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An explicit symmetric DGLA model of a triangle

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Abstract

We give explicit formulae for a differential graded Lie algebra (DGLA) model of the triangle which is symmetric under the geometric symmetries of the cell. This follows the work of Lawrence-Sullivan on the (unique) DGLA model of the interval and of Gadish-Griniasty-Lawrence on an explicit symmetric model of the bi-gon. As in the case of the bi-gon, the essential intermediate step is the construction of a symmetric point. Although in this warped geometry of points given by solutions of the Maurer-Cartan equation and lines given by a gauge transformation by Lie algebra elements of grading zero, the medians of a triangle are not concurrent, various other geometric constructions can be carried out. The construction can similarly be applied to give symmetric models of arbitrary k-gons.

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1. Introduction

For a regular cell complex X, it is possible to associate a DGLA model $A = A(X)$ over $\mathbb{Q}$ satisfying the following conditions

- (i) as a Lie algebra, $A(X)$ is freely generated by a set of generators, one for each cell in X and whose grading is one less than the geometric degree of the cell;
- (ii) vertices (that is 0-cells) in X give rise to generators a which satisfy the Maurer-Cartan equation $\partial a + \frac{1}{2}[a, a] = 0$ (a flatness condition);
- (iii) for a cell x in X, the part of ∂x without Lie brackets is the geometric boundary $\partial_0 x$ (where an orientation must be fixed on each cell);
- (iv) (locality) for a cell x in X, ∂x lies in the Lie algebra generated by the generators of $A(X)$ associated with cells of the closure $\bar{x}$.

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The existence and general construction of such a model was demonstrated by Sullivan in the appendix to [9]. By [1], there exist consistent (even symmetric) towers of models of simplices, and such towers are unique up to (exact) DGLA isomorphism. The model of an interval is unique [6]. In [5], an explicit symmetric model of the bi-gon (exhibiting the dihedral symmetry of the bi-gon) was given, the main intermediate step being the construction of a 'symmetric point' in the model of the boundary of the bi-gon, invariant under the full symmetries of the bi-gon. The main theorem of this paper is Theorem 3.8, which provides an explicit construction of a model of a single triangle (one 2-cell, three 1-cells and three 0-cells) which is invariant under the action of the symmetry group S_3 of the triangle.

While the inspiration for the construction of such models came from rational homotopy theory ([7], [8]), their application may be to diverse fields where such infinity structures enter, from deformation theory to discretisation of differential equations, to be discussed in future work.

In this section we collect some general facts about DGLAs and models of cell complexes (see [6]). In section 2, we focus on the triangle and its boundary giving some asymmetric models of a triangle as well as conditions on data from which a symmetric model may be constructed, the main element being what we call a 'symmetric point'[1]. In section 3, we complete the construction by showing how to construct such symmetric data, while in section 4 we show how a similar procedure can be applied for an n-gon, $n > 3$.

General DGLAs. For simplicity we will work over $k = \mathbb{Q}$, though the discussion also holds for any field of characteristic zero. Recall that a DGLA over k is a vector space A over k with $\mathbb{Z}$-grading $A = \oplus_{n\in\mathbb{Z}} A_n$ along with a bilinear map $[.,.]\colon A \times A \longrightarrow A$ (bracket, respecting the grading) and a linear map $\partial\colon A \longrightarrow A$ (differential, grading shift -1) for which $\partial^2 = 0$ while

- *symmetry of bracket*: $[b,a] = -(-1)^{|a||b|}[a,b]$;
- *Jacobi identity*: $(-1)^{|a||b|}[[b,c],a] + (-1)^{|b||c|}[[c,a],b] + (-1)^{|c||a|}[[a,b],c] = 0$;
- *Leibnitz rule*: $\partial[a,b] = [\partial a, b] + (-1)^{|a|}[a, \partial b]$;

for all homogeneous $a, b, c \in A$. Defining the *adjoint* action of A on itself by $\mathrm{ad}_e(a) = [e,a]$, the operator $\mathrm{ad}_e : A \longrightarrow A$ has grading shift $|e|$, for homogeneous $e \in A$. The Jacobi identity and Leibnitz rule can now be reformulated as operator equalities

- *Jacobi identity:* $\mathrm{ad}_{[a,b]} = [\mathrm{ad}_a, \mathrm{ad}_b]$;
- *Leibniz rule:* $\mathrm{ad}_{\partial a} = [\partial, \mathrm{ad}_a]$;

in terms of the graded operator commutator, $[A,B] \equiv A \circ B - (-1)^{|A||B|} B \circ A$. Since the relations all preserve the number of brackets, it is meaningful to define an additional grading by the number of (lie) brackets; in particular, for $x \in A$, let $x^{[m]}$ denote the part of x containing precisely m brackets.

Points and localisation. An element $a \in A_{-1}$ is called a *point* (or said to be *flat*) in the model, if it satisfies the Maurer-Cartan equation $\partial a + \frac{1}{2}[a,a] = 0$. For any point $a \in A_{-1}$, define the *twisted differential* ∂_a by $\partial_a \equiv d + \mathrm{ad}_a$; the fact that $\partial_a^2 = 0$ is guaranteed by the Maurer-Cartan condition. By the *localisation* of A to a point a, denoted $A(a)$, we will mean the DGLA which as a graded Lie algebra is

$$(\ker \partial_a|_{A_0}) \oplus \bigoplus_{n>0} A_n$$

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[1] For an independent non-constructive existence proof of a symmetric point in a triangle see [2], independent and simultaneous work.

with the induced bracket from A and the differential ∂_a. This contains only non-negative gradings. Leibnitz guarantees that $\ker \partial_a|_{A_0}$ is closed under Lie bracket.

Edges and flows. Any element $e \in A_0$ defines a *flow* on A by

$$\frac{dx}{dt} = \partial e - \mathrm{ad}_e(x) \quad \text{on} \quad A_{-1}\,, \qquad \frac{dx}{dt} = -\mathrm{ad}_e(x) \quad \text{on} \quad A_{\neq -1}\,. \tag{1}$$

This flow is called the *flow by* e, and preserves the grading. (To define this rigorously, one may work in a space quotiented by all expressions involving $N+1$ Lie brackets, as in [6], effectively truncating to the space of linear combinations of terms involving at most N Lie brackets, whose coefficients are polynomials in t with rational coefficients. Then one considers the tower of spaces as N increases. Equivalently, one may choose a basis for the finite-dimensional space of expressions involving exactly N Lie brackets and then allowed expressions are formal combinations of these basis elements, over all N, with coefficients which are polynomials in t. While we talk about functions of t and their derivatives, these are well-defined for rational t, with derivatives being well-defined since all the coefficients are polynomial functions of t.)

Lemma 1.1. *For any $e \in A_0$, the flow by e in grading -1 preserves flatness. That is, if $x(t) \in A_{-1}$ satisfies (1) with initial condition $x(0)$ satisfying the Maurer-Cartan condition, then at any (rational) time t, also $x(t)$ satisfies Maurer-Cartan.*

Proof. As in the proof of Theorem 1 in [6], consider the curvature $f(t) \in A_{-2}$ defined by $f \equiv \partial x + \frac{1}{2}[x,x]$. It satisfies

$$\begin{aligned}\frac{df}{dt} &= \partial\frac{dx}{dt} + \left[x, \frac{dx}{dt}\right] = -\partial(\mathrm{ad}_e x) + [x, \partial e] - [x, \mathrm{ad}_e(x)]\\ &= -\partial \circ \mathrm{ad}_e(x) + ad_{\partial e}(x) + (\mathrm{ad}_x)^2 e = -\mathrm{ad}_e \circ \partial(x) + \tfrac{1}{2} ad_{[x,x]} e = -ad_e f\,,\end{aligned}$$

a first order homogeneous linear ode for $f(t)$ with initial condition $f(0) = 0$, since $x(0)$ satisfies the Maurer-Cartan condition. Thus $f(t) = 0$ for all t, as required. □

Linearity of the differential equations (1) in e, ensures that flowing by e for time t is equivalent to flowing by te for a unit time. Denote the result of flowing by e from $a \in A_{-1}$ for unit time, by $u_e(a)$, so that the solution of the first equation in (1) is $x(t) = u_{te}(x(0))$. Explicitly

$$u_e(a) = e^{-\mathrm{ad}_e} a + \frac{1 - e^{-\mathrm{ad}_e}}{\mathrm{ad}_e} \partial e\,,$$

where the meaning of the second term on the right hand side is the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!} (\mathrm{ad}_e)^{n-1} (\partial e)$.

Lemma 1.2. *For a point a, the condition that $u_e(a) = a$ is equivalent to $\partial_a e = 0$, that is $e \in A(a)_0$ (e is localised at a). This is a linear condition on e and therefore in this case the flow by e fixes a at all time (not only after unit time).*

Lemma 1.3. *(see [5], Lemma 2.2) If e flows from a point a to a point b in unit time, then $\partial_b \circ \exp(-\mathrm{ad}_e) = \exp(-\mathrm{ad}_e) \circ \partial_a$ so that $\exp(-\mathrm{ad}_e)$ intertwines the localisation $A(a)$ to the localisation $A(b)$.*

Example 1.4. The unique DGLA model, $A(I)$, of an interval has three generators; a, b of grading -1 (the endpoints) and e of grading 0 (the 1-cell). The differential is given by the condition $u_e(a) = b$ (see [6]). Explicitly

$$\partial e = (\mathrm{ad}_e)b + \sum_{i=0}^{\infty} \frac{B_i}{i!}(\mathrm{ad}_e)^i(b-a) = \frac{E}{1-e^E}a + \frac{E}{1-e^{-E}}b\,,$$

where $E \equiv \mathrm{ad}_e$, B_i denotes the i^{th} Bernoulli number defined as coefficients in the expansion $\frac{x}{e^x-1} = \sum\limits_{n=0}^{\infty} B_n \frac{x^n}{n!}$, and the expressions in E are considered as formal power series.

Example 1.5. In any DGLA model $A(X)$ of a regular cell complex X, for any 1-cell e in X with endpoints a, b, there is a natural DGLA homomorphism $A(I) \longrightarrow A(X)$, while $u_e(a) = b$.

Baker-Campbell-Hausdorff formula and BCH. For non-commuting indeterminates x and y, there are unique homogeneous (non-commuting) polynomials $F_n(x,y)$ of degree n, for $n \in \mathbb{N}$, such that, as formal series

$$\exp(x).\exp(y) = \exp\left(\sum_{n=1}^{\infty} F_n(x,y)\right).$$

In particular, $F_1(x,y) = x+y$ and it is a classical result that for $n > 1$, $F_n(x,y)$ lies in the free Lie algebra on the two generators x, y, that is, it can be expressed as a linear combination of iterated brackets of x, y; see [4] for a short proof. The formula for $\sum_{n=1}^{\infty} F_n(x,y)$ is known as the Baker-Campbell-Hausdorff formula, and we will denote it by $\mathrm{BCH}(x,y)$; see [3] for a computational formula.

Properties 1.6. (a) The first few terms of $\mathrm{BCH}(x,y)$ are

$$\begin{aligned}\mathrm{BCH}(x,y) &= x + y + \frac{1}{2}[x,y] + \frac{1}{12}(X^2y + Y^2x) - \frac{1}{24}XYXy \\ &-\frac{1}{720}(X^4y + Y^4x) + \frac{1}{120}(X^2Y^2x + Y^2X^2y) + \frac{1}{360}(XY^3x + YX^3y) + \cdots\end{aligned}$$

where X, Y denote $\mathrm{ad}_x, \mathrm{ad}_y$.

(b) The formula is universal and thus also applies to the operators ad_x, ad_y for $x, y \in A$, in any Lie algebra A. By the Jacobi identity, $\mathrm{BCH}(\mathrm{ad}_x, \mathrm{ad}_y) = \mathrm{ad}_{\mathrm{BCH}(x,y)}$ and so in $\mathrm{Aut}(A)$, $(\exp \mathrm{ad}_x) \circ (\exp \mathrm{ad}_y) = \exp \mathrm{ad}_{\mathrm{BCH}(x,y)}$.

(c) Uniqueness implies associativity of BCH, that is $\mathrm{BCH}\big(\mathrm{BCH}(x,y),z\big) = \mathrm{BCH}\big(x, \mathrm{BCH}(y,z)\big)$ for any symbols x, y, z. Denote the combined BCH of n symbols $x_1, \dots x_n \in A$ by $\mathrm{BCH}(x_1, \dots, x_n)$ so that

$$\exp \mathrm{BCH}(x_1, \dots, x_n) = (\exp x_1) \cdots (\exp x_n)\,,$$

in the (completed) universal enveloping algebra of A and

$$\exp \mathrm{BCH}(\mathrm{ad}_{x_1}, \dots, \mathrm{ad}_{x_n}) = (\exp \mathrm{ad}_{x_1}) \cdots (\exp \mathrm{ad}_{x_n}) \in \mathrm{Aut}(A)\,.$$

Again $\mathrm{BCH}(x_1, \dots, x_n)$ will be a formal sum of terms, the zeroth order being $x_1 + \cdots + x_n$ and higher orders being linear combinations of (repeated) Lie brackets of the x_i's.

(d) Uniqueness similarly implies that $\mathrm{BCH}(x,-x)=0$ while

$$\mathrm{BCH}(-x_1,\dots,-x_n)=-\mathrm{BCH}(x_n,\dots,x_1)\,.$$

(e) $\mathrm{BCH}(x,y,-x)=(\exp \mathrm{ad}_x)y$.

(f) $\mathrm{BCH}(\exp(\mathrm{ad}_e)x,\exp(\mathrm{ad}_e)y))=\exp(\mathrm{ad}_e)\mathrm{BCH}(x,y)$.

Lemma 1.7. *There is a homomorphism from the group A_0 considered with operation* BCH*, to the group* $\mathrm{Aut}(A)$*, defined by mapping $e\in A_0$ to the flow (in unit time) as defined on all gradings in A by equations (1).*

Proof. By [6] Lemma 3, and the explicit formula given for $u_e(a)$ above, it follows that

$$u_{e_2}\left(u_{e_1}(a)\right)=u_{\mathrm{BCH}(e_1,e_2)}(a)\;,$$

for any $a\in A_{-1}$. Thus, on elements of grading -1, a flow by e_1 for unit time followed by a flow by e_2 for unit time is equivalent to a flow by $\mathrm{BCH}(e_1,e_2)$ for unit time. Note that the flow for unit time by e acting on A_n for $n\neq-1$, is just the exponential operator $\exp(-ad_e)$ for which it is immediate that $\exp(-ad_{e_2})\circ\exp(-ad_{e_1})=\exp(-ad_{\mathrm{BCH}(e_1,e_2)})$. □

Definition 1.8. By a *piecewise linear path* γ in A, is meant a sequence of points $a_i\in A_{-1}$ $(0\le i\le m)$ along with elements $e_i\in A_0$ $(1\le i\le m)$, called *edges*, which are such that the edges define flows between the respective points, that is $u_{e_i}(a_{i-1})=a_i$ for all $1\le i\le m$. For such a path, we denote by $\mathrm{BCH}(\gamma)\in A_0$ the iterated BCH of the edges, $\mathrm{BCH}(\gamma)\equiv\mathrm{BCH}(e_1,\dots,e_m)$. A piecewise linear path in A is called a *loop* if its initial and final points agree, that is $a_0=a_m$.

Lemma 1.9. *(see [1]) If X has c connected components and $\{a_1,\dots,a_c\}$ is a choice of basepoints, one in each connected component, then the set of points in $A(X)$ is*

$$\bigcup_{i=1}^{c}\left\{u_e(a_i)\;\middle|\;e\in A_0\right\}\cup\left\{u_e(0)\;\middle|\;e\in A_0\right\}\,.$$

For each i, the map $\pi_i\colon e\mapsto u_e(a_i)$ is a 'fibration', with fibre $\pi_i^{-1}(a_i)$ generated as a vector space by $\{\mathrm{BCH}(\gamma)|\gamma\in\pi_1(X,a_i)\}$, while the map $\pi_0\colon e\mapsto u_e(0)$ is injective.

2. The triangle

Let $\bar{\Delta}$ be the triangle, with three 0-cells, three 1-cells and one 2-cell. We denote a corresponding model (DGLA) by $\bar{A}$; as a Lie algebra it will be generated freely by a,b,c (grading -1), e,f,g (grading 0) and h (grading 1) corresponding to the 0,1,2-cells respectively in $\bar{\Delta}$; see Figure 1, center.

The geometric symmetry group of the triangle, S_3, acts on $\bar{A}$ by permuting the vertices and thus also the corresponding generators a,b,c. Such a permutation induces a permutation of the edges and thus on the corresponding generators e,f,g, possibly with signs coming from changes in orientation of the edge; it is immediate that the signs of the images of all these generators are the same, namely the sign of the permutation. Finally it acts on the generator h, corresponding to the face, by mapping it to $\pm h$, the sign being that of the permutation.

By the Leibnitz rule, the differential ∂ is determined by its values on generators. On vertices, ∂ is fixed by the Maurer-Cartan condition, namely

$$\partial a=-\tfrac{1}{2}[a,a],\qquad \partial b=-\tfrac{1}{2}[b,b],\qquad \partial c=-\tfrac{1}{2}[c,c]\,.$$

On 1-cells, ∂ is also unique (see Examples 1.4 and 1.5), for example

$$\partial e = \frac{E}{1-e^{E}} b + \frac{E}{1-e^{-E}} c \,,$$

and similarly for ∂f, ∂g by permuting a, b, c cyclically. The only freedom in $\bar{A}$ is in $\partial h \in A_0$, which to give a valid model of $\bar{\Delta}$ must be such that $\partial^2(h) = 0$, while $(\partial h)^{[0]}$ must coincide with the topological boundary $\partial_0 h = e + f + g$. The purpose of this paper is to give a formula for ∂h which is symmetric under the S_3 action of the symmetries of the triangle.

Let Δ denote $\bar{\Delta}$ with the 2-cell removed, and A its corresponding model, which is unique, $A = \langle a, b, c, e, f, g \rangle \subset \bar{A}$.

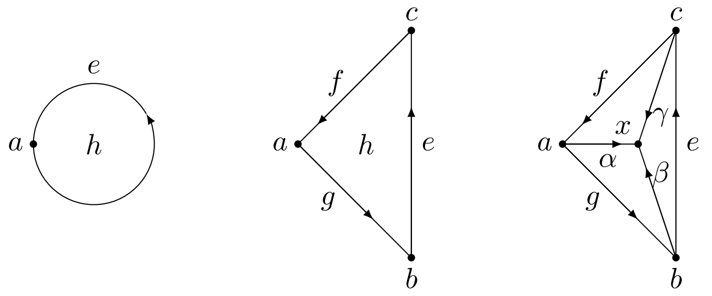

Figure 1: *Left,* The complex $\bar{X}^1$, can be sub-divided into $\bar{\Delta}$ (*center*), however the derived algebra would not be symmetric under S_3. *Right,* the symmetric model $\bar{\Delta}$ is based at the central point x, which can be connected to the vertices by α, β, γ and are permuted under S_3.

Explicit non-symmetric models of the triangle. The 2-cell with one vertex $\bar{X}^1$, has a model $\bar{A}^1$ with one generator in each degree -1,0,1, say a, e, h respectively with $\partial_0 e = 0$, $\partial_0 g = e$. The explicit model is given by

$$\partial a = -\tfrac{1}{2}[a, a] \,, \qquad \partial e = [e, a] \,, \qquad \partial h = e - [a, h] \,.$$

Equivalently, $\partial_a e = 0$ and $\partial_a g = e$. Using the functoriality of the construction $X \mapsto A(X)$ under subdivision of intervals, one obtains a model of $\bar{\Delta}$ (Figure 1) in which

$$\partial h = \mathrm{BCH}(g, e, f) - [a, h] \tag{2}$$

This is not symmetric under the symmetries of the triangle (although it is invariant under the reflection in the median from a). We could describe this model as 'based' at a, and will denote it $\bar{A}_a$. Similarly there are models based at the other vertices of the triangle

$$\begin{aligned} \bar{A}_b: &\quad \partial h = \mathrm{BCH}(e, f, g) - [b, h] \,, \\ \bar{A}_c: &\quad \partial h = \mathrm{BCH}(f, g, e) - [c, h] \,. \end{aligned}$$

The symmetries of the triangle permute a, b and c. Similarly they permute e, f, g with an added sign (the sign of the permutation). These symmetries preserve A, which was after all the unique model of the triangle boundary Δ. However they permute the three models $\bar{A}_a$, $\bar{A}_b$ and $\bar{A}_c$.

Data for symmetric triangle model. The aim of this work is to provide an explicit fully symmetric model of $\bar{\Delta}$. As in [5], this will be done by finding a symmetric point x in A (that is, a flat element of A_{-1} which is invariant under the S_3 action) and then producing a model 'based' at x, given by

$$\partial h = q - [x, h], \tag{3}$$

where $q \in A_0$. The condition $q \in \ker \partial_x|_{A_0}$ guarantees that $\partial^2 = 0$.

By Lemma 1.3, a path from a to x (in A) whose BCH is say $\alpha \in A_0$, allows an identification of $A(x)$ with $A(a)$. By Lemma 1.9, $\ker \partial_a|_{A_0}$ is one-dimensional, generated by the BCH of the one generating loop in Δ, namely $\mathrm{BCH}(g, e, f)$. Thus $\ker \partial_x|_{A_0}$ is also one-dimensional, generated by $\exp(-\mathrm{ad}_\alpha)\mathrm{BCH}(g, e, f) = \mathrm{BCH}(-\alpha, g, e, f, \alpha)$, whose zeroth order part (no Lie brackets) is $e + f + g$.

Lemma 2.1. *An element $\alpha \in A_0$ for which $x = u_\alpha(a)$ is a symmetric point in A, will generate a symmetric model of $\bar{\Delta}$ given by (3) with $q = \mathrm{BCH}(-\alpha, g, e, f, \alpha)$ so long as q is totally antisymmetric under the action of S_3.*

Proof. The only condition remaining to check is symmetry. Under the action of $\epsilon \in S_3$, h changes to $\mathrm{sgn}(\epsilon) \cdot h$ while x remains unchanged. Relation (3) thus transforms to

$$\mathrm{sgn}(\epsilon) \cdot \partial h = \epsilon(q) - [x, \mathrm{sgn}(\epsilon) \cdot h],$$

which is equivalent to (3) so long as $\epsilon(q) = \mathrm{sgn}(\epsilon) \cdot q$. □

3. Construction of symmetric data for the triangle

In this section we work exclusively in the model, A, of the triangle boundary Δ (triangle with 2-cell removed).

Flattening the triangle. For any graph, Γ, by a *realisation* of Γ in A we mean an assignment of points in A_{-1} to vertices of Γ and elements of A_0 to (oriented) edges of Γ in such a way that the relation $u_e(a) = b$ holds for every edge of Γ, where e is assigned to the edge and the vertices it connects are assigned the points a and b. A realisation will be said to be *flat*, if the BCH of any loop in the realisation vanishes, in the sense of Definition 1.8. A flat realisation of a connected graph Γ is uniquely determined by the label on one vertex, a (an arbitrary point in A) and an assignation of elements of A_0 to edges in such a way that $\mathrm{BCH}(\gamma) = 0$ for all loops γ in Γ based at that vertex (it suffices to check that this holds for a collection of generators γ of $\pi_1(\Gamma, a)$). For, given an edge labelling, and the label on the vertex a, the labels on the other vertices may be defined using the flows along paths from a. This is always well-defined by the flatness condition.

We will construct a symmetric point x in A, along with an element α for which $u_\alpha(a) = x$, as limits of sequences of points and BCH's of paths, respectively, on finer and finer subdivisions of the triangle. It will be important to use flat realisations on the graphs in order to establish the symmetry properties, since it allows any path in a graph to be replaced by any other path with the same endpoints, and still lead to equivalent elements in A.

Example 3.1. The graph Γ_0 with three vertices and three edges has a natural realisation in A, where a, b, c label the vertices, e, f, g label the edges, $u_e(b) = c$, $u_f(c) = a$ and $u_g(a) = b$. This is not flat, since there are no relations between e, f, g and in particular $\mathrm{BCH}(e, f, g) \neq 0$.

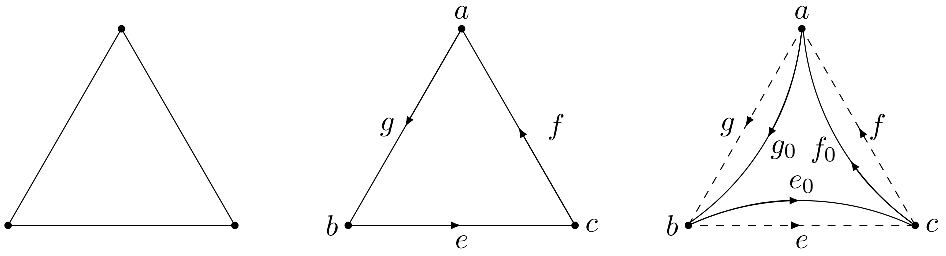

Figure 2: *Left,* The triangle graph Γ_0 has a natural (non-flat) realisation in A (*center*). *Right,* A flat realisation of the same graph in A.

Define $e_0, f_0, g_0 \in A_0$ by

$$\begin{aligned} e_0 &= \mathrm{BCH}\left(-\tfrac{1}{3}\mathrm{BCH}(e,f,g), e\right) \ , \\ f_0 &= \mathrm{BCH}\left(-\tfrac{1}{3}\mathrm{BCH}(f,g,e), f\right) \ , \\ g_0 &= \mathrm{BCH}\left(-\tfrac{1}{3}\mathrm{BCH}(g,e,f), g\right) \ . \end{aligned}$$

Lemma 3.2. $u_{e_0}(b) = c\,.$

Proof. e, f and g, in that order, compose a loop based at b, thus $u_{\mathrm{BCH}(e,f,g)}(b) = b$ and by Lemma 1.2, $\mathrm{BCH}(e,f,g) \in \ker \partial_b$. Hence also $-\frac{1}{3}\mathrm{BCH}(e,f,g) \in \ker \partial_b$ so that $u_{-\frac{1}{3}\mathrm{BCH}(e,f,g)}(b) = b$ by Lemma 1.2 again. Thus by Lemma 1.7, $u_{e_0}(b) = u_e\left(u_{-\frac{1}{3}\mathrm{BCH}(e,f,g)}(b)\right) = u_e(b) = c$, as required. ☐

Similarly, $u_{f_0}(c) = a$ and $u_{g_0}(a) = b$.

Lemma 3.3. $\mathrm{BCH}(e_0, f_0, g_0) = 0\,.$

Proof. By properties 1.6 (e),(c) of BCH,

$$\exp(-\mathrm{ad}_e)\mathrm{BCH}(e,f,g) = \mathrm{BCH}(-e, \mathrm{BCH}(e,f,g), e) = \mathrm{BCH}(f,g,e)\,.$$

Since $\exp(-ad_e)$ is linear, $\exp(-\mathrm{ad}_e)\left(-\frac{1}{3}\mathrm{BCH}(e,f,g)\right) = -\frac{1}{3}\mathrm{BCH}(f,g,e)$ and so e_0 can also be written as $e_0 = \mathrm{BCH}\left(e, -\frac{1}{3}\mathrm{BCH}(f,g,e)\right)$. Thus

$$\mathrm{BCH}(e_0, f_0) = \mathrm{BCH}\left(e, -\tfrac{2}{3}\mathrm{BCH}(f,g,e), f\right) = \mathrm{BCH}\left(e, f, -\tfrac{2}{3}\mathrm{BCH}(g,e,f)\right) \ ,$$

and combining with g_0, $\mathrm{BCH}(e_0, f_0, g_0) = \mathrm{BCH}(e, f, -\mathrm{BCH}(g,e,f), g) = 0$. ☐

Combining Lemmas 3.2 and 3.3 gives the following.

Proposition 3.4. *There is a flat realisation of* Γ_0 *in which the vertices are assigned* a, b, c *while the edges are assigned* e_0, f_0, g_0*, as in Figure 2.*

Iterative step - subdividing a flat triangle. The graph, Γ_1, obtained from Γ_0 by adding midpoints to the edges and joining the three midpoints, will have six vertices and nine edges. From any flat realisation of Γ_0, say with edges labelled by e_0, f_0, g_0, there can be constructed according to Figure 3, a flat realisation of Γ_1 in which the corners are labelled by the same points as the given realisation. To verify flatness, it suffices to verify the condition for the four generating loops around the four smaller triangles in Γ_1. Verification for the outer triangles is immediate from the definition, while for the inner triangle

$$\mathrm{BCH}\left(\mathrm{BCH}(\tfrac{1}{2}f_0, \tfrac{1}{2}g_0), \mathrm{BCH}(\tfrac{1}{2}g_0, \tfrac{1}{2}e_0), \mathrm{BCH}(\tfrac{1}{2}e_0, \tfrac{1}{2}f_0)\right) = \mathrm{BCH}(\tfrac{1}{2}f_0, g_0, e_0, \tfrac{1}{2}f_0),$$

which vanishes since $\mathrm{BCH}(e_0, f_0, g_0) = 0$, by flatness of the given realisation.

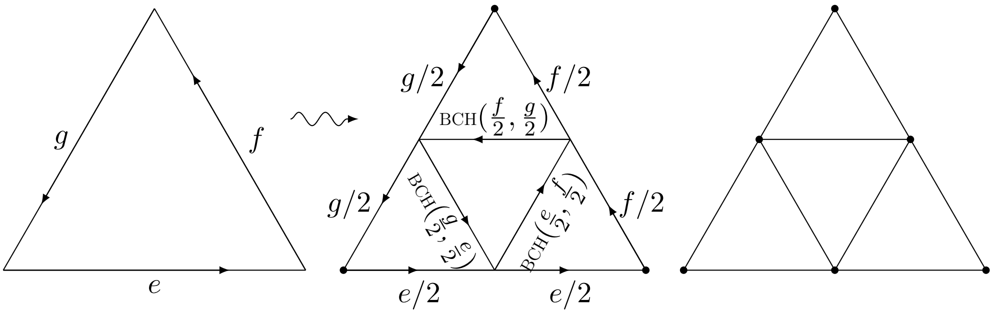

Figure 3: *Left,* A flat realisation of Γ_0 generating a flat realisation (*center*) of Γ_1 (*right*).

Iterative construction. Iteratively define e_n, f_n, g_n for non-negative integers n, starting with e_0, f_0, g_0 defined above, by

$$e_{n+1} = \mathrm{BCH}\left(\tfrac{1}{2}f_n, \tfrac{1}{2}g_n\right), f_{n+1} = \mathrm{BCH}\left(\tfrac{1}{2}g_n, \tfrac{1}{2}e_n\right), g_{n+1} = \mathrm{BCH}\left(\tfrac{1}{2}e_n, \tfrac{1}{2}f_n\right).$$

Let Γ_n be the graph obtained from Γ_0 by repeatedly subdividing the inner triangle, n times, each subdivision of the innermost triangle according as the replacement of Γ_0 by Γ_1. As in the previous paragraph, starting with a flat realisation of Γ_0, we obtain a flat realisation of Γ_n with the same labels on the corners as the original realisation, and in which the innermost triangle has edges labelled by e_n, f_n, g_n. Let a_n, b_n, c_n be the points labelling the vertices of the innermost triangle in Γ_n. In particular, $a_0 = a$, $b_0 = b$, $c_0 = c$.

Pick any path in Γ_n from a_0 to a_n and let $\alpha_n \in A_0$ denote its BCH in the realisation; this is well-defined since the realisation is flat.

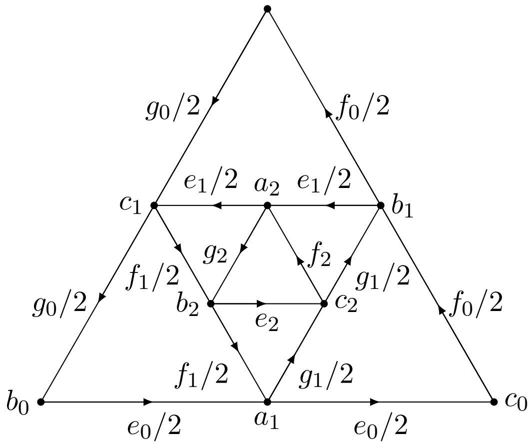

Figure 4: The constructed flat realisation of Γ_2.

Convergence

Lemma 3.5. *$e_n, f_n, g_n \longrightarrow 0$ in A_0 as $n \to \infty$. In other words, for all $m \geq 0$, $e_n^{[m]} \longrightarrow 0$ as $n \to \infty$ (and similarly for f, g).*

Proof. Applying the iterative construction above n times to the initial condition e_1, f_1, g_1 (in place of e_0, f_0, g_0) will arrive at $e_{n+1}, f_{n+1}, g_{n+1}$. Consequently $e_{n+1}, f_{n+1}, g_{n+1}$ can be obtained from e_n, f_n, g_n by the replacement $e_0 \to e_1$, $f_0 \to f_1$, $g_0 \to g_1$. Recall that $\mathrm{BCH}(e_0, f_0, g_0) = 0$ and so there is a unique Lie algebra expression for e_n as a linear combination of Lie words in e_0, f_0. Indeed, e_n, f_n can be obtained from e_0, f_0 by iterating n times the substitution

$$\begin{aligned} e_0 &\longmapsto e_1 = \mathrm{BCH}\left(\tfrac{1}{2}f_0, \tfrac{1}{2}g_0\right) = \mathrm{BCH}\left(\tfrac{1}{2}f_0, -\tfrac{1}{2}\mathrm{BCH}(e_0, f_0)\right) \ , \\ f_0 &\longmapsto f_1 = \mathrm{BCH}\left(\tfrac{1}{2}g_0, \tfrac{1}{2}e_0\right) = \mathrm{BCH}\left(-\tfrac{1}{2}\mathrm{BCH}(e_0, f_0), \tfrac{1}{2}e_0\right) \ . \end{aligned}$$

Let B be the free Lie algebra on two generators e_0, f_0, and consider it embedded in A_0 in the natural way. The above substitution induces a linear map $\tau : B \longrightarrow B$ which is non-decreasing on the number of Lie brackets and for which $e_n = \tau^n(e_0)$, $f_n = \tau^n(f_0)$. For each $m \geq 0$, choose a basis for the finite dimensional vector space $B^{[m]}$. With respect to the basis for B obtained from the union of these bases, the matrix for τ is a lower triangular (partitioned) matrix. Since $\tau(e_0)^{[0]} = -\frac{1}{2}e_0$ and $\tau(f_0)^{[0]} = -\frac{1}{2}f_0$, thus the diagonal blocks in the matrix of τ are multiples of the identity matrix with factor $(-2)^{-r}$ on the r-th block (dealing with terms with precisely $r-1$ Lie brackets). The truncated (finite-dimensional) matrix of the first $m \times m$ blocks gives the matrix of $\tau^{[<m]}$, the induced action of τ on $B/B^{[\geq m]}$. It has eigenvalues $(-2)^{-r} \in (-1, 1)$ for $1 \leq r \leq m$, and thus $\left(\tau^{[<m]}\right)^n \longrightarrow \mathbf{0}$ as $n \to \infty$ for all m. Applying this to e_0, f_0 gives $e_n^{[<m]} \longrightarrow 0$ and $f_n^{[<m]} \longrightarrow 0$ as $n \to \infty$; in other words $e_n, f_n \longrightarrow 0$ in B and hence also in A_0. Since $g_n = -\mathrm{BCH}(e_n, f_n)$, it follows from continuity of BCH that $g_n \longrightarrow 0$. □

Lemma 3.6. *The sequence (α_n) converges in A_0.*

Proof. By the flat realisation of Γ_n constructed above, it follows that

$$\alpha_{3n+1} = \mathrm{BCH}\left(\alpha_{3n}, \tfrac{1}{2}g_{3n}, f_{3n+1}\right), \quad \alpha_{3n+2} = \mathrm{BCH}\left(\alpha_{3n}, \tfrac{1}{2}g_{3n}, -\tfrac{1}{2}e_{3n+1}\right).$$

Hence by Lemma 3.5, it suffices to show that the subsequence (α_{3n}) converges. Now,

$$\alpha_{3n+3} = \mathrm{BCH}\left(\alpha_{3n}, \tfrac{1}{2}g_{3n}, \tfrac{1}{2}f_{3n+1}, \tfrac{1}{2}e_{3n+2}\right).$$

Let σ be the Lie algebra homomorphism $B \longrightarrow B$ defined on the generators by

$$\begin{aligned} e_0 &\longmapsto g_1 = \mathrm{BCH}\left(\tfrac{1}{2}e_0, \tfrac{1}{2}f_0\right) \ , \\ f_0 &\longmapsto e_1 = \mathrm{BCH}\left(\tfrac{1}{2}f_0, -\tfrac{1}{2}\mathrm{BCH}(e_0, f_0)\right) \ . \end{aligned}$$

This is the composition of τ (defined in the proof of Lemma 3.5) with a rotation. Then $\sigma(e_n) = g_{n+1}$, $\sigma(f_n) = e_{n+1}$ while $\sigma(g_n) = f_{n+1}$ and $\alpha_{3n} = \mathrm{BCH}\left(\frac{g_0}{2}, \sigma(\frac{g_0}{2}), \ldots, \sigma^{3n-1}(\frac{g_0}{2})\right)$. Thus it is enough to show that the sequence

$$\left(\mathrm{BCH}\left(\frac{g_0}{2}, \sigma(\frac{g_0}{2}), \ldots, \sigma^{n-1}(\frac{g_0}{2})\right)\right)$$

(which contains $\{\alpha_{3n}\}$ as a subsequence) converges, which we do by proving that for any natural number m its projection onto the finite-dimensional vector space $B/B^{[\geq m]}$ converges.

Matrix of σ. We use the same notation as in the proof of the previous lemma. The matrix of σ is a lower-triangular block matrix. Since $\sigma(e_0)^{[0]} = \frac{1}{2}(e_0 + f_0)$ and $\sigma(f_0)^{[0]} = -\frac{1}{2}e_0$, thus the block in the (1,1) position of the partitioned matrix for σ is

$$\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}.$$

This is diagonalisable with eigenvalues $-\frac{1}{2}\omega, -\frac{1}{2}\omega^2$ where ω is a cube-root of unity. Choose a basis for $B^{[0]}$ which diagonalises the (1,1) block of σ there.

Diagonal blocks of σ. Let σ' denote the linear map $B \longrightarrow B$ defined as the Lie algebra homomorphism defined on the generators by

$$\begin{aligned} e_0 &\longmapsto \tfrac{1}{2}(e_0 + f_0)\ , \\ f_0 &\longmapsto -\tfrac{1}{2}e_0\ . \end{aligned}$$

The matrix of σ' will be block diagonal and these diagonal blocks will agree with those in σ. Via the map

$$\begin{aligned} \left(B^{[0]}\right)^{\otimes r} &\longrightarrow B^{[r-1]}, \\ v_1 \otimes \cdots \otimes v_r &\longmapsto [v_1, \ldots, [v_{r-1}, v_r] \ldots], \end{aligned}$$

we can consider $B^{[r-1]}$ as a quotient of $(B^{[0]})^{\otimes r}$ by the ideal I_r generated by Jacobi relations. The action of σ' on $B^{[0]}$ induces one on $(B^{[0]})^{\otimes r}$ preserving I_r and the action on the quotient is precisely the action of σ' on $B^{[r-1]}$, described by the (r,r) block in the matrix of σ' (or of σ). By the previous paragraph, $\sigma'|_{B^{[0]}}$ is diagonalisable with eigenvalues $-\frac{1}{2}\omega, -\frac{1}{2}\omega^2$ and thus the induced action on $(B^{[0]})^{\otimes r}$ is also diagonalisable with eigenvalues which all have absolute value 2^{-r}. The (r,r) block of the matrix for σ is a quotient of this and thus also diagonalisable with eigenvalues which all have absolute value 2^{-r}.

Bound on matrix entries in powers of $\sigma^{[<m]}$. Fix m. We consider only the induced actions on $B/B^{[\geq m]}$, that is the first $m \times m$ blocks in the matrix representations; let $\sigma^{[<m]}$ denote this induced action from σ. Choose a basis for $B^{[r-1]}$ which diagonalises the (r,r) block in σ for $1 \leq r \leq m$. Let C be the absolute value of the largest matrix entry in $\sigma^{[<m]}$. Let $d_r = \dim B^{[r-1]}$ be the size of the r-th block.

For any natural number n, the matrix for σ^n will be a lower triangular block matrix, because σ is a lower triangular block matrix; the diagonal blocks will be diagonal and the entries will have absolute values 2^{-rn} in the (r,r) block. The (a,b) entry in the (i,j) block $(i > j)$ of σ^n is

$$\sum_{i \geq i_1 \geq \cdots \geq i_{n-1} \geq j} \sum_{p_1=1}^{d_{i_1}} \cdots \sum_{p_{n-1}=1}^{d_{i_{n-1}}} (\sigma_{ii_1})_{ap_1} (\sigma_{i_1i_2})_{p_1p_2} \ldots (\sigma_{i_{n-1}j})_{p_{n-1}b}\ ,$$

where σ_{ij} denotes the (i,j) block of the partitioned matrix for σ. For any $i \geq i_1 \geq \cdots \geq i_{n-1} \geq j$, let $s_1, \ldots, s_k$ denote the points at which steps occur, that is those s ($1 \leq s \leq n$, in increasing order) for which $i_{s-1} > i_s$ (counting $i_0 \equiv i$ and $i_n \equiv j$). In particular, $i_{s_1-1} = i$ while $i_{s_k} = j$. The maximum number of steps k is $i - j$. For a particular sequence of steps (that is, where they occur $s_1, \ldots, s_k$ and what are their values $j_1 \equiv i_{s_1}, \ldots, j_{k-1} \equiv i_{s_{k-1}}$), the contribution to the above sum is bounded by

$$(2^{-i})^{s_1-1} C (2^{-j_1})^{s_2-s_1-1} C \cdots (2^{-j_{k-1}})^{s_k-s_{k-1}-1} C (2^{-j})^{n-s_k} \cdot d_{j_1} \cdots d_{j_{k-1}}\ ,$$

since σ_{ii} is diagonal. For fixed $k \leq i-j$ and $j_1, \ldots, j_{k-1}$,

$$\sum_{1 \leq s_1 < \cdots < s_k \leq n} (2^{-i})^{s_1-1}(2^{-j_1})^{s_2-s_1-1} \cdots (2^{-j})^{n-s_k} \leq \binom{n}{k} (2^{-j})^{n-k-1}\,.$$

So, if $d = \max\{d_1, \ldots, d_m\}$, an arbitrary entry in the (i,j) block of σ^n is bounded by

$$\sum_{k=1}^{i-j} C^k d^{k-1} \binom{i-j-1}{k-1} \binom{n}{k} (2^{-j})^{n-k-1}\,.$$

Since $i-j \leq m-1$ and $j \geq 1$, this bound is at most 2^{-n} times a polynomial in n of degree at most $m-1$ and hence all matrix entries in $(\sigma^{[<m]})^n$ can be bounded by $C'(2/3)^n$ for some C' (dependent on m).

Bound on coordinates of $v_n \equiv \left(\sigma^n(\frac{1}{2}g_0)\right)^{[<m]}$. As above,

$$\left(\sigma^n\left(\tfrac{1}{2}g_0\right)\right)^{[<m]} = (\sigma^{[<m]})^n \left(\tfrac{1}{2}g_0^{[<m]}\right),$$

which we denote by $v_n \in B^{[<m]}$. The matrix elements in the power of $\sigma^{[<m]}$ are all bounded by a multiple of $(2/3)^n$ while the vector $g_0^{[<m]}$ is constant. Thus, in any chosen basis for $B^{[<m]}$, v_n has all coordinates (and thus also their sum) bounded by a constant (dependent on m) times $(2/3)^n$.

Coefficients in BCH. From now onwards we will revert to a basis for $B^{[r]}$ in which the basis elements are (a subset of) Lie monomials in e_0, f_0 with r brackets. The formula for $\mathrm{BCH}(x,y)$ is an element of the (completed) free Lie algebra on x and y. Since $g_0 = -\mathrm{BCH}(e_0, f_0)$, the coefficients in the formula are given precisely by the coordinates of $-g_0$ with respect to the chosen basis. Denote these coefficients $h_j^{[r]} \in \mathbf{Q}$, so that

$$\mathrm{BCH}(e_0, f_0) = \sum_{r=0}^{\infty} \sum_{j=1}^{d^{[r]}} h_j^{[r]} \mathbf{e}_j^{[r]},$$

where $\mathbf{e}_j^{[r]}$ is the j-th basis vector in $B^{[r]}$ and $d^{[r]} \equiv d_{r+1}$ is the dimension of $B^{[r]}$. For example, $d^{[0]} = 2$, take $\mathbf{e}_1^{[0]} = e_0$, $\mathbf{e}_2^{[0]} = f_0$ as basis for $B^{[0]}$, and then $h_1^{[0]} = h_2^{[0]} = 1$. Similarly $d^{[1]} = 1$, $\mathbf{e}_1^{[1]} = [e_0, f_0]$ and $h_1^{[1]} = \frac{1}{2}$. For second order brackets, $d^{[2]} = 2$, use $\mathbf{e}_1^{[2]} = [e_0, [e_0, f_0]]$, $\mathbf{e}_2^{[2]} = [f_0, [e_0 . f_0]]$ and then $h_1^{[2]} = -h_2^{[2]} = \frac{1}{12}$.

Bound on growth of BCH. Since BCH is non-decreasing on the number of Lie brackets, it induces a well-defined (associative) binary operation on $B/B^{[\geq m]}$. Define a metric on $B/B^{[\geq m]}$ by

$$\left\| \sum_{r=0}^{\infty} \sum_{j=1}^{d^{[r]}} a_j^{[r]} \mathbf{e}_j^{[r]} \right\| = \sum_{r=0}^{m-1} \sum_{j=1}^{d^{[r]}} |a_j^{[r]}|.$$

Let D denote the maximum norm of all Lie monomials in e_0, f_0 with at most $m-1$ brackets. For $a \in B$, denote by $a^{[r]} \in B^{[r]}$ the part of a with r Lie brackets. Then for any $a, b \in B$,

$$(\mathrm{BCH}(a,b))^{[r]} = \sum_{i=0}^{r} \sum_{j=1}^{d^{[i]}} h_j^{[i]} \left(\mathbf{e}_j^{[i]}(a,b)\right)^{[r]},$$

where $\mathbf{e}(a,b)$ is the result of substituting a, b in place of e_0, f_0 in the Lie monomial $\mathbf{e} \in B$. For example

$$\begin{aligned}
&\mathrm{BCH}(a,b)^{[0]} = a^{[0]} + b^{[0]}\,, \\
&\mathrm{BCH}(a,b)^{[1]} = a^{[1]} + b^{[1]} + \tfrac{1}{2}[a^{[0]}, b^{[0]}]\,, \\
&\mathrm{BCH}(a,b)^{[2]} = a^{[2]} + b^{[2]} + \tfrac{1}{2}[a^{[0]}, b^{[1]}] + \tfrac{1}{2}[a^{[1]}, b^{[0]}] + \tfrac{1}{12}[a^{[0]}, [a^{[0]}, b^{[0]}]] - \tfrac{1}{12}[b^{[0]}, [a^{[0]}, b^{[0]}]]\,.
\end{aligned}$$

But for any monomial $\mathbf{e} \in B$ involving k times e_0 and l times f_0 $(k, l > 0)$,

$$||\mathbf{e}(a,b)^{[<m]}|| \leq D||a||^k ||b||^l\,,$$

since substituting monomials for e_0, f_0 in a monomial will produce another monomial, which will have norm at most D. Thus there exist homogeneous polynomials p_r in two variables, of degree $r+1$, such that for all a, b,

$$||\mathrm{BCH}(a,b) - a - b|| \leq \sum_{r=1}^{m-1} p_r(||a||, ||b||),$$

with $p_1(x,y) = Dxy/2$, $p_2(x,y) = Dxy(x+y)/12$ and furthermore $p_r(x,y)$ is divisible by xy for all r. So in particular,

$$||\mathrm{BCH}(a,b) - a|| \leq ||b|| Q(||a||, ||b||),$$

for a suitable polynomial Q in two variables of degree $m-1$.

BCH-*Cauchy.* By the previous paragraphs, we have a sequence of vectors $v_n \in B/B^{[\geq m]}$ satisfying $||v_n|| \leq D(2/3)^n$ for all n (some constant D) and the proof of the lemma will be complete once it is shown that the sequence

$$\left(\mathrm{BCH}(v_0, v_1, \ldots, v_{n-1})^{[<m]}\right)$$

converges in $B^{[<m]}$. Let X denote the maximum value of $Q(x,y)$ when $0 \leq x, y \leq D$. By the previous paragraph,

$$||\mathrm{BCH}(a,b) - a|| \leq X||b|| \quad \text{whenever} \quad ||a||, ||b|| \leq D\,.$$

Choose N sufficiently large that $(3/2)^N \geq 1 + 2X$.

Fact. For arbitrary $k \geq N$, $||\mathrm{BCH}(v_k, \ldots, v_i)|| \leq D$ for any $i \geq k$.

Proof. By induction on i. When $i = k$, the statement holds as $||\mathbf{v}_k|| \leq D(2/3)^k \leq D$. Assuming it holds for all $k \leq i < n$, then

$$||\mathrm{BCH}(v_k, \ldots, v_i, v_{i+1}) - \mathrm{BCH}(v_k, \ldots, v_i)|| \leq X||v_{i+1}|| \leq DX(2/3)^{i+1}.$$

Combining with the triangle inequality for $i = k, k+1, \ldots, n-1$,

$$||\mathrm{BCH}(v_k, \ldots, v_n)|| \leq ||v_k|| + \sum_{i=k}^{n-1} DX(2/3)^{i+1} \leq D(2/3)^k(1 + 2X) \leq D(2/3)^{k-N},$$

which is at most D, proving the inductive step. □

Thus, for any $n \geq k \geq N$,

$$\begin{aligned}&||\mathrm{BCH}(v_N, \ldots, v_n) - \mathrm{BCH}(v_N, \ldots, v_k)||\\ &\quad = ||\mathrm{BCH}(\mathrm{BCH}(v_N, \ldots, v_k), \mathrm{BCH}(v_{k+1}, \ldots, v_n)) - \mathrm{BCH}(v_N, \ldots, v_k)||\\ &\quad \leq X||\mathrm{BCH}(v_{k+1}, \ldots, v_n)|| \leq XD(2/3)^{k+1-N},\end{aligned}$$

and therefore the sequence $\left(\mathrm{BCH}(v_N, \ldots, v_n)^{[<m]}\right)$ is a Cauchy sequence in $B^{[<m]}$ and hence converges. Since $\mathrm{BCH}^{[<m]}$ is continuous, taking a BCH of the sequence with $\mathrm{BCH}(v_0, \ldots, v_{N-1})$, will produce a convergent sequence also, namely $\left(\mathrm{BCH}(v_0, \ldots, v_n)^{[<m]}\right)$, as required. ☐

Denote the limit of the sequence $\{\alpha_n\}$ by α. Set $x = u_\alpha(a)$. Since $a_n = u_{\alpha_n}(a)$, it follows that $a_n \longrightarrow x$.

Symmetry. The symmetry group S_3 of the triangle permutes the vertices a, b, c and the edges (with signs) e, f, g. By construction, e_0, f_0, g_0 will be identically permuted (with signs) as e, f, g and the symmetry of the iterative step guarantees that this holds also for e_n, f_n, g_n for all n and finally that a_n, b_n, c_n will be permuted amongst themselves, and similarly for $\alpha_n, \beta_n, \gamma_n$.

Since $b_n = u_{g_n}(a_n)$, $g_n \to 0$ (Lemma 3.5) and $a_n \to x$ (Lemma 3.6), it follows that $b_n \longrightarrow x$. Similarly $c_n \longrightarrow x$. Since S_3 permutes a_n, b_n, c_n, it follows that x is invariant under this action, that is, x is a symmetric point.

Similarly to Lemma 3.6, $\{\beta_n\}$ and $\{\gamma_n\}$ are convergent sequences; denote their limits by $\beta, \gamma \in A_0$. Since S_3 permutes $\alpha_n, \beta_n, \gamma_n$, it also permutes α, β, γ. Applying S_3 to the equality $u_\alpha(a) = x$ we obtain that $u_\beta(b) = u_\gamma(c) = x$.

Lemma 3.7. *A flat realisation of the tetrahedral graph T is obtained in which the outer vertices are labelled a, b, c and the outer edges e_0, f_0, g_0 with the central vertex labelled x and interior edges labelled by α, β, γ, as in Figure 5.*

Proof. The previous paragraphs suffice to show that the given labelling is a realisation of T. It remains to prove flatness of the realisation, that is, vanishing of the BCH of closed loops in T, and in particular that the BCHs of each of the three generating loops vanish. Note that $\mathrm{BCH}(g_0, \beta_n, -g_n, -\alpha_n) = 0$, since it is represented by a loop based at a on the flat realisation of Γ_n constructed above. In the limit $n \to \infty$, the equality gives $\mathrm{BCH}(g_0, \beta, -\alpha) = 0$. Similarly for the other faces. ☐

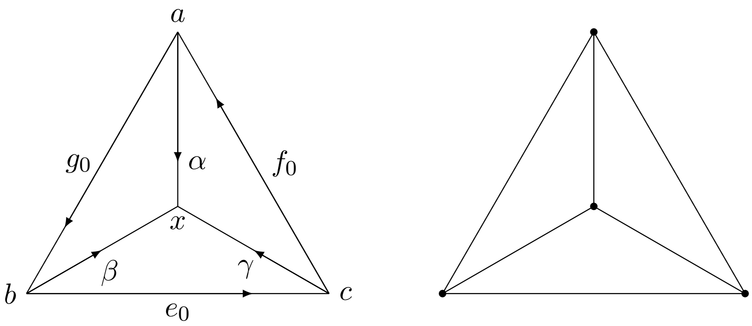

Figure 5: *Left,* The constructed flat realisation of the tetrahedron T (*right*).

Symmetric model of the triangle.

Theorem 3.8. *A symmetric model of* $\bar{\Delta}$ *is given by*

$$\partial h = \mathrm{BCH}(-\alpha, g, e, f, \alpha) - [x, h]\,.$$

Proof. It is already known that x is a symmetric point and so by Lemma 2.1, it remains only to prove that $q = \mathrm{BCH}(-\alpha, g, e, f, \alpha)$ is anti-symmetric under the S_3 action, for which it is enough to check the action under generators of S_3.

Reflection in the median through a acts by fixing a, interchanging b, c, changing the sign of e, interchanging $f, -g$. This fixes α and interchanges β, γ. This reverses the sign of $\mathrm{BCH}(g, e, f)$ and thus also of q.

Rotation cycles between a, b, c and similarly between e, f, g and between α, β, γ. Thus q transforms to $\mathrm{BCH}(-\beta, e, f, g, \beta)$. Since $\beta = \mathrm{BCH}(-g_0, \alpha)$, thus

$$\mathrm{BCH}(-\beta, e, f, g, \beta) = \mathrm{BCH}(-\alpha, g_0, e, f, g, -g_0, \alpha) = q,$$

where the last step follows, using the definition of g_0, from

$$\mathrm{BCH}(g_0, e, f, g, -g_0) = \mathrm{BCH}(-\tfrac{1}{3}\mathrm{BCH}(g, e, f), g, e, f, \tfrac{1}{3}\mathrm{BCH}(g, e, f)) = \mathrm{BCH}(g, e, f)\,. \qquad \square$$

4. Generalisations

Computations. By iteratively solving the condition $\sigma(\alpha) = \mathrm{BCH}(-g_0/2, \alpha)$ along with the requirement that β is obtained from α (and γ from β) under the rotation $e_0 \longmapsto f_0$, $f_0 \longmapsto -\mathrm{BCH}(e_0, f_0)$, one can calculate α, β, γ in terms of e_0, f_0. The result is

$$\begin{aligned}
\alpha &= -\frac{1}{3}(e_0 + 2f_0) - \frac{1}{6}[e_0, f_0] - \frac{1}{54}[e_0, [e_0, f_0]] + \frac{1}{36}[f_0, [e_0, f_0]] + \cdots\,,\\
\beta &= \frac{1}{3}(2e_0 + f_0) + \frac{1}{6}[e_0, f_0] + \frac{1}{36}[e_0, [e_0, f_0]] - \frac{1}{54}[f_0, [e_0, f_0]] + \cdots\,,\\
\gamma &= \frac{1}{3}(f_0 - e_0) - \frac{1}{108}[e_0 + f_0, [e_0, f_0]] + \cdots\,.
\end{aligned}$$

Remark 4.1. Note that α, β freely generate $B = \langle e_0, f_0\rangle$ and so γ can be written as a universal Lie word in α, β, say $\gamma = f(\alpha, \beta)$. The symmetry constraints imply that $f(\beta, \alpha) = f(\alpha, \beta)$ while $f(\alpha, f(\alpha, \beta)) = \beta$. In fact $f(\alpha, \beta) = -\alpha - \beta + \cdots$ where the first non-trivial term has at four Lie brackets:

$$\frac{17}{2^2 \cdot 3^3 \cdot 5 \cdot 11}\left(A^4\beta + B^4\alpha - A^2B^2\alpha - B^2A^2\beta + \frac{1}{2}(AB^3\alpha + BA^3\beta)\right)\,.$$

Here $A \equiv \mathrm{ad}_\alpha$ and $B \equiv \mathrm{ad}_\beta$.

***k*-gons** The arguments of this paper can be applied to any k-gon, where the iterative operation is to replace a k-gon by inscribing another k-gon joining the edge midpoints. The only slight complication is in the convergence argument. For example, for a square, τ is replaced by an automorphism of the free Lie algebra on three generators given by

$$e \longmapsto \mathrm{BCH}\left(\frac{e}{2}, \frac{f}{2}\right), \quad f \longmapsto \mathrm{BCH}\left(\frac{f}{2}, \frac{g}{2}\right), \quad g \longmapsto \mathrm{BCH}\left(\frac{g}{2}, -\frac{1}{2}\mathrm{BCH}(e, f, g)\right).$$

To zeroth order, this is $e \longmapsto \frac{1}{2}(e + f)$, $f \longmapsto \frac{1}{2}(f + g)$, $g \longmapsto -\frac{1}{2}(e + f)$ which has eigenvalues $0, \frac{1}{2}(-1 \pm i)$ which still all have absolute value less than 1.

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