(Q, I))$.

In $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ the cokernel of a morphism $\phi : L \to M$ is never reduced to $\{0\}$ and maps always surjectively onto the cokernel of the identity map Id_M and $\mathrm{cok}(\mathrm{Id}_M)$ is the projection $p : M \to M^\sigma$. In fact, Proposition 6.20 states a canonical isomorphism $(\mathrm{Coker}(\phi))^\sigma \simeq M^\sigma$. Thus when we consider the functor $F := \mathrm{Hom}(Q, -)$ and test the functoriality of $F(w'')$ as in (88), we can first replace the involved cokernels $\mathrm{Coker}(F(c''))$ and $\mathrm{Coker}(F(\alpha''))$ by their null subsemimodules and test the functoriality there.

Lemma 8.6. *(i) Let N, M be two objects of $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$. Then one has a canonical isomorphism*

$$r : \underline{\mathrm{Hom}}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(N, M)^\sigma \simeq Hom_{\mathbb{B}}(N^\sigma, M^\sigma).$$

(ii) Let Q be an object of $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ and $F := \underline{\mathrm{Hom}}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(Q, -)$. Let $w_j : C \to I$ be two extensions of the morphism $v : C' \to I'$ for short doubly exact sequences as in (87). Then the restrictions to the null objects $\mathrm{Coker}(F(c''))^\sigma$ of the two morphisms $F(w_1''), F(w_2'')$ are equal.

Proof. (i) For any $\phi \in \mathrm{Hom}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(N, M)$ the restriction of ϕ to N^σ gives an element $r(\phi) \in \mathrm{Hom}_{\mathbb{B}}(N^\sigma, M^\sigma)$. When $\phi \in \underline{\mathrm{Hom}}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(N, M)^\sigma$ one has $\phi = \phi \circ \sigma = \sigma \circ \phi$ and thus $\phi = r(\phi) \circ p$. Moreover for any $\psi \in \mathrm{Hom}_{\mathbb{B}}(N^\sigma, M^\sigma)$ one has $\psi \circ p \in \underline{\mathrm{Hom}}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(N, M)^\sigma$ which shows that r is an isomorphism.

(ii) Proposition 6.20 gives canonical isomorphisms

$$F(C'')^\sigma \to \mathrm{Coker}(F(c''))^\sigma, \quad F(I'')^\sigma \to \mathrm{Coker}(F(\alpha''))$$

and using (i) one obtains, by composition, canonical isomorphisms

$$\mathrm{Hom}_{\mathbb{B}}(Q^\sigma, C''^\sigma) \to \mathrm{Coker}(F(c''))^\sigma, \quad \mathrm{Hom}_{\mathbb{B}}(Q^\sigma, I''^\sigma) \to \mathrm{Coker}(F(\alpha'')).$$

Under these isomorphisms the restrictions to the null objects $\mathrm{Coker}(F(c''))^\sigma$ of the two morphisms $F(w_1''), F(w_2'')$ are given by composition on the left with the restrictions $w_j'' : C''^\sigma \to I''^\sigma$ which give maps

$$\mathrm{Hom}_{\mathbb{B}}(Q^\sigma, C''^\sigma) \ni \phi \mapsto w_j'' \circ \phi \in \mathrm{Hom}_{\mathbb{B}}(Q^\sigma, I''^\sigma).$$

Thus in order to prove (ii) it is enough to show that the restrictions $w_j'' : C''^\sigma \to I''^\sigma$ are equal. Since the sequence c is short exact, the term $C \overset{c''}{\to} C''$ is the cokernel of the first term and Proposition 6.20 applies to give a canonical isomorphism $C''^\sigma \simeq C^\sigma$. The same result applies to the short doubly exact sequence i and gives a canonical isomorphism $I''^\sigma \simeq I^\sigma$. In the two short doubly exact sequences c and i the left term is the kernel of the last one, and thus contains the null elements. This gives natural inclusions $C^\sigma \subset C'$ and $I^\sigma \subset I'$. Moreover by construction the maps w_j both induce the same map $v : C' \to I'$ and thus the same map on the subsemimodules $C^\sigma \subset C'$. This shows that the restrictions $w_j'' : C''^\sigma \to I''^\sigma$ are equal. $\square$

It follows from Lemma 8.6 that if we let $p : F(I'') \to F(I'')^\sigma$ be the projection, one has $p \circ F(w_1'') = p \circ F(w_2'')$, since for any morphism $\psi : L \to M$ in $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ the composition $p \circ \psi = \psi \circ p$ is determined by the restriction to L^σ. This equality suffices to get $F(w_1'')(u) = F(w_2'')(u)$ when the $F(w_j'')(u)$ belong to the image of $F(\alpha'')$, since by Proposition 6.19 the cokernel coincides with the projection p on that image. Thus the interesting case to consider is when for a $u \in F(C'') = \underline{\mathrm{Hom}}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(Q, C'')$ the compositions $w_j'' \circ u \in \underline{\mathrm{Hom}}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(Q, I'')$ do not lift to $\underline{\mathrm{Hom}}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(Q, I)$.

Let us take the notations of Section 7.3.2.

Theorem 8.7. *The short doubly exact sequence $s : J \rightarrowtail \mathbb{B}^3 \overset{\phi}{\twoheadrightarrow} S$ satisfies condition (a) of [18] (i.e. (88)) with respect to the functor $H := \underline{\mathrm{Hom}}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(S, -)$ and all endomorphisms of $J = Ker(\phi)$.*

Proof. By Lemma 8.6 it is enough to show that if $v \in \mathrm{End}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(J)$ admits more than one extension to $\mathbb{B}^3$ then for any extension w of v to $\mathbb{B}^3$ the action of w'' by left multiplication on $\mathrm{Coker}(H(\phi))$ is null. By (iv) of Proposition 7.21 the restriction to null elements of w'' fails to be surjective and thus the same holds for any $w'' \circ u$, $\forall u \in \mathrm{End}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(S)$ which shows that the range of left multiplication by w'' is contained in $\overline{\mathrm{Im}}(H(\phi))$ and is hence null in $\mathrm{Coker}(H(\phi))$. $\square$

The next step is to extend Theorem 8.7 to general morphisms from a short doubly exact sequence c as in (87). We first determine the freedom in extending the morphism $v : C' \to J$ to a morphism $w : C \to \mathbb{B}^3$.

Lemma 8.8. *Let $C' \overset{c'}{\rightarrowtail} C \overset{c''}{\twoheadrightarrow} C''$ be a short doubly exact sequence and $v : C' \to J$ be a morphism. The extensions of v to a morphism $w : C \to \mathbb{B}^3$ are uniquely determined by the element $L_w \in C^*$ defined by*

$$L_w = \epsilon_a \circ w, \qquad \epsilon_a(z) = 0 \iff z \le b + c.$$

Proof. The three elements $\epsilon_a, \epsilon_b, \epsilon_c$ generate the dual of $\mathbb{B}^3$ thus w is uniquely determined by the composition of these elements with w. But $\epsilon_c = \epsilon_a \circ \sigma$ so that $\epsilon_a \circ w$ determines $\epsilon_c \circ w = L_w \circ \sigma$. Let us show that $\epsilon_b \circ w$ is uniquely determined by v. One has $\epsilon_b \circ \sigma = \epsilon_b$ and thus $\epsilon_b \circ w = \epsilon_b \circ w \circ p$ where $p(x) = x + \sigma(x)$ is the projection on C^σ. One has $C^\sigma = C'^\sigma$ and the restriction of w to C^σ is uniquely determined by v. $\square$

Lemma 8.8 provides a second interpretation of the proof Proposition 7.21 (iv). Indeed, one considers the short doubly exact sequence $C' \overset{c'}{\rightarrowtail} C \overset{c''}{\twoheadrightarrow} C''$ identical to s. The issue of extending elements of $J^* = \mathrm{Hom}_{\mathbb{B}}(J,\mathbb{B})$ to elements of $(\mathbb{B}^3)^*$ is related to the map $\phi : \mathbb{B}^3 \to S$, using the natural isomorphism $J^* \simeq S$ visible in Figures 8 and 9. It follows that the only element of $J^* = \mathrm{Hom}_{\mathbb{B}}(J,\mathbb{B})$ which does not extend uniquely to $\mathbb{B}^3$ is the maximal element τ which takes the value $1 \in \mathbb{B}$ on any non-zero element of J. Thus from Lemma 8.8 one derives that the only $v \in \mathrm{End}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(J)$ which admit more than one extension to $\mathbb{B}^3$ fulfills $\epsilon_a \circ v = \tau$. This entails that the restriction of v to J^σ cannot be surjective.

8.4 Reduction of condition (a) to endomorphisms. We consider a short doubly exact sequence $\iota : I' \overset{i'}{\rightarrowtail} I \overset{i''}{\twoheadrightarrow} I''$ where we now make the following "Frobenius[3] hypothesis" that the middle term I is an object of $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ which is *injective and projective*. We consider the functor $F := \underline{\mathrm{Hom}}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(I'', -)$ and we use the fact that the object I is projective as follows

Lemma 8.9. *With the above notations, let* $C' \overset{c'}{\rightarrowtail} C \overset{c''}{\twoheadrightarrow} C''$ *be a short doubly exact sequence and let* $x \in \mathrm{Coker}(F(c''))$. *Then there exists a morphism of short doubly exact sequences* $\xi : \iota \to c$ *such that*

$$F(\xi'')(\mathrm{cok}(F(i'')(Id_{I''})) = x, \quad F(\xi'') : \mathrm{Coker}(F(i'')) \to \mathrm{Coker}(F(c'')). \tag{89}$$

Proof. Since $\mathrm{cok}(F(c''))$ is surjective let $z \in F(C'')$ be such that $\mathrm{cok}(F(c''))(z) = x$. One has $z \in F(C'') = \underline{\mathrm{Hom}}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(I'', C'')$. Since I is projective and the morphism c'' is surjective one can lift the morphism $z \circ i'' : I \to C''$ to a morphism $\xi : I \to C$ such that $c'' \circ \xi = z \circ i''$. For $u \in \mathrm{Ker}(i'')$ one has that $c'' \circ \xi(u)$ is null and thus $\xi(u) \in \mathrm{Ker}(c'')$ so that ξ restricts to a morphism $\xi' : I' \to C'$ and with $\xi'' := z$ one obtains a morphism of short doubly exact sequences in $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ of the form

$$\begin{tikzcd} I' \arrow[r, tail, "i'"] \arrow[d, "\xi'"] & I \arrow[r, two heads, "i''"] \arrow[d, "\xi"] & I'' \arrow[d, "\xi''=z"] \\ C' \arrow[r, tail, "c'"] & C \arrow[r, two heads, "c''"] & C'' \end{tikzcd} \tag{90}$$

This gives a commutative square using the functor $F = \underline{\mathrm{Hom}}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(I'', -)$

$$\begin{tikzcd} F(I) \arrow[r, "F(i'')"] \arrow[d, "F(\xi)"'] & \underline{\mathrm{Hom}}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(I'', I'') \arrow[r, "\mathrm{cok}(F(i''))"] \arrow[d, "\xi''\circ -"] & \mathrm{Coker}(F(i'')) \arrow[d, dashed] \\ F(C) \arrow[r, "F(c'')"] & \underline{\mathrm{Hom}}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(I'', C') \arrow[r, "\mathrm{cok}(F(c''))"] & \mathrm{Coker}(F(c'')) \end{tikzcd} \tag{91}$$

The commutativity of the diagram given by the left square in (91) insures that the dotted vertical arrow $F(\xi'') : \mathrm{Coker}(F(i'')) \to \mathrm{Coker}(F(c''))$ exists and that the diagram given by the right square is commutative. Moreover the image of $\mathrm{cok}(F(i'')(\mathrm{Id}_{I''})$ by $F(\xi'')$ is then the same as the image by $\mathrm{cok}(F(c''))$ of $\xi'' \circ \mathrm{Id}_{I''} = z$ and is thus equal to x which gives (89). $\square$

Theorem 8.10. *Let* $\iota : I' \overset{i'}{\rightarrowtail} I \overset{i''}{\twoheadrightarrow} I''$ *be a short doubly exact sequence such that the middle term* I *is an object of* $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ *which is* injective and projective. *Then the functor* $F := \underline{\mathrm{Hom}}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(I'', -)$

[3] By reference to the notion of Frobenius algebra, *i.e.* of algebras for which projective modules are the same as injective modules. Note that in $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ this property holds for finite objects.

satisfies condition (a) *with respect to an arbitrary morphism of short doubly exact sequences* $c \to \iota$, *provided this holds for all endomorphisms of* I'.

Proof. Let $c : C' \overset{c'}{\rightarrowtail} C \overset{c''}{\twoheadrightarrow} C''$ be a short doubly exact sequence, $v : C' \to I'$ be a morphism, and w_j be extensions of v to morphisms of short doubly exact sequences from c to ι. We need to show the equality

$$F(w_1'') = F(w_2'') : \mathrm{Coker}(F(c'')) \to \mathrm{Coker}(F(i'')). \tag{92}$$

Let $x \in \mathrm{Coker}(F(c''))$. Then by Lemma 8.9, there exists a morphism of short doubly exact sequences $\xi : \iota \to c$ such that (89) holds, *i.e.* $F(\xi'')(\mathrm{cok}(F(i'')(\mathrm{Id}_{I''})) = x$. We now consider the composition $w_j \circ \xi$ of the morphism $\xi : \iota \to c$ with either of the extensions w_j of v to morphisms of short doubly exact sequences from c to ι. These provide two extensions of the endomorphism $v \circ \xi' : I' \to I'$.

$$\begin{tikzcd} I' \arrow[r, tail, "i'"] \arrow[d, "\xi'"] & I \arrow[r, two heads, "i''"] \arrow[d, "\xi"] & I'' \arrow[d, "\xi''=z"] \\ C' \arrow[r, tail, "c'"] \arrow[d, "v"] & C \arrow[r, two heads, "c''"] \arrow[d, shift right, "w_1"'] \arrow[d, shift left, "w_2"] & C'' \arrow[d, shift right, "w_1''"'] \arrow[d, shift left, "w_2''"] \\ I' \arrow[r, tail, "i'"] & I \arrow[r, two heads, "i''"] & I'' \end{tikzcd} \tag{93}$$

Since by hypothesis the short doubly exact sequence ι satisfies condition (a) with respect to all endomorphisms of I', we get the equality $F(w_1'' \circ \xi'') = F(w_2'' \circ \xi'') : \mathrm{Coker}(F(i'')) \to \mathrm{Coker}(F(i''))$. We apply this equality to the element $\mathrm{cok}(F(i'')(\mathrm{Id}_{I''}) \in \mathrm{Coker}(F(i''))$. By (89) we derive $F(\xi'')(\mathrm{cok}(F(i'')(\mathrm{Id}_{I''})) = x$ and hence we get $F(w_1'')(x) = F(w_2'')(x)$ so that (92) holds. □

Theorem 8.11. *Let S be as in Theorem 8.7. The satellite functor SH of $H := \underline{\mathrm{Hom}}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(S, -)$ is non-null and $SH(J)$ is the cokernel of the morphism $H(\phi) : H(\mathbb{B}^3) \to H(S)$ of Proposition 7.20, (iv).*

Proof. Let $\mathrm{Sh}_{\mathrm{small}}$ be any small subcategory of the category $\mathrm{Sh}(\mathbb{B}\mathrm{mod}^{\mathfrak{s}})$ of short doubly exact sequences in the homological category $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$, with $\mathrm{Sh}_{\mathrm{small}}$ large enough to contain all short doubly exact sequences of finite objects. By construction $H : \mathbb{B}\mathrm{mod}^{\mathfrak{s}} \longrightarrow \mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ is a covariant functor and $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ is a cocomplete homological category. By (82) the following colimit makes sense for any object X of $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$, and defines the covariant functor SH

$$SH : \mathbb{B}\mathrm{mod}^{\mathfrak{s}} \longrightarrow \mathbb{B}\mathrm{mod}^{\mathfrak{s}}, \quad SH(X) := \varinjlim_{I} \mathrm{Coker}(H(a'')), \quad I = \mathrm{Sh}_{\mathrm{small}}(\mathbb{B}\mathrm{mod}^{\mathfrak{s}}) \downarrow_{P'} X \tag{94}$$

We proceed as in the proof of Theorem 4.2.2 of [18] and show that the object

$$j = \left(J \overset{\mathrm{Id}}{\leftarrow} J \overset{\subset}{\rightarrowtail} \mathbb{B}^3 \overset{\phi}{\twoheadrightarrow} S \right)$$

is a final object for the functor H in the comma category $I = \mathrm{Sh}_{\mathrm{small}}(\mathbb{B}\mathrm{mod}^{\mathfrak{s}}) \downarrow_{P'} J$. Indeed, since $\mathbb{B}^3$ is injective, for any object $J \overset{v}{\leftarrow} C' \overset{c'}{\rightarrowtail} C \overset{c''}{\twoheadrightarrow} C''$ of I one gets a morphism in I

$$\begin{tikzcd} & C' \arrow[r, tail, "c'"] \arrow[dl, "v"'] \arrow[d, "v"] & C \arrow[r, two heads, "c''"] \arrow[d, "w"] & C'' \arrow[d, "w''"] \\ J & J \arrow[l, "\mathrm{Id}"'] \arrow[r, tail, "\subset"] & \mathbb{B}^3 \arrow[r, two heads, "\phi"] & S \end{tikzcd} \tag{95}$$

where the choice of w is not unique in general but where, by Theorem 8.10 combined with Theorem 8.7, the induced action $H(w''): \mathrm{Coker}(H(c'')) \to \mathrm{Coker}(H(\phi))$ is unique. It follows that the colimit (94) is simply the evaluation on the final object, *i.e.* is given by $\mathrm{Coker}(H(\phi))$ and Proposition 7.20, (iv) shows that it is non-null. $\square$

9. The cokernel of the diagonal

In Proposition 5.12 (ii), we have shown that the functor F associated to the cokernel of a morphism of $\mathbb{B}$-semimodules $f: L \to M$ becomes representable *i.e.* of the form $\underline{\mathrm{Hom}}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(Q, -)$ in the category $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$. In this section we prove that the satellite functor Ext of the functor F is non null for the simplest natural example where the morphism $f: L \to M$ is the diagonal $\Delta: \mathbb{B} \to \mathbb{B} \times \mathbb{B}$.

The cokernel of the diagonal enters in the construction of the Čech version of sheaf cohomology, as the dual of the differential of the Koszul resolution $d: \wedge^2 \to \wedge$. In the classical case of sheaves of abelian groups on a topological space X, one conceptual understanding of the Čech complex associated to a covering $\mathcal{U}$ is as $\mathrm{Hom}(K(\mathcal{U})_*, \bullet)$ where $K(\mathcal{U})_*$ is the Koszul chain complex canonically associated to the covering. It involves the antisymmetric powers of $K(\mathcal{U})_0$ and the differential d. In the simplest case and at a point belonging to two open sets U, V of the covering $\mathcal{U}$, $K(\mathcal{U})_0$ is simply $\mathbb{Z} \oplus \mathbb{Z}$ and the dual d^* of the Koszul chain complex reduces to the following short exact sequence

$$0 \to \mathbb{Z} \stackrel{\Delta}{\to} \mathbb{Z} \oplus \mathbb{Z} \stackrel{d^*}{\to} \wedge^2(\mathbb{Z} \oplus \mathbb{Z}) \to 0$$

where Δ is the diagonal and d is the Koszul differential. Thus the dual d^* of the Koszul differential appears naturally as the cokernel of the diagonal. In this section we study this cokernel of the diagonal in our framework in which the category of abelian groups is replaced by the category $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$.

The steps which allow us to compute the satellite functor SF of the cokernel of the diagonal are the following:

1. In §9.1 we determine the object $Q = \mathrm{Coker}(\mathfrak{s}\Delta)$ of $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ representing the cokernel of the diagonal $\Delta: \mathbb{B} \to \mathbb{B} \times \mathbb{B}$.
2. In Lemma 9.3 we show that, with $F = \underline{\mathrm{Hom}}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(Q, -)$, the cokernel $\mathrm{Coker}(F(\mathrm{cok}(\mathfrak{s}\Delta)))$ is non-null.
3. In §9.2 we determine $\mathrm{Coker}(F(\mathrm{cok}(\mathfrak{s}\Delta)))$ using duality.
4. In §9.3 we analyze the elements of the comma category $\mathcal{I} = \mathrm{Sh}(\mathbb{B}\mathrm{mod}^{\mathfrak{s}}) \downarrow_{P'} K$ associated to endomorphisms of the kernel of $\mathrm{cok}(\mathfrak{s}\Delta)$.
5. The correspondence underlying the multiple extensions of endomorphisms of the kernel is discussed in §9.4 as a preparation for the proof of property (a) for endomorphisms.

The main result which computes the satellite functor SF of the functor $F := \underline{\mathrm{Hom}}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(Q, -)$ as the cokernel $\mathrm{Coker}(F(\alpha''))$ is obtained in §9.5, Theorem 9.15.

9.1 The cokernel pair of $\Delta: \mathbb{B} \to \mathbb{B} \times \mathbb{B}$ as a representable functor. The cokernel pair of the diagonal $\Delta: \mathbb{B} \to \mathbb{B} \times \mathbb{B}$ is first comprehended as the covariant functor (43) of Proposition 4.16

$$F(X) = \mathbb{B}^2/\mathbb{B}(X) := \{(f, g) \in \mathrm{Hom}_{\mathbb{B}\mathrm{mod}^2}(\mathbb{B}^2, X) \mid f(x) = g(x), \ \forall x \in \Delta\}.$$

The pair $(f,g) \in \mathrm{Hom}_{\mathbb{B}\mathrm{mod}^2}(\mathbb{B}^2, X)$ is characterized by the 4 elements of X

$$a = f((1,0)),\ b = f((0,1)),\ c = g((1,0)),\ d = g((0,1))$$

and the only constraint is that $a+b=c+d$. Given a morphism $(h,k) \in \mathrm{Hom}_{\mathbb{B}\mathrm{mod}^2}(X, X')$, one obtains the morphism $(f',g') \in \mathrm{Hom}_{\mathbb{B}\mathrm{mod}^2}(\mathbb{B}^2, X')$ corresponding to

$$a' = h(a)+k(c),\ b' = h(b)+k(d),\ c' = h(c)+k(a),\ d' = h(d)+k(b).$$

By Proposition 5.5 the functor F is represented in $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ and the object $\mathrm{Coker}(\mathfrak{s}\Delta)$ representing F is the quotient Q of $\mathbb{B}^2 \times \mathbb{B}^2$ by the equivalence relation (44). We view the elements of $\mathbb{B}^2 \times \mathbb{B}^2$ as sums over the subsets of the set $\{\alpha, \beta, \gamma, \delta\}$. One finds that the first ten elements of the list

$$0, \alpha, \beta, \gamma, \delta, \alpha+\beta, \alpha+\gamma, \alpha+\delta, \beta+\gamma, \beta+\delta,$$

$$\gamma+\delta, \alpha+\beta+\gamma, \alpha+\beta+\delta, \alpha+\gamma+\delta, \beta+\gamma+\delta, \alpha+\beta+\gamma+\delta$$

represent all the elements in the quotient, while all the others project on $\alpha+\beta \sim \gamma+\delta$.

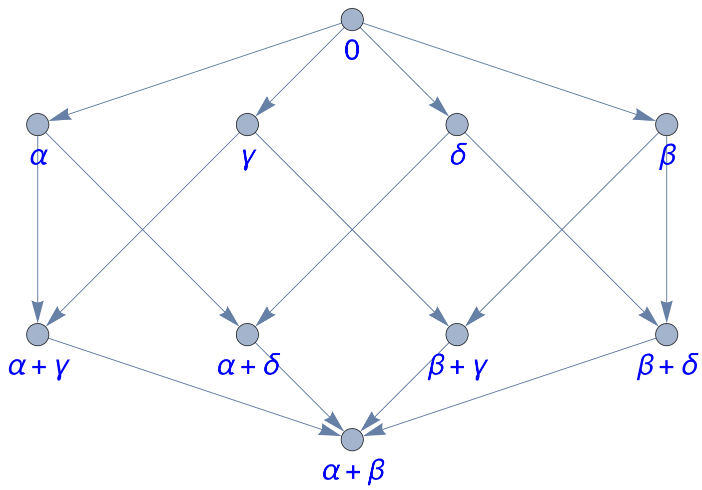

Figure 10: The structure of the $\mathbb{B}$-semimodule Q

The involution σ is given by $\sigma(\alpha) = \gamma$, $\sigma(\beta) = \delta$. Its fixed points in Q are $0, \alpha+\beta, \alpha+\gamma, \beta+\delta$ and the preimages of the fixed points are the following 10 elements of $\mathbb{B}^2 \times \mathbb{B}^2$

$$0, \alpha+\beta, \alpha+\gamma, \beta+\delta, \gamma+\delta, \alpha+\beta+\gamma, \alpha+\beta+\delta, \alpha+\gamma+\delta, \beta+\gamma+\delta, \alpha+\beta+\gamma+\delta$$

which form the kernel $K := \mathrm{Ker}(\mathrm{cok}(\mathfrak{s}\Delta))$. The intersection of this kernel with the sums not invoking γ or δ is as expected the diagonal $\{0, \alpha+\beta\} \subset \mathbb{B}^2$. By Proposition 5.12 (ii), one derives the following short doubly exact sequence

$$K \overset{\ker(\mathrm{cok}(\mathfrak{s}\Delta))}{\rightarrowtail} \mathbb{B}^2 \times \mathbb{B}^2 \overset{\mathrm{cok}(\mathfrak{s}\Delta)}{\twoheadrightarrow} Q. \tag{96}$$

We want to detect the obstruction to lift back from Q to $\mathbb{B}^2 \times \mathbb{B}^2$. Thus we consider the represented functor $F := \underline{\mathrm{Hom}}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(Q, -)$ which we view as a covariant endofunctor $\mathbb{B}\mathrm{mod}^{\mathfrak{s}} \longrightarrow \mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ (using the natural internal Hom). We need to understand

$$F(\mathbb{B}^2 \times \mathbb{B}^2) = \underline{\mathrm{Hom}}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(Q, \mathbb{B}^2 \times \mathbb{B}^2).$$

Any given $\phi \in \underline{\mathrm{Hom}}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(Q, \mathbb{B}^2 \times \mathbb{B}^2)$ is uniquely determined by $a = \phi(\alpha)$ and $b = \phi(\beta)$ which must fulfill the condition that $a + b$ is fixed under σ. There are only 4 elements of $\mathbb{B}^2 \times \mathbb{B}^2$ which are invariant under σ and they are: $0,\ \alpha + \gamma,\ \beta + \delta,\ \alpha + \beta + \gamma + \delta$.

Lemma 9.1. *The object* $\underline{\mathrm{Hom}}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(Q, \mathbb{B}^2 \times \mathbb{B}^2)$ *of* $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ *is the square of* $Q^* = \underline{\mathrm{Hom}}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(Q, \mathfrak{s}\mathbb{B})$.

Proof. As an object of $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$, $\mathbb{B}^2 \times \mathbb{B}^2$ is the product of the two copies of $\mathfrak{s}\mathbb{B}$ given by

$$\mathfrak{s}\mathbb{B} \sim \{0, \alpha, \gamma, \alpha + \gamma\},\ \sigma(\alpha) = \gamma, \quad \mathfrak{s}\mathbb{B}' \sim \{0, \beta, \delta, \beta + \delta\},\ \sigma(\beta) = \delta.$$

□

By Lemma 5.3 one has a canonical isomorphism in $\mathbb{B}\mathrm{mod}$

$$\pi : \mathrm{Hom}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(Q, \mathfrak{s}\mathbb{B}) \to \mathrm{Hom}_{\mathbb{B}}(I(Q), \mathbb{B}). \tag{97}$$

A morphism $\phi \in \mathrm{Hom}_{\mathbb{B}}(I(Q), \mathbb{B})$ is given by 4 elements $a, b, c, d \in \mathbb{B}$ such that $a \vee b = c \vee d$. Either all are 0 or $a \vee b = c \vee d = 1$, and this condition has 9 solutions. One thus gets the ten possibilities given by the lines of the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

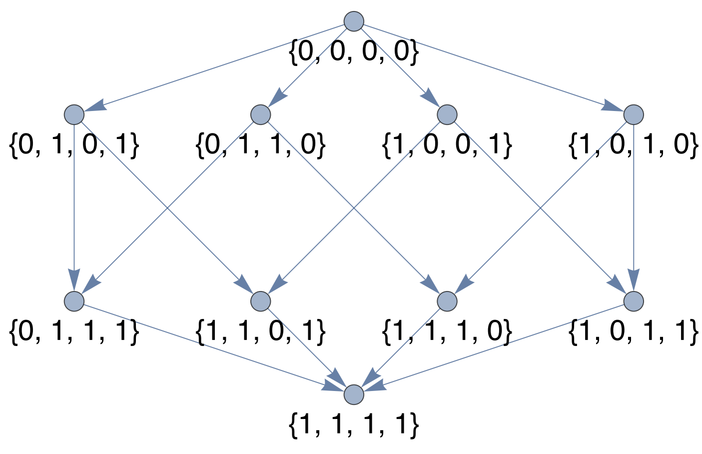

Figure 11: The structure of the $\mathbb{B}$-semimodule Q^*

For an object X of $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$, $\phi \in \mathrm{Hom}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(Q, X)$ is specified by the pair $(\phi(\alpha), \phi(\beta)) \in X \times X$, with the only constraint that $\phi(\alpha) + \phi(\beta)$ is fixed under σ. For $X = \mathfrak{s}\mathbb{B}$ we thus specify two elements $\phi(\alpha), \phi(\beta) \in \mathfrak{s}\mathbb{B}$. The map π of (97) is such that

$$\pi(\phi) = (a, b, c, d) \iff \phi(\alpha) = (a, c), \quad \phi(\beta) = (b, d).$$

The involution on $\underline{\mathrm{Hom}}_{\mathbb{B}\mathrm{mod}}(I(Q),\mathbb{B})$ coming from the internal Hom structure of $\underline{\mathrm{Hom}}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(Q,\mathfrak{s}\mathbb{B})$ by the isomorphism (97), is given by composition with σ_Q and hence by the action of the permutation $(a,b,c,d)\mapsto(c,d,a,b)$ on the labels. Thus while the $\mathbb{B}$-semimodules underlying Q and Q^* are isomorphic, the objects Q and Q^* of $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ are distinct since the involutions are not the same (the two elements $(1,0,1,0)$ and $(0,1,0,1)$ are fixed under the involution).

Next, we investigate the map $F(\mathrm{cok}(\mathfrak{s}\Delta))$ obtained by composition with $\mathrm{cok}(\mathfrak{s}\Delta)$

$$F(\mathrm{cok}(\mathfrak{s}\Delta)):\underline{\mathrm{Hom}}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(Q,\mathbb{B}^2\times\mathbb{B}^2)\to\underline{\mathrm{End}}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(Q),\quad \phi\mapsto \mathrm{cok}(\mathfrak{s}\Delta)\circ\phi=F(\mathrm{cok}(\mathfrak{s}\Delta))(\phi).$$

We know that this map cannot be surjective since the identity $\mathrm{Id}\in\underline{\mathrm{End}}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(Q)$ cannot be lifted. We view $F(\mathrm{cok}(\mathfrak{s}\Delta))$ as a morphism in $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ and want to compute its cokernel. An element $\psi\in\underline{\mathrm{End}}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(Q)$ is encoded by the values $(\psi(\alpha),\psi(\beta))\in Q\times Q$ and the only constraint on the pair is that $\psi(\alpha)+\psi(\beta)$ is σ-invariant *i.e.* belongs to $\{0,\alpha+\beta,\alpha+\gamma,\beta+\delta\}$. One obtains 70 solutions which are listed below. The list $L_{\alpha,\gamma}$ of nine solutions with $\psi(\alpha)+\psi(\beta)=\alpha+\gamma$ is

$$((0,\alpha+\gamma),(\alpha,\gamma),(\alpha,\alpha+\gamma),(\gamma,\alpha),(\gamma,\alpha+\gamma),(\alpha+\gamma,0),(\alpha+\gamma,\alpha),(\alpha+\gamma,\gamma),(\alpha+\gamma,\alpha+\gamma))$$

The list $L_{\beta,\delta}$ of nine solutions with $\psi(\alpha)+\psi(\beta)=\beta+\delta$ is

$$((0,\beta+\delta),(\beta,\delta),(\beta,\beta+\delta),(\delta,\beta),(\delta,\beta+\delta),(\beta+\delta,0),(\beta+\delta,\beta),(\beta+\delta,\delta),(\beta+\delta,\beta+\delta))$$

Lemma 9.2. *$\psi\in End_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(Q)$ can be lifted to $\phi\in Hom_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(Q,\mathbb{B}^2\times\mathbb{B}^2)$ if and only if the pair $(\psi(\alpha),\psi(\beta))\in Q\times Q$ is a sum of two elements of the lists $L_{\alpha,\gamma}\cup\{(0,0)\}$ and $L_{\beta,\delta}\cup\{(0,0)\}$.*

Proof. Notice first that each element ψ of the two lists is uniquely liftable to $\phi\in\mathrm{Hom}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(Q,\mathbb{B}^2\times\mathbb{B}^2)$, where $\phi(\alpha)=\psi(\alpha)$ and $\phi(\beta)=\psi(\beta)$. This follows from the σ-invariance in $\mathbb{B}^2\times\mathbb{B}^2$ of the elements $\alpha+\gamma$ and $\beta+\delta$. Thus any element of $\mathrm{End}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(Q)$ which is a sum of elements of the lists $L_{\alpha,\gamma}\cup\{(0,0)\}$ and $L_{\beta,\delta}\cup\{(0,0)\}$ can be lifted. Conversely, Lemma 9.1 shows that any element of $\mathrm{Hom}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(Q,\mathbb{B}^2\times\mathbb{B}^2)$ belongs to the sums of the two lists and thus only these sums in $\mathrm{End}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(Q)$ are liftable. □

The elements of $\mathrm{End}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(Q)$ which do not belong to $L_{\alpha,\gamma}\cup L_{\beta,\delta}\cup\{(0,0)\}$ are those coming from the solutions in $Q\times Q$ of the equation $\psi(\alpha)+\psi(\beta)=\alpha+\beta$. There are 51 of them

$$\begin{gathered}(0,\alpha+\beta),\ (\alpha,\beta),\ (\alpha,\alpha+\beta),\ (\alpha,\beta+\gamma),\ (\alpha,\beta+\delta),\ (\beta,\alpha),\ (\beta,\alpha+\beta),\\ (\beta,\alpha+\gamma),\ (\beta,\alpha+\delta),\ (\gamma,\delta),\ (\gamma,\alpha+\beta),\ (\gamma,\alpha+\delta),\ (\gamma,\beta+\delta),\ (\delta,\gamma),\\ (\delta,\alpha+\beta),\ (\delta,\alpha+\gamma),\ (\delta,\beta+\gamma),\ (\alpha+\beta,0),\ (\alpha+\beta,\alpha),\ (\alpha+\beta,\beta),\ (\alpha+\beta,\gamma),\\ (\alpha+\beta,\delta),\ (\alpha+\beta,\alpha+\beta),\ (\alpha+\beta,\alpha+\gamma),\ (\alpha+\beta,\alpha+\delta),\ (\alpha+\beta,\beta+\gamma),\ (\alpha+\beta,\beta+\delta),\\ (\alpha+\gamma,\beta),\ (\alpha+\gamma,\delta),\ (\alpha+\gamma,\alpha+\beta),\ (\alpha+\gamma,\alpha+\delta),\ (\alpha+\gamma,\beta+\gamma),\ (\alpha+\gamma,\beta+\delta),\\ (\alpha+\delta,\beta),\ (\alpha+\delta,\gamma),\ (\alpha+\delta,\alpha+\beta),\ (\alpha+\delta,\alpha+\gamma),\ (\alpha+\delta,\beta+\gamma),\ (\alpha+\delta,\beta+\delta),\\ (\beta+\gamma,\alpha),\ (\beta+\gamma,\delta),\ (\beta+\gamma,\alpha+\beta),\ (\beta+\gamma,\alpha+\gamma),\ (\beta+\gamma,\alpha+\delta),\ (\beta+\gamma,\beta+\delta),\\ (\beta+\delta,\alpha),\ (\beta+\delta,\gamma),\ (\beta+\delta,\alpha+\beta),\ (\beta+\delta,\alpha+\gamma),\ (\beta+\delta,\alpha+\delta),\ (\beta+\delta,\beta+\gamma).\end{gathered}$$

Among them only the following 23 are liftable as one sees using Lemma 9.2

$$(0,\alpha+\beta),(\alpha,\alpha+\beta),(\beta,\alpha+\beta),(\gamma,\alpha+\beta),(\delta,\alpha+\beta),(\alpha+\beta,0),(\alpha+\beta,\alpha),$$

$$(\alpha+\beta,\beta),(\alpha+\beta,\gamma),(\alpha+\beta,\delta),(\alpha+\beta,\alpha+\beta),(\alpha+\beta,\alpha+\gamma),(\alpha+\beta,\alpha+\delta),$$
$$(\alpha+\beta,\beta+\gamma),(\alpha+\beta,\beta+\delta),(\alpha+\gamma,\alpha+\beta),(\alpha+\gamma,\beta+\delta),(\alpha+\delta,\alpha+\beta),$$
$$(\alpha+\delta,\beta+\gamma),(\beta+\gamma,\alpha+\beta),(\beta+\gamma,\alpha+\delta),(\beta+\delta,\alpha+\beta),(\beta+\delta,\alpha+\gamma).$$

Thus the range of $F(\mathrm{cok}(\mathfrak{s}\Delta)) : \underline{\mathrm{Hom}}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(Q, \mathbb{B}^2\times\mathbb{B}^2) \to \underline{\mathrm{End}}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(Q)$ consists of $1+9+9+23$ elements and for instance fails to contain the identity map.

Next, we investigate $\mathrm{Coker}(F(\mathrm{cok}(\mathfrak{s}\Delta)))$. We apply Proposition 6.17 to show that it is not a null object of $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$.

Lemma 9.3. *The class of $Id \in \underline{\mathrm{End}}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(Q)$ is non-null in the cokernel* $\mathrm{Coker}(F(\mathrm{cok}(\mathfrak{s}\Delta)))$.

Proof. We apply Proposition 6.17 to the element $\xi = \mathrm{Id}_Q \in \mathrm{End}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(Q)$. It is given by the pair (α,β) and does not lift to an element of $\mathrm{Hom}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(Q,\mathbb{B}^2\times\mathbb{B}^2)$ because $\alpha+\beta$ is not σ-invariant in $\mathbb{B}^2\times\mathbb{B}^2$. Thus $\xi \notin F(\mathrm{cok}(\mathfrak{s}\Delta))(\mathrm{Hom}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(Q,\mathbb{B}^2\times\mathbb{B}^2))$. The element $\sigma(\xi)$ corresponds to the pair (γ,δ) and in fact it represents the endomorphism $\sigma \in \mathrm{End}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(Q)$. It remains to show that ξ is indecomposable. The only non-trivial way to write the pair (α,β) as a sum in $Q\times Q$ involves $(\alpha,0)$ or $(0,\beta)$ but none of the two terms comes from an element of $\mathrm{End}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(Q)$ since it does not fulfill the condition of σ-invariance. $\square$

In fact, by applying Proposition 6.16 one determines all elements of $\underline{\mathrm{End}}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(Q)$ whose class is non-null (*i.e.* not σ-invariant) in the cokernel $\mathrm{Coker}(F(\mathrm{cok}(\mathfrak{s}\Delta)))$. They form the list

$$\begin{aligned} C = \{&(\alpha,\beta),(\alpha,\beta+\delta),(\beta,\alpha),(\beta,\alpha+\gamma),(\gamma,\delta),(\gamma,\beta+\delta),\\ &(\delta,\gamma),(\delta,\alpha+\gamma),(\alpha+\gamma,\beta),(\alpha+\gamma,\delta),(\beta+\delta,\alpha),(\beta+\delta,\gamma)\}. \end{aligned} \tag{98}$$

The complement of this list is the subsemimodule $\overline{\mathrm{Im}}(F(\mathrm{cok}(\mathfrak{s}\Delta))) \subset \underline{\mathrm{End}}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(Q)$.

Lemma 9.3 shows that the short doubly exact sequence (96), *i.e.* $K \overset{\subset}{\rightarrowtail} \mathbb{B}^2\times\mathbb{B}^2 \overset{\mathrm{cok}(\mathfrak{s}\Delta)}{\twoheadrightarrow} Q$, gives rise after applying the functor $F = \underline{\mathrm{Hom}}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(Q,-)$ to a non-null cokernel for the map $F(a'')$ with $a'' = \mathrm{cok}(\mathfrak{s}\Delta)$. This yields an element of the satellite functor $SF(K)$. Since F is a Hom-functor the first satellite functor here is the analogue of $\mathrm{Ext}^1(Q,K)$. It remains to show that this element remains non-null in the colimit (83), with respect to a suitable small class of exact sequences. Here for instance one can take the class $\mathrm{Sh}_{\mathrm{small}}$ of exact sequences only involving finite objects of $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$. This ensures the smallness of the involved categories.

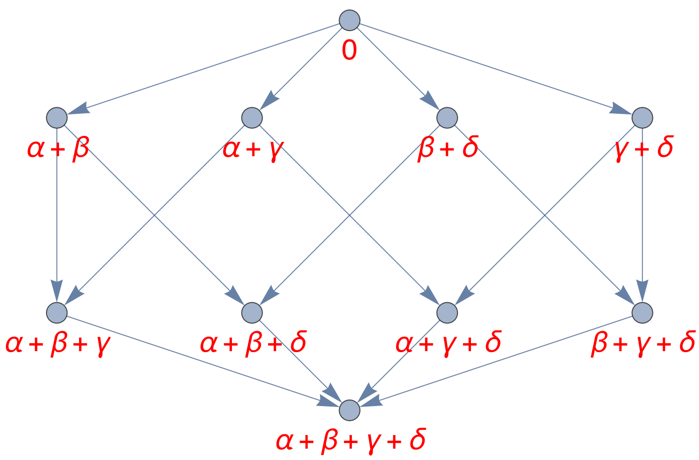

Figure 12: The structure of $K = \mathrm{Ker}(\mathrm{cok}(\mathfrak{s}\Delta))$

The structure of $K = \mathrm{Ker}(\mathrm{cok}(\mathfrak{s}\Delta))$ is displayed in Figure 12. In particular one sees that it is isomorphic to the dual Q^* of Q (see Figure 11) since the involution fixes $\alpha+\gamma$ and $\beta+\delta$ and interchanges $\alpha+\beta$ with $\gamma+\delta$. We are thus dealing with a specific element of $\mathrm{Ext}^1(Q, Q^*)$. Inside $K = \mathrm{Ker}(\mathrm{cok}(\mathfrak{s}\Delta))$ one has the range of the diagonal map $\Delta : \mathbb{B} \to \mathbb{B} \times \mathbb{B}$ once lifted to the square. Here it gives the subsemimodule generated by $\alpha+\beta$ and $\gamma+\delta$. It follows from Proposition 6.17 that $K = \mathrm{Range}(\Delta) + K^\sigma$.

Lemma 9.4. *(i) The projection $p : K \to K^\sigma$, $p(x) := x + \sigma(x)$ admits only one non-trivial fiber which is $p^{-1}(\{\tau\}) = \{\tau\} \cup (K^\sigma)^c$, where $\tau = \alpha+\beta+\gamma+\delta$.*

(ii) The fiber of the projection $p : End_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(K) \to End_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(K)^\sigma$, $p(\phi) := p \circ \phi = \phi \circ p$ over $\psi \in End_{\mathbb{B}\mathrm{mod}}(K^\sigma)$ is reduced to $\psi \circ p$ if $\psi(\tau) \neq \tau$.

(iii) Let $\psi \in End_{\mathbb{B}\mathrm{mod}}(K^\sigma)$ with $\psi(\tau) = \tau$, then the following map is bijective

$$\{\phi \in End_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(K) \mid p(\phi) = \psi\} \to p^{-1}(\{\tau\}), \quad \phi \mapsto \phi(\alpha+\beta).$$

Proof. (i) Using Figure 12 one sees that there is only one non-trivial fiber and it has seven elements.

(ii) $\alpha+\beta$, $\sigma(\alpha+\beta)$ generate K when taken together with K^σ. Thus $\phi \in \mathrm{End}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(K)$ is uniquely determined by $\psi = p \circ \phi$ and $\phi(\alpha+\beta) \in p^{-1}(\{\psi(\tau)\})$. By (i) this shows that ψ uniquely determines ϕ when $\psi(\tau) \neq \tau$.

(iii) By the proof of (ii) it follows that ϕ is uniquely determined by ψ and by $\phi(\alpha+\beta) \in p^{-1}(\{\tau\})$ using $\psi(\tau) = \tau$. Conversely, any value $\phi(\alpha+\beta) = \xi \in p^{-1}(\{\tau\})$ defines an extension of ψ as an endomorphism of K such that $\phi(\gamma+\delta) = \sigma(\xi)$. $\square$

It follows from Lemma 9.4 that $\mathrm{End}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(K)$ has 70 elements among which 7 correspond to the seven elements of $\mathrm{End}_{\mathbb{B}\mathrm{mod}}(K^\sigma)$ such that $\psi(\tau) \neq \tau$ and $63 = 9 \times 7$ correspond to the nine elements of $\mathrm{End}_{\mathbb{B}\mathrm{mod}}(K^\sigma)$ such that $\psi(\tau) = \tau$.

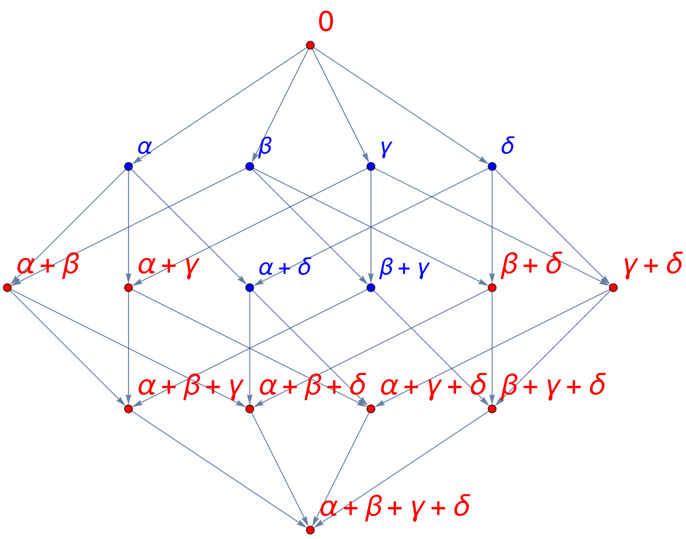

Figure 13: The inclusion $K = \mathrm{Ker}(\mathrm{cok}(\mathfrak{s}\Delta)) \subset \mathbb{B}^2 \times \mathbb{B}^2$

9.2 Duality and the cokernel $\mathrm{Coker}(F(\mathrm{cok}(\mathfrak{s}\Delta)))$**.** In order to show that the satellite functor $SF(K)$ is non-trivial we use a dual theory. We develop it in the explicit example of $\mathrm{Coker}(F(\mathrm{cok}(\mathfrak{s}\Delta)))$.

We have seen that there are twelve elements of $\underline{\mathrm{End}}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(Q)$ whose image in $\mathrm{Coker}(F(\mathrm{cok}(\mathfrak{s}\Delta)))$ are non-null, but this does not imply that they are all distinct in the quotient. To determine $\mathrm{Coker}(F(\mathrm{cok}(\mathfrak{s}\Delta)))$ one has to find all linear forms $\phi \in \mathrm{Hom}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(\underline{\mathrm{End}}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(Q), \mathfrak{s}\mathbb{B})$ whose restriction to the subsemimodule $\overline{\mathrm{Im}}(F(\mathrm{cok}(\mathfrak{s}\Delta))) \subset \underline{\mathrm{End}}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(Q)$ is null. One notices that this restriction is entirely specified by a linear form $\ell \in \mathrm{Hom}_{\mathbb{B}}(E, \mathbb{B})$, where $E = (\underline{\mathrm{End}}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(Q))^{\sigma} = \mathrm{End}_{\mathbb{B}\mathrm{mod}}(Q^{\sigma})$. Any linear form ϕ is uniquely determined by $p_1 \circ \phi$, where $p_1 : \mathfrak{s}\mathbb{B} \to \mathbb{B}$ is the first projection. Moreover $p_1 \circ \phi$ is determined by ℓ and the 4 values

$$p_1 \circ \phi((\alpha, \beta)) \in \mathbb{B},\ p_1 \circ \phi((\beta, \alpha)) \in \mathbb{B},\ p_1 \circ \phi((\gamma, \delta)) \in \mathbb{B},\ p_1 \circ \phi((\delta, \gamma)) \in \mathbb{B}$$

since the values of $p_1 \circ \phi$ on the remaining 8 elements of the complement of $\overline{\mathrm{Im}}(F(\mathrm{cok}(\mathfrak{s}\Delta))) \subset \underline{\mathrm{End}}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(Q)$ are then uniquely determined using the relations

$$\left(\begin{array}{ccc}
\begin{pmatrix} \alpha & \beta \\ 0 & 0 \\ \alpha & \beta \end{pmatrix} & \begin{pmatrix} \alpha & \beta \\ 0 & \beta+\delta \\ \alpha & \beta+\delta \end{pmatrix} & \begin{pmatrix} \alpha & \beta \\ \alpha+\gamma & 0 \\ \alpha+\gamma & \beta \end{pmatrix} \\
\begin{pmatrix} \beta & \alpha \\ 0 & 0 \\ \beta & \alpha \end{pmatrix} & \begin{pmatrix} \beta & \alpha \\ 0 & \alpha+\gamma \\ \beta & \alpha+\gamma \end{pmatrix} & \begin{pmatrix} \beta & \alpha \\ \beta+\delta & 0 \\ \beta+\delta & \alpha \end{pmatrix} \\
\begin{pmatrix} \gamma & \delta \\ 0 & 0 \\ \gamma & \delta \end{pmatrix} & \begin{pmatrix} \gamma & \delta \\ 0 & \beta+\delta \\ \gamma & \beta+\delta \end{pmatrix} & \begin{pmatrix} \gamma & \delta \\ \alpha+\gamma & 0 \\ \alpha+\gamma & \delta \end{pmatrix} \\
\begin{pmatrix} \delta & \gamma \\ 0 & 0 \\ \delta & \gamma \end{pmatrix} & \begin{pmatrix} \delta & \gamma \\ 0 & \alpha+\gamma \\ \delta & \alpha+\gamma \end{pmatrix} & \begin{pmatrix} \delta & \gamma \\ \beta+\delta & 0 \\ \beta+\delta & \gamma \end{pmatrix}
\end{array}\right)$$

One can then determine which among the 16×16 possible choices give rise to a linear form and one finds 28 solutions whose values on the $16 + 12$ elements of $E \cup C$ are the columns of the large matrix displayed below.

Lemma 9.5. *The cokernel of* $F(\mathrm{cok}(\mathfrak{s}\Delta))$ *is the map* $\tilde{p} : \underline{\mathrm{End}}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(Q) \to E \cup C$ *which is the identity on* C *and the projection* $p(\xi) := \xi + \sigma(\xi)$ *on the complement of* $C \subset \underline{\mathrm{End}}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(Q)$.

Proof. The map $\tilde{p}$ is well defined and surjective by construction, and what is non-trivial is the statement that

$$(\xi, \eta) \mapsto \{\tilde{p}(u + v) \mid \tilde{p}(u) = \xi,\ \tilde{p}(v) = \eta\} \tag{99}$$

is single valued. This is clear when $\xi, \eta \in E$ by linearity of p and also for $\xi, \eta \in C$. One can thus assume that $\xi \in C$ and $\eta \in E$. One then has to show that

$$\#\{\tilde{p}(u + \xi) \mid u \notin C,\ p(u) = \eta\} = 1. \tag{100}$$

If $u+\xi \notin C$ then $\tilde{p}(u+\xi) = p(u+\xi) = p(u)+p(\xi) = \eta+p(\xi)$. Thus the case one needs to consider with care is when $u + \xi \in C$. But in our case, the only values of u such that $C \cap (C + u) \neq \emptyset$ are $(\alpha + \gamma, 0)$, $(0, \alpha + \gamma)$, $(\beta + \delta, 0)$, $(0, \beta + \delta)$ and all these elements belong to E and the fiber $p^{-1}(\{u\})$ in the complement of C is reduced to the single element $u \in E$. This proves (100) and shows that the operation (99) is well defined. It is automatically associative and commutative and $\tilde{p}$ is a morphism by construction. $\square$

The huge matrix reported in the following page describes the 28 linear forms $\phi \in \mathrm{Hom}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(\underline{\mathrm{End}}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(Q), \mathfrak{s}\mathbb{B})$ whose restriction to the subsemimodule $\overline{\mathrm{Im}}(F(\mathrm{cok}(\mathfrak{s}\Delta))) \subset \underline{\mathrm{End}}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(Q)$ is null. They are defined by the composite $p_1 \circ \phi$ which form the columns of the matrix.

$$\begin{pmatrix}
(0,0) & 0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0 \\
(0,\alpha+\beta) & 0,0,0,0,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1 \\
(0,\alpha+\gamma) & 0,0,0,0,0,0,0,0,0,0,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1 \\
(0,\beta+\delta) & 0,0,0,0,1,1,1,1,1,1,0,0,0,0,0,0,1,1,1,1,1,1,1,1,1,1,1,1 \\
(\alpha+\beta,0) & 0,1,1,1,0,1,1,1,1,1,0,1,1,1,1,1,0,1,1,1,1,1,1,1,1,1,1,1 \\
(\alpha+\beta,\alpha+\beta) & 0,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1 \\
(\alpha+\beta,\alpha+\gamma) & 0,1,1,1,0,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1 \\
(\alpha+\beta,\beta+\delta) & 0,1,1,1,1,1,1,1,1,1,0,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1 \\
(\alpha+\gamma,0) & 0,0,1,1,0,0,1,1,1,1,0,0,1,1,1,1,0,0,0,0,1,1,1,1,1,1,1,1 \\
(\alpha+\gamma,\alpha+\beta) & 0,0,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1 \\
(\alpha+\gamma,\alpha+\gamma) & 0,0,1,1,0,0,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1 \\
(\alpha+\gamma,\beta+\delta) & 0,0,1,1,1,1,1,1,1,1,0,0,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1 \\
(\beta+\delta,0) & 0,1,0,1,0,1,0,1,1,1,0,1,0,1,1,1,0,1,1,1,0,0,0,1,1,1,1,1 \\
(\beta+\delta,\alpha+\beta) & 0,1,0,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1 \\
(\beta+\delta,\alpha+\gamma) & 0,1,0,1,0,1,0,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1 \\
(\beta+\delta,\beta+\delta) & 0,1,0,1,1,1,1,1,1,1,0,1,0,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1 \\
(\alpha,\beta) & 0,0,1,1,1,1,1,1,1,1,0,0,1,0,1,1,1,0,1,1,1,1,1,0,1,1,1,1 \\
(\alpha,\beta+\delta) & 0,0,1,1,1,1,1,1,1,1,0,0,1,0,1,1,1,1,1,1,1,1,1,1,1,1,1,1 \\
(\beta,\alpha) & 0,1,0,1,0,1,0,0,1,1,1,1,1,1,1,1,1,1,1,1,0,1,1,1,0,1,1,1 \\
(\beta,\alpha+\gamma) & 0,1,0,1,0,1,0,0,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1 \\
(\gamma,\delta) & 0,0,1,1,1,1,1,1,1,1,0,0,1,1,0,1,1,1,0,1,1,1,1,1,1,0,1,1 \\
(\gamma,\beta+\delta) & 0,0,1,1,1,1,1,1,1,1,0,0,1,1,0,1,1,1,1,1,1,1,1,1,1,1,1,1 \\
(\delta,\gamma) & 0,1,0,1,0,1,0,1,0,1,1,1,1,1,1,1,1,1,1,1,1,0,1,1,1,1,0,1 \\
(\delta,\alpha+\gamma) & 0,1,0,1,0,1,0,1,0,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1 \\
(\alpha+\gamma,\beta) & 0,0,1,1,1,1,1,1,1,1,0,0,1,1,1,1,1,0,1,1,1,1,1,1,1,1,1,1 \\
(\alpha+\gamma,\delta) & 0,0,1,1,1,1,1,1,1,1,0,0,1,1,1,1,1,1,0,1,1,1,1,1,1,1,1,1 \\
(\beta+\delta,\alpha) & 0,1,0,1,0,1,0,1,1,1,1,1,1,1,1,1,1,1,1,1,0,1,1,1,1,1,1,1 \\
(\beta+\delta,\gamma) & 0,1,0,1,0,1,0,1,1,1,1,1,1,1,1,1,1,1,1,1,1,0,1,1,1,1,1,1
\end{pmatrix}$$

Thus one is in fact computing a kernel in the dual theory. A more refined study is needed to state the non-vanishing of the classes in the colimit which defines the satellite functor SF. The natural framework for discussing the duality is as in Lemma 8.4, where we assume now that both categories $\mathcal{E}$ and $\mathcal{B}$ are endowed with a duality which given by a contravariant functor of the form $X \longrightarrow X^* = \mathrm{Hom}(X, \beta)$, where β is an injective object so that the functor is exact. We then associate to the covariant functor $F : \mathcal{E} \longrightarrow \mathcal{B}$ the new covariant functor obtained by conjugation

Definition 9.6. *The functor $F^* : \mathcal{E} \longrightarrow \mathcal{B}$ is defined as the composition $F^*(X) := (F(X^*))^*$.*

In the case of the functor $F = \underline{\mathrm{Hom}}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(Q, -)$ the guess for the conjugate functor is $G := Q \otimes -$, in view of the adjunction

$$G(M)^* = \underline{\mathrm{Hom}}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(Q \otimes M, \mathfrak{s}\mathbb{B}) \simeq \underline{\mathrm{Hom}}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(Q, \underline{\mathrm{Hom}}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(M, \mathfrak{s}\mathbb{B})) = F(M^*) \tag{101}$$

One cannot assert that the conjugate of F is G except for finite objects since for these the duality is involutive as shown in §7.1. Thus we should expect that the conjugate functor of $F = \underline{\mathrm{Hom}}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(Q, -)$ admits the analogue of the Tor functor as a left satellite. The short doubly exact sequence (96)

$$K \leftarrow \mathrm{Ker}(\mathrm{cok}(\mathfrak{s}\Delta)) \overset{\subset}{\rightarrowtail} \mathbb{B}^2 \times \mathbb{B}^2 \overset{\mathrm{cok}(\mathfrak{s}\Delta)}{\twoheadrightarrow} Q$$

together with the isomorphism $K \simeq Q^*$ provides an element of the comma category $\mathcal{I} = \mathrm{Sh}_{\mathrm{small}}(\mathbb{B}\mathrm{mod}^{\mathfrak{s}}) \downarrow_{P'} K$ which enters in the construction of $SF(K)$. However, we can view the short doubly exact sequence (96) as self-dual and use it together with the identity map $\mathrm{Id} : Q \to Q$ as

$$Q^* \overset{\subset}{\rightarrowtail} \mathbb{B}^2 \times \mathbb{B}^2 \overset{\mathrm{cok}(\mathfrak{s}\Delta)}{\twoheadrightarrow} Q \overset{\mathrm{Id}}{\longleftarrow} Q$$

as an element of the comma category $\mathcal{J} = Q \downarrow_{P''} \mathrm{Sh}_{\mathrm{small}}(\mathbb{B}\mathrm{mod}^{\mathfrak{s}})$ which enters in the construction of $SG(Q)$. We note that the short doubly exact sequence

$$\alpha : Q^* \simeq K \overset{\alpha'}{\rightarrowtail} \mathbb{B}^2 \times \mathbb{B}^2 \overset{\alpha''}{\twoheadrightarrow} Q, \quad \alpha' = \subset, \quad \alpha'' = \mathrm{cok}(\mathfrak{s}\Delta) \tag{102}$$

is both a left semiresolution of Q and a right semiresolution of Q^* in the sense of [18], 4.2.1, (4.27).

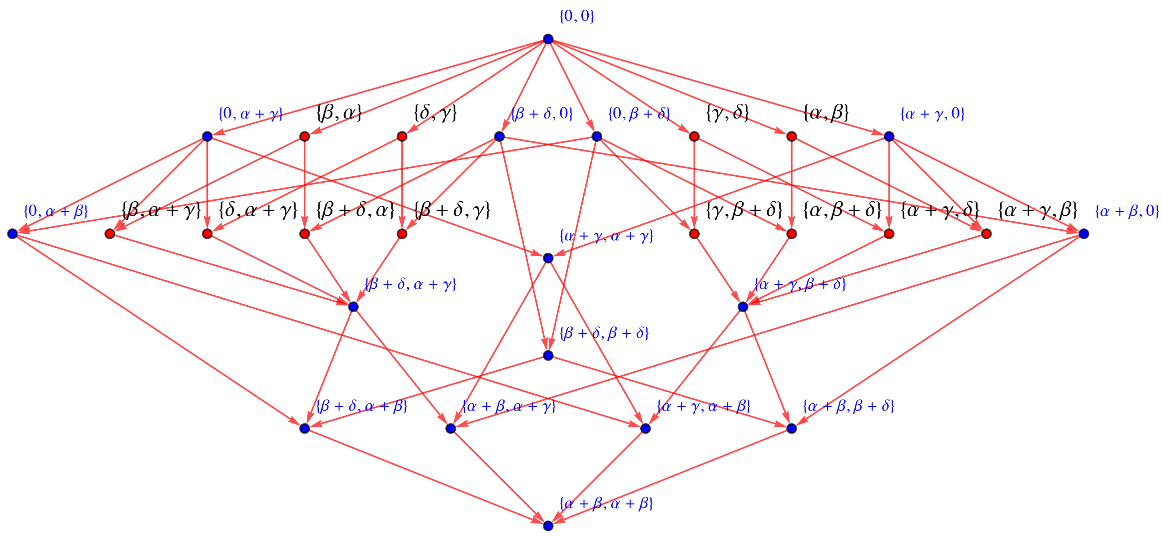

Figure 14: Graph of the cokernel $\mathrm{Coker}(F(\alpha''))$.

Indeed, the object $\mathbb{B}^2 \times \mathbb{B}^2$ of $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ is injective and projective. If one could apply Theorem 4.2.2 of [18], §4.2.1, one would conclude that both satellite functors SF and SG can be computed using these semiresolutions as

$$SF(K) = \mathrm{Coker}(F(\alpha'')), \quad SG(Q) = \mathrm{Ker}(G(\alpha')).$$

This raises the problem of proving that condition (a) of Theorem 4.2.2 of [18] is valid. This issue already arises with the short doubly exact sequence α as in (102) for which the problem of checking condition (a) arises for both functors F and G. We first consider F and the comma category $\mathcal{I} = \mathrm{Sh}_{\mathrm{small}}(\mathbb{B}\mathrm{mod}^{\mathfrak{s}}) \downarrow_{P'} K$ which enters in the construction of $SF(K)$. The first question is whether one can determine an action of the group $\mathrm{Aut}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(K)$ on the object $\mathrm{Coker}(F(\alpha''))$. This action should reflect the functoriality of the satellite functor SF. The group $\mathrm{Aut}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(K)$ is given by the permutations of the 4 generators $\alpha+\beta, \alpha+\gamma, \beta+\delta, \gamma+\delta$ which respect the partition in the two subsets of σ-fixed points, *i.e.* $\alpha+\gamma, \beta+\delta$ and its complement. This gives the following permutations

$$\begin{pmatrix} \alpha+\beta & \alpha+\gamma & \beta+\delta & \gamma+\delta \\ \alpha+\beta & \beta+\delta & \alpha+\gamma & \gamma+\delta \\ \gamma+\delta & \alpha+\gamma & \beta+\delta & \alpha+\beta \\ \gamma+\delta & \beta+\delta & \alpha+\gamma & \alpha+\beta \end{pmatrix}$$

These permutations extend to the automorphisms of $\mathbb{B}^2 \times \mathbb{B}^2$ associated to the permutations

$$\begin{pmatrix} \alpha & \beta & \gamma & \delta \\ \beta & \alpha & \delta & \gamma \\ \gamma & \delta & \alpha & \beta \\ \delta & \gamma & \beta & \alpha \end{pmatrix} \tag{103}$$

which form a subgroup of order 4 of the group $\mathrm{Aut}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(\mathbb{B}^2 \times \mathbb{B}^2)$ which is of order 8 and consists of all permutations of $(\alpha, \beta, \gamma, \delta)$ which commute with σ. There are 8 such permutations

$$\begin{pmatrix} \alpha & \beta & \gamma & \delta \\ \alpha & \delta & \gamma & \beta \\ \beta & \alpha & \delta & \gamma \\ \beta & \gamma & \delta & \alpha \\ \gamma & \beta & \alpha & \delta \\ \gamma & \delta & \alpha & \beta \\ \delta & \alpha & \beta & \gamma \\ \delta & \gamma & \beta & \alpha \end{pmatrix} \tag{104}$$

An element of the group $\mathrm{Aut}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(Q)$ is determined by its action on the generators (see Figure 5) and the associated permutation of $(\alpha, \beta, \gamma, \delta)$ must commute with σ and also respect the equation $\alpha + \beta = \gamma + \delta$. This reduces the group (104) to its subgroup (103). We thus get:

Lemma 9.7. *(i) The functors P' and P'' establish isomorphisms $P' : \mathrm{Aut}(\alpha) \to \mathrm{Aut}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(K)$ and $P'' : \mathrm{Aut}(\alpha) \to \mathrm{Aut}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(Q)$ where α is the exact sequence (102).*

(ii) This determines a canonical functorial action $SF(u) \in \mathrm{Aut}(\mathrm{Coker}(F(\alpha''))$ for $u \in \mathrm{Aut}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(K)$.

(iii) Similarly one gets a canonical functorial action $SG(u) \in \mathrm{Aut}(Ker(G(\alpha'))$ for $u \in \mathrm{Aut}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(Q)$.

Proof. (i) The subgroup (103) of $\mathrm{Aut}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(\mathbb{B}^2 \times \mathbb{B}^2)$ preserves globally the subobject of $\mathbb{B}^2 \times \mathbb{B}^2$, $K = \mathrm{Ker}(\mathrm{cok}(\mathfrak{s}\Delta)) \simeq Q^*$, and thus acts as automorphisms of the short doubly exact sequence α. The above computation shows that the functors P' and P'' establish isomorphisms.

(ii) To compute $SF(u) \in \mathrm{Aut}(\mathrm{Coker}(F(\alpha''))$ one applies the permutations of (103) to the twelve elements of C of (9.1). One checks that one obtains a subgroup of order 4 of $\mathrm{Aut}(\mathrm{Coker}(F(\alpha''))$ whose non-trivial elements act without fixed points on C.

(iii) The proof is the same as for (ii). □

The short doubly exact sequence α is self-dual, *i.e.* $\alpha^* = \alpha$, where the duality is defined by applying the functor $\mathrm{Hom}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(-, \mathfrak{s}\mathbb{B})$. Moreover, the satellite functors SF and SG ought to be related by the duality of (101).

9.3 The comma category $\mathcal{I} = \mathrm{Sh}(\mathbb{B}\mathrm{mod}^{\mathfrak{s}}) \downarrow_{P'} K$. Next, we investigate the more delicate aspect of functoriality involving all endomorphisms $u \in \mathrm{End}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(K)$. The general idea is to use the injectivity of $\mathbb{B}^2 \times \mathbb{B}^2$ in the short doubly exact sequence α to show that (α, Id_K) is a weakly final object in the comma category $\mathcal{I} = \mathrm{Sh}_{\mathrm{small}}(\mathbb{B}\mathrm{mod}^{\mathfrak{s}}) \downarrow_{P'} K$ which enters in the construction of $SF(K)$. Thus let c be an object of the comma category

$$c = \{K \xleftarrow{v} C' \overset{c'}{\rightarrowtail} C \xrightarrow{c''} \!\!\!\!\rightarrow C''\}.$$

Since $\mathbb{B}^2 \times \mathbb{B}^2$ is injective, one can extend the morphism $\alpha' \circ v : C' \to \mathbb{B}^2 \times \mathbb{B}^2$ to a morphism $w : C \to \mathbb{B}^2 \times \mathbb{B}^2$ such that $w \circ c' = \alpha' \circ v$.

$$\begin{tikzcd}
K \arrow[d, "\mathrm{Id}"'] & C' \arrow[l, "v"'] \arrow[r, tail, "c'"] \arrow[d, "v"] & C \arrow[r, two heads, "c''"] \arrow[d, "w"] & C'' \arrow[d, dashed] \\
K & K \arrow[l, "\mathrm{Id}_K"'] \arrow[r, tail, "\alpha'"] & \mathbb{B}^2 \times \mathbb{B}^2 \arrow[r, two heads, "\alpha''"] & Q
\end{tikzcd} \tag{105}$$

In order to obtain a morphism $c \to (\alpha, \mathrm{Id}_{T^*})$ in the comma category it remains to fill the dotted vertical arrow of the diagram (105). In fact this is automatic using the proof of Theorem 4.2.2 of [18].

Lemma 9.8. *The object (α, Id_K) is weakly final in the comma category $\mathcal{I} = \mathrm{Sh}(\mathbb{B}\mathrm{mod}^{\mathfrak{s}}) \downarrow_{P'} K$.*

Proof. One just needs to fill the dotted vertical arrow of the diagram (105), but since the map c'' is surjective it is enough to show that $\alpha'' \circ w : C \to Q$ is compatible with the equivalence relation associated to c''. Since c is a short doubly exact sequence this compatibility is equivalent to showing that $\alpha'' \circ w \circ c'$ is null, and since $w \circ c' = \alpha' \circ v$ this follows from the nullity of $\alpha'' \circ \alpha'$. $\square$

We now apply this result to test the functoriality of SF with respect to the endomorphisms $\mathrm{End}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(K)$. Lemma 9.8 shows that any endomorphism $v \in \mathrm{End}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(K)$ lifts to an endomorphism $w \in \mathrm{End}(\alpha)$ of the short doubly exact sequence α. For any of the lifts $w \in P'^{-1}(\{v\})$ one obtains an associated element of $\mathrm{End}(\mathrm{Coker}(F(\alpha'')))$. More precisely the endomorphism $w \in \mathrm{End}(\alpha)$ induces an endomorphism $w'' \in \mathrm{End}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(Q)$ making the following diagram commutative

$$\begin{tikzcd}
\mathbb{B}^2 \times \mathbb{B}^2 \arrow[r, two heads, "\alpha''"] \arrow[d, "w"'] & Q \arrow[d, "w''"] \\
\mathbb{B}^2 \times \mathbb{B}^2 \arrow[r, two heads, "\alpha''"] & Q
\end{tikzcd} \tag{106}$$

This gives a commutative square using the functor $F = \underline{\mathrm{Hom}}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(Q, -)$

$$\begin{tikzcd}
F(\mathbb{B}^2 \times \mathbb{B}^2) \arrow[r, "F(\alpha'')"] \arrow[d, "F(w)"'] & \underline{\mathrm{Hom}}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(Q, Q) \arrow[r, "\mathrm{cok}(F(\alpha''))"] \arrow[d, "w'' \circ -"] & \mathrm{Coker}(F(\alpha'')) \arrow[d, dashed] \\
F(\mathbb{B}^2 \times \mathbb{B}^2) \arrow[r, "F(\alpha'')"] & \underline{\mathrm{Hom}}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(Q, Q) \arrow[r, "\mathrm{cok}(F(\alpha''))"] & \mathrm{Coker}(F(\alpha''))
\end{tikzcd} \tag{107}$$

and one needs to compute the dotted vertical arrow whose existence follows as in Lemma 9.8. Note that this vertical arrow is uniquely determined by the functorial action $F(w'')$ of w'' on $F(Q) = \underline{\mathrm{Hom}}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(Q, Q)$, and thus it only depends upon w'' rather than on w. Thus to compute the functoriality one just needs the correspondence from $v \in \mathrm{End}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(K)$ to $w'' \in \mathrm{End}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(Q)$. An explicit computation shows that there are 19 elements of $\mathrm{End}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(K)$ which give ambiguous values of w'' and the w'' is unique for the remaining 51 elements of $\mathrm{End}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(K)$. We provide below the list of the ambiguous elements of $\mathrm{End}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(K)$ by specifying how they act on the three elements $\alpha+\beta$, $\alpha+\gamma$ and $\beta+\delta$. On the right hand side we give the associated choices for w'' by their action on the two elements α, β of Q.

$$\{\alpha+\gamma,\alpha+\gamma,\alpha+\gamma\} \to \begin{pmatrix} \alpha & \gamma \\ \alpha & \alpha+\gamma \\ \gamma & \alpha \\ \gamma & \alpha+\gamma \\ \alpha+\gamma & \alpha \\ \alpha+\gamma & \gamma \\ \alpha+\gamma & \alpha+\gamma \end{pmatrix}$$

$$\{\beta+\delta,\beta+\delta,\beta+\delta\} \to \begin{pmatrix} \beta & \delta \\ \beta & \beta+\delta \\ \delta & \beta \\ \delta & \beta+\delta \\ \beta+\delta & \beta \\ \beta+\delta & \delta \\ \beta+\delta & \beta+\delta \end{pmatrix}$$

$$\{\alpha+\beta+\gamma,\alpha+\gamma,\alpha+\beta+\gamma+\delta\} \to \begin{pmatrix} \alpha & \alpha+\beta \\ \alpha & \beta+\gamma \\ \gamma & \alpha+\beta \\ \alpha+\gamma & \alpha+\beta \\ \alpha+\gamma & \beta+\gamma \end{pmatrix}$$

$$\{\alpha+\beta+\gamma,\alpha+\beta+\gamma+\delta,\alpha+\gamma\} \to \begin{pmatrix} \alpha+\beta & \alpha \\ \alpha+\beta & \gamma \\ \alpha+\beta & \alpha+\gamma \\ \beta+\gamma & \alpha \\ \beta+\gamma & \alpha+\gamma \end{pmatrix}$$

$$\{\alpha+\beta+\gamma,\alpha+\beta+\gamma+\delta,\alpha+\beta+\gamma+\delta\} \to \begin{pmatrix} \alpha+\beta & \alpha+\beta \\ \alpha+\beta & \beta+\gamma \\ \beta+\gamma & \alpha+\beta \end{pmatrix}$$

$$\{\alpha+\beta+\delta,\beta+\delta,\alpha+\beta+\gamma+\delta\} \to \begin{pmatrix} \beta & \alpha+\beta \\ \beta & \alpha+\delta \\ \delta & \alpha+\beta \\ \beta+\delta & \alpha+\beta \\ \beta+\delta & \alpha+\delta \end{pmatrix}$$

$$\{\alpha+\beta+\delta,\alpha+\beta+\gamma+\delta,\beta+\delta\} \to \begin{pmatrix} \alpha+\beta & \beta \\ \alpha+\beta & \delta \\ \alpha+\beta & \beta+\delta \\ \alpha+\delta & \beta \\ \alpha+\delta & \beta+\delta \end{pmatrix}$$

$$\{\alpha+\beta+\delta,\alpha+\beta+\gamma+\delta,\alpha+\beta+\gamma+\delta\}\to\begin{pmatrix}\alpha+\beta & \alpha+\beta\\ \alpha+\beta & \alpha+\delta\\ \alpha+\delta & \alpha+\beta\end{pmatrix}$$

$$\{\alpha+\gamma+\delta,\alpha+\gamma,\alpha+\beta+\gamma+\delta\}\to\begin{pmatrix}\alpha & \alpha+\beta\\ \gamma & \alpha+\beta\\ \gamma & \alpha+\delta\\ \alpha+\gamma & \alpha+\beta\\ \alpha+\gamma & \alpha+\delta\end{pmatrix}$$

$$\{\alpha+\gamma+\delta,\alpha+\beta+\gamma+\delta,\alpha+\gamma\}\to\begin{pmatrix}\alpha+\beta & \alpha\\ \alpha+\beta & \gamma\\ \alpha+\beta & \alpha+\gamma\\ \alpha+\delta & \gamma\\ \alpha+\delta & \alpha+\gamma\end{pmatrix}$$

$$\{\alpha+\gamma+\delta,\alpha+\beta+\gamma+\delta,\alpha+\beta+\gamma+\delta\}\to\begin{pmatrix}\alpha+\beta & \alpha+\beta\\ \alpha+\beta & \alpha+\delta\\ \alpha+\delta & \alpha+\beta\end{pmatrix}$$

$$\{\beta+\gamma+\delta,\beta+\delta,\alpha+\beta+\gamma+\delta\}\to\begin{pmatrix}\beta & \alpha+\beta\\ \delta & \alpha+\beta\\ \delta & \beta+\gamma\\ \beta+\delta & \alpha+\beta\\ \beta+\delta & \beta+\gamma\end{pmatrix}$$

$$\{\beta+\gamma+\delta,\alpha+\beta+\gamma+\delta,\beta+\delta\}\to\begin{pmatrix}\alpha+\beta & \beta\\ \alpha+\beta & \delta\\ \alpha+\beta & \beta+\delta\\ \beta+\gamma & \delta\\ \beta+\gamma & \beta+\delta\end{pmatrix}$$

$$\{\beta+\gamma+\delta,\alpha+\beta+\gamma+\delta,\alpha+\beta+\gamma+\delta\}\to\begin{pmatrix}\alpha+\beta & \alpha+\beta\\ \alpha+\beta & \beta+\gamma\\ \beta+\gamma & \alpha+\beta\end{pmatrix}$$

$$\{\alpha+\beta+\gamma+\delta,\alpha+\gamma,\alpha+\beta+\gamma+\delta\}\to\begin{pmatrix}\alpha & \alpha+\beta\\ \gamma & \alpha+\beta\\ \alpha+\gamma & \alpha+\beta\end{pmatrix}$$

$$\{\alpha+\beta+\gamma+\delta,\beta+\delta,\alpha+\beta+\gamma+\delta\}\to\begin{pmatrix}\beta & \alpha+\beta\\ \delta & \alpha+\beta\\ \beta+\delta & \alpha+\beta\end{pmatrix}$$

$$\{\alpha+\beta+\gamma+\delta,\alpha+\beta+\gamma+\delta,\alpha+\gamma\}\to\begin{pmatrix}\alpha+\beta & \alpha\\ \alpha+\beta & \gamma\\ \alpha+\beta & \alpha+\gamma\end{pmatrix}$$

$$\{\alpha+\beta+\gamma+\delta,\alpha+\beta+\gamma+\delta,\beta+\delta\}\to\begin{pmatrix}\alpha+\beta & \beta\\ \alpha+\beta & \delta\\ \alpha+\beta & \beta+\delta\end{pmatrix}$$

Besides the above 18 ambiguous elements there is a unique element of $\mathrm{End}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(K)$ which extends in 49 different ways as an endomorphism of the short doubly exact sequence α. But it gives rise to only 7 choices of w'' as follows

$$\{\alpha+\beta+\gamma+\delta,\alpha+\beta+\gamma+\delta,\alpha+\beta+\gamma+\delta\}\to\begin{pmatrix}\alpha+\beta & \alpha+\beta\\ \alpha+\beta & \alpha+\delta\\ \alpha+\beta & \beta+\gamma\\ \alpha+\delta & \alpha+\beta\\ \alpha+\delta & \beta+\gamma\\ \beta+\gamma & \alpha+\beta\\ \beta+\gamma & \alpha+\delta\end{pmatrix}$$

We now describe in our case the explicit computation of the action of $w''\in\mathrm{End}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(Q)$ by naturality on the cokernel $\mathrm{Coker}(F(\alpha''))$ as in (106).

One gets seven fibers of seven elements

$$\begin{pmatrix}\alpha+\beta & \alpha+\beta\\ \alpha+\beta & \alpha+\delta\\ \alpha+\beta & \beta+\gamma\\ \alpha+\delta & \alpha+\beta\\ \alpha+\delta & \beta+\gamma\\ \beta+\gamma & \alpha+\beta\\ \beta+\gamma & \alpha+\delta\end{pmatrix},\begin{pmatrix}\alpha+\beta & \alpha\\ \alpha+\beta & \gamma\\ \alpha+\beta & \alpha+\gamma\\ \alpha+\delta & \gamma\\ \alpha+\delta & \alpha+\gamma\\ \beta+\gamma & \alpha\\ \beta+\gamma & \alpha+\gamma\end{pmatrix},\begin{pmatrix}\alpha+\beta & \beta\\ \alpha+\beta & \delta\\ \alpha+\beta & \beta+\delta\\ \alpha+\delta & \beta\\ \alpha+\delta & \beta+\delta\\ \beta+\gamma & \delta\\ \beta+\gamma & \beta+\delta\end{pmatrix},\begin{pmatrix}\alpha & \alpha+\beta\\ \alpha & \beta+\gamma\\ \gamma & \alpha+\beta\\ \gamma & \alpha+\delta\\ \alpha+\gamma & \alpha+\beta\\ \alpha+\gamma & \alpha+\delta\\ \alpha+\gamma & \beta+\gamma\end{pmatrix},$$

$$\begin{pmatrix}\beta & \alpha+\beta\\ \beta & \alpha+\delta\\ \delta & \alpha+\beta\\ \delta & \beta+\gamma\\ \beta+\delta & \alpha+\beta\\ \beta+\delta & \alpha+\delta\\ \beta+\delta & \beta+\gamma\end{pmatrix},\begin{pmatrix}\alpha & \gamma\\ \alpha & \alpha+\gamma\\ \gamma & \alpha\\ \gamma & \alpha+\gamma\\ \alpha+\gamma & \alpha\\ \alpha+\gamma & \gamma\\ \alpha+\gamma & \alpha+\gamma\end{pmatrix},\begin{pmatrix}\beta & \delta\\ \beta & \beta+\delta\\ \delta & \beta\\ \delta & \beta+\delta\\ \beta+\delta & \beta\\ \beta+\delta & \delta\\ \beta+\delta & \beta+\delta\end{pmatrix}$$

and the homomorphism is injective on the remaining 21 elements of $\mathrm{End}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(Q)$.

In the next subsection we shall describe how things work for the short doubly exact sequence α in order to understand, using Lemma 8.6, why condition (a) holds there. In fact it is important to distinguish two levels.

9.4 The correspondence $v\mapsto w''$ at the level of α. In this Section we describe the correspondence between $\mathrm{End}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(K)$ and $\mathrm{End}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(Q)$ coming from the ambiguity in the extension of an endomorphism of K to an endomorphism of the short doubly exact sequence α. We ignore the functor F and work directly at the level of the short doubly exact sequence α using the two morphisms

$$\mathrm{End}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(K)\overset{res}{\leftarrow}\mathrm{End}(\alpha)\overset{quot}{\rightarrow}\mathrm{End}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(Q)\qquad(108)$$

where $K\simeq Q^*$, $\mathrm{End}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(K)$ is determined by Lemma 9.4, $\mathrm{End}(\alpha)$ is the subobject of the endomorphisms $\mathrm{End}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(\mathbb{B}^2\times\mathbb{B}^2)$ which map K to K, the left arrow is the restriction to K and the right arrow is the quotient action. Let $X=\{\alpha,\beta,\gamma,\delta\}$ be the set of minimal elements of

$\mathbb{B}^2\times\mathbb{B}^2$. An element $z\in\mathbb{B}^2\times\mathbb{B}^2$ is given uniquely by the subset $Z\subset X$ such that $z=\sum_Z \epsilon$. We denote by $|z|$ the cardinality of Z. An element $\phi\in \mathrm{End}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(\mathbb{B}^2\times\mathbb{B}^2)$ is uniquely determined by the two subsets $A,B\subset X$ such that $\phi(\alpha)=\sum_A \epsilon$ and $\phi(\beta)=\sum_B \epsilon$. The restriction of ϕ to K (if it exists) determines $A\cup B$, $A\cup\sigma(A)$ and $B\cup\sigma(B)$ using $\phi(\alpha+\beta)$, $\phi(\alpha+\gamma)$, $\phi(\beta+\delta)$. If $A\cup\sigma(A)$ or $B\cup\sigma(B)$ is empty the union $A\cup B$ determines both A and B uniquely. To deal with the lack of uniqueness of the extension of $\phi\in\mathrm{End}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(K)$ as an endomorphism in $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ of $\mathbb{B}^2\times\mathbb{B}^2$ we can thus assume that $\phi(\tau_j)\neq 0\ \forall j$, where we let $\tau_1=\alpha+\gamma\in K^\sigma$, $\tau_2=\beta+\delta\in K^\sigma$. We encode the elements $\phi\in\mathrm{End}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(K)$ by the triple $(\phi(\alpha+\beta),\phi(\tau_1),\phi(\tau_2))$ which fulfills the conditions of Lemma 9.4 (so that in particular $\phi(\tau_j)\in K^\sigma=\{0,\tau_1,\tau_2,\tau\}$).

Lemma 9.9. *(i) Among the seven elements of $\mathrm{End}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(K)$ such that $\phi(\tau)\neq\tau$ only two, namely (τ_j,τ_j,τ_j) for $j=1,2$, admit more than one extension to $\mathbb{B}^2\times\mathbb{B}^2$.*

(ii) Let $\phi\in\mathrm{End}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(K)$ with $\phi(\tau)=\tau$. Then ϕ admits more than one extension to $\mathbb{B}^2\times\mathbb{B}^2$ if and only if either $\{|\phi(\tau_1)|,|\phi(\tau_2)|\}=\{2,4\}$ and $\phi(\alpha+\beta)>\phi(\tau_j)\mid|\phi(\tau_j)|=2$, or if $\phi(\tau_j)=\tau$, $\forall j$, and $|\phi(\alpha+\beta)|\geq 3$.

Proof. (i) Under the hypothesis (i), the σ-invariant set $A\cup B\cup\sigma(A)\cup\sigma(B)$ is $\neq X$ and is thus a single σ-orbit. If $A\cup B$ has one element this uniquely determines A and B thus the only cases left are when $A\cup B=A\cup\sigma(A)=B\cup\sigma(B)$ is a single orbit. There are two such cases given by $\phi(\alpha+\beta)=\phi(\alpha+\gamma)=\phi(\beta+\delta)=\alpha+\gamma$ and $\phi(\alpha+\beta)=\phi(\alpha+\gamma)=\phi(\beta+\delta)=\beta+\delta$. In each case the extensions correspond to the seven choices of two non-empty subsets whose union is a given set with two elements.

(ii) Under the hypothesis (ii), the σ-invariant set $A\cup B\cup\sigma(A)\cup\sigma(B)$ is X. Let us show that if $A\cup B$ has two elements the extension is unique. Indeed $A\cup B$ intersects each σ-orbit in a single point and thus one has $A=(A\cup B)\cap(A\cup\sigma(A))$ and $B=(A\cup B)\cap(B\cup\sigma(B))$. Similarly if $A\cup\sigma(A)$ and $B\cup\sigma(B)$ are both $\neq X$ they must be disjoint orbits, thus $A\cap B=\emptyset$ and one has $A=(A\cup B)\cap(A\cup\sigma(A))$ and $B=(A\cup B)\cap(B\cup\sigma(B))$. We have thus shown that if ϕ admits more than one extension one has $|\phi(\alpha+\beta)|\geq 3$ and either $\{|\phi(\tau_1)|,|\phi(\tau_2)|\}=\{2,4\}$ or $\phi(\tau_j)=\tau$, $\forall j$. It remains to exclude the case where $A\cup\sigma(A)$ is a single σ-orbit and is not contained in $A\cup B$. In that case $A=(A\cup\sigma(A))\cap(A\cup B)$ is uniquely determined and has one element. Also $B\cap\sigma(A)=\emptyset$ since otherwise one would have $A\cup\sigma(A)\subset A\cup B$. Moreover since $\#(A\cup B)\geq 3$ and $B\cup\sigma(B)=X$ one gets $B=\sigma(A)^c$. This shows the uniqueness of the extension in that case. ∎

We thus obtain the list of the 19 elements of $\mathrm{End}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(K)$ with multiple extensions

$$\begin{pmatrix}
\alpha+\gamma & \alpha+\gamma & \alpha+\gamma \\
\beta+\delta & \beta+\delta & \beta+\delta \\
\alpha+\beta+\gamma & \alpha+\gamma & \alpha+\beta+\gamma+\delta \\
\alpha+\beta+\gamma & \alpha+\beta+\gamma+\delta & \alpha+\gamma \\
\alpha+\beta+\gamma & \alpha+\beta+\gamma+\delta & \alpha+\beta+\gamma+\delta \\
\alpha+\beta+\delta & \beta+\delta & \alpha+\beta+\gamma+\delta \\
\alpha+\beta+\delta & \alpha+\beta+\gamma+\delta & \beta+\delta \\
\alpha+\beta+\delta & \alpha+\beta+\gamma+\delta & \alpha+\beta+\gamma+\delta \\
\alpha+\gamma+\delta & \alpha+\gamma & \alpha+\beta+\gamma+\delta \\
\alpha+\gamma+\delta & \alpha+\beta+\gamma+\delta & \alpha+\gamma \\
\alpha+\gamma+\delta & \alpha+\beta+\gamma+\delta & \alpha+\beta+\gamma+\delta \\
\beta+\gamma+\delta & \beta+\delta & \alpha+\beta+\gamma+\delta \\
\beta+\gamma+\delta & \alpha+\beta+\gamma+\delta & \beta+\delta \\
\beta+\gamma+\delta & \alpha+\beta+\gamma+\delta & \alpha+\beta+\gamma+\delta \\
\alpha+\beta+\gamma+\delta & \alpha+\gamma & \alpha+\beta+\gamma+\delta \\
\alpha+\beta+\gamma+\delta & \beta+\delta & \alpha+\beta+\gamma+\delta \\
\alpha+\beta+\gamma+\delta & \alpha+\beta+\gamma+\delta & \alpha+\gamma \\
\alpha+\beta+\gamma+\delta & \alpha+\beta+\gamma+\delta & \beta+\delta \\
\alpha+\beta+\gamma+\delta & \alpha+\beta+\gamma+\delta & \alpha+\beta+\gamma+\delta
\end{pmatrix}$$

In each of these cases one finds 7 extensions of ϕ to $\mathbb{B}^2\times\mathbb{B}^2$, except for the last one which admits 49 extensions. To obtain the w'' we recall that the cokernel $\mathrm{cok}:\mathbb{B}^2\times\mathbb{B}^2\to Q$ of the inclusion $K\subset\mathbb{B}^2\times\mathbb{B}^2$ is obtained (using Proposition 6.19) as the identity on the first ten elements of the list

$$0,\alpha,\beta,\gamma,\delta,\alpha+\beta,\alpha+\gamma,\alpha+\delta,\beta+\gamma,\beta+\delta,$$

$$\gamma+\delta,\alpha+\beta+\gamma,\alpha+\beta+\delta,\alpha+\gamma+\delta,\beta+\gamma+\delta,\alpha+\beta+\gamma+\delta$$

while all the others project on $\alpha+\beta\sim\gamma+\delta$. Thus the quotient map in (108) is obtained by applying the map cok to the values $\phi(\alpha),\phi(\beta)$ for endomorphisms of $\mathbb{B}^2\times\mathbb{B}^2$ which preserve K globally. To see what happens we review the various cases of Lemma 9.9. For the case (i), one finds that the operation of passing from w to w'' is injective and one gets (for τ_1) the corresponding 7 distinct elements of $\mathrm{End}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(Q)$

$$\begin{pmatrix}
\alpha & \gamma \\
\alpha & \alpha+\gamma \\
\gamma & \alpha \\
\gamma & \alpha+\gamma \\
\alpha+\gamma & \alpha \\
\alpha+\gamma & \gamma \\
\alpha+\gamma & \alpha+\gamma
\end{pmatrix}$$

In the case (ii), the first type is $\phi=(\alpha+\beta+\gamma,\alpha+\gamma,\alpha+\beta+\gamma+\delta)$, which admits the seven extensions

$$\begin{pmatrix} \alpha & \beta+\gamma \\ \alpha & \alpha+\beta+\gamma \\ \gamma & \alpha+\beta \\ \gamma & \alpha+\beta+\gamma \\ \alpha+\gamma & \alpha+\beta \\ \alpha+\gamma & \beta+\gamma \\ \alpha+\gamma & \alpha+\beta+\gamma \end{pmatrix}$$

but when one takes the associated w'' the elements $(\gamma, \alpha+\beta)$ and $(\gamma, \alpha+\beta+\gamma)$ for instance give the same result so one finally obtains only five values as follows

$$\begin{pmatrix} \alpha & \alpha+\beta \\ \alpha & \beta+\gamma \\ \gamma & \alpha+\beta \\ \alpha+\gamma & \alpha+\beta \\ \alpha+\gamma & \beta+\gamma \end{pmatrix}$$

The second type is $\phi = (\alpha+\beta+\gamma, \tau, \tau)$ which admits the seven extensions

$$\begin{pmatrix} \alpha+\beta & \beta+\gamma \\ \alpha+\beta & \alpha+\beta+\gamma \\ \beta+\gamma & \alpha+\beta \\ \beta+\gamma & \alpha+\beta+\gamma \\ \alpha+\beta+\gamma & \alpha+\beta \\ \alpha+\beta+\gamma & \beta+\gamma \\ \alpha+\beta+\gamma & \alpha+\beta+\gamma \end{pmatrix}$$

but when one takes the associated w'' one obtains only three endomorphisms of Q

$$\begin{pmatrix} \alpha+\beta & \alpha+\beta \\ \alpha+\beta & \beta+\gamma \\ \beta+\gamma & \alpha+\beta \end{pmatrix}$$

The third type is $\phi = (\tau, \alpha+\gamma, \tau)$ which admits the seven extensions

$$\begin{pmatrix} \alpha & \beta+\gamma+\delta \\ \alpha & \alpha+\beta+\gamma+\delta \\ \gamma & \alpha+\beta+\delta \\ \gamma & \alpha+\beta+\gamma+\delta \\ \alpha+\gamma & \alpha+\beta+\delta \\ \alpha+\gamma & \beta+\gamma+\delta \\ \alpha+\gamma & \alpha+\beta+\gamma+\delta \end{pmatrix}$$

and which induce the following three w''

$$\begin{pmatrix} \alpha & \alpha+\beta \\ \gamma & \alpha+\beta \\ \alpha+\gamma & \alpha+\beta \end{pmatrix}$$

Finally, there remains the element $\phi = (\tau, \tau, \tau)$ which admits 49 extensions which correspond to all choices of subsets $A, B \subset X$ such that $A \cup B = A \cup \sigma(A) = B \cup \sigma(B) = X$. One checks that they induce the following seven w''

$$\begin{pmatrix} \alpha+\beta & \alpha+\beta \\ \alpha+\beta & \alpha+\delta \\ \alpha+\beta & \beta+\gamma \\ \alpha+\delta & \alpha+\beta \\ \alpha+\delta & \beta+\gamma \\ \beta+\gamma & \alpha+\beta \\ \beta+\gamma & \alpha+\delta \end{pmatrix}$$

We have now completed the description of the correspondence (108) and shown that it is univalent on 51 elements of $\mathrm{End}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(K)$ and $1 \to 7$, $1 \to 5$ and $1 \to 3$ on the remaining 19 elements of $\mathrm{End}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(K)$.

9.5 The functor F and the action on the cokernel. The discussion of 9.4 is independent of the functor F and the key issue is to show that the action of the various w'' in the same multivalued piece of the correspondence (108) all act in the same way on the cokernel $\mathrm{Coker}(F(\alpha''))$ of figure 14. One needs to find a conceptual reason why the multivalued pieces all act by null transformations *i.e.* by the sixteen null transformations among the 28 described in Figure 15. The required independence will then follow from Lemma 8.6 since a morphism ϕ in $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$ whose range is contained in null elements is determined uniquely as $\psi \circ p$ where ψ is the restriction of ϕ to null elements. In order to simplify the verification that the multivalued pieces all act by null transformations we first investigate the effect of a natural transformation $\mu : F \to F'$ on the computation of the action of w'' on the cokernels. With the notations of (87), we start with a morphism of short doubly exact sequences in $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$

$$\begin{tikzcd} C' \arrow[r, tail, "c'"] \arrow[d, "v"] & C \arrow[r, two heads, "c''"] \arrow[d, "w"] & C'' \arrow[d, "w''"] \\ I' \arrow[r, tail, "\alpha'"] & I \arrow[r, two heads, "\alpha''"] & I'' \end{tikzcd} \tag{109}$$

and compare the induced actions on cokernels

$$F(w'') : \mathrm{Coker}(F(c'')) \to \mathrm{Coker}(F(\alpha'')), \quad F'(w'') : \mathrm{Coker}(F'(c'')) \to \mathrm{Coker}(F'(\alpha'')). \tag{110}$$

Lemma 9.10. *(i) Let $w : C \to I$ be a morphism of short doubly exact sequences as in (109), and $\mu : F \to F'$ be a natural transformation. The morphisms induced by $F(w''), F'(w''), \mu_{C''}, \mu_{I''}$ form a commutative square.*

(ii) Let $F = F' := \underline{\mathrm{Hom}}(Q, -)$ and $\mu : F \to F'$ be given by right composition with $\rho \in \mathrm{End}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(Q)$. Then the induced morphism $\tilde\rho : \mathrm{Coker}(F(\alpha'')) \to \mathrm{Coker}(F(\alpha''))$ commutes with the morphism $F(w'') : \mathrm{Coker}(F(\alpha'')) \to \mathrm{Coker}(F(\alpha''))$ for any endomorphism w of the short doubly exact sequence α.

Proof. (i) One has, using the natural transformation μ, the commutative diagram

$$\begin{tikzcd}
& F(C) \arrow[rr, "F(c'')"] \arrow[dl, "\mu_C"'] \arrow[dd, "F(w)" near start] & & F(C'') \arrow[rr] \arrow[dl, "\mu_{C''}"'] \arrow[dd, "F(w'')" near start] & & \mathrm{Coker}(F(c'')) \arrow[dl, dashed] \arrow[dd] \\
F'(C) \arrow[rr, "F'(c'')" near start] \arrow[dd, "F'(w)"'] & & F'(C'') \arrow[rr] \arrow[dd, "F'(w'')" near end] & & \mathrm{Coker}(F'(c'')) \arrow[dd] & \\
& F(I) \arrow[rr, "F(\alpha'')" near start] \arrow[dl, "\mu_I"'] & & F(I'') \arrow[rr] \arrow[dl, "\mu_{I''}"'] & & \mathrm{Coker}(F(\alpha'')) \arrow[dl, dashed] \\
F'(I) \arrow[rr, "F'(\alpha'')"] & & F'(I'') \arrow[rr] & & \mathrm{Coker}(F'(\alpha'')) &
\end{tikzcd}$$

where the dashed arrows are induced by the natural transformations $\mu_{C''} : F(C'') \to F'(C'')$ and $\mu_{I''} : F(I'') \to F'(I'')$.

(ii) Follows from (i) since right composition with ρ defines a morphism of functors. $\square$

Lemma 9.11. *(i) For any $\phi \in End_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(Q)$ such that $\phi \notin \overline{\mathrm{Im}}(F(\alpha''))$ the restriction $\psi \in End_{\mathbb{B}\mathrm{mod}}(Q^\sigma)$ of ϕ to Q^σ is an automorphism.*

(ii) Let $v \in End_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(K)$ admit more than one extension to $\mathbb{B}^2 \times \mathbb{B}^2$. Then for any of these extensions w the induced morphism $w'' \in End_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(Q)$ fulfills $w'' \in \overline{\mathrm{Im}}(F(\alpha''))$.

Proof. (i) The twelve elements $\phi \in \mathrm{End}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(Q)$ which do not belong to $\overline{\mathrm{Im}}(F(\alpha''))$ are given by the list (9.1), *i.e.*

$$(\alpha,\beta),(\alpha,\beta+\delta),(\beta,\alpha),(\beta,\alpha+\gamma),(\gamma,\delta),(\gamma,\beta+\delta),$$

$$(\delta,\gamma),(\delta,\alpha+\gamma),(\alpha+\gamma,\beta),(\alpha+\gamma,\delta),(\beta+\delta,\alpha),(\beta+\delta,\gamma). \tag{111}$$

The corresponding restrictions $\psi \in \mathrm{End}_{\mathbb{B}\mathrm{mod}}(Q^\sigma)$ are given by applying the projection $p(u) = u+\sigma(u)$ to each term of the list and one gets, with $\tau_1 = \alpha+\gamma$, $\tau_2 = \beta+\delta$ the list $(\tau_1,\tau_2),(\tau_1,\tau_2)$, (τ_2,τ_1), (τ_2,τ_1),(τ_1,τ_2), (τ_1,τ_2), (τ_2,τ_1), (τ_2,τ_1), (τ_1,τ_2), (τ_1,τ_2), (τ_2,τ_1), (τ_2,τ_1) which gives in all cases an automorphism of Q^σ.

(ii) Let $v \in \mathrm{End}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(K)$ admit more than one extension to $\mathbb{B}^2\times\mathbb{B}^2$. Then by Lemma 9.9 the restriction of v to K^σ fails to be surjective. For any extension w of v to $\mathbb{B}^2\times\mathbb{B}^2$ the restriction to the null elements is the same and this also holds for the induced morphisms w'' on Q as follows from Proposition 6.20. Thus (i) shows that the induced morphism $w'' \in \mathrm{End}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(Q)$ fulfills $w'' \in \overline{\mathrm{Im}}(F(\alpha''))$. $\square$

Theorem 9.12. *The short doubly exact sequence $\alpha : K \overset{\ker(\mathrm{cok}(\mathfrak{s}\Delta))}{\rightarrowtail} \mathbb{B}^2 \times \mathbb{B}^2 \overset{\mathrm{cok}(\mathfrak{s}\Delta)}{\twoheadrightarrow} Q$ satisfies condition (a) with respect to the functor F and all endomorphisms of $K \simeq Q^*$.*

Proof. By Lemma 8.6 it is enough to show that if $v \in \mathrm{End}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(K)$ admits more than one extension to $\mathbb{B}^2\times\mathbb{B}^2$ then for any extension w of v to $\mathbb{B}^2\times\mathbb{B}^2$ the action of w'' by left multiplication on $\mathrm{Coker}(F(\alpha''))$ is null. By (ii) of Lemma 9.11 the restriction to null elements of w'' fails to be surjective and thus the same holds for any $w''\circ u$ $\forall u \in \mathrm{End}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(Q)$ which shows that the range of left multiplication by w'' is contained in $\overline{\mathrm{Im}}(F(\alpha''))$ and is hence null in $\mathrm{Coker}(F(\alpha''))$. $\square$

Remark 9.13. One may also check directly that none of the w'' in the long list of the multiple values for the correspondence (108) is one of the twelve elements of the complement of $\overline{\mathrm{Im}}(F(\alpha''))$. One then applies Lemma 9.10 to conclude using the commutation of right multiplications by elements of $\mathrm{End}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(Q)$ to reduce the verification that the left multiplication by multiple values is always null to its verification on $\mathrm{Id} \in \mathrm{End}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(Q)$.

Remark 9.14. Note that $\mathrm{Coker}(F(\alpha''))$ (see Figure 14) fulfills the condition of Corollary 7.12. Indeed, consider a non-trivial fiber of p, such as $p^{-1}((\alpha+\gamma, \beta+\delta))$ and let $N = \{0\} \cup p^{-1}((\alpha+\gamma, \beta+\delta))$. Then N has 4 maximal ideals, which correspond to the tuples involving each three different letters, and the remaining ideals are obtained by pairwise intersections of the maximal ones. The fact that the dual of $\mathrm{Coker}(F(\alpha''))$ is not generated by its minimal elements is obtained by considering the 4 maximal elements (see Figure 14) of this $\mathbb{B}$-semimodule and observing that all of them are fixed under σ. This shows that the same holds for any element obtained as the top element of the intersection of the corresponding hereditary subsemimodules. Equivalently the subsemimodule of the dual of $\mathrm{Coker}(F(\alpha''))$ which is generated by the minimal elements is null.

Let $Q, K \simeq Q^*$ be as in Theorem 9.12.

Theorem 9.15. *The satellite functor SF of the functor $F := \underline{\mathrm{Hom}}_{\mathbb{B}\mathrm{mod}^{\mathfrak{s}}}(Q, -)$ is non-null and $SF(K)$ is the cokernel $\mathrm{Coker}(F(\alpha''))$.*

Proof. The proof is identical to the proof of Theorem 8.11, with the role of Theorem 8.7 replaced here by Theorem 9.12. In the same way $\mathrm{Sh}_{\mathrm{small}}$ is any small subcategory of the category $\mathrm{Sh}(\mathbb{B}\mathrm{mod}^{\mathfrak{s}})$ of short doubly exact sequences in the homological category $\mathbb{B}\mathrm{mod}^{\mathfrak{s}}$, with $\mathrm{Sh}_{\mathrm{small}}$ large enough to contain all short doubly exact sequences of finite objects. $\square$

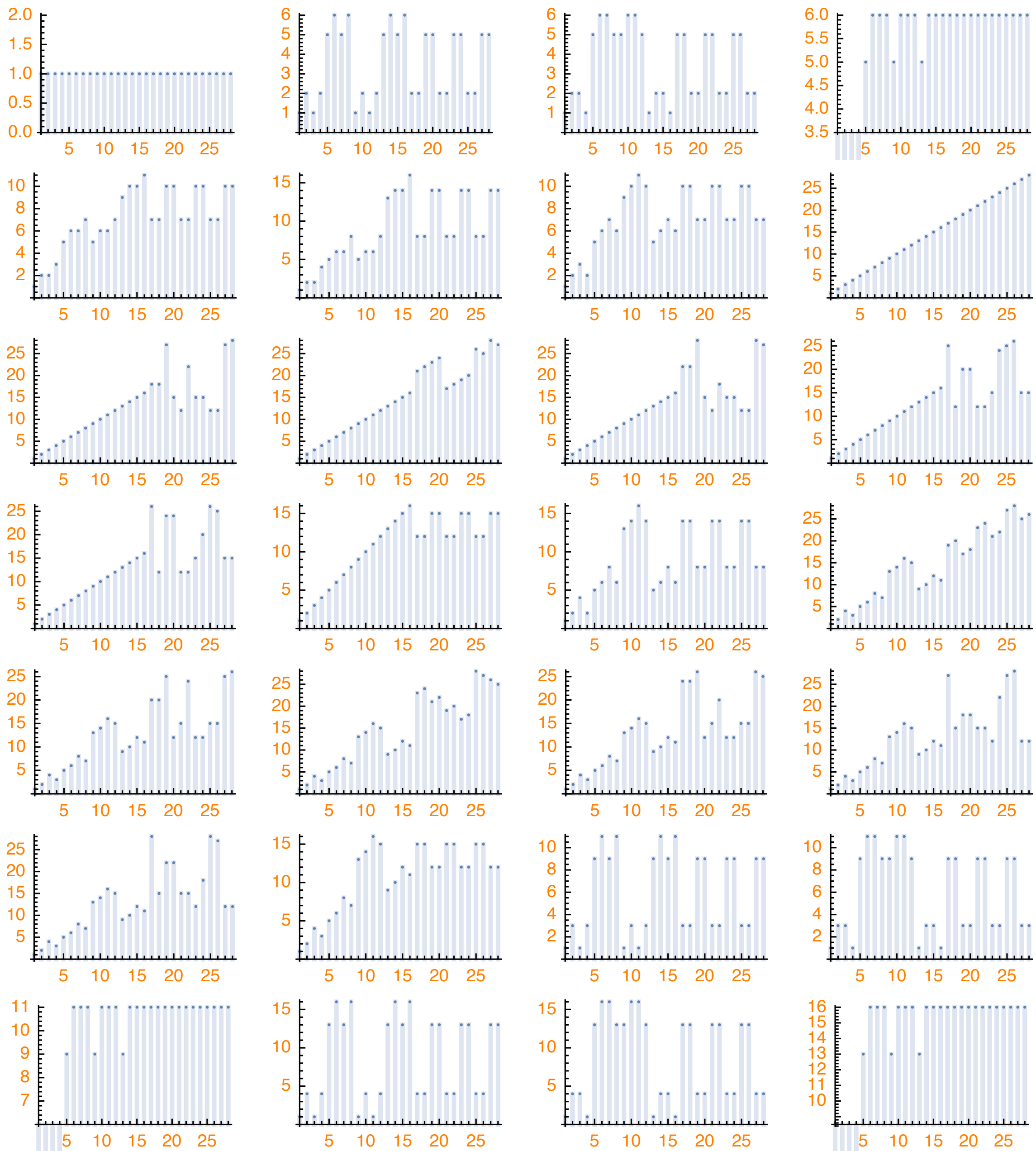

Figure 15: Graphs of the 28 transformations of the cokernel induced by endomorphisms of the short doubly exact sequence α. The elements of the cokernel labeled in $\{1, 2, \ldots, 16\}$ are the null elements. Each transformation maps null elements to null elements. There are 16 null transformations, *i.e.* those whose range is formed of null elements.

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References

[1] M. Akian, S. Gaubert, A. Guterman, *Tropical Cramer determinants revisited.* (English summary) Tropical and idempotent mathematics and applications, 1–45, Contemp. Math., **616**, Amer. Math. Soc., Providence, RI, 2014.

[2] R. M. Amadio, P-L. Curien, *Domains and lambda-calculi.* Cambridge Tracts in Theoretical Computer Science, **46**. Cambridge University Press, Cambridge, 1998.

[3] M. Barr, C. Wells, *Toposes, triples and theories.* Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], **278**. Springer-Verlag, New York, 1985.

[4] N. Bourbaki, *Topological vector spaces.* Chapter I–V. Springer

[5] T. Buhler *Exact categories*, Expo. Math. **28** (2010), no. 1, 1–69.

[6] H. Cartan, S. Eilenberg *Homological algebra.* Princeton University Press, Princeton, N. J., 1956.

[7] G. Cohen, S. Gaubert, J.P. Quadrat, *Duality and separation theorems in idempotent semimodules.* Tenth Conference of the International Linear Algebra Society. Linear Algebra Appl. **379** (2004), 395–422.

[8] A. Connes, C. Consani, *The Arithmetic Site*, Comptes Rendus Mathematique Ser. I **352** (2014), 971–975.

[9] A. Connes, C. Consani, *Geometry of the Arithmetic Site*, Advances in Mathematics **291** (2016) 274–329.

[10] A. Connes, C. Consani, *The Scaling Site*, C.R. Mathematique, Ser. I **354** (2016) 1–6.

[11] A. Connes, C. Consani, *Geometry of the Scaling Site*, Selecta Math. (N.S.) **23** (2017), no. 3, 1803–1850.

[12] B. A. Davey, H. A. Priestley, *Introduction to Lattices and Order* (second ed.). Cambridge University Press. (2002).

[13] C. Ehresmann, *Sur une notion générale de cohomologie*, C. R. Acad. Sci. Paris **259** (1964) 2050–2053.

[14] P. Gabriel, F. Ulmer, *Lokal Praesentierbare Kategorien*, Springer Lecture Notes in Mathematics **221**, Berlin, 1971

[15] S. Gaubert, *Théorie des systèmes linéaires dans les diodes.* Thèse, École des mines de Paris, (1992).

[16] G. Gierz, K. Hofmann, K. Keimel, J. Lawson, M. Mislove, D. Scott, *Continuous lattices and domains.* Encyclopedia of Mathematics and its Applications, **93**. Cambridge University Press, Cambridge, 2003.

[17] M. Grandis, *A Categorical Approach to Exactness in Algebraic Topology*, V International Meeting on Topology in Italy (Italian) (Lecce, 1990/Otranto, 1990). Rend. Circ. Mat. Palermo (2) Suppl. No. **29** (1992), 179–213.

[18] M. Grandis, *Homological algebra in strongly non-abelian settings.* World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2013. xii+343 pp.

[19] A. Grothendieck *Sur quelques points d'algèbre homologique*, I. Tohoku Math. J. (2) **9** (1957), no. 2, 119–221

[20] R. Lavendhomme, La notion d'idéal dans la théorie des catégories, Ann. Soc. Sci. Bruxelles, Sér. 1 **79** (1965) 5–25.

[21] S. Mac Lane, *Categories for the working mathematician.* Second edition. Graduate Texts in Mathematics, **5**. Springer-Verlag, New York, 1998.

[22] S. Mac Lane, I Moerdijk, *Sheaves in geometry and logic. A first introduction to topos theory.* Corrected reprint of the 1992 edition. Universitext. Springer-Verlag, New York, 1994.

[23] P. Moller. *Théorie algébrique des Systèmes à évènements Discrets.* Thèse, Ecole des Mines de Paris, 1988.

[24] A. Patchkoria *Extensions of semimodules by monoids and their cohomological characterization*, Bull. Georgian Acad. Sci. **86**, No.1 (1977), 21–24 (in Russian).

[25] A. Patchkoria *Cohomology of monoids with coefficients in semimodules*, Bull. Georgian Acad. Sci. **86**, No.3 (1977), 545–548 (in Russian).

[26] A. Patchkoria *On monoid cohomology*, Proc. A.Razmadze Math. Inst. **91** (1988), 36–43 (in Russian).

[27] L. Rowen, *Algebras with a negation map* arXiv:1602.00353

[28] C. Weibel, *An introduction to homological algebra.* Cambridge Studies in Advanced Mathematics, **38**. Cambridge University Press, Cambridge, 1994.