H IGHER
STRUCTURES

L-infinity and A-infinity structures

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Abstract

Looking back over 55 years of higher homotopy structures, I reminisce as I recall the early days and ponder how they developed and how I now see them. From the history of A_∞-structures and later of L_∞-structures and their progeny, I hope to highlight some old results which seem not to have garnered the attention they deserve as well as some tantalizing new connections.

Dedicated to the memory of Masahiro Sugawara and John Coleman Moore

1. Introduction

Since 1931 (Dirac's magnetic monopole), but especially in the last six decades, there has been increased use of cohomological and even homotopy theoretical techniques in mathematical physics. It all began with Gauss in 1833, if not sooner with Kirchof's laws. The cohomology referred to in Gauss was that of differential forms, div, grad, curl and especially Stokes Theorem (the de Rham complex). I'll mention some of the more 'sophisticated' tools now being used.

As I tried to bring this survey reasonably up to date, one thing led to another, reinforcing my earlier image not of a tree but of a spider web. I finally had to quit pursuit before I became trapped! My apologies if your favorite strand is not mentioned.

I have included bits of history with dates which are often for the published work, not for the earlier arXiv post or samizdat. In addition to my early survey [167], among later ones often referred to are Keller's [104] and Lefévre-Hasegawa's [120].

Signs for many formulas are not specified; the interested reader may consult the cited literature.

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2. Once upon a time: A_∞-spaces, algebras, maps

For me, the study of associativity began as an undergraduate at Michigan. I was privileged to have a course in classical projective geometry from the eminent relativist George Yuri Rainich[1]. A later course "The theory of invariants" emphasized the invariants of Maxwell's equations. Rainich included secondary invariants, preparing me well when I later encountered secondary cohomology operations.

2.1 A_∞-**spaces.** The history of A_∞-structures begins, implicitly, in 1957 with the work of Masahiro Sugawara [177, 178]. He showed that, with a generalized notion of fibration, the Spanier-Whitehead condition for a space F to be an H-space:

   the existence of a fibration with fibre F contractible in the total space

is necessary and sufficient. Sugawara goes on to obtain similar criteria for F to be a homotopy associative H-space or a loop space.

   When I was a graduate student at Princeton, John Moore suggested I look at when a primitive cohomology class $u \in H^n(X, \pi)$ of a topological group or loop space X was the suspension of a class in $H^{n+1}(BX, \pi)$. In other words, when was an H-map $X \to K(\pi, n)$ induced as the loops on a map $BX \to K(\pi, n + 1)$. Here, for a topological group or monoid, BX refers to a "classifying space".

   In part inspired by the work of Sugawara, I attacked the problem in terms of a filtration of the classifying bundle $EX \to BX$ by the projective spaces $XP(n)$. Sugawara's work was for $XP(3)$ and $XP(\infty)$. His criteria consisted of an infinite sequence of conditions, generalizations of homotopy associativity. He included conditions involving homotopy inverses which I was able to avoid via mild restrictions.

Remark 2.1. Any H-space has a projective 'plane' $XP(2)$ and homotopy associativity implies the existence of $XP(3)$. Contrast this with classical projective geometry where the existence of a projective 3-space implies strict associativity.
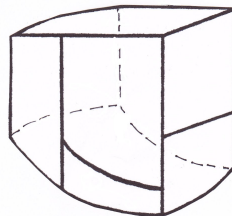
To systematize the higher homotopies, I defined [165]:

Definition 2.2. An A_n *space* X consists of a space X together with a coherent set of maps

$$m_k : K_k \times X^k \to X \ \ for \ \ 2 \le k \le n$$

where K_k is the (by now) well known $(k-2)$-dimensional associahedron.

   My original realization had little symmetry, being based on parameterizations, following Sugawara.

   The *"so-called"* Stasheff polytope was in fact constructed by Tamari in 1951 [182, 183, 184], a full decade before my version.

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[1]Born Yuri Germanovich Rabinovich https://www.lib.umich.edu/faculty-history/faculty/george-yuri-rainich
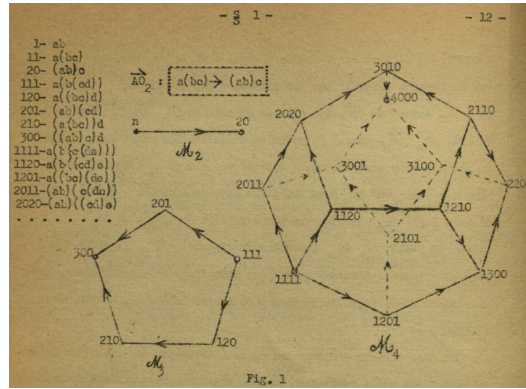
Figure 1: from Tamari's thesis [182]

Many other realizations are now popular and have been collected by Forcey [60] along with other relevant "hedra".

Tamari's point of view was much different from mine, but just as inspiring for later work, primarily in combinatorics. The book [137] has a wealth of offspring.

My multi-indexing of the cells of the associahedra was awkward, but the technology of those years made indexing by trees unavailable. Boardman and Vogt use spaces of binary trees with interior edges given a length in $[0, 1]$, producing a cubical subdivision of the associahedra [29]. Alternatively, the associahedron K_n can be realized as the compactification of the space of n distinct points in $[0, 1]$ [132] Example 4.36, [173] Appx B. For n distinct points on the circle, the compactification is of the form $S^1 \times W_n$, where W_n is called the cyclohedron (see Section 7).

Just as one construction of a classifying space BG for a group G is as a quotient of the disjoint union of pieces $\Delta^n \times G^n$ where Δ^n is the n-simplex, I constructed BX as a quotient of the disjoint union of pieces $K_n \times X^n$.

I was still working on Moore's problem when I went to Oxford as a Marshall Scholar. I was automatically assigned to J.H.C. Whitehead as supervisor, then transferred to Michael Barratt whom I knew from Princeton. When Barratt moved to Manchester, Ioan James took on my supervision. Though Frank Adams was not at Oxford, he did visit and had a significant impact on my research. Ultimately, this resulted in my theses for Oxford and Princeton. I needed to refer to what would be my Princeton thesis as a prequel to the Oxford one. Oxford had a strange rule that I could not include anything I *had submitted* elsewhere for a degree, even with attribution, so I made sure to submit to Oxford before returning to the US and finishing the formalities at Princeton. For the record, the 1963 published versions *Homotopy Associativity of H-spaces I and II* [165, 166] correspond respectively to the topology of the Princeton thesis and the homological algebra of the Oxford one.

2.2 A_∞-algebras.

Thinking of the cellular structure of the associahedra led to:

Definition 2.3. Given a graded vector space $A = \{A_n\}$, an A_∞-algebra structure on A is a coherent set of maps $m_k : A^{\otimes k} \to A$ of degree $k - 2$.

Here *coherence* refers to the relations

$$0 = \sum_{i+j=n+1} \sum_{k=1} \pm m_i\big(a_1 \otimes \cdots \otimes a_{k-1} \otimes m_j(a_k \otimes \cdots \otimes a_{k+j-1}) \otimes a_{k+j} \otimes \cdots \otimes a_n\big)$$

For an ordinary dga (A, d, m), Massey constructed secondary operations now called *Massey products* (Massey's d was cohomological, i.e. of degree $+1$). These products generalize easily to an A_∞-algebra, using m_3, etc.

For an ordinary associative algebra (A, m), there is the *bar constuction BA*, a differential graded coalgeba with differential determined by the multiplication m. For an A_∞-algebra, there is a completely analogous construction using all the m_k. The most effective defintion of an A_∞-morphism from A_1 to A_2 is as a morphism of dg coalgebras $BA_1 \to BA_2$. There is an important but subtle relation between the differentials of the *bar construction spectral sequence* and (higher) Massey products (see [32]).

2.3 A_∞-morphisms. Morphisms of A_∞-spaces are considerably more subtle. Again the way was led by Sugawara [179], who presented *strongly homotopy multiplicative maps* of strictly associative H-spaces.

Definition 2.4. A *strongly homotopy multiplicative map $X \to Y$* of associative H-spaces consists of a coherent family of maps $I^n \times X^n \to Y$.

The analog for dg associative algebras is straightforward.

When X and Y are A_∞-spaces, the parameter spaces for the higher homotopies, now known as the *multiplihedra* (often denoted $\mathcal{J}_n$ or J_n), are considerably more complicated and realization as convex polytopes took much longer to appear. An excellent illustrated collection of these and other related polyhera is given by Forcey [59], see also [103]. My early picture for $n = 4$ was not really a convex polytope though it appears as a subdivided pentagonal cylinder. Iwase and Mimura in [93] gave the first detailed definition of the multiplihedra and describe their combinatorial properties. If the range Y is strictly associative, then the multiplihedron $\mathcal{J}_n$ collapses to the associahedron K_{n+1} [167]. On the other hand, Forcey [58] observed that if the domain X is strictly associative, then the multiplihedron $\mathcal{J}_n$ collapses to a *composihedron* he created and denotes $\mathcal{C}K(n)$, new for $n \geq 4$.

The analog for A_∞-algebras is much easier to write down, convexity not being an issue. The study of A_∞ spaces and algebras continues. There are interesting questions about the extension of A_n-maps, as in [85] and about the transfer of A_∞ structure through these maps, as in [130].

3. Iterated loop spaces and operads

In terms of the development of higher homotopy structures, perhaps my most important result was this characterization of spaces of the homotopy type of loop spaces in terms of an A_∞-structure:

Theorem 3.1. *A 'nice' connected space X has the homotopy type of a based loop space ΩY for some Y if and only if X admits the structure of an A_∞-space.*

Here 'nice' means of the homotopy type of a CW-complex with a non-degenerate base point. For the standard description of ΩY in terms of loops parameterized by $[0, 1]$, the maps are generalization of the usual one for homotopy associativity.

An alternative is to use the strictly associative *Moore space of loops* $\mathbf{\Omega}Y$ with loops parameterized by intervals $[0, r]$ for $r \geq 0$ [136]. Apparently Moore never formally published this major contribution, but, thanks to the internet, his seminar of 1955/56 is available at

http://faculty.tcu.edu/gfriedman/notes/aht23.pdf. The homotopy equivalences of X and ΩY and $\boldsymbol{\Omega} Y$ are indeed A_∞-morphisms.

From characterizing loop spaces, I went on to characterizing loop spaces on H-spaces by constructing a multiplication on a classifying space BX. This led to issues of homotopy commutativity with formulas getting out of hand [164]. Soon after, Clark [40] investigated the subtleties of comparing $\boldsymbol{\Omega}(X \times X)$ with $\boldsymbol{\Omega} X \times \boldsymbol{\Omega} X$.

In 1967 at the University of Chicago, prompted by a visit from Frank Adams, a seminar *Iterated homotopies and the bar construction* was organized by Adams and Mac Lane. Thanks to Rainer Vogt, we have a great record of that seminar [194]. The audience was exceptional; see Vogt's listing as well as his recollection of the lectures. He writes that the seminar had a great influence on his work with Mike Boardman, as it did on several of the participants, myself included.

In 1968 Boardman and Vogt [28, 29] (1973), motivated by the many infinite loop spaces then of interest thanks to Bott periodicity and that seminar, emphasized the point of view of *homotopy invariant algebraic structures* to characterize such infinite loop spaces.

A key idea was the passage from a strict algebraic structure to one that was homotopy invariant but still of the same homotopy type, suitably interpreted. Since retiring from the University of North Carolina - Chapel Hill, I have been a long time guest at the University of Pennsylvania, where the language of algebraic geometry is dominant, puzzling me as to what 'derived' referred. To paraphrase Monsieur Jourdain, I was happy to discover that I had been speaking 'derived' all my life, i.e. speaking in a related homotopy category (see Section 6.3 for an alternative meaning).

Then around 1970, Peter May blew the subject wide open with his development of operads [133] to handle the complexity of homotopy symmetries to characterize iterated loop spaces of any level. Of particular importance early on for Boardman and Vogt and for May was the little n-cubes operad. Somewhat later, Getzler [75] introduced the little n-disks operad and its framed version. Since then there has been a proliferation of operads with additional structure as well as many generalizations of the concept, [126, 132].

Operads were crucial for studying an important issue in the ∞-version of commutative algebras: whether to relax the commutativity up to homotopy or to keep the strict symmetry but relax the associativity or relax both.

As emphasized by Kontsevich [108], the triumvirate of A_∞-, L_∞- and C_∞-algebras play a dominant role. By C_∞-algebra is meant what is also known as a balanced A_∞-algebra, that is, a strictly graded commutative A_∞-algebra defined in terms of a coherent set of n-ary products which vanish on shuffles. C_∞-algebras and L_∞-algebras are in an adjoint relationship just as are strict associative commutative algebras and Lie algebras. The next most prominent might be E_∞-algebras, dubbed *homotopy everything*. That is a bit of a misnomer, though they are very important for the study of infinite loop spaces.

As readers of this journal are well aware, there has been a proliferation of strong homotopy or ∞-structure versions of classical algebras such as Gerstenhaber, Batalin-Vilkovisky, Froebenius and on and on (see Sections 6.3 and 6.5) as well ∞ 'spaces' and even ∞ 'group/oids'. The latter occur in attempting 'integration' as in integration of a Lie algebra to a Lie group [86] regarded as a simplicial set or Kan complex. There is also *derived* A_∞ for use over a ring rather than a field [155, 39].

4. L_∞-algebras

In contrast to homotopy associativity, which was considered long before the higher homotopies were recognized, algebras with just 'Jacobi up to homotopy' appeared only after the full set of higher Jacobi homotopies were incorporated in L_∞-algebras (aka strong homotopy Lie algebras or sh-Lie algebras), which in turn had waited many years to be introduced for lack of applications. However, 'Jacobi up to homotopy' was implicit in the many proofs of the graded Lie algebra structure of the Whitehead product since, as Massey said [83]:

> This question was 'in the air' among homotopy theorists in the early 1950's, I don't believe you can point to any one person and say that he or she raised this.

Retakh and Allday were the first to define Lie-Massey operations, both in 1977: [149] in Russian and [6]. A preferable name might be *Massey brackets*. For Retakh, they appeared as obstructions to deformations of complex singularities (compare [53, 148] and see Section 8). Retakh's *n*-homotopy multiplicative maps are a special case of L_∞-morphisms. Both Retakh and Allday emphasize applications to the Quillen spectral sequence and to rational Whitehead products. For higher Whitehead products, see also [22].

Clear exposition of the Massey brackets and their connection to rational homotopy theory is in chapter V of Tanré [186].

Higher homotopies for Jacobi led to:

Definition 4.1. [118, 117] An L_∞-structure on a graded vector space V is a collection of skew graded symmetric linear brackets $l_n : \bigotimes^n V \longrightarrow V$

$$\ell_n = [\ ,\dots,\ ] : \Lambda^n V \to V$$

for $n \geq 1$ of degree $2 - n$ (for cochain complexes, and $n - 2$ for chain complexes) such that

$$\sum_{i+j=n+1} \pm l_i(l_j(v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(j)}) \otimes v_{\sigma(j+1)} \otimes \cdots \otimes v_{\sigma(n)}) = 0$$

where σ runs through all $(j, n - j)$ unshuffles and the signs are in the graded sense.

Equivalently, an L_∞-*algebra* is a graded vector space $L = \{L_i\}$ with a coderivation differential of degree ± 1 on the graded symmetric coalgebra $C(L)$ on the *shifted* graded vector space sL. For an ordinary Lie algebra, this is the classical Chevalley-Eilenberg *chain* complex.

The map l_1 is a differential: $(l_1)^2 = 0$ and l_2 may be compared to an ordinary (graded) Lie bracket $[\ ,\ ]$. When l_1 or $l_3 = 0$, the definition yields the usual (graded) Jacobi identity. In general, l_3 is a homotopy between the Jacobi expression and 0 while the other l_n's are known as *higher homotopies* or *higher brackets*. If $l_3 = 0$, there still may be non-trivial higher l_n, e.g. on the homology of a dg Lie algebra (see Section 8.4). Physicists like to say products, but we have consistently used *'brackets'*. There are alternative notations:

$$l_n(v_1, \cdots, v_n) = [v_1, \cdots, v_n]\ (math) \quad = [v_1 \cdots v_n]\ (physics).$$

Remark 4.2. Note the ambiguity as to the degree ± 1 of d in defining an L_∞-algebra. The binary operation is always of degree 0; sometimes the 'manifest' grading in examples is **not** the right one; see examples below. The shift of the bracket now has the same degree as the shift of ℓ_1. Notice also this bracket extends to an action of the degree 0 piece on the piece of degree 1 (or -1 respectively), as for a module over an algebra (contrast BBvD structures in Section 4.4).

Belatedly, Sullivan's models for rational homotopy types were recognized as being of the form $C(L)$ for the L_∞-algebra of rational homotopy groups [180]. The 2-bracket corresponds to the Whitehead or Samelson product and higher order brackets to higher order Whitehead products [22].

4.1 L_∞-structures in physics. Giovanni Felder was all too briefly my colleague at UNC. Later he joined the L_∞ club, remarking: *"the ∞-virus had a long incubation time and the outbreak came after I left the infection zone."* Since then, it has expanded to epidemic proportions in the field theoretic physics community.

In 1982, L_∞-algebras appeared in disguise in gravitational physics in work of D'Auria and Fré. Unfortunately they referred to their algebras as free differential algebras; to be precise, their FDA is a dgca (free as a gca ignoring the differential).

Around 1984, Gerett Burgers visited Henk Van Dam at UNC and we discussed parts of his thesis; again it looked like an L_∞ structure was lurking there. Later this was confirmed by Fulp, Lada and Stasheff. The essential idea in Berends, Burgers and van Dam [33, 23] was a novel attack on particles of higher spin by letting the gauge parameters act in a *field dependent* way.

In 1987, the formulas of the BRST operator in the construction of Batalin-Fradkin-Vilkovisky for constrained Hamiltonian systems [15, 61, 62] could be recognized as corresponding to an L_∞-algebra [168, 171, 174] as did the Batalin-Vilkovisky corresponding Lagrangian formulas [175].

In 1989, the L_∞ structure of CSFT (Closed String Field Theory) was first identified when Zwiebach fortuitously gave a talk in Chapel Hill at the last GUT (Grand Unification Theory) Workshop [206, 208].

By 1993, it seemed appropriate to provide an *Introduction to sh Lie algebras for physicists* [118].

In 1998, Roytenberg and Weinstein, building on Roytenberg's thesis, showed that Courant algebroids give rise to (small) L_∞-algebras (see Section 4.6 and [205]).

L_∞-algebras are continuing to be useful (and popular !) in physics[2] in two ways:

- Solution of a physical problem leads to a structure which later is recognized as that of an L_∞-algebra.
- Solution of a physical problem is attacked using knowledge of L_∞-algebras.

There are some famous 'no go' theorems that rule out certain physical models, e.g. for higher spin particles (see Section 4.4). What are ruled out are only models in terms of representations of strict Lie algebras (compare Section 9).

4.2   'Small' L_∞-algebras. CSFT requires the full panoply of higher brackets of all orders, but many other examples of L_∞-algebras with at most 3 pieces in the grading have appeared in physics. More generally, the name *Lie n-algebra* uses the n-categorical language and refers to an L_∞-algebra L with $L_k = 0$ for $k < 0$ and for $k > n - 1$ or for $k > 0$ and $k < -n + 1$. Some authors refer to these as 'truncated' L_∞-algebras.

Zwiebach and other physicists had asked about small examples of L_∞-algebras, (physicists' 'toy' models) leading to work of Tom Lada and his student Marilyn Daily. She classified all 3-dimensional L_∞-algebras: 2-graded with one 1-dimensional component and one 2-dimensional component; 3-graded where each component is 1-dimensional [43].

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[2]See https://ncatlab.org/nlab/print/L-infinity+algebras+in+physics for an extensive, annotated chronological list - thanks to Urs Schrieber- but beware the linguistic problems.

⬦ **Warning!!** There is a real problem of nomenclature. There are also notions of n-Lie algebra which have a k-ary bracket *only* for $k = n$ (see Section 5.1).

In 2017, Hohm and Zwiebach [91] introduced a wide class of field theories they call *standard-form*, which included, by assumption, a (usually small) L_∞-structure; again field dependence was crucial. They went beyond Berends, Burgers and van Dam by including a space of 'field equations' in their L_∞-algebras. Together with Andreas Deser, Irina Kogan and Tom Lada, we are adding such field equations to the Berends, Burgers and van Dam approach to show that again field dependence implies the L_∞-structure.

Following Hohm and Zwiebach, Blumenhagen, Brunner, Kupriyanov and Lüst [26] attacked the existence of an appropriate L_∞-structure with given initial terms by what they call a boot-strap approach. By this they mean an inductive argument for the n-bracket by *solving* the equations for the L_∞ relations. They succeed for non-commutative Chern-Simons and for non-commutative Yang-Mills using properties of the *star product* and various string field theories combining Chern-Simons and Yang-Mills by careful computation (see Section 4.3 for uniqueness of their approach).

Contrast the inductive argument for BFV and BV (see Section 4.5) where a solution follows from the auxiliary acyclic resolution.

4.3 L_∞-morphisms. L_∞-morphisms $V \to W$ can most efficiently be described as morphisms of dgcas $C(V) \to C(W)$. They are particularly important in Kontsevich's proof of the Formality Conjecture in relation to deformation quantization of Poisson manifolds [110]. Here the crucial L_∞-morphism was from the Lie algebra of multivector fields to the dg Lie of multidifferential operators. Since his initial success, there have been many formality theorems upgraded to other settings, for example, for BV algebras [34] or for a cyclic version [201].

A special case is comprised of the Seiberg-Witten maps (SW maps) [162] between "non-commutative" gauge field theories, compatible with their gauge structures; in particular, between non-commutative versions of theories with a gauge freedom. When these theories exist, they are consistent deformations of their commutative counterparts.

Blumenhagen, Brinkmann, Kupriyanov and Traube [27] establish uniqueness of their results (up to gauge equivalence) by constructing SW maps. Aschieri and Deser [11] construct specific examples in terms of $U(n)$-vector bundles on two-dimensional tori given by globally defined Seiberg-Witten maps (induced from the plane to the torus).

4.4 BBvD L_∞-algebras. In contrast to Lie n-algebras as defined above, it is possible to have an L_∞-algebra concentrated in degrees 0 to $n - 1$ with d of degree $+1$ as for the higher spin algebras of Berends, Burgers and van Dam: They start with a given space of 'fields' Φ which is a module over a Lie algebra Ξ of gauge symmetries.

By a field dependent gauge transformation of Ξ on Φ, they mean a polynomial (or power series) map $\Xi \otimes \Lambda^* \Phi \to \Phi$:

$$\delta_\xi(\phi) = T(\xi, \phi) = \Sigma_{i \geq 0} T_i(\xi, \phi)$$

where T_i is linear in ξ and polynomial of homogeneous degree i in ϕ. Note the operation $T_0 : \Xi \to \Phi$ from 'algebra' to 'module', in contrast to Lie n-algebras.

They have a corresponding field dependent generalization of a Lie algebra structure on Ξ: a polynomial (or power series) map $\Xi \otimes \Xi \otimes \Lambda^* \Phi \to \Xi$

$$[\xi, \eta](\phi) = C(\xi, \eta, \phi) = \Sigma_{i \geq 0} C_i(\xi, \eta, \phi)$$

where C_i is bilinear in ξ and η and of homogeneous degree i in ϕ.

These operations obey consistency relations which Fulp, Lada and I identified as structure relations of an L_∞ algebra [68].

Remark 4.3. Note the BBvD structure gives rise to an L_∞-algebra structure on the direct sum of the space of fields and the space of gauge parameters, **not** of the form of an L_∞-algebra and its module, rather a Lie 2-algebra. This is similar to what occurs in the BFV and BV formalisms.

4.5 The BFV and BV dg algebra formalisms.
For the related but separate notion of a BV algebra, see Definition 6.5.

Work of Batalin-Fradkin-Fradkina-Vilkovisky (BFV) from 1975-1985 [15, 61, 62] concerned reduction of constrained Hamiltonian systems. The constraints formed a Lie algebra and generated a commutative ideal. Their construction combined a Koszul-Tate resolution with a Chevalley-Eilenberg complex. The sum of the two differentials no longer squared to 0 but required terms of higher order. These were shown to exist using the acyclicity of the Koszul-Tate resolution. In 1988 [169], I was able to understand those terms in the context of homological perturbation theory (HPT, a common technique in ∞-theories) and representations up to (strong) homotopy (RUTHs) (see Section 9). My construction was extended by Schätz [157] who showed an equivalence with an associated strong homotopy Lie algebroid (see Definitions 4.4 and 4.5).

When a technique dating back to at least Weishu Shih [163] [31] and Ronnie Brown [31] was formalized and named by Gugenheim in terms of perturbations [79] (see also [80, 81]), the word seemed to evoke only an analogy with perturbations in physics; now it plays a major role in several aspects of mathematical physics.

For the Lagrangian version, Batalin and Vilkovisky [16, 17, 14] constructed the analogous dg algebra by a similar method [175]. The two formalisms BV and BFV were combined on a manifold with boundary by Cattaneo, Mnev and Reshetikhin [37]. They have also a quantum version.

These constructions suggested an L_∞-algebra and module, but things were not so simple. My student Lars Kjeseth in his UNC thesis [105, 106] showed that the appropriate structure was that of a *strong homotopy* version of a Lie-Rinehart algebra [150]. This concept then lay dormant until resurrected around 2013 in the work of Huebschmann [92] and of Vitagliano [193].

More generally, Barnich, Fulp, Lada and Stasheff [13] constructed an L_∞-algebra on any homological resolution of a Lie algebra.

4.6 Some special L_∞-algebras.
Special L_∞-algebras arise from Courant algebroids, Double Field Theory (DFT) and multisymplectic manifolds. In the last two decades, what is called T-duality in string theory and supergravity led to a formulation of differential geometry on a generalized tangent bundle:

$$0 \longrightarrow T^*M \longrightarrow E \xrightarrow{\pi} TM \longrightarrow 0. \tag{1}$$

Locally, the bundle E looks like $TM \oplus T^*M$.

Courant algebroids. Courant algebroids are structures which include as examples the doubles of Lie bialgebras and bundles $TM \oplus T^*M$. They are named for T. Courant [42] who introduced them in his study of Dirac structures.

Given a bilinear skew-symmetric operation $[\ ,\ ]$ on a vector space V, its *Jacobiator* J is the trilinear operator on V:

$$J(e_1, e_2, e_3) = [[e_1, e_2], e_3] + [[e_2, e_3], e_1] + [[e_3, e_1], e_2],$$

$e_1, e_2, e_3 \in V$. The Jacobiator is obviously skew-symmetric. Of course, in a Lie algebra $J \equiv 0$.

Definition 4.4. A *Lie algebroid* is a sequence $E \xrightarrow{\rho} TM \xrightarrow{p} M$ of vector bundle maps with a Lie bracket on the space of sections $\Gamma(E)$ such that

$$[X, fY] = \rho(X)(f)Y + f[X, Y]$$

for $X, Y \in \Gamma(E), f \in C^\infty(M)$. (It follows that $\rho[X, Y] = [\rho(X), \rho(Y)]$.)

Remark 4.5. The purely algebraic analog of a Lie algebroid is a *Lie-Rinehart algebra* over a commutative algebra more general than $C^\infty(M)$ [150].

Roytenberg's definition [152, 154] of a Courant algebroid is approximately that of an algebroid with anchor $\rho : E \to TM$ equipped with a nondegenerate symmetric bilinear form $\langle \cdot\ \cdot \rangle$ on the bundle E, a skew-symmetric *Courant* bracket $[\ ,\ ]$ on $\Gamma(E)$ and a map $\mathcal{D} : C^\infty(M) \longrightarrow \Gamma(E)$ satisfying many properties of which the most relevant is that the Courant brackets on $\Gamma(E)$ satisfy the Jacobi identity up to a $\mathcal{D}$-exact term: For any $e_1, e_2, e_3 \in \Gamma(E)$,

$$J(e_1, e_2, e_3) = \mathcal{D}T(e_1, e_2, e_3)$$

where $T(e_1, e_2, e_3)$ is the function on the base M defined by:

$$T(e_1, e_2, e_3) = 1/6 \sum_{cyclic} \langle [e_1, e_2], e_3 \rangle. \tag{2}$$

Remark 4.6. A related structure is due to Dorfman [51] for which the bracket is not skew-symmetric but satisfies the Loday (also called Leibniz) version of the Jacobi identity [124] which is expressed as a derivation from the left (see Definition 6.2). The name *Dorfman bracket* appears in [153] which discussed Courant-Dorfman structures. The Courant bracket is the skew symmetrization of the Dorfman bracket [115].

Roytenberg showed in his PhD thesis [152] the equivalence of this structure with a specific L_∞ algebra X, in which l_1 is determined by the de Rham differential, l_2 by the Lie bracket and the operation l_3 contains "flux"-degrees of freedom (e.g. a three-form known as H-flux). Here X is a resolution of $\mathcal{H} = coker\mathcal{D}$:

$$X_2 = \ker \mathcal{D} \xrightarrow{d_2} X_1 = C^\infty(M) \xrightarrow{d_1} X_0 = \Gamma(E) \longrightarrow \mathcal{H} \longrightarrow 0, \tag{3}$$

where with $d_1 = \mathcal{D}$ and d_2 is the inclusion $\iota : \ker \mathcal{D} \hookrightarrow C^\infty(M)$.

The Courant brackets on $\mathcal{H}$ come from Courant brackets on $\Gamma(E)$. Roytenberg and Weinstein [154] use this to extend the Courant bracket to an L_∞-structure on all of their resolution X, manifestly a Lie 3-algebra.

Further generalizations known as *higher Courant algebroids* [205] are locally $TM \oplus \Lambda^k(T^*M)$.

Double Field Theory (DFT). Aldazabal, Marqués and Núñez write in *Double Field Theory: A Pedagogical Review* [5]:

> Double Field Theory (DFT) is a proposal to incorporate T-duality, a distinctive symmetry of string theory, as a symmetry of a field theory defined on a *double configuration space*.

Indeed, as originally introduced in physics, DFT (Double Field Theory) refers *not* to a doubling of fields but rather to doubling of an underlying structure such as double vector bundles and Drinfel'd doubles. The original Drinfel'd double occurred in the contexts of quantum groups and of Lie bialgebras. Following Roytenberg [152], Deser and I [49] interpret the gauge algebra of DFT in terms of Poisson brackets on a suitable generalized Drinfel'd double. In DFT, it is the coordinates that are doubled and 'double fields' refer to fields which depend on both sets of coordinates. Thus we also refer to double functions, double vector fields, double forms, etc. In the physics literature, reference is made to a *C-bracket* to distinguish it from a Poisson bracket, but we showed that the C-bracket is essentially the Poisson bracket on $T^*[1]TM$. (Here [1] denotes the shift in degree of the *fibre* coordinates of TM.)

Similarly, Deser and Sämann in [48] used the analog of a Poisson bracket on $T^*[n]T[1]M$ to define an action of a Lie n-algebra on the algebra of functions $\mathcal{F} := \mathcal{C}^\infty(T^*[n]T[1]M)$. In particular, what is called "section condition" in the physics of DFT was identified as the requirement for a specific Lie 2-algebra to act on $\mathcal{F}$ in a well-defined way.

As a bundle over TM with a local trivialization with respect to a covering $\mathcal{U} = \{U_\alpha\}$, we have transition functions $a_{\alpha\beta} \in GL(2d, \mathbf{R})$ satisfying the usual cocycle condition, but, in terms of the local splitting $TM \oplus T^*M$, there is a higher order 'twist' τ depending on a 2-form ω. Now the cocycle condition fails, the failure depending on ω:

$$g_{\alpha\beta}\, g_{\beta\gamma} \neq g_{\alpha\gamma} \quad \text{on} \quad U_\alpha \cap U_\beta \cap U_\gamma.$$

This is often described as a failure of associativity, but it is more accurately failure to correspond to a representation. Since $g_{\alpha\beta}$, $g_{\beta\gamma}$ and $g_{\alpha\gamma}$ can be expressed in terms of 1-forms, it can be that the difference is an exact form $d\lambda_{\alpha\beta\gamma}$ for some function $\lambda_{\alpha\beta\gamma}$ on $U_\alpha \cap U_\beta \cap U_\gamma$. One would then say the transition functions form a representation up to homotopy (RUTH) or one can speak in terms of 'gerbes' (see Section 10 for higher order generalizations).

Multisymplectic manifolds

Definition 4.7. [151] A *multisymplectic* or, more specifically *n-plectic*, manifold is one equipped with a closed nondegenerate differential form of degree $n + 1$.

As shown by Chris Rogers, [151], just as a symplectic manifold gives rise to a Poisson algebra of functions, any n-plectic manifold gives rise to a Lie n-algebra of differential forms with multi-brackets specified via the n-plectic structure. The underlying graded vector space consists of a subspace of $(n-1)$-forms he calls *Hamiltonian* together with all p-forms for $0 \le p \le n-2$:

$$C^\infty(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{n-2}(M) \xrightarrow{d} \Omega^{n-1}_{Ham}. \tag{4}$$

5. Generalized Jacobi identities and Nambu-Poisson algebras

5.1 Identities. There are two important ways to generalize to n-variables the Jacobi identity for a Lie algebra written as a left derivation (see Definition 6.2):

$$[a, [b, c]] = [[a, b], c] + [b, [a, c]].$$

In older literature, these generalized Jacobi relations are referred to as *fundamental identities*, but, as suggested by de Azcarraga and Izquierdo [45], a better name might be *characteristic identities*.

One identity is the corresponding L_∞-relation for a bracket of just n variables:

$$\sum \pm l_n(l_n(v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)} \otimes v_{\sigma(j+1)} \otimes \cdots \otimes v_{\sigma(2n-1)}) = 0.$$

It has been studied quite independently of my work and of each other by Hanlon and Wachs [82] (combinatorial algebraists), by Gnedbaye [76] (of Loday's school) and by de Azcarraga and Bueno [44] (physicists).

On the other hand, the characteristic identity for a *Filippov algebra* [57] says $[X_1, X_2, \ldots, X_{n-1},\quad]$ acts as a left derivation.

$$[X_1, X_2, \ldots, X_{n-1}, [Y_1, Y_2, \ldots, Y_n]] =$$
$$[[X_1, X_2, \ldots, X_{n-1}, Y_1], Y_2, \ldots, Y_n] + \cdots + [Y_1, \ldots, Y_{n-1}, [X_1, \ldots, X_{n-1}, Y_n]].$$

That identity was known also to Sahoo and Valsakumar [156]. Unfortunately, both versions are called n-Lie algebras, hence my attempt (probably futile) to rename his as Filippov's.

As I recall, I first learned of this other (Filippov) identity from Alexander Vinogradov when we met at the Conference on Secondary Calculus and Cohomological Physics, Moscow, August 1997. (See A. and M. Vinogradov's [192] for a comparison of these two distinct generalizations of the ordinary Jacobi identity to n-ary brackets.)

Yet again this identity arose independently by yet another name in work of Tamara Friedmann et al [64, 65]. The article [45] by de Azcarraga and Izquierdo is a very thorough survey of even more n-ary algebras.

All these algebras are important in geometry and in physics where the corresponding structures are on vector bundles over a smooth manifold (see [199] and references therein).

5.2 Nambu-Poisson n-Hamiltonian mechanics. Nambu's original work [138] was a generalization to an n-ary bracket of Hamiltonian mechanics with its binary Poisson bracket. Often the literature refers to Nambu-Poisson structures, which emphasizes the setting of C^∞ functions on a smooth manifold as in traditional Hamiltonian mechanics. Just as the latter can be extended to sections of a Lie algebroid, Nambu-Poisson structures on manifolds can be extended to the context of Lie algebroids [199]. The Nambu bracket satisfies the Filippov identity [181].

One expects there to be "∞"-versions with the full panoply of applications as for L_∞-structures, e.g. homological reduction of constrained Nambu-Poisson algebras (cf. Section 4.5).

6. Derived bracket and brace and BV algebras

6.1 Differential operators. An increasingly popular approach to L_∞-structure is the use of *derived brackets* (see Definition 6.4) introduced by Kosmann-Schwarzbach in [113], inspired by

unpublished notes by Jean-Louis Koszul [116]. She traces their origin to work of Buttin and of A. Vinogradov in the context of unification of brackets in differential geometry, though physics (of integrable systems for instance) was not far behind, especially in work of Irina Dorfman [51, 52]. See [114] for an excellent survey and history.

Koszul's point of view was that of an algebraic characterization of the order of a differential operator. Multilinear operators with arguments that are left multiplication operators ℓ_a rather than elements a of an algebra may have appeared first in various definitions of the order of a differential operator. For example, the *order* of a differential operator Δ can be defined as

Definition 6.1. [2]   A linear operator Δ on an algebra A is a *differential operator of order $\leq r$* if an inductively defined $(r+1)$-linear form Φ_Δ^{r+1} with values in A is identically zero.

or, more simply:

An operator Δ on an algebra A is of order at most n if, for all left multipliers ℓ_a for $a \in A$, the commutator $[\Delta, \ell_a]$ is of order at most $n - 1$.

This approach can be traced back to 1967: Grothendieck [78], then developed by Koszul [116]. This was carried forward in 1997 by Akman [2] who defines higher order differential operators on a general *noncommutative, nonassociative* graded algebra $\mathcal{A}$. Akman and Ionescu [4] compare and show equivalence of several definitions of order when the underlying algebra A is classical, i.e., graded commutative and associative.

The brackets that are now called "higher derived" (as defined below) can be represented in a form similar to that for defining "order", but are of interest for providing L_∞-structure.

6.2   *Loday$_\infty$*-algebras. Not only L_∞-algebras are relevant, but also *Loday*-algebras and even *Loday$_\infty$*-algebras. Loday (1946 - 2012) originally called them *Leibniz algebras*, but he deserves the credit. Recall:

Definition 6.2. A (left) *Loday-algebra* $(V, [\,,\,])$ consists of a vector space V with a *Loday bracket* $[\,,\,]$ satisfying the left Leibnitz identity:

$$[a, [b, c]] = [[a, b], c] + [b, [a, c]].$$

Definition 6.3. [125] A (left) *Loday$_\infty$-algebra* (V, π) consists of a graded vector space V with a sequence $\pi = (\pi_1, \pi_2, \dots)$ of multivariable coderivations π_i on the free graded coalgebra co-generated by V shifted with a rather unusual coproduct using unshuffles that preserve the (last) element. The sequence π is to satisfy $[\pi, \pi] = 0$.

Apparently this was defined first by Livernet [123] for right *Loday$_\infty$*-algebras, later independently in a different setting by Ammar and Poncin [7] with further work by Peddie [143].

6.3   Derived brackets. Kosmann-Schwarzbach defined *derived brackets* in 1996 [113]:

Definition 6.4. If $(V, [\,,\,], D)$ is a graded differential Lie or Loday algebra with a bracket $[\,,\,]$ of degree n, the *derived bracket* of $[\,,\,]$ by D is denoted

$$[\,,\,]_D : V \otimes V \to V$$

and defined by

$$[a, b]_D = (-1)^{|a|+n+1}[Da, b] \,,$$

for a and $b \in V$.

Often D is itself given as $D = [\Delta, \ ]$ for an element $\Delta \in V$.

In this generality, the derived bracket is graded Loday and respected by the differential. To obtain a differential graded Lie bracket, additional actions are necessary.

Continuing the iteration of Lie brackets led in 1996 to Bering, Damgaard and Alfaro defining *higher BV anti-brackets* in physics [25]. Somewhat later, independently in math, T. Voronov defined *higher derived brackets* [198]. This was followed quickly by [24] and further proliferation. Apparently the membrane between math and physics is osmotic: information travels one way faster than the other.

The definition closest to my interests is:

Definition 6.5. *Higher derived brackets*

$$B_\Delta^r(a_1, \ldots, a_r)$$

are defined as

$$[\ldots [\,[\Delta, a_1], a_2], \ldots, a_r]$$

where $(L, [-, -])$ is a Lie algebra with $\Delta, a_1, \ldots, a_r \in L$.

In terms of operators D:

Definition 6.6 (Alternative)**.** *Higher derived brackets*

$$B_D^r(a_1, \ldots, a_r)$$

are defined as

$$[\ldots [Da_1], a_2], \ldots, a_r]$$

where $(L, [-, -])$ is a Lie algebra with $a_1, \ldots, a_r \in L$ and $D : L \to L$.

The major uses of higher derived brackets are for describing/constructing L_∞-algebras and other ∞-algebras. An important case is the derived bracket built with an odd, square-zero differential operator Δ of order ≤ 2, which Akman uses to define a *generalized Batalin-Vilkovisky algebra* structure on an algebra A (see Section 6.5). The emphasis on order ≤ 2 is historical, dating back to Batalin-Vilkovisky and their applications in physics.

T. Voronov's *higher derived brackets* [198] first occurred in the context of a Lie algebra L with an abelian sub-algebra A. The higher derived brackets on A were derived on L and then projected back to A. More general higher derived brackets are introduced by Bandiera [12].

Derived (binary and higher) brackets are also important for describing and understanding infinitesimal symmetry actions relevant in physics. The Roytenberg-Weinstein L_∞-structure can be expressed and generalized in terms of derived brackets [152]. Most recently, Deser and Sämann [47] show binary derived brackets underlie the symmetries of Double Field Theory (Section 4.6). They suggest adopting as a guiding principle:

> Whenever we are seeking an infinitesimal action of a fundamental symmetry, we try to find the corresponding derived bracket description.

One should also seek higher derived brackets as in [198] and hence L_∞-structures as called for in [91] (see Section 4.1).

6.4 Braces. Brace algebras, like operads, are all about composition. As Akman says: it all depends on what you want to achieve and how far you want to go.

Definition 6.7. [74, 197] A *brace algebra* is a graded vector space with a collection of braces $x\{x_1, \ldots, x_n\}$ of degree $\pm n$ satisfying the identities

$$x\{x_1, \ldots, x_m\}\{y_1, \ldots, y_n\}$$
$$= \sum \pm x\{y_1, \cdots, y_{i_1}, x_1\{y_{i_1+1}, \cdots, y_{j_1}\}, y_{j_1+1}, \cdots$$
$$\cdots, y_{i_m}, x_m\{y_{i_m+1}, \ldots, y_{j_m}\}, y_{j_m+1}, \ldots, y_n\} \quad (5)$$

where the sum is over $0 \le i_1 \le \cdots \le i_m \le n$ and the sign is due to the x_i's passing through the y_j's in the shuffle.

Thus brace algebras are flexible enough to accommodate insertion of several arguments at one time, but not necessarily filling all slots (an operad does fill all slots).

Braces first occurred in work of Kadeishvili [94] who called them higher $\smile_1$ products. In later but independent work, Gerstenhaber and A. Voronov [74, 197] named them *braces*, which is now the common term used by several authors. There is a corresponding *brace operad* with subtle use of trees; [50] explains its relation to Deligne's 'conjecture' for A_∞-algebras.

(Higher) derived brackets can be defined in terms of braces (special compositions of maps).

6.5 BV algebras

Warning!! Batalin and Vilkovisky have made two *distinct* contributions to gauge field theory:

- the formalism of the Lagrangian dg construction with antifields and ghosts etc. (see Section 4.5).
- graded algebras with a special differential operator of order 2.

Unfortunately both are sometimes refered to as the BV *formalism*, which preferably should be used for their dg construction.

Recall a *Gerstenhaber algebra* $(A, \cdot, [\;,\;])$ is the 'odd' analog of a graded Poisson algebra, $[\;,\;]$ being called a Gerstenhaber bracket. The first example occurred in his paper on Hochschild cohomology of an associative algebra [71].

Definition 6.8. A *Batalin-Vilkovisky algebra*, or *BV algebra*, is a Gerstenhaber algebra where the bracket $[\;,\;]$ is obtained from an odd, square zero, second order differential operator Δ:

$$[a, b] = (-1)^{|a|}\Delta(ab) - (-1)^{|a|}\Delta(a)b - a\Delta(b)$$

Batalin and Vilkovisky introduced the algebras now named in their honor in their study of conformal field theroies and quantization [14].

There is a notion of *generalized BV algebra* due to Akman [2] in which the bracket $[\;,\;]$ is obtained from an odd, square zero, second order differential operator Δ as above, though the algebra doesn't have to be associative nor commutative, but the bracket still measures the deviation of Δ from being a derivation. She also considers *differential BV algebras*. For Akman, a major motivation is the work of Lian-Zuckerman [121] related to VOAs (vertex operator algebras). The restriction to 'second order' is just the classical Batalin-Vilkovisky case.

Of course, there is a 'higher' notion of BV_∞ algebra. The article by Gálvez-Carrillo, Tonks and Vallette [70] should be the canonical reference. They point out that several authors call a "homotopy BV-algebra" what is a *commutative* homotopy BV-algebra, that is one for which the product is commutative (compare remarks at the end of Section 3). In addition to giving an operadic description (see also [55]), they give four equivalent definitions of a homotopy BV-algebra and provide applications in four different categories. In particular, they develop the deformation theory and homotopy theory of BV-algebras and of BV_∞ algebras.

7. Deligne's question and cyclicity

In his seminal work [71], Gerstenhaber showed the Hochschild cohomology $HH^\bullet(A)$ of an associative algebra carries what is now known as a Gerstenhaber algebra structure. Moreover, he constructed homotopies on the Hochschild cochains $CH^\bullet(A)$ to yield the relevant identities on $HH^\bullet(A)$. Thirty years later in 1993 in a letter to several of us, Deligne *asked*: Does the Hochschild cochain complex of an associative ring have a natural action by suitable chains of the small squares operad?

This lead to *many* 'higher structure' papers resolving this 'conjecture' and generalizations, in many of which the little disks operad was invoked instead and braces played a key role. The chains involved were various 'combinatorial' chains models for the topological operad. The generalization to the Hochschild cochains when A is an A_∞-algebra followed soon after. To follow this development further (including for dg-categories) means entering a labyrinth and trying to choose a path (see [134, 100, 185, 19] among others). That of [111] emphasizes relations to deformation theory (see Section 8).

Let me instead pay attention to the cyclic Deligne 'conjecture' which, on the one hand, refers to the Hochschild cochain complex for an algebra A with an invariant inner product and, on the other hand, to the *framed little disks operad* [75]. Solutions involve *cacti* [196, 102], *spineless cacti* [100] and Sullivan chord diagrams [190, 189] and of course compactified moduli spaces. For a combinatorial proof, see [18]. The solution is developed further in an A_∞ context and treated particularly well by Ben Ward [200]. Although the latter clearly involves associativity and cyclicity, it is intriguing to see the approach of Kaufmann and Schwell [101] involves both the associahedron and the cyclohedron W_n, which was introduced by Bott and Taubes [30] though only later named in [173]. Progressing one step further, Tradler considers an ∞-inner product [188].

8. L_∞ in deformation theory

Deformations of complex structure go back to Riemann, but just for one complex dimension. In higher dimension, even a proper definition of infinitesimal deformation was a problem, resolved by Nijenhuis and Frölicher [66] by identifying it as an class in the cohomology H^1 of the sheaf of germs of holomorphic tangent vectors. Kodaira and Spencer took over from there [107]. The primary obstruction to extending the infinitesimal δ was a class in H^2 given by the product $\delta \smile \delta$. Higher obstructions drew the attention of Douady [53], who related them to Massey 'products' . Unnoticed for a while, this approach was carried further by Retakh [149].

8.1 What is deformation theory?. Based on the history for complex structures, Gerstenhaber offered the first general description:

A deformation theory seems to have at least the following aspects:

- A definition of the class of objects within which deformation takes place, and identification of the infinitesimal deformations of a given object with the elements of a suitable cohomology group.
- A theory of the obstructions to the integration of an infinitesimal deformation.
- A parameterization of the set of objects obtainable by deformation from a fixed one, and the construction of a fiber space over this space, the fibers of which are the objects.
- A determination of the natural automorphisms of the parameter space (the modular group of the theory) and determination of the rigid objects. In some cases almost all points of a parameter space will represent the same rigid object, degenerating in various ways to objects admitting proper deformations.

By analogy with Riemann's original work, the associated parameter spaces are referred to as *moduli spaces*. In contrast to classical results, the moduli space need not be in the class of objects being studied.

From 1963 to 1968, Gerstenhaber studied deformations of associative algebras in terms of Hochschild cochains and cohomology [72]. This had major impacts: for deformation theory, for physics (see Section 8.3) and for higher homotopy theory (see Section 7). His initial work was soon followed by work of Nijenhuis and Richardson [140, 139]. They were the first to articulate something close to the current "metatheorem", adding dg Lie algebra as a crucial ingredient:

The deformation theory of any mathematical object, e.g., an associative algebra, a complex manifold, etc., can be described starting from a certain differential graded (dg) Lie algebra associated to the mathematical object in question.

This provides a natural setting in which to pursue the obstruction method for trying to integrate "infinitesimal deformations". The deformation equation is known as the Master Equation in the physics and physics inspired literature and now most commonly as the Maurer-Cartan equation. It is also the equation for a *twisting cochain* used in describing twisted tensor products [81, 176].

As in the above examples, some authors refer to the corresponding *cohomology* as controlling the deformations.

In 1986, inspired instead by Goldman-Millson [77], Deligne stated this as a philosophy in a letter to Millson [46].

Meanwhile in 1967, Lichtenbsum and Schlessinger [122] expressed deformation theory in terms of a *cotangent complex*; think differential forms on the moduli space with the *tangent complex* corresponding to a Lie algebra of derivations. This was carried further in Mike's thesis [158], written from the viewpoint of representability of certain functors.

The extension of deformation theory to a further variety of algebraic objects is well surveyed by Gerstenhaber and Schack [73]. On the other hand, deformations in the material sense are more geometric; Oh and Park [142] have a treatment for deformations of coisotropic submanifolds of a symplectic manifold which relates the deformation space to L_∞-algebroid cohomology (see Definitions 4.4 and 4.5).

8.2   The (not quite) metatheorem. In his MR review of [111], A. Voronov writes with respect to the 'metatheorem':

The uncomfortable generality of the statement might have prevented anybody from making it a theorem and proving it. A route chosen by many was usually the following one: given a mathematical object A, construct a dg Lie algebra L and answer as many questions about the deformation theory of A in terms of L as you can.

There have been several theorems of great but not complete generality.

Markl [129] develops a deformation theory phrased as a controlling cohomology theory for k-algebras over a k-linear equationally given category, e.g. over operads and PROPs for bialgebras. He invites comparison to traditional cotangent cohomology of a commutative algebra (in characteristic zero) based on the free differential graded algebra resolution of the algebra under consideration [160, 158]. In [131], he provides explicit constructions in terms of L_∞-deformation theory, enhanced further in [63]; see an alternative [135] for representations of prop(erad)s.

Others are expressed in 'languages the muse did not sing at my cradle'. Kontsevich and Soibleman [111] develop the existence of deformation theory for an algebra over any (colored) operad, but in a more geometric language of formal dg manifolds. This refers to cofree cocommutative coalgebras with differential in disguise. Thus again L_∞-algebras are doing the controlling. They give further insight in the comparison of the algebriac and geometric version of A_∞-structures in [112]. The one closest to my native language e.g. of L_∞-algebras, is that of Pridham [146, 147], which is the most general, going beyond characteristic 0. Closer to (derived) algebraic geometry is that of Lurie [127].

8.3 Deformation quantization. Especially important was the eponymous *Gerstenhaber bracket* [71] which was later identified by interpreting the Hochschild complex in terms of coderivations [172, 3]. His work led to an algebraic description of *deformation quantization* [20, 21], a term derived from physics.

Given a Poisson algebra $(A, \{ \ , \ \})$, a deformation quantization is an associative unital $\star$ product on the algebra of formal power series $A[[\hbar]]$ subject to the following two axioms:

$$f \star g = fg + \mathcal{O}(\hbar) \tag{6}$$
$$f \star g - g \star f = \hbar\{f, g\} + \mathcal{O}(\hbar^2). \tag{7}$$

In the physics context, the Poisson bracket is given in terms of differential operators as are the sought after terms of higher order (in which the devil resides!).

8.4 L_∞-algebras in rational homotopy theory. L_∞-algebras arose by 1977 in my work with Mike Schlesinger [160, 161, 159] on deformation theory of rational homotopy types. Mike and I extended the yoga of control by a dg Lie algebra (See 8.1) to similar control by an L_∞-algebra, for example, on the homology of a huge strict dg Lie algebra.

Let $\mathcal{H}$ be a simply connected graded commutative algebra of finite type and $(\Lambda Z, d) \to \mathcal{H}$ a filtered model. Differential graded Lie algebras provide a natural setting in which to pursue the obstruction method for trying to integrate "infinitesimal deformations", elements of $H^1(Der\Lambda Z)$, to full perturbations. In that regard, $H^*(Der\Lambda Z)$ appears not only as a graded Lie algebra (in the obvious way) but also as an L_∞-algebra.

Our main result compares the *moduli space* set of augmented homotopy types of dgca's $(A, i : \mathcal{H} \approx H(A))$ with the path components of $C(L)$ where L is a sub Lie algebra of $Der\,\Lambda Z$ consisting of the weight decreasing derivations. We were inspired by the work of Kadeishvili

on the A_∞-structure on the homology of a dg associative algebra. This again used the 'higher structure' machinery of HPT (Homological Perturbation Theory).

Independently, in 1988, correspondence between Drinfel'd and Schechtman [54, 187] develops L_∞-algebra under the name *Sugawara - Lie algebra* for the needs of deformations theory. As with 'Jacobi up to homotopy', the need for L_∞-algebras was 'in the air' (see Section 4). "A letter from Kharkov to Moscow" [54] deserves further attention.

The disparity in the dates recalls the lack of communication between the Soviet Union and the West. The situation was much better two years later when Gerstenhaber and I were able to participate in person at the Euler International Mathematical Institute's Workshop on Quantum Groups, Deformation Theory, and Representation Theory [56]. Interactions there were to prove very fruitful (see Section 11.2).

9. Representations up to homotopy/RUTHs

A good bit of group theory has been carried over to A_∞-algebras, but far from all. Most recently, the analog of Sylow theorems appeared [145]. It was late in the game before higher homotopy representations received much attention. Although the language is slightly different, an ordinary representation of a group or algebra is equivalent to a morphism to the endomorphisms of another object. This appeared early on in homotopy theory (I learned it as a grad student from Hilton's *Introduction to Homotopy Theory* [88] - the earliest textbook on the topic) in terms of the action of the based loop space ΩB on the fiber F of a fibration $F \to E \to B$:

$$\Omega B \times F \to F \quad or \quad \Omega B \to Haut(F)$$

where $Haut(F)$ denotes the monoid of self homotopy equivalences $F \to F$. The existence of $\Omega B \times F \to F$ follows from the covering homotopy property, but with uniqueness only 'up to homotopy'.

Initially, this was referred to as a *homotopy action*, meaning only that $(f\lambda)\mu$ was homotopic to $f(\lambda)f(\mu)$, with no higher structure. The map $\Omega B \to Haut(F)$ was only an H-map; the full equivalence between such fibrations and A_∞-maps $\Omega B \to Haut(F)$ had to wait until such maps were available. The corresponding terminology is that of (strong or ∞) homotopy action, which has further variants under a variety of names. The definition is simplified if we use the Moore space of loops $\mathbf{\Omega}B$ [136] (see Section 3).

Definition 9.1. A *representation up to homotopy* of $\mathbf{\Omega}B$ on a fibration $E \to B$ is an A_∞-morphism (or shm-morphism [179]) from $\mathbf{\Omega}B$ to $End_B(E)$ (see Definition 2.4).

In 1971, Nowlan [141] considered fibrations $F \to E \to B$ with associative H-space F as fibre, acting fibrewise on E, but the action being associative only up to higher homotopies. His main result shows that a fibration $\Omega Y \to E \to B$ is fibre homotopy equivalent to one induced by a map of $B \to Y$ if and only if E admits an A_∞-action of ΩY.

A_∞-actions occur also for relative loop spaces [90, 191].

In the case of a smooth vector bundle $E \to B$, the corresponding notion is parallel transport, usually determined by a *choice* of connection, hence unique. In physics, various action functionals for quantum field theories correspond to *higher parallel transport* in bundles with graded vector space fibers, corresponding to the higher homotopies of the strong homotopy action.

Switching to algebra, there is the corresponding notion of *representation up to homotopy* of an associative dg algebra on a dg vector space V. This seems to have first occurred in [174] in

the context of a Poisson algebra $(P, \{\ ,\ \})$ with a commutative ideal I closed under $\{\ ,\ \}$. Such a structure arises in physics with $A = C^\infty(W)$ for some symplectic manifold, W (see Section 4.5 [15, 61, 62]).

There are analogous notions of RUTHs of Lie structures. In particular, in open-closed string field theory (see Section 11.3), the CSFT acts up to homotopy in the strong sense on the OSFT and similalry for OCHAs with the L_∞-algebra acting on the A_∞-algebra by ∞-homotopy derivations. In 2011, RUTHs of Lie algebroids was developed by Abad and Crainic [1][3]. In particular, they use representations up to homotopy to define the adjoint representation of a Lie algebroid to control deformations of structure (see Section 8).

10. A_∞-functors, A_∞-categories and ∞-geometry

Since associativity is a key property of categories, it is not surprising that A_∞-categories were eventually defined. In 1993, Fukaya [67] defined them to handle Morse theoretic homology.

Just as one considers A_∞-morphisms of A_∞-algebras, one can consider A_∞-functors (also known as *homotopy coherent functors*) between A_∞-categories. Such functors were first considered for ordinary strict but topological categories in the context of classification of fibre spaces. For fibrations which are locally homotopy trivial with respect to a good open cover $\{U_\alpha\}$ of the base, one can define transition functions

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to Haut(F),$$

but instead of the cocycle condition for fibre bundles, one obtains only that $g_{\alpha\beta}g_{\beta\gamma}$ is homotopic to $g_{\alpha\gamma}$ as a map of $U_\alpha \cap U_\beta \cap U_\gamma$ into $Haut(F)$.

In 1965, Wirth [203, 202] showed how a set of coherent higher homotopies arise on multiple intersections. He calls that set a *homotopy transition cocycle*. The disjoint union $\coprod U_\alpha$ can be given a rather innocuous structure of a topological category U, i.e., Ob $U = \coprod U_\alpha$ and Mor $U = \coprod U_\alpha \cap U_\beta$. Regarding $Haut(F)$ as a category with one object in the standard way, Wirth shows the transition cocycle web of higher homotopies is precisely equivalent to a homotopy coherent functor. For 'good' spaces, the usual classification of such fibrations is effected by the realization of this functor via a map $BU \to BHaut(F)$.

Having come this far in the A_∞ world, what about '∞-*geometry*'?

Recall Roytenberg and others define a *dg manifold* as a locally ringed space $(M, \mathcal{O}_M)$ (in dg commutative algebras over $\mathbb{R}$), which is locally isomorphic to $(U, \mathcal{O}_U)$, where $\mathcal{O}_U = C^\infty(U) \otimes S(V^\bullet)$ where $\{U\}$ is an open cover of M, $V^\bullet$ is a dg vector space and $S(V^\bullet)$ is the free graded commutative algebra. Again the transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to Aut\ S(V^\bullet)$ with respect to an open cover $\{U_\alpha\}$ of M satisfy the cocycle condition.

Notice that 'manifold' is irrelevant, a hold-over from the early days, but useful intuition for those trained in differential geometry; at most a topological space with a good open cover is needed. The coherent homotopy generalization of the definition of a dg-manifold is straightforward, but requires a coherent homotopy cocycle condition, as proposed here:

Definition 10.1. A *dg ∞-manifold* or *sh-manifold* is a locally ringed space $(M, \mathcal{O}_M)$ (in dg commutative algebras over $\mathbb{R}$), which is *locally* homotopy equivalent (as dcga's) to $(U, \mathcal{O}_U)$, where $\mathcal{O}_U = C^\infty(U) \otimes S(V^\bullet)$ with $\{U\}$ an open cover of M and $(S(V^\bullet), d)$ is free as graded commutative algebra.

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[3]As of this writing, the wiki is not up to date.

The higher analogs of classical transition functions with a cocycle condition are exactly Wirth's homotopy transition cocycles.

11. A_∞-structures and L_∞-structures in physics

There has not been as much presence of A_∞-structures in physics as of L_∞-structure, though Zwiebach followed with A_∞-algebras for open string field theory in 1997 [69], a few years after his L_∞-algebra for closed string field theory. Then, for open-closed string field theory, A_∞ and L_∞ were combined.

On the other hand, discovery by physicists of 'mirror symmetry' among Hodge numbers of dual Calabi-Yau manifolds led Kontsevich to propose *homological mirror symmetry* [109] in terms of A_∞-categories such as were presented by Fukaya [67] (see Section 10). His suggestions included extended moduli spaces and deformations of A_∞-categories. Calabi-Yau A_∞-structures are studied in the context of open topological string theory by Herbst, Lazaroiu and Lerche [87] and more generally by Costello [41].

11.1   String algebra.   In OSFT, strings are considered as paths in a manifold with interactions handled by "joining" two strings to form a third. This is often pictured in one of three ways (see [170] for graphics):

- E: endpoint interaction: (most familiar in mathematics as far back as the study of the fundamental group) occurs only when the end of one string agrees with the beginning of the other and the parameterization is adjusted appropriately,
- M: midpoint or half overlap interaction: occurs only when the last half of one agrees with the first half of the other as parameterized,
- V: variable overlap interaction: occurs only when, as parametrized, a last portion of one agrees with the corresponding first portion of the other.

The endpoint case E comes in two flavors:

- with paths parameterized by a fixed interval $[0, 1]$ in math or $[0, \pi]$ or $[0, 2\pi]$ in physics,
- with paths parameterized by intervals $[0, r]$ for $r \geq 0$ (see Section 3).

The midpoint case M was considered by Lashof [119] and later by Witten [204]; it has the advantage of being associative when defined. Note for the iterated composite of three strings the middle one disappears in the composite! For Lashof and topologists generally, the interaction was regarded as $a + b = c$ whereas Witten and physicists generally treat the interaction of three strings more symmetrically, cf. by reversing the orientation of the "composite": $a + b = -c$ or $a + b + c = 0$. One could call this a 'cyclic associative algebra'. According to Kontsevich, his reading of [170] led to defining *cyclic A_∞-algebras*.

The variable case V seems to have occurred first in physics in the work of Kaku [99]. Surprisingly, it is associative only up to homotopy but the pentagon relation holds on the nose! In physspeak, that is said as 3-string and 4-string vertices suffice [84].

The images of closed string interaction and open-closed string interaction are much more subtle.

11.2   String field theories.   But those are for strings; string fields are functions or forms on the space of strings and they form an algebra under the convolution product where the comultiplication on a string is the set of *decompositions* at arbitrary points in the parameterizing

interval. (An excellent and extensive 'bilingual' (math and physics) survey is given by Kajiura [96].)

One of the striking aspects of A_∞- and L_∞-algebras in physics is the use of an inner product $<,>$ for the *action functional*, an integral of a real or complex valued function of the fields. I first noticed this in Zwiebach's CSFT [206] for the classical (genus 0) action

$$\sum \int (\phi_0, \phi_1, \cdots, \phi_n)$$

where the ϕ's are string fields and the integrand is cyclically symmetric (up to sign).

The corresponding L_∞-structure is determined by

$$< v_0, \ l_n(v_1, \cdots, v_n) > = \ (v_0, v_1, \cdots, v_n).$$

Cyclic L_∞-algebras were formalized by Penkava [144].

As best I can determine, it was Kontsevich who first considered A_∞-algebras with an invariant inner product, known as *cyclic A_∞-algebras*. Retakh recalls he and Feigin discussed one of the talks at the Euler Workshop on Quantum Groups, Deformation Theory, and Representation Theory and a text associated with the talk and showed it to Kontsevich or as Kontsevich says:

B. Feigin, V. Retakh and I had tried to understand a remark of
J. Stasheff on open string theory and higher associative algebras.

This led to his [108]. "There's an operad for that"; see [200].

11.3 Open-Closed Homotopy Algebra and string field theory. Having considered both A_∞- and L_∞-algebras, plain and fancy, we come to the combination known as *OCHA* for Open-Closed Homotopy Algebra [97, 98]. Inspired by open-closed string field theories [207], OCHAs involve an L_∞-algebra acting by derivations (up to strong homotopy) on an A_∞-algebra but having an additional piece of structure corresponding to a closed string opening to an open string. The relevant operad (rather a colored operad with two colors - one for open and one for closed) is known as the "Swiss cheese operad" (see graphics in [195] for explanation of the name). The details are quite complicated in the original papers, but, just as other "∞" algebras can be characterized by a single coderivation on an appropriate dgc coalgebra, the same has been achieved for OCHAs by Hoefel [89]. A small example appears in [95].

A rich class of examples are provided by Landau-Ginzburg models, e.g. in the context of open topological string theory [36, 35].

Remark 11.1. Physicists will refer to 'topological' theories, usually meaning without a metric or with a metric but results independent of the choice of metric.

In addition, string theory and string field theory have inspired both string topology, initiated by Chas and Sullivan [38] and a further variety of ∞-algebras.

11.4 Scattering amplitudes. Scattering amplitudes in gauge theories are important 'observables'. Arkani-Hamed and his colleagues [10, 9, 8] were led to yet another polyhdedron, the *amplituhedron*, which is a generalization of the 'positive Grassmannian'. As we have seen, trees are often a starting point leading to more general structures and tree scattering amplitudes are a good place to start. In [8], they present a "novel construction of the associahedron in kinematic space".

12. Now and future

That brings us only somewhat up to date; there is much work in progress "as we come on the air" even in this small part of the space of higher structures.

I find in Manin's *Mathematics, Art, Civilization* [128]:

> With the advent of polycategories, enriched categories, A_∞-categories, and similar structures, we are beginning to speak a language. . . .

Now I find this delightfully ironic since, when I first submitted my theses for publication in AJM, they were deemed too narrow and essentially of no relation to other parts of math!

Perhaps Heraclitus was right: All is flux, nothing stays still. ☺

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