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Quillen-Segal Algebras and Stable Homotopy Theory

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Abstract

Let $\mathscr{M}$ be a monoidal model category that is also combinatorial. If $\mathcal{O}$ is a monad, operad, properad, or a PROP; following Segal's ideas we develop a theory of Quillen-Segal $\mathcal{O}$-algebras and show that we have a Quillen equivalence between usual $\mathcal{O}$-algebras and Quillen-Segal $\mathcal{O}$-algebras. We also introduce Quillen-Segal theories and we use them to obtain the *stable homotopy category* by a similar method to that of Hovey.

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1. Introduction

This paper is part of a project that aims to develop a homotopy theory of *weak algebraic structures* encoded by objects such as operads, properads, PROP's or monads. We hope that this theory can be useful to understand in some way the algebra of higher categories and its applications. The notion of symmetric operad was introduced by May [71] as a solution to the *delooping problem* in stable homotopy theory. Segal [86] gave a solution to the same problem, which is of a simplicial nature, with the notion of Γ-space. Both solutions have inspired different directions in the development of *Higher Category Theory*, and we shall refer the reader to the book of Simpson [87] for a detailed account on the subject.

We choose a general formalism that we hope will somehow be "a bridge" between the operadic and the simplicial method for the delooping problem. Let $\mathscr{M}$ be a model category and let $\text{Arr}(\mathscr{M})$ be its category of morphisms (or arrows). For $n \in \mathbb{N}$, define inductively $\text{Arr}^{n+1}(\mathscr{M}) = \text{Arr}(\text{Arr}^n(\mathscr{M}))$, the category of *hyper-cubes* in $\mathscr{M}$; with $\text{Arr}^0(\mathscr{M}) = \mathscr{M}$. In order to simplify the treatment of the homotopy theory in "Segal situations" we start with the following definition.

Definition. Let $\mathcal{C}$ be an arbitrary category.

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1. An $\mathscr{M}$-*valued Quillen-Segal theory* $\mathcal{T}$ on $\mathcal{C}$ is a family of functors called *Segal data functors*: $\{\mathcal{T}_i : \mathcal{C} \longrightarrow \mathrm{Arr}(\mathscr{M})\}_{i\in I}$, for some small set I.
2. Say that an object $c \in \mathcal{C}$ satisfies the *generalized Segal conditions*, if for every $i \in I$ $\mathcal{T}_i(c) \in \mathrm{Arr}(\mathscr{M})$ is a weak equivalence in $\mathscr{M}$.
3. Inductively, an *n-fold* $\mathscr{M}$-*valued Quillen-Segal theory* $\mathcal{T}$ on $\mathcal{C}$ is a family of functors, called *Segal n-data functors*: $\{\mathcal{T}_i : \mathcal{C} \longrightarrow \mathrm{Arr}^n(\mathscr{M})\}_{i\in I}$, for some small set I.

In the original paper of Segal [86], $\mathcal{C}$ is the category of Γ-spaces, $\mathscr{M} = \mathbf{Top}$ and the theory is given by the family functors $\{\mathcal{T}_n : \mathcal{C} \longrightarrow \mathrm{Arr}(\mathbf{Top})\}_{n\geq 1}$, where $\mathcal{T}_n(\mathcal{A})$ is the nth *Segal map*: $p_n : \mathcal{A}(n) \longrightarrow \mathcal{A}(1)\times_{\mathcal{A}(0)}\cdots\times_{\mathcal{A}(0)}\mathcal{A}(1)$. The same formula defines the theory for *Segal spaces* as in Rezk [79], and classical Segal categories (see [18, 77, 87, 94]). These theories have been extended to Segal n-categories and Segal enriched $\mathscr{M}$-categories (see [6, 87]). One can iterate the process *à la Simpson-Tamsamani* to consider a $\mathcal{C}$-valued theory on a category $\mathcal{D}$: $\{\mathcal{T}'_j : \mathcal{D} \longrightarrow \mathrm{Arr}(\mathcal{C})\}_{j\in J}$ to get a 2-fold $\mathscr{M}$-valued theory $\{\mathrm{Arr}(\mathcal{T}_i)\circ \mathcal{T}'_j : \mathcal{D} \longrightarrow \mathrm{Arr}^2(\mathscr{M})\}_{(i,j)\in I\times J}$ and so on.

There are many categories that are equipped with a relevant Quillen-Segal theory, and some of them will be reviewed later. But in this paper we want to extend Segal's formalism to the following situations.

1. Let $\mathcal{C} = \mathcal{S}\mathrm{p}_{\Omega}(\mathbb{N}, \mathbf{sSet}_*)$ be the category of Ω-prespectra with the theory given by the functors $\mathcal{T}_n : \mathcal{C} \longrightarrow \mathrm{Arr}(\mathbf{sSet}_*)$ that take a prespectrum X to the connecting morphism $X_n \longrightarrow \Omega(X_{n+1})$ which is adjoint to the map $S^1 \wedge X_n \longrightarrow X_{n+1}$. Then a fibrant prespectrum satisfying the Segal conditions is simply an Ω-spectrum, thus a generalized cohomology theory.
2. Let $\mathcal{C} = (\mathscr{M} \downarrow \mathcal{U})$ be the comma category associated to a (right Quillen) functor $\mathcal{U} : \mathscr{A} \longrightarrow \mathscr{M}$. We remind the reader that an object of $(\mathscr{M} \downarrow \mathcal{U})$ is a triple $[\mathcal{F}] = [\mathcal{F}_0, \mathcal{F}_1, \pi_{\mathcal{F}} : \mathcal{F}_0 \longrightarrow \mathcal{U}(\mathcal{F}_1)] \in \mathscr{M} \times \mathscr{A} \times \mathrm{Arr}(\mathscr{M})$. The theory is given by the functor that projects the maps $\pi_{\mathcal{F}}$:
$$\Pi_{\mathrm{Arr}} : (\mathscr{M} \downarrow \mathcal{U}) \longrightarrow \mathrm{Arr}(\mathscr{M}), \quad \text{with} \quad \Pi_{\mathrm{Arr}}([\mathcal{F}]) = \pi_{\mathcal{F}}.$$

If $\mathcal{O}$ is an operad, monad, or a PROP; a *Quillen-Segal* $\mathcal{O}$-*algebra* is an object $[\mathcal{F}] \in \mathcal{C} = (\mathscr{M} \downarrow \mathcal{U})$, that satisfies the Segal condition for the forgetful functor $\mathcal{U} : \mathcal{O}\text{-}Alg(\mathscr{M}) \longrightarrow \mathscr{M}$. The first example of Quillen-Segal algebras comes from *co-Segal algebras* and their generalizations to co-Segal categories [6]. Recall that loop spaces $\Omega_*(X)$ are the first examples of genuine Segal 1-categories with one object (Segal algebras). It turns out that $\Omega_*(X)$ is also a Quillen-Segal algebra (see Example 3.4).

Before discussing further the theory of Quillen-Segal algebras we have the following theorem on Quillen-Segal theories in general (see Theorem 2.7).

Theorem. *Let* $\mathscr{M}$ *be a cofibrantly generated model category and let* $\mathcal{T} = (\mathcal{T}_i)$ *be a Quillen-Segal theory on* $\mathcal{C}$*. Assume that one the following conditions holds.*

1. $\mathcal{C}$ *has a model structure* $(\mathscr{W}, \mathbf{cof}, \mathbf{fib})$ *which is cellular and left proper such that every* $\mathcal{T}_i \in \mathrm{Hom}(\mathcal{C}, Arr(\mathscr{M})_{inj})$ *is right Quillen.*
2. $\mathcal{C}$ *has a model structure* $(\mathscr{W}, \mathbf{cof}, \mathbf{fib})$ *which is combinatorial and left proper such that every* $\mathcal{T}_i \in \mathrm{Hom}(\mathcal{C}, Arr(\mathscr{M})_{inj})$ *is right Quillen.*

Then there is a new model structure $\mathcal{C}_B(\mathcal{T}) = (\mathscr{W}_B, \mathbf{cof}_B, \mathbf{fib}_B)$ *on* $\mathcal{C}$ *which is a left Bousfield localization, such that:*

- $\mathcal{T} = \{\mathcal{T}_i : \mathcal{C} \longrightarrow Arr(\mathscr{M})_{\mathrm{inj}}\}_{i\in I}$ *defines an Quillen-Segal theory such that every* $\mathcal{T}_i$ *is also right Quillen with respect to the model structure* $\mathcal{C}_B(\mathcal{T})$*;*

- *every fibrant object in this new model structure satisfies the generalized Segal conditions.*

Here $\text{Arr}(\mathscr{M})_{inj}$ is the *injective model* structure which can be found for example in Hovey [43]. There is also a *projective model* structure on $\text{Arr}(\mathscr{M})$ denoted by $\text{Arr}(\mathscr{M})_{proj}$. We will review them later in a more general context. We also have the following theorem if the cofibrations in $\mathscr{M}$ are generated by a set of maps between cofibrant objects. *Tractable* model categories in the sense of Barwick [8] are examples. The following theorem is valid outside tractable model categories (see Theorem 2.9).

Theorem. *Let $\mathscr{M}$ be a tractable model category and let $\mathcal{T} = (\mathcal{T}_i)$ be a Quillen-Segal theory on $\mathcal{C}$. Assume that one the following conditions holds.*

1. *$\mathcal{C}$ has a model structure $(\mathscr{W}, \mathbf{cof}, \mathbf{fib})$ which is cellular and left proper such that every $\mathcal{T}_i \in \text{Hom}(\mathcal{C}, Arr(\mathscr{M})_{proj})$ is right Quillen.*
2. *$\mathcal{C}$ has a model structure $(\mathscr{W}, \mathbf{cof}, \mathbf{fib})$ which is combinatorial and left proper such that every $\mathcal{T}_i \in \text{Hom}(\mathcal{C}, Arr(\mathscr{M})_{proj})$ is right Quillen.*

Then there is a new model structure $\mathcal{C}_B(\mathcal{T}) = (\mathscr{W}_B, \mathbf{cof}_B, \mathbf{fib}_B)$ on $\mathcal{C}$ which is a left Bousfield localization, such that:

- *$\mathcal{T} = \{\mathcal{T}_i : \mathcal{C} \longrightarrow Arr(\mathscr{M})_{\text{proj}}\}_{i \in I}$ defines an Quillen-Segal theory such that every $\mathcal{T}_i$ is also right Quillen with respect to the model structure $\mathcal{C}_B(\mathcal{T})$;*
- *every fibrant object in this new model structure satisfies the generalized Segal conditions.*

These theorems require the left properness of the model structure on $\mathcal{C}$. However it is possible to have a localization of a theory without left properness using a result of Beke [12] which is itself a consequence of a theorem of Jeff Smith. This is what we do for the comma category $\mathscr{M}_{\mathcal{U}}[\mathscr{A}] := (\mathscr{M} \downarrow \mathcal{U})$ and in particular for Quillen-Segal algebras when $\mathscr{M}$ is a combinatorial model category.

First we prove that:

1. There is an injective model structure on the comma category $\mathscr{M}_{\mathcal{U}}[\mathscr{A}] := (\mathscr{M} \downarrow \mathcal{U})$ (Theorem 5.12)
2. There is a projective model structure on $\mathscr{M}_{\mathcal{U}}[\mathscr{A}] := (\mathscr{M} \downarrow \mathcal{U})$ (Theorem 5.19). These model structures are classical but we include them because they play an important role and we want to make the paper almost self contained.
3. If $\mathcal{U} = \text{Id}$ we recover the injective and projective model structure on $\text{Arr}(\mathscr{M})$.

Category theory provides an embedding $\iota : \mathscr{A} \longrightarrow (\mathscr{M} \downarrow \mathcal{U})$ with $\iota(\mathcal{P}) = [\mathcal{U}(\mathcal{P}), \mathcal{P}, \text{Id}_{\mathcal{U}(\mathcal{P})}]$ (see Proposition 4.1); and this is how usual strict algebras are QS-algebras ($\text{Id}_{\mathcal{U}(\mathcal{P})}$ is always a weak equivalence). We have a functor $\Pi_1 : (\mathscr{M} \downarrow \mathcal{U}) \longrightarrow \mathscr{A}$ that is simultaneously a left adjoint and a retraction for ι. The object $\mathcal{F}_1$ is a usual $\mathcal{O}$-algebra and with the weak equivalence $\pi_{\mathcal{F}} : \mathcal{F}_0 \longrightarrow \mathcal{U}(\mathcal{F}_1)$, one may want to lift the $\mathcal{O}$-algebra structure to $\mathcal{F}_0$. This is a classical problem of *homotopy invariance for algebras* that goes back to Boardman and Vogt [22], Dwyer, Kan and Smith [33] and others. There are many results in this direction which can be found for example in Berger-Moerdijk [13], Johnson-Yau [48] and the many references therein. We will discuss this in a future work.

One of the central results on the homotopy theory of $(\mathscr{M} \downarrow \mathcal{U})$ is that we can localize directly the injective and projective model structure along the functor $\Pi_1 : (\mathscr{M} \downarrow \mathcal{U}) \longrightarrow \mathscr{A}$. We have the following result (see Theorem 5.17 and Theorem 5.23).

Theorem. *Let $\mathcal{U} : \mathscr{A} \leftrightarrows \mathscr{M} : \mathbf{F}$ be a Quillen adjunction between combinatorial model categories where $\mathcal{U}$ is right adjoint. Then the following hold.*

1. *There is a model structure on the category* $(\mathscr{M} \downarrow \mathcal{U}) = \mathscr{M}_{\mathcal{U}}[\mathscr{A}]$ *such that:*
   (a) *fibrant objects satisfy the Segal condition and*
   (b) *we have a Quillen equivalence* $\iota : \mathscr{A} \leftrightarrows \mathscr{M}_{\mathcal{U}}[\mathscr{A}] : \Pi_1$, *where* ι *is right Quillen and* $\Pi_1([\mathcal{F}]) = \mathcal{F}_1$.
2. *If the Quillen pair* $\mathcal{U} : \mathscr{A} \leftrightarrows \mathscr{M} : \mathbf{F}$ *is a Quillen equivalence then the functor*

$$\Pi_0 : \mathscr{M}_{\mathcal{U}}[\mathscr{A}] \longrightarrow \mathscr{M}$$

is a right Quillen equivalence and $\mathcal{U}$ *is a composite of Quillen equivalences:*

$$\mathscr{A} \xrightarrow{\iota} \mathscr{M}_{\mathcal{U}}[\mathscr{A}] \xrightarrow{\Pi_0} \mathscr{M}.$$

If $\mathscr{M}$ is tractable we can show that this localization is in fact a left Bousfield localization. The proof is just a consequence of Ken Brown's lemma. We get the homotopy theory of Quillen-Segal algebras as a corollary of the previous theorem as follows (see Theorem 6.1).

Theorem. *Let* $\mathscr{M}$ *be a combinatorial model category and let* $\mathcal{O}$ *be an operad enriched over* $\mathscr{M}$ *or a monad on* $\mathscr{M}$. *Let* $\mathcal{U} : \mathcal{O}\text{-}Alg(\mathscr{M}) \longrightarrow \mathscr{M}$ *be the forgetful functor. Then the following hold.*

1. *The transferred model structure on* $\mathcal{O}\text{-}Alg(\mathscr{M})$ *exists if and only if the projective and the injective model structure on* $\mathscr{M}_{\mathcal{U}}[\mathcal{O}\text{-}Alg(\mathscr{M})] := (\mathscr{M} \downarrow \mathcal{U})$ *exist.*
2. *In the latter case there is a model structure on* $\mathscr{M}_{\mathcal{U}}[\mathcal{O}\text{-}Alg(\mathscr{M})]$ *such that:*
   (a) *fibrant objects are Quillen-Segal* $\mathcal{O}$*-algebras*
   (b) *the adjunction* $\iota : \mathcal{O}\text{-}Alg(\mathscr{M}) \leftrightarrows \mathscr{M}_{\mathcal{U}}[\mathcal{O}\text{-}Alg(\mathscr{M})] : \Pi_1$, *is a Quillen equivalence where* ι *is right Quillen.*

Other interesting situations for Theorem 5.17 and Theorem 5.23 occur when $\mathcal{U}$ is a not simply a forgetful functor but an arbitrary right Quillen functor $\mathcal{O}\text{-}Alg(\mathscr{M}) \longrightarrow \mathcal{O}'\text{-}Alg(\mathscr{M})$. Among such functors, we have the ones that arise from a map of operads $\Phi : \mathcal{O}' \longrightarrow \mathcal{O}$. Indeed, given such map we have a functor $\mathcal{U} = \Phi^\star : \mathcal{O}\text{-}Alg(\mathscr{M}) \longrightarrow \mathcal{O}'\text{-}Alg(\mathscr{M})$ and the previous theorems apply (see Remark 6.3).

The relation with Stable Homotopy Theory comes when $\mathscr{A} = \mathscr{M}$ and we consider the product $\prod_{n\in\mathbb{Z}} \mathscr{M}_{\mathcal{U}}[\mathscr{M}]$ for an endofunctor $\mathcal{U} : \mathscr{M} \longrightarrow \mathscr{M}$ such as the loop space functor $\Omega : \mathbf{sSet}_* \longrightarrow \mathbf{sSet}_*$. This product is in fact a functor category $\operatorname{Hom}(\mathbb{Z}_{\text{disc}}, \mathscr{M}_{\mathcal{U}}[\mathscr{M}])$ where $\mathbb{Z}_{\text{disc}}$ is the set of integers regarded as a discrete category and *not* as posetal category. The category $\mathcal{S}\mathrm{p}_{\mathcal{U}}(\mathbb{Z}, \mathscr{M})$, of $\mathcal{U}$-prespectra, is equivalent to a category of objects in $\operatorname{Hom}(\mathbb{Z}_{\text{disc}}, \mathscr{M}_{\mathcal{U}}[\mathscr{M}])$ that are *linked* in the sense of Definition 7.7. This link condition is just a 2-pullback condition, which can be considered as a *descent* condition in some sense.

What is important and original here is the approach that we choose to get the stable model structure. The result itself is well known and is about the existence of a model structure that is Quillen equivalent to the model structure of Bousfield and Friedlander. More precisely, we put to use the formalism of Quillen-Segal theories to get this model structure by a similar method as Hovey [45], but still different. Our goal here was to show that spectra satisfy some generalized Segal conditions and that the existing methods used to produce the homotopy theory of Segal categories also work for Ω-spectra.

In particular the analogue of a *Segalification functor* produces here an Ω*-spectrification*. We show that if we proceed by a *dévissage* then the *strict projective model structure* of Bousfield and Friedlander [24], follows directly from the projective (=Reedy) model structure on $(\mathscr{M} \downarrow \mathcal{U}) = \mathscr{M}_{\mathcal{U}}[\mathscr{M}]$ in Theorem 5.19.

It is worth noticing that working with $\text{Hom}(\mathbb{Z}_{\text{disc}}, \mathscr{M}_{\mathcal{U}}[\mathscr{M}])$ presents some advantages. Both spectra and chain complexes can be defined as objects of $\text{Hom}(\mathbb{Z}_{\text{disc}}, \mathscr{M}_{\mathcal{U}}[\mathscr{M}])$ that satisfies some conditions: the link conditions for spectra and the chain conditions for chain complexes (Definition 7.27). Moreover we can restrict to finite ordinals $\mathbb{O} \subset \mathbb{Z}$ and consider for future purposes *perfect complexes* (see [95]). We can put model structures on finite linked sequences such that the fibrant objects satisfy (some finite) Segal conditions; and we know by May's recognition theorem that these fibrant objects are in some cases E_n-algebras. Applications for such considerations included iterated weak enrichment over spectra (spectral categories) and over the category of chain complexes (dg-categories). But for the moment we set up first the foundations. These applications will appear in the subsequent papers.

We have a functor $P : \mathcal{S}\text{p}_{\mathcal{U}}(\mathbb{Z}, \mathscr{M}) \longrightarrow \text{Hom}(\mathbb{Z}_{\text{disc}}, \mathscr{M}_{\mathcal{U}}[\mathscr{M}])$ that forgets the links and this functor has a left adjoint (Proposition 7.12). We use this (monadic) adjunction to get various model structures on the category of $\mathcal{U}$-prespectra. If $\mathscr{M}$ is locally presentable, there are four model structures on $\mathcal{S}\text{p}_{\mathcal{U}}(\mathbb{Z}, \mathscr{M})$ that follow from the injective and projective model structures on $\text{Hom}(\mathbb{Z}_{\text{disc}}, \mathscr{M}_{\mathcal{U}}[\mathscr{M}]_{inj})$ and $\text{Hom}(\mathbb{Z}_{\text{disc}}, \mathscr{M}_{\mathcal{U}}[\mathscr{M}]_{proj})$. All of them are combinatorial and left proper if $\mathscr{M}$ is in addition left proper. In this paper we restrict to the case where $\mathscr{M}$ is combinatorial and left proper to keep the paper short. We will discuss in a different paper the case where $\mathscr{M}$ is cellular. The arguments remain essentially the same.

We show in virtue of Theorem 2.9 that:

Theorem. *Let $\mathcal{C} = \mathcal{S}p_{\Omega}(\mathbb{N}, \mathbf{sSet}_*)$ be the category of prespectra in simplicial sets with the Quillen-Segal theory $\mathcal{T}_n : \mathcal{C} \longrightarrow Arr(\mathbf{sSet}_*)$ given by $\mathcal{T}_n(X) = [X_n \longrightarrow \Omega(X_{n+1})]$.*

1. *Then there is a model structure on $\mathcal{C}$ such that the fibrant objects are the Ω-spectra X that are level-wise fibrant (Kan).*
2. *The homotopy category is equivalent to the stable homotopy category of Bousfield-Friedlander obtained by Hovey [45] and Schwede [84].*

Applications. There are various applications of the theory that is being developed here. We outline for the moment some directions of interest. They will appear in the subsequent papers.

1. Our interest in comma categories comes when we consider a right Quillen $\mathcal{U} : \mathscr{A} \longrightarrow \mathscr{M}$ that is a Quillen equivalence. An important example is the homotopy coherent nerve of Cordier and Porter [28] from simplicial categories to quasicategories. Joyal showed that this right Quillen functor is in fact a Quillen equivalence (see Bergner [17, 19]). Our theorem shows that the comma category, which lives in between, is Quillen equivalent to both simplicial categories and quasicategories. On the one hand, Lurie and Joyal gave a significant amount of work on quasicategories. And on the other hand, simplicial categories are **sSet**-enriched categories, and there is a lot in the literature on the subject. A classical reference is the book of Kelly [54]. We would like to understand how the existing constructions for enriched categories such as weighted limits, colimits, adjunctions and Cauchy completions, interact with the corresponding notions introduced by Joyal and Lurie.
2. Following the ideas of Goerss-Jardine [37], Jardine [46], Hirshowitz-Simpson [41], Morel-Voevodsky [74], Toën-Vezzozi [98] and others, the study of presheaves on a Grothendieck site with coefficients in $\text{Hom}(\mathbb{Z}_{\text{disc}}, \mathscr{M}_{\mathcal{U}}[\mathscr{M}])$ has applications in motivic homotopy theory. Indeed we can obtain presheaves of spectra this way.
3. The category $\mathscr{M}_{\mathcal{U}}[\mathscr{M}]$ and more generally $\mathscr{M}_{\mathcal{U}}[\mathscr{A}]$ can be endowed with different monoidal structures if $\mathcal{U}$ is (lax) monoidal. When $\mathcal{U}$ is the identity we have the point-wise product

an and the pushout-product that are considered in Hovey [44]. These constructions will be transported to functor categories $\mathrm{Hom}(\mathbb{Z}_{\mathrm{disc}}, \mathscr{M}_{\mathcal{U}}[\mathscr{M}])$ and $\mathrm{Hom}(\mathbb{O}_{\mathrm{disc}}, \mathscr{M}_{\mathcal{U}}[\mathscr{M}])$ for future applications.

4. The theory of operads, properads, PROP's and algebras over them has applications in several fields of mathematics (see for example Loday-Vallette [63], Markl-Shnider-Stasheff [70]). Many of the existing constructions for usual algebras can be transported to Quillen-Segal algebras. When we consider a general model category, the problem of homotopy invariance for algebras does not always have a solution. It turns out that this problem is intimately related to a conjecture of Simpson about *weak units* in Higher categories (see Joyal-Kock [50]). We will address this later.
5. One of the consequences of this work is a *strictification* theorem for co-Segal algebras and co-Segal categories. This is the analogue of a theorem of Bergner [16] for the strictification of Segal categories.

Organization of the paper

- Section 2 is about Quillen-Segal theories. The main result is the "Bousfield localization of the theory".
- We introduce Quillen-Segal algebras in Section 3.
- Section 4 contains the category theory of comma constructions.
- In Section 5, we develop the homotopy theory on the comma category $(\mathscr{M} \downarrow \mathcal{U})$. We follow closely the same method as Hovey [43] to get an *injective* and *projective* model structure. Then we localize them to get the correct model structure in which the fibrant objects satisfy the Segal condition
- In Section 6 we apply the results of Section 5 to have the homotopy theory for Quillen-Segal algebras (QS-algebras for short).
- In Section 7, we use the formalism of Quillen-Segal theories to get the stable model structure.

Related works. Model structures on comma categories have been discussed by Stanculescu [88, Section 6.5], Toën [96] and others. However they consider instead the (equivalent) comma categories $(\mathcal{U} \downarrow \mathscr{M})$, where $\mathcal{U}$ is a left Quillen functor. Considering other motivations, Stanculescu focused on the dual notion of what we call a Quillen-Segal object, in that one would demand $\mathcal{U}(\mathcal{F}_0) \longrightarrow \mathcal{F}_1$ to be a weak equivalence. But in the case of algebras, we will capture the homotopy theory of free algebras. He also mentioned that it could be interesting to study the comma category $(\mathscr{M} \downarrow \mathcal{U})$ and consider what we call a Quillen-Segal object.

The general philosophy of this paper is also close to that of Hirschowitz-Simpson [41], Hollander [42], Jardine [46], Joyal-Tierney [51], Stanculescu [89] and many others. By this we mean that the theory of *higher stacks* is a theory of fibrant objects, and these fibrant objects satisfy some generalized Segal conditions with respect to a certain Quillen-Segal theory.

There is a lot that is left to be done on Quillen-Segal algebras. For example we know that operads, properads , PROPs, are themselves algebras over a specific monad $\mathscr{P}$ (see for example Garner-Hirschowitz [36]). It could be interesting to study the link between Quillen-Segal $\mathscr{P}$-algebras and the notions of ∞-operads, dendroidal sets and Segal operads that have been introduced lately (see [21, 27, 65, 73]).

When we have a Quillen-Segal theory $\mathcal{T} = \{\mathcal{T}_i\}_{i \in I}$ on $\mathcal{C}$, depending on the context, we might not want to impose the Segal conditions for the whole set I but only with respect to a subset

$I' \subseteq I$. For example, when we consider the nerve of a classical small 1-category, it satisfies the Segal conditions at every stage, but we do not need to require them for all $n \geq 2$ to capture the category structure. This is due to the presence of simplicial operations together with the fact that the Segal maps are isomorphisms of sets. Another important example is in the theory of simplicial presheaves as in Jardine [46]. There are different Quillen-Segal theories given by Čech nerves and hypercovers (see Example 2.4).

2. Quillen-Segal theories

The goal of this section is to provide a general framework that is commonly used to produce a model structure on some category $\mathcal{C}$ such that the fibrant objects satisfy *some generalized Segal conditions*.

Let $\mathscr{M}$ be a general model category, which will in many cases be a monoidal model category in the sense of Hovey [44]. We will denote by $\text{Arr}(\mathscr{M})$ its category of morphisms, also called the *arrow category*. Hovey [43] provides the **injective and projective** model structures on $\text{Arr}(\mathscr{M})$. In both model structure the weak equivalences are the objective-wise weak equivalences. We will denote them by $\text{Arr}(\mathscr{M})_{\text{inj}}$ and $\text{Arr}(\mathscr{M})_{\text{proj}}$ and they will be reviewed in Section 2.2.1 and later in Section 5. The category $\text{Arr}(\mathscr{M})$ is a diagram category and these model structures are particular case of Reedy model structures which have been widely discussed in the literature (see [30, 38, 40, 87]).

Definition 2.1. Let $\mathcal{C}$ be an arbitrary category.

1. An $\mathscr{M}$*-valued Quillen-Segal theory* $\mathcal{T}$ on $\mathcal{C}$ is a family of functors called *Segal data functors*: $\{\mathcal{T}_i : \mathcal{C} \longrightarrow \text{Arr}(\mathscr{M})\}_{i\in I}$, for some small set I.
2. Say that an object $c \in \mathcal{C}$ satisfies the *generalized Segal conditions*, if for every $i \in I$ $\mathcal{T}_i(c) \in \text{Arr}(\mathscr{M})$ is a weak equivalence in $\mathscr{M}$.
3. Inductively, an *n-fold $\mathscr{M}$-valued Quillen-Segal theory* $\mathcal{T}$ on $\mathcal{C}$ is a family of functors, called *Segal n-data functors*: $\{\mathcal{T}_i : \mathcal{C} \longrightarrow \text{Arr}^n(\mathscr{M})\}_{i\in I}$, for some small set I.

Example 2.2. The first example comes of course from Γ-spaces and Segal n-categories. We shall outline the idea for classical Segal categories. Let **sSet** be the category of simplicial sets and set $I = \{n \in \text{Ob}(\Delta), n \geq 1\}$ and let $\mathcal{C} = \text{Hom}(\Delta^{op}, \mathbf{sSet})$ be the category of simplicial spaces. For $n \geq 1$, let $\mathcal{T}_n : \mathcal{C} \longrightarrow \text{Arr}(\mathbf{sSet})$ be the functor that takes a simplicial space $\mathcal{A}$ to the nth Segal map: $p_n : \mathcal{A}(n) \longrightarrow \mathcal{A}(1) \times_{\mathcal{A}(0)} \cdots \times_{\mathcal{A}(0)} \mathcal{A}(1)$. As mentioned before, the same formulas apply for Segal spaces [79].

Example 2.3. Leinster [60] introduced up-to-homotopy monoids as colax monoidal functors $\mathcal{X} : (\Delta^+, +, 0) \longrightarrow (\underline{M}, \otimes, I)$ such that for every $n, m \in \Delta^+$ the colaxity map $\varphi_{n,m} : \mathcal{X}(n+m) \longrightarrow \mathcal{X}(n) \otimes \mathcal{X}(m)$ and $\varphi_0 : \mathcal{X}(\mathbf{0}) \longrightarrow I$ are weak equivalences. In this case the theory is given by the functors $\mathcal{T}_{n,m}$ defined by $\mathcal{T}_{n,m}(\mathcal{X}) = \varphi_{n,m}$ and $\mathcal{T}_0(\mathcal{X}) = \varphi_0$. We have generalized this notion to Segal enriched categories [4]. Moreover, the homotopy theory is difficult because the functors $\mathcal{T}_{n,m}$ are not right adjoints for general tensor products $\otimes \neq \times$. We will discuss this in [5].

Example 2.4. Let $\mathcal{C} = SPr(\mathscr{X})$ be the category of simplicial presheaves on a Grothendieck site $\mathscr{X}$ as in Jardine [46]. Given an object U with a covering $\mathcal{U} = \{U_l \longrightarrow U\}$, we can form the corresponding Čech nerve $C_\bullet(\mathcal{U})$ (see Artin-Mazur [3], Jardine [47]). Then we have a Quillen-Segal theory $\mathcal{T} = \{\mathcal{T}_{C_\bullet(\mathcal{U})} : \mathcal{C} \longrightarrow \text{Arr}(\mathbf{sSet})\}_{C_\bullet(\mathcal{U})}$ that maps a simplicial presheaf $\mathcal{F}$ to the map:

$$\mathcal{F}(U) \longrightarrow (ho) \lim \mathcal{F}(C_\bullet(\mathcal{U})).$$

An object $\mathcal{F}$ that satisfies the Segal conditions in the sense of Definition 2.1 is sometimes called a *higher stack*. We have a similar definition for hyper-coverings.

2.1 Injective and projective theories

Definition 2.5. Let $\mathcal{C}$ be a model category.

1. An **injective $\mathscr{M}$-valued Quillen-Segal theory** on $\mathcal{C}$ is a family of right Quillen functors, called *Segal data functors*:
$$\{\mathcal{T}_i : \mathcal{C} \longrightarrow \mathrm{Arr}(\mathscr{M})_{\mathrm{inj}}\}_{i \in I}$$
for some small set I.
2. A **projective $\mathscr{M}$-valued Quillen-Segal theory** on $\mathcal{C}$ is a family of right Quillen functors, called *Segal data functors*:
$$\{\mathcal{T}_i : \mathcal{C} \longrightarrow \mathrm{Arr}(\mathscr{M})_{\mathrm{proj}}\}_{i \in I}$$
for some small set I.
3. We will say that there is an $\mathscr{M}$-valued Quillen-Segal theory on $\mathcal{C}$ if there is a projective or an injective $\mathscr{M}$-valued Quillen-Segal theory on $\mathcal{C}$.

We remind the reader that part of being a right Quillen functor means that there is a left adjoint $\Upsilon_i : \mathrm{Arr}(\mathscr{M}) \longrightarrow \mathcal{C}$ to each of the Segal data functor $\mathcal{T}_i$. It also implies that Υ_i is automatically left Quillen, in that it preserves the cofibrations and the trivial cofibrations in $\mathrm{Arr}(\mathscr{M})$.

Notation 2.6. If $\mathcal{C}$ is a model category we will denote by $\mathscr{W}$, **cof**, **fib**, the three classes of weak equivalences, cofibrations, and fibrations, respectively.

Theorem 2.7. *Let $\mathscr{M}$ be a cofibrantly generated model category and let $\mathcal{C}$ be a category. Assume that one the following conditions holds.*

1. *$\mathcal{C}$ has a model structure $(\mathscr{W}, \mathbf{cof}, \mathbf{fib})$ which is cellular and left proper.*
2. *$\mathcal{C}$ has a model structure $(\mathscr{W}, \mathbf{cof}, \mathbf{fib})$ which is combinatorial and left proper.*

Then for any injective Quillen-Segal theory $\mathcal{T} = (\mathcal{T}_i)$ on $\mathcal{C}$, there is a new model structure $\mathcal{C}_B(\mathcal{T}) = (\mathscr{W}_B, \mathbf{cof}_B, \mathbf{fib}_B)$ on $\mathcal{C}$ which is a left Bousfield localization, such that:

- *$\mathcal{T} = \{\mathcal{T}_i : \mathcal{C} \longrightarrow \mathrm{Arr}(\mathscr{M})_{\mathrm{inj}}\}_{i \in I}$ defines an injective Quillen-Segal theory with respect to the model structure $\mathcal{C}_B(\mathcal{T})$ and;*
- *every fibrant object in this new model structure satisfies the generalized Segal conditions.*

We will give the proof at the end of the section. For a projective Quillen-Segal theory, we need to recall the definition of a tractable model category as in Barwick [8].

Definition 2.8. A cofibrantly generated model category $\mathscr{M}$ is **tractable** if $\mathscr{M}$ is locally presentable and both cofibrations and trivial cofibrations are generated by a set of morphisms between cofibrant objects.

Theorem 2.9. *Let $\mathscr{M}$ be a tractable model category and let $\mathcal{C}$ be a category. Assume that one the following conditions holds.*

1. *$\mathcal{C}$ has a model structure $(\mathscr{W}, \mathbf{cof}, \mathbf{fib})$ which is cellular and left proper.*

2. *$\mathcal{C}$ has a model structure $(\mathscr{W}, \mathbf{cof}, \mathbf{fib})$ which is combinatorial and left proper.*

Then for any projective theory $\mathcal{T} = \{\mathcal{T}_i : \mathcal{C} \longrightarrow Arr(\mathscr{M})_{\text{proj}}\}_{i \in I}$ on $\mathcal{C}$ with respect to the model structure $(\mathscr{W}, \mathbf{cof}, \mathbf{fib})$, there is a new model structure $\mathcal{C}_B(\mathcal{T}) = (\mathscr{W}_B, \mathbf{cof}_B, \mathbf{fib}_B)$ on $\mathcal{C}$ which is a left Bousfield localization such that:

- *$\mathcal{T} = \{\mathcal{T}_i : \mathcal{C} \longrightarrow Arr(\mathscr{M})_{\text{proj}}\}_{i \in I}$ defines a projective theory with respect to the model structure $\mathcal{C}_B(\mathcal{T})$ and;*
- *every fibrant object in this new model structure satisfies the generalized Segal conditions.*

The guiding principle in proving Theorem 2.7 and Theorem 2.9 is to develop some techniques that allow one to perform two tasks: the first task is to detect when an object satisfies the Segal conditions; and the second task is to have a functorial process that takes an object c and creates an object $\mathscr{S}(c)$ that satisfies the Segal conditions. The first task amounts to knowing when a morphism $f = \mathcal{T}_i(c) \in \text{Arr}(\mathscr{M})$ is a weak equivalence in $\mathscr{M}$. There are three known techniques to detect this when $\mathscr{M}$ is cofibrantly generated and we outline them briefly hereafter.

1. f will be a weak equivalence if we can show that f has the RLP with respect to the generating set of cofibrations $\mathbf{I}$ of $\mathscr{M}$. In that case f is in fact a trivial fibration (see for example Hovey [44]).
2. If f is a map between fibrant objects, f will be a weak equivalence if we can show that it satisfies the *Homotopy Extension Lifting Property (HELP)* with respect to the elements of $\mathbf{I}$ (see for example Simpson [87], Vogt [102]).
3. If $f : X \longrightarrow Y$, we can use *function complexes* and check whether $\text{Map}(C, f) : \text{Map}(C, X) \longrightarrow \text{Map}(C, Y)$ is a weak equivalence of simplicial sets, as C runs through the set of domains and codomains of maps in $\mathbf{I}$ (see for example Hovey [45]).

With these techniques in mind, it suffices to provide a set of maps $\mathbf{K_I}$ and consider the left Bousfield localization with respect to $\mathbf{K_I}$ such that by adjunction, being $\mathbf{K_I}$-local as in [40, Definition 3.2.4] forces our map $f = \mathcal{T}_i(c)$ to be a weak equivalence in $\mathscr{M}$ through one of the above techniques. A $\mathbf{K_I}$-localization in the sense of [40, Definition 3.3.11] will allow by *the small object argument* of Quillen to perform the second task, that is, to produce a functor $\mathscr{S}$ with a natural transformation $\text{Id} \longrightarrow \mathscr{S}$ (a fibrant replacement functor) such that $\mathscr{S}(c)$ satisfies the generalized Segal conditions.

These techniques are now standard in Homotopy theory and go back to Bousfield-Friedlander [24], Jardine [46], Joyal [49], Kan [53] and others. Simpson [87, Chapter 7.7] used the first two techniques for Segal categories where the functor $\mathscr{S}$ is a *Segalification functor*. Hovey [45, Proposition 3.2] used the third technique for spectra, where the functor $\mathscr{S}$ is weakly equivalent to Ω-*spectrification* as in Bousfield-Friedlander [24].

In this paper we use the first two techniques like Simpson to prove Theorem 2.7 and Theorem 2.9. As a consequence we get, among other things, a generalized *Segalification functor* that encompasses the Ω-*spectrification*. Hovey's method can also be used in our context and we will explain later the connection between his approach and ours. We will produce two sets of maps $\mathbf{K_I}_{inj}$ and $\mathbf{K_I}_{proj}$ in $\text{Arr}(\mathscr{M})$ that will be used in the small object argument to *force and to detect*, by adjunction, the RLP and the HELP against elements of $\mathbf{I}$. The set $\mathbf{K_I}_{inj}$ is suitable for an injective Quillen-Segal theory, and $\mathbf{K_I}_{proj}$ is for a projective Quillen-Segal theory.

2.2 Lifting properties and detection of the Segal conditions

2.2.1 Homotopy theory of the arrow category. Let us review briefly the **injective** and the **projective** model structure on the category $\mathrm{Arr}(\mathscr{M})$ which can be found in Hovey [43]. Let $\mathbb{I} = [0 \longrightarrow 1]$ be a the walking morphism category. Then $\mathrm{Arr}(\mathscr{M})$ is the functor category $\mathrm{Hom}(\mathbb{I}, \mathscr{M})$ and these model structures are special cases of Reedy model structures (see [44]). Indeed, one can consider the category $\mathbb{I} = [0 \longrightarrow 1]$ as an inverse category, and in that case the Reedy structure is the injective model structure. We can also consider $\mathbb{I} = [0 \longrightarrow 1]$ as a direct category and we get the projective model structure (see [43],[44] for details). It is important to observe that $\mathscr{M}$ need not be cofibrantly generated to get the Reedy model structure.

Given two objects of $\mathrm{Arr}(\mathscr{M})$, $f : X_0 \longrightarrow X_1$ and $g : Y_0 \longrightarrow Y_1$, a map $\alpha : f \longrightarrow g$ in $\mathrm{Arr}(\mathscr{M})$ consists of two morphisms $\alpha_i : X_i \longrightarrow Y_i$ such that we have a commutative square

$$\begin{array}{ccc} X_0 & \xrightarrow{\alpha_0} & Y_0 \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ X_1 & \xrightarrow{\alpha_1} & Y_1 \end{array}$$

in $\mathscr{M}$.

Definition 2.10. Let $\alpha : f \longrightarrow g$ be a map in $\mathrm{Arr}(\mathscr{M})$. With the previous notation we will say that:

1. α is a **injective cofibration** if α_0 and α_1 are cofibrations in $\mathscr{M}$.
2. α is a **projective cofibration** if
   - α_0 is a cofibration in $\mathscr{M}$ and
   - the unique induced morphism $X_1 \cup^{X_0} Y_0 \longrightarrow Y_1$ is a cofibration in $\mathscr{M}$.
3. α is a **level-wise weak equivalence** (resp. **level-wise fibration**) if α_0 and α_1 are weak equivalences (resp. fibrations) in $\mathscr{M}$.
4. α is an **injective fibration** if $\alpha_1 : X_1 \longrightarrow Y_1$ is a fibration and if the induced map
$$X_0 \longrightarrow X_1 \times_{Y_1} Y_0$$
is a fibration.

The homotopy theory of $\mathrm{Arr}(\mathscr{M})$ is given by the following theorem.

Theorem 2.11. *Let $\mathscr{M}$ be a model category. Then with the previous definitions, the following hold.*

1. *The three classes of injective fibrations, injective cofibrations and level-wise weak equivalences determine a model structure on $\mathrm{Arr}(\mathscr{M})$ called **the injective model structure**.*
2. *The three classes of projective fibrations, projective cofibrations and level-wise weak equivalences determine a model structure on $\mathrm{Arr}(\mathscr{M})$ called **the injective model structure**.*
3. *If $\mathscr{M}$ is cofibrantly generated (resp combinatorial) then so are the injective and projective model structures on $\mathrm{Arr}(\mathscr{M})$.*

Proof. See Hovey [43]. □

2.2.2 Localizing sets. We borrow here some notation from Hovey [43].

Notation 2.12. Let $\mathscr{M}$ be a cofibrantly generated model category with $\mathbf{I}_{\mathscr{M}}$ and $\mathbf{J}_{\mathscr{M}}$ the respective generating sets of cofibrations and trivial cofibrations.

1. For any morphism $s : A \longrightarrow B$ of $\mathscr{M}$ we will denote by $\alpha_s : s \longrightarrow \mathrm{Id}_B$ the map in $\mathrm{Arr}(\mathscr{M})$ corresponding to the following commutative square.

$$\begin{array}{ccc} A & \xrightarrow{\ s\ } & B \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle \mathrm{Id}} \\ B & \xrightarrow{\ \mathrm{Id}\ } & B \end{array}$$

2. We will denote by $\alpha_{\mathbf{I}_{\mathscr{M}}}$ the set of all α_i for $i \in \mathbf{I}_{\mathscr{M}}$, that is, $\alpha_{\mathbf{I}_{\mathscr{M}}} = \{\alpha_i\}_{i \in \mathbf{I}_{\mathscr{M}}}$.
3. Similarly will denote by $\alpha_{\mathbf{J}_{\mathscr{M}}}$ the set of all α_j for $j \in \mathbf{J}_{\mathscr{M}}$.

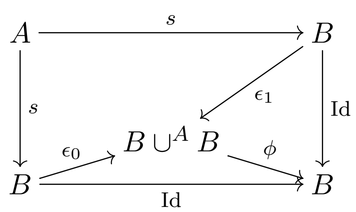
Definition 2.13. Define the **localizing injective set** as the set $\alpha_{\mathbf{I}_{\mathscr{M}}}$.

Proposition 2.14. *Let $s : A \longrightarrow B$ be a morphism of $\mathscr{M}$ and let $\mathrm{Id}_\emptyset$ be the identity morphism of the initial object of $\mathscr{M}$.*

1. *If $s : A \longrightarrow B$ is a weak equivalence in $\mathscr{M}$ then $\alpha_s : s \longrightarrow \mathrm{Id}_B$ is a level-wise weak equivalence in Arr($\mathscr{M}$).*
2. *If $s : A \longrightarrow B$ is a (trivial) cofibration in $\mathscr{M}$ then $\alpha_s : s \longrightarrow \mathrm{Id}_B$ is a level-wise (trivial) cofibration in Arr($\mathscr{M}$).*
3. *If A is cofibrant in $\mathscr{M}$, and s is a cofibration, then the unique map $\mathrm{Id}_\emptyset \longrightarrow s$ is a level-wise cofibration in Arr($\mathscr{M}$).*

Proof. Like any isomorphism of $\mathscr{M}$, the identity Id_B is a simultaneously a cofibration, a trivial cofibration, and a weak equivalence. The maps s and Id_B are the components defining the map α_s. Therefore α_s is a level-wise weak equivalence (resp. (co)fibration) if and only if s is a weak equivalence (resp. (co) fibration). The third assertion is clear because B is also cofibrant since by composition the unique map $\emptyset \longrightarrow A \xrightarrow{s} B$ is a cofibration. □

Relative cylinder object For a cofibration $s : A \longrightarrow B$ the map α_s is related to a *relative cylinder object* for the cofibration s (see [87, Chapter 9.4]). Consider the pushout of s along itself with the pushout data $B \xleftarrow{s} A \xrightarrow{s} B$ and write $B \cup^A B$ for the pushout object. From the commutative square that defines the map $\alpha_s : s \xrightarrow{(s,\mathrm{Id}_B)} \mathrm{Id}_B$, we get a unique map $\phi : B \cup^A B \longrightarrow B$ using the universal property of the pushout. In particular everything commutes below.



Now use the axiom of the model category $\mathscr{M}$ to factor the map ϕ as a cofibration followed by a trivial fibration:

$$B \cup^A B \overset{\delta}{\hookrightarrow} E_s \xrightarrow[\sim]{p} B.$$

This diagram determines a *relative cylinder object* for the cofibration $s : A \longrightarrow B$. The maps ϵ_1 and ϵ_2 are cofibrations because cofibrations are closed under cobase change, and by composition, $\delta \circ \epsilon_0$ and $\delta \circ \epsilon_1$ are also cofibrations. Moreover by 3-for-2 with respect to the equality $p \circ [\delta \circ \epsilon_i] = \mathrm{Id}_B$, they are also weak equivalences. It follows that $\delta \circ \epsilon_1$ and $\delta \circ \epsilon_2$ are trivial cofibrations.

Notation 2.15. Let $s : A \longrightarrow B$ be a cofibration as previously.

1. Denote by j_0 the composite $\delta \circ \epsilon_0$, and let $j_1 = \delta \circ \epsilon_1$. Both are trivial cofibrations.
2. Let $\zeta_s : s \xrightarrow{(s,j_0)} j_1$ be the map in $\mathrm{Arr}(\mathscr{M})$ given by the commutative square:

$$\begin{array}{ccc} A & \xrightarrow{s} & B \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle \sim\, j_1} \\ B & \xrightarrow[\sim]{j_0} & E_s \end{array}$$

3. Let $\ell_s : j_1 \xrightarrow{(\mathrm{Id}_B, p)} \mathrm{Id}_B$ be the level-wise weak equivalence in $\mathrm{Arr}(\mathscr{M})$ given by the commutative square:

$$\begin{array}{ccc} B & \xrightarrow{\mathrm{Id}_B} & B \\ {\scriptstyle j_1}\downarrow & & \downarrow{\scriptstyle \mathrm{Id}_B} \\ E_s & \underset{\sim}{\overset{p}{\twoheadrightarrow}} & B \end{array}$$

Definition 2.16. Define the **localizing projective set** as $\zeta_{\mathbf{I}_{\mathscr{M}}} = \{\zeta_i\}_{i \in \mathbf{I}_{\mathscr{M}}}$.

Proposition 2.17. *With the previous notations the following hold.*

1. *For every cofibration s, ζ_s is a cofibration in $Arr(\mathscr{M})_{\mathrm{proj}}$.*
2. *For every s we have $\alpha_s = \ell_s \circ \zeta_s$.*
3. *If $s : A \longrightarrow B$ is a cofibration such that A is cofibrant, then ℓ_s is a level-wise weak equivalence between cofibrant objects in $Arr(\mathscr{M})_{\mathrm{proj}}$, and thus in $Arr(\mathscr{M})_{\mathrm{inj}}$.*
4. *Let $\Upsilon : Arr(\mathscr{M}) \longrightarrow \mathcal{C}$ be a left Quillen functor with respect to either $Arr(\mathscr{M})_{inj}$ or $Arr(\mathscr{M})_{proj}$. If s is a cofibration between cofibrant objects, then $\Upsilon(\ell_s)$ is a weak equivalence in $\mathcal{C}$.*
5. *Let $\Upsilon : Arr(\mathscr{M}) \longrightarrow \mathcal{C}$ be a left Quillen functor with respect to either $Arr(\mathscr{M})_{inj}$ or $Arr(\mathscr{M})_{proj}$. Then for any cofibration s between cofibrant objects, $\Upsilon(\alpha_s)$ is a weak equivalence if and only if $\Upsilon(\zeta_s)$ is a weak equivalence.*

Proof. Assertions (1) and (2) are immediate by construction. Observe that the cofibrant objects in the projective model structure $\mathrm{Arr}(\mathscr{M})_{proj}$ are the cofibrations $s : A \longrightarrow B$ whose domain is cofibrant. Moreover any projective cofibration is also an injective cofibration, thus s is also a cofibrant object in the injective model structure. Now clearly ℓ_s is a level-wise weak equivalence. This gives Assertion (3).

Assertion (4) is a consequence of Ken Brown's Lemma [44, Lemma 1.1.12]. Indeed, s and Id_B are cofibrant and ℓ_s is weak equivalence between cofibrant objects, so by Ken Brown's Lemma any left Quillen functor must send it to a weak equivalence between cofibrant objects.

Assertion (5) is a consequence of Assertion (4) and the 3-for-2 property of weak equivalences in the model category $\mathcal{C}$ applied to the equality $\Upsilon(\alpha_s) = \Upsilon(\ell_s) \circ \Upsilon(\zeta_s)$. □

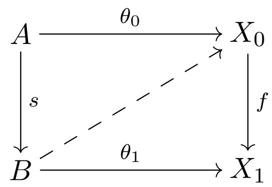
Definition 2.18. Let $s : A \longrightarrow B$ be a cofibration and let u and v be two elements in $\mathrm{Hom}_{\mathscr{M}}(B, X)$. Say that u and v are *homotopic relative to* s if there is a map $h : E_s \longrightarrow X$ such that $h \circ j_0 = u$ and $h \circ j_1 = v$.

Remark 2.19. It is important to observe the following.

1. Note that the maps j_0 and j_1 restrict to the same map on A. Therefore we have an equality $h \circ (j_0 \circ s) = h \circ (j_1 \circ s)$.
2. Any map $h : E_s \longrightarrow X$ defines tautologically a homotopy between $u := h \circ j_0$ and $v := h \circ j_1$.

Homotopy lifting problem

Definition 2.20. Let $f : X_0 \longrightarrow X_1$ be an arbitrary morphism and let $s : A \longrightarrow B$ be a cofibration in $\mathscr{M}$. Consider a lifting problem of solid arrows defined by s and f through a map $\theta : s \xrightarrow{(\theta_0,\theta_1)} f$ in $\mathrm{Arr}(\mathscr{M})$ as follows.

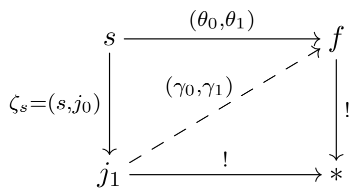


A solution **up-to-homotopy** to this problem consists of a map $r : B \dashrightarrow X_0$ such that:

1. $r \circ s = \theta_0$, that is, r "solves" strictly the upper triangle; and
2. $f \circ r$ and θ_1 are homotopic relative to s, in that, there exists a map $h : E_s \longrightarrow X_1$ such that $h \circ j_0 = \theta_1$ and $h \circ j_1 = f \circ r$. In other words, r "solves" the lower triangle up to a relative homotopy.

Say that f has the *Homotopy Extension Lifting Property* with respect to s if there is a solution up-to-homotopy to any lifting problem defined by s and f.

Proposition 2.21. *Let $f : X_0 \longrightarrow X_1$ be an object in $Arr(\mathscr{M})$ and let $f \longrightarrow *$ be the unique map to the terminal object. Consider a lifting problem of solid arrows defined by ζ_s and $f \longrightarrow *$ as follows.*



Then we have an equivalence between the following.

1. *A solution $\gamma = (\gamma_0, \gamma_1) : j_1 \longrightarrow f$ to this lifting problem.*
2. *An up-to-homotopy solution to the lifting problem in $\mathscr{M}$ defined by s and f by the map $\theta : s \xrightarrow{(\theta_0,\theta_1)} f$.*

Proof. Given a solution $\gamma = (\gamma_0, \gamma_1) : j_1 \longrightarrow f$, we get a solution up-to-homotopy by letting $r = \gamma_0$ and $h = \gamma_1$. Conversely given a solution up-to-homotopy $r : B \longrightarrow X_0$ and $h : E_s \longrightarrow X_1$ we get a map $\gamma = (r, h) : j_1 \longrightarrow f$ that is a solution to the lifting problem of the proposition. □

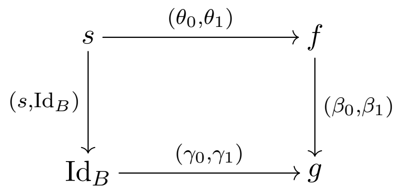
2.2.3 Detecting the Segal conditions. With the sets of maps $\alpha_{\mathbf{I}_{\mathscr{M}}}$ and $\alpha_{\mathbf{J}_{\mathscr{M}}}$ we are able to detect in $\mathrm{Arr}(\mathscr{M})$ whether a map $f : X_0 \longrightarrow X_1$ is a trivial fibration or fibration by lifting properties just like we do with the sets $\mathbf{I}_{\mathscr{M}}$ and $\mathbf{J}_{\mathscr{M}}$ in $\mathscr{M}$. Hovey [43, Proposition 2.2] showed that $\alpha_{\mathbf{I}_{\mathscr{M}}}$ is a subset of the generating set of cofibrations in the **injective model structure** on $\mathrm{Arr}(\mathscr{M})$.

Lemma 2.22. *Let $\mathscr{M}$ be a model category and let S be a set of maps.*

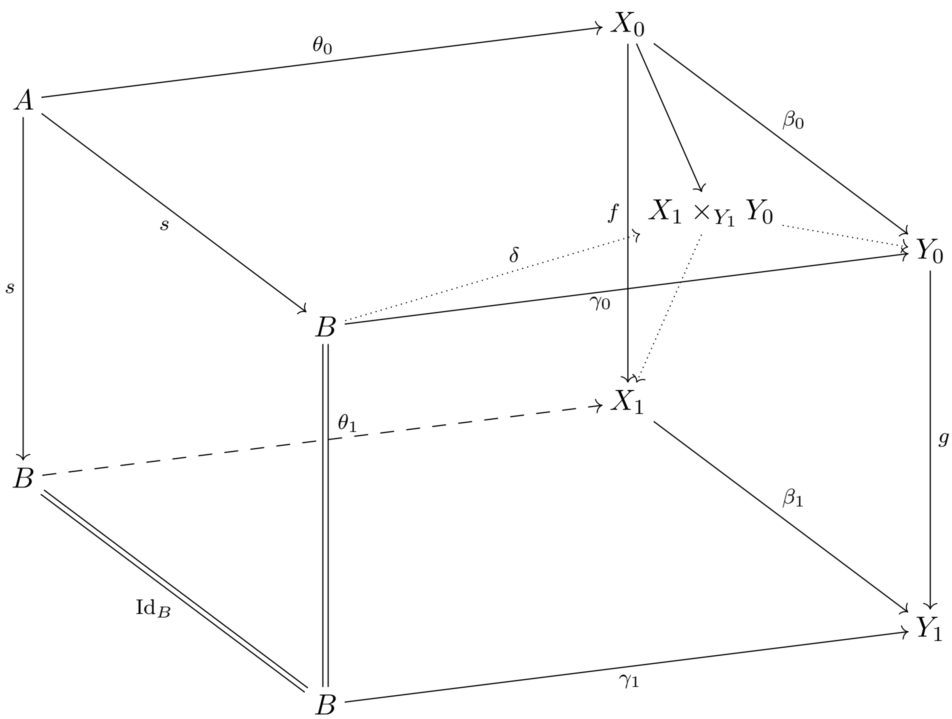
1. *A morphism $f : X_0 \longrightarrow X_1$ has the RLP in $\mathscr{M}$ with respect to S, if and only if the unique map $f \longrightarrow *$ to the terminal object has the RLP in Arr($\mathscr{M}$) with respect to the set $\alpha_S = \{\alpha_s\}_{s\in S}$.*
2. *More generally, if $g : Y_0 \longrightarrow Y_1$, then a map $\beta : f \longrightarrow g$ in Arr($\mathscr{M}$) has the RLP with respect to $\alpha_S = \{\alpha_s\}_{s\in S}$, if and only if the induced map $X_0 \longrightarrow X_1 \times_{Y_1} Y_0$ has the RLP in $\mathscr{M}$ with respect to the set S.*
3. *In particular a morphism $f : X_0 \longrightarrow X_1$ is a trivial fibration if and only if $f \longrightarrow *$ has the RLP with respect to the set $\alpha_{\mathbf{I}_{\mathscr{M}}}$.*

The last assertion is also mentioned in Rosický [83, Proof of Proposition 3.3].

Proof. It suffices to prove Assertion (2). Consider a lifting problem defined by α_s and β:



This commutative square corresponds to a commutative cube in $\mathscr{M}$. And if we unfold it, we see that the maps $\gamma_0 : B \longrightarrow Y_0$ and $\theta_1 : B \longrightarrow X_1$ determine a commutative square with the maps g and β_1, in that $\beta_1 \circ \theta_1 = g \circ \gamma_0$. The universal property of the pullback square gives a unique map $\delta : B \longrightarrow X_1 \times_{Y_1} Y_0$ that makes everything compatible. In particular θ_1 factors through δ. If we put this in the original cube, everything below commutes.



As one can see from this last diagram, we get a commutative square that corresponds to a lifting problem defined by the map $s : A \longrightarrow B$ and the map $X_0 \longrightarrow X_1 \times_{Y_1} Y_0$:

$$\begin{array}{ccc} A & \xrightarrow{\theta_0} & X_0 \\ {\scriptstyle s}\downarrow & & \downarrow \\ B & \xrightarrow{\delta} & X_1 \times_{Y_1} Y_0 \end{array}$$

Now it suffices to observe that this lifting problem has a solution if and only if our original lifting problem has a solution. Indeed if $\chi : B \longrightarrow X_0$ is a solution to the previous lifting problem, then the map $(\chi, \theta_1) : \mathrm{Id}_B \longrightarrow f$ in $\mathrm{Arr}(\mathscr{M})$ determines a solution to the original lifting problem. Conversely, given a solution to the original lifting problem $(\chi, \theta_1) : \mathrm{Id}_B \longrightarrow f$, then the component $\chi : B \longrightarrow X_0$ is a solution to the lifting problem defined by s and $X_0 \longrightarrow X_1 \times_{Y_1} Y_0$. □

HELP Lemma There are various versions of the HELP Lemma and we refer the reader to Boardman-Vogt [22], May [72], Simpson [87], Vogt [102] and the references therein. We will use the version used by Simpson [87, Lemma 7.5.1]. This version is for tractable model categories but the argument remains exactly the same for a cofibrantly generated model category where the cofibrations are generated by a set of maps between cofibrant objects.

Lemma 2.23. *Let $\mathscr{M}$ be a tractable model category with a generating set of cofibrations $\mathbf{I}_{\mathscr{M}}$. Then any morphism f between fibrant objects and that satisfies the HELP with respect to every element in $\mathbf{I}_{\mathscr{M}}$ is a weak equivalence.*

Equivalently we have:

Lemma 2.24. *Let $\mathscr{M}$ be a tractable model category with a generating set of cofibrations $\mathbf{I}_{\mathscr{M}}$. If f is a morphism in $\mathscr{M}$ such that the unique map $f \longrightarrow *$ is a level-wise fibration in Arr($\mathscr{M}$) and has the RLP with respect to every element in $\zeta_{\mathbf{I}_{\mathscr{M}}}$, then f is a weak equivalence.*

Proof. See [87, Lemma 7.5.1]. □

2.3 Localizing a Quillen-Segal theory.

We now give the proof of Theorem 2.7 and Theorem 2.9.

Definition 2.25. Let $\mathcal{T} = \{\mathcal{T}_i : \mathcal{C} \longrightarrow \mathrm{Arr}(\mathscr{M})\}_{i \in I}$ be an $\mathscr{M}$-valued Quillen-Segal theory on $\mathcal{C}$ such that for every $i \in I$ we have a left adjoint $\Upsilon_i : \mathrm{Arr}(\mathscr{M}) \longrightarrow \mathcal{C}$.

- Define the injective localizing set for $\mathcal{T}$ as:

$$\mathbf{K}_{\mathbf{I}_{inj}}(\mathcal{T}) := \coprod_{i \in I} \Upsilon_i(\alpha_{\mathbf{I}_{\mathscr{M}}})$$

- Define the projective localizing set for $\mathcal{T}$ as:

$$\mathbf{K}_{\mathbf{I}_{proj}}(\mathcal{T}) := \coprod_{i \in I} \Upsilon_i(\zeta_{\mathbf{I}_{\mathscr{M}}})$$

Proposition 2.26. *With the previous notation we have the following.*

1. *If $\mathcal{T}$ is an injective theory then every element in $\mathbf{K}_{\mathbf{I}_{inj}}$ is a cofibration in $\mathcal{C}$.*

2. If $\mathcal{T}$ is a projective theory then every element in $\mathbf{K}_{\mathbf{I}_{proj}}$ is a cofibration in $\mathcal{C}$.

Proof. Indeed, every element in $\alpha_{\mathbf{I}_{\mathscr{M}}}$ (resp. $\zeta_{\mathbf{I}_{\mathscr{M}}}$) is a injective (reps. projective) cofibration in $\mathrm{Arr}(\mathscr{M})$; therefore its image by the left Quillen functor Υ_i is a cofibration in $\mathcal{C}$. □

Proof of Theorem 2.7. The hypotheses of the theorem allow us to consider the left Bousfield localization with respect to the set $\mathbf{K}_{\mathbf{I}_{inj}}$ (see Barwick [8] and Hirschhorn [40]). Then thanks to Proposition 2.26 every element in $\mathbf{K}_{\mathbf{I}_{inj}}$ is a cofibration and a weak equivalence, thus a new trivial cofibration. It follows that for any fibrant object c in this new model structure the unique map $c \longrightarrow *$ in $\mathcal{C}$ must be $\Upsilon_i(\alpha_{\mathbf{I}_{\mathscr{M}}})$-injective for every $i \in I$. And by adjunction we find that the map $\mathcal{T}_i(c) \longrightarrow *$ in $\mathrm{Arr}(\mathscr{M})$ is $\alpha_{\mathbf{I}_{\mathscr{M}}}$-injective and $\mathcal{T}_i(c)$ is a trivial fibration by Lemma 2.22. In particular for every $i \in I$, $\mathcal{T}_i(c)$ is a weak equivalence (between fibrant objects), thus c satisfies the generalized Segal conditions. □

Proof of Theorem 2.9. The proof is the same as in the injective case. Consider the left Bousfield localization with respect to the set $\mathbf{K}_{\mathbf{I}_{proj}}$. Then thanks to Proposition 2.26 every element in $\mathbf{K}_{\mathbf{I}_{proj}}$ is a cofibration and a weak equivalence, thus a new trivial cofibration. It follows that for any fibrant object c in this new model structure the unique map $c \longrightarrow *$ in $\mathcal{C}$ is a fibration and must be $\Upsilon_i(\zeta_{\mathbf{I}_{\mathscr{M}}})$-injective for every $i \in I$. By adjunction we find that the map $\mathcal{T}_i(c) \longrightarrow *$ in $\mathrm{Arr}(\mathscr{M})$ is $\zeta_{\mathbf{I}_{\mathscr{M}}}$-injective, and it is a level-wise fibration because $\mathcal{T}_i$ is right Quillen (it preserves the fibrations). Then $\mathcal{T}_i(c)$ is a weak equivalence by the HELP Lemma 2.24. In particular for every $i \in I$, $\mathcal{T}_i(c)$ is a weak equivalence (between fibrant objects), thus c satisfies the generalized Segal conditions. □

3. Quillen-Segal algebras

3.1 Comma categories. We recall here some definitions and properties on comma categories. These results are well known in Category Theory (see for example [23, 68, 88]). We include some of them for completeness.

Definition 3.1. Let $\mathcal{U} : \mathscr{A} \longrightarrow \mathscr{M}$ be a functor. The *comma category* $(\mathscr{M} \downarrow \mathcal{U})$ is the category described as follows.

1. Objects are triples $[\mathcal{F}] = [\mathcal{F}_0, \mathcal{F}_1, \pi_{\mathcal{F}} : \mathcal{F}_0 \longrightarrow \mathcal{U}(\mathcal{F}_1)] \in \mathscr{M} \times \mathscr{A} \times \mathrm{Arr}(\mathscr{M})$
2. Given two objects $[\mathcal{F}] = [\mathcal{F}_0, \mathcal{F}_1, \pi_{\mathcal{F}}]$ and $[\mathcal{G}] = [\mathcal{G}_0, \mathcal{G}_1, \pi_{\mathcal{G}}]$, a map $\sigma : [\mathcal{F}] \longrightarrow [\mathcal{G}]$ is a pair $\sigma = [\sigma_0, \sigma_1] \in \mathrm{Hom}_{\mathscr{M}}(\mathcal{F}_0, \mathcal{G}_0) \times \mathrm{Hom}_{\mathscr{A}}(\mathcal{F}_1, \mathcal{G}_1)$ such that we have a commutative square in $\mathscr{M}$:

$$\begin{array}{ccc} \mathcal{F}_0 & \xrightarrow{\sigma_0} & \mathcal{G}_0 \\ \Big\downarrow{\scriptstyle \pi_{\mathcal{F}}} & & \Big\downarrow{\scriptstyle \pi_{\mathcal{G}}} \\ \mathcal{U}(\mathcal{F}_1) & \xrightarrow{\mathcal{U}(\sigma_1)} & \mathcal{U}(\mathcal{G}_1) \end{array}$$

That is:

$$\mathrm{Hom}_{(\mathscr{M}\downarrow\mathcal{U})}([\mathcal{F}], [\mathcal{G}]) = \{[\sigma_0, \sigma_1] \in \mathrm{Hom}_{\mathscr{M}}(\mathcal{F}_0, \mathcal{G}_0) \times \mathrm{Hom}_{\mathscr{A}}(\mathcal{F}_1, \mathcal{G}_1) \quad | \quad \pi_{\mathcal{G}} \circ \sigma_0 = \mathcal{U}(\sigma_1) \circ \pi_{\mathcal{F}}\}$$

We have three obvious functors:

- $\Pi_0 : (\mathscr{M} \downarrow \mathcal{U}) \longrightarrow \mathscr{M}$, with $\Pi_0([\mathcal{F}]) = \mathcal{F}_0$;
- $\Pi_1 : (\mathscr{M} \downarrow \mathcal{U}) \longrightarrow \mathscr{A}$, with $\Pi_1([\mathcal{F}]) = \mathcal{F}_1$;
- $\Pi_{\mathrm{Arr}} : (\mathscr{M} \downarrow \mathcal{U}) \longrightarrow \mathrm{Arr}(\mathscr{M})$, with $\Pi_{\mathrm{Arr}}([\mathcal{F}]) = \pi_{\mathcal{F}}$.

3.2 Quillen-Segal algebras and objects. We are now able to formulate the definition of Quillen-Segal algebras and Quillen-Segal objects.

Definition 3.2. Let $\mathscr{M}$ be a model category and let $\mathcal{U} : \mathscr{A} \longrightarrow \mathscr{M}$ be a functor. An object $[\mathcal{F}] = [\mathcal{F}_0, \mathcal{F}_1, \pi_{\mathcal{F}}] \in (\mathscr{M} \downarrow \mathcal{U})$ is a *Quillen-Segal $\mathcal{U}$-object* if $[\mathcal{F}]$ satisfies the Segal condition, that is, if $\pi_{\mathcal{F}} : \mathcal{F}_0 \longrightarrow \mathcal{U}(\mathcal{F}_1)$ is a weak equivalence in $\mathscr{M}$.

Definition 3.3. Let $\mathscr{M}$ be a model category and let $\mathcal{O}$ be a monad on $\mathscr{M}$ or an operad enriched over $\mathscr{M}$. Denote by $\mathscr{A} = \mathcal{O}\text{-}Alg(\mathscr{M})$ and by $\mathcal{U} : \mathscr{A} \longrightarrow \mathscr{M}$ the forgetful functor. A *Quillen-Segal $\mathcal{O}$-algebra* is a Quillen-Segal $\mathcal{U}$-object $[\mathcal{F}] = [\mathcal{F}_0, \mathcal{F}_1, \pi_{\mathcal{F}}] \in (\mathscr{M} \downarrow \mathcal{U})$.

In the previous definition, $\mathcal{O}$ is not restricted to be a monad or an operad. Indeed, $\mathcal{O}$ can be a PROP, properad, cyclic operad etc. These objects can be viewed themselves as algebras over some monad $\mathcal{O}'$ defined on another category $\mathscr{M}'$ (see [69, 100, 104]). Moreover if we fix such an "operator-object" $\mathcal{O}$, its category of algebras with coefficient in $\mathscr{M}$, is equivalent to the category of algebras of a monad defined on another category $\mathscr{M}'$, where $\mathscr{M}'$ is in many cases a product of copies of $\mathscr{M}$ (see for example [39]) or more generally a diagram category $\text{Hom}(\mathcal{C}, \mathscr{M})$.

3.3 Examples of Quillen-Segal algebras

Example 3.4. Let $(X, *)$ be a pointed space. Denote by $\Omega_*(X)$ its loop space and by $\Omega_*^M(X)$ its Moore loop space. As shown in Carlsson-Milgram [25, Proposition 5.1], $\Omega_*(X)$ is a deformation retract of $\Omega_*^M(X)$. In particular the inclusion $\Omega_*(X) \hookrightarrow \Omega_*^M(X)$ is a homotopy equivalence, thus a weak equivalence. The Moore loop space is a topological monoid, thus an $\mathcal{O}$-algebra for the associative topological operad **Ass**. So if we equip $\Omega_*(X)$ with the map $\Omega_*(X) \hookrightarrow \Omega_*^M(X)$ we get a Quillen-Segal **Ass**-algebra. Finally we know from Stasheff [90] that $\Omega_*(X)$ is an $\mathbf{A}_\infty$-algebra.

Example 3.5. Let A be a dg-algebra over a ring R and let $H^\bullet(A)$ bet its cohomology R-module. If $H^\bullet(A)$ is a free R-module, for example if R is a field; then we can define a cycle representative map $H^\bullet(A) \longrightarrow A$, and this map becomes a quasi-isomorphism of complexes, if we endow $H^\bullet(A)$ with zero differentials. In this case $H^\bullet(A)$ equipped with the previous map defines a Quillen-Segal dg-algebra. Just like in the previous example for loop spaces, it has been shown by Kadeishvili [52] that $H^\bullet(A)$ carries an $\mathbf{A}_\infty$-structure if A is graded by the non negative integers.

The clear analogy between the linear and non linear case suggests that there is a more general picture when we consider a general operad $\mathcal{O}$ other than the associative operad **Ass**. Moreover, there should be an operad $\mathcal{O}_\infty$ playing the role of $\mathbf{A}_\infty$.

Remark 3.6. The definition of Quillen-Segal algebras makes sense in any category equipped with a class of morphisms called weak equivalences.

4. Properties of comma constructions

We give here the required definitions and properties on comma categories that are necessary to deploy the homotopy theory on $(\mathscr{M} \downarrow \mathcal{U})$. Most of the material is classical in category theory and many results are straightforward.

Notation and Hypotheses

1. $\mathcal{U} : \mathscr{A} \longrightarrow \mathscr{M}$ is a functor that will be in general a right adjoint.
2. $\mathscr{M}_{\mathcal{U}}[\mathscr{A}] := (\mathscr{M} \downarrow \mathcal{U})$ and both notations will be used when some clarity is needed. Especially when we want to outline the presence of the category $\mathscr{A}$.
3. $\mathbf{F} : \mathscr{M} \longrightarrow \mathscr{A}$ will denote a left adjoint to $\mathcal{U}$ when it exists.
4. The notation $[\mathcal{F}]$ (with square bracket) denotes an object of $(\mathscr{M} \downarrow \mathcal{U})$. We will freely use it from now on and in particular throughout Section 5.
5. The categories $\mathscr{A}$ and $\mathscr{M}$ will be in general complete and cocomplete and sometimes locally presentable but we will mention these hypotheses explicitly.

4.1 Embedding.

A direct consequence of the definition is the following:

Proposition 4.1. *We have a functor $\iota : \mathscr{A} \hookrightarrow \mathscr{M}_{\mathcal{U}}[\mathscr{A}]$ defined as follows.*

1. *For $\mathcal{P} \in \mathscr{A}$, $\iota(\mathcal{P}) = [\mathcal{U}(\mathcal{P}), \mathcal{P}, \mathrm{Id}_{\mathcal{U}(\mathcal{P})}]$.*
2. *The functor ι takes a map $\theta : \mathcal{P} \longrightarrow \mathcal{Q}$ of $\mathscr{A}$ to the map $\iota(\theta) = [\mathcal{U}(\theta), \theta]$.*
3. *The composite $\Pi_1 \circ \iota$ is the identity. In other words $\mathscr{A}$ is a retract of $\mathscr{M}_{\mathcal{U}}[\mathscr{A}]$ in the category* **Cat** *of small categories.*
4. *The functor ι is injective on objects and we have an isomorphism of hom-sets*

$$\mathrm{Hom}_{\mathscr{A}}(\mathcal{P}, \mathcal{Q}) \cong \mathrm{Hom}_{\mathscr{M}_{\mathcal{U}}[\mathscr{A}]}(\iota(\mathcal{P}), \iota(\mathcal{Q}))$$

In particular the functor $\iota : \mathscr{A} \longrightarrow \mathscr{M}_{\mathcal{U}}[\mathscr{A}]$ exhibits $\mathscr{A}$ as a full subcategory of $\mathscr{M}_{\mathcal{U}}[\mathscr{A}]$.

Proof. The fact that ι is injective on objects is clear since $\mathcal{P}$ appears alone in $[\mathcal{U}(\mathcal{P}), \mathcal{P}, \mathrm{Id}_{\mathcal{U}(\mathcal{P})}]$. On morphisms we have $\iota(\theta_1) = \iota(\theta_2) \Leftrightarrow [\mathcal{U}(\theta_0), \theta_0] = [\mathcal{U}(\theta_1), \theta_1] \Leftrightarrow \theta_1 = \theta_2$ and $\mathcal{U}(\theta_0) = \mathcal{U}(\theta_1)$, which means that ι is also injective on morphisms ($= \iota$ is faithful). Let us now show that ι is also surjective on morphisms ($= \iota$ is full). If $\sigma = [\sigma_1, \sigma_1]$ is a map in $\mathscr{M}_{\mathcal{U}}[\mathscr{A}]$ from $\iota(\mathcal{P})$ to $\iota(\mathcal{Q})$, we have

$$\mathrm{Id}_{\mathcal{U}(\mathcal{Q})} \circ \sigma_0 = \mathcal{U}(\sigma_1) \circ \mathrm{Id}_{\mathcal{U}(\mathcal{P})} \Leftrightarrow \sigma_0 = \mathcal{U}(\sigma_1) \Leftrightarrow \sigma = [\mathcal{U}(\sigma_1), \sigma_1] = \iota(\sigma_1)$$

with $\sigma_1 : \mathcal{P} \longrightarrow \mathcal{Q}$. □

4.2 Arrow-category as comma category.

Recall that $\mathbb{I} = [0 \longrightarrow 1]$ is the *walking-morphism category* and therefore $\mathrm{Arr}(\mathscr{M})$ is the functor category $\mathrm{Hom}(\mathbb{I}, \mathscr{M})$. If we put together this fact and the previous definition of comma category we get the following obvious results.

Proposition 4.2. *Let $\mathcal{U} : \mathscr{M} \longrightarrow \mathscr{M}$ be the identity functor. Then the following hold.*

1. *The category of morphisms $Arr(\mathscr{M})$ is isomorphic to the comma category $\mathscr{M}_{\mathcal{U}}[\mathscr{M}] := (\mathscr{M} \downarrow \mathcal{U})$. And the functor $\Pi_{Arr} : (\mathscr{M} \downarrow \mathcal{U}) \longrightarrow Arr(\mathscr{M})$ is an isomorphism.*
2. *The category of morphisms $Arr(\mathscr{M})$ is isomorphic to the functor category $\mathrm{Hom}(\mathbb{I}, \mathscr{M})$.*
3. *The functor $\Pi_0 : \mathscr{M}_{\mathcal{U}}[\mathscr{M}] \longrightarrow \mathscr{M}$ is the source-functor and corresponds to the evaluation functor $Ev_0 : \mathrm{Hom}(\mathbb{I}, \mathscr{M}) \longrightarrow \mathscr{M}$.*
4. *The functor $\Pi_1 : \mathscr{M}_{\mathcal{U}}[\mathscr{M}] \longrightarrow \mathscr{M}$ is the target-functor and corresponds to the evaluation functor $Ev_1 : \mathrm{Hom}(\mathbb{I}, \mathscr{M}) \longrightarrow \mathscr{M}$.* □

If we apply Proposition 4.1 for the functor $\mathcal{U} = \mathrm{Id}_{\mathscr{M}}$, then $\mathscr{M}_{\mathcal{U}}[\mathscr{M}]$ is the category of $\mathrm{Arr}(\mathscr{M})$. In particular one gets the following results that can also be found in Hovey [43, Lemma 1.1].

Proposition 4.3. *Let $\mathscr{M}$ be any category. Then the following hold.*

1. *We have an embedding* $\iota_{\mathscr{M}} : \mathscr{M} \hookrightarrow Arr(\mathscr{M})$ *which takes an object* $X \in \mathscr{M}$ *to the identity* Id_X*, and a morphism* $f : X \longrightarrow X'$ *to the map* $\mathrm{Id}_X \longrightarrow \mathrm{Id}_{X'}$ *whose components are both equal to* f*.*
2. *The functor* $\iota_{\mathscr{M}}$ *is left adjoint to the functor* $Ev_0 : Arr(\mathscr{M}) \longrightarrow \mathscr{M}$*. We will denote by* $L_0 = \iota_{\mathscr{M}}$*.*
3. *If there is an initial object* $\emptyset \in \mathscr{M}$*, then the functor* $Ev_1 : Arr(\mathscr{M}) \longrightarrow \mathscr{M}$ *has also a left adjoint* $L_1 : \mathscr{M} \longrightarrow Arr(\mathscr{M})$*. We have* $L_1(X) = \emptyset \longrightarrow X$*.*

4.3 Adjunctions, limits and colimits

4.3.1 Adjunctions. We give here the adjunctions that will be needed to get the various model structures on our comma category. The constructions and the results that will follow are also classical in category theory. We include the proof for completeness. Recall that we have an embedding $\iota : \mathscr{A} \longrightarrow \mathscr{M}_{\mathcal{U}}[\mathscr{A}]$ and a functor $\Pi_1 : \mathscr{M}_{\mathcal{U}}[\mathscr{A}] \longrightarrow \mathscr{A}$. Our goal in this section is to establish the following theorem.

Theorem 4.4. *Let* $\mathcal{U} : \mathscr{A} \longrightarrow \mathscr{M}$ *be a functor. With the previous notation the following hold.*

1. *The functor* $\Pi_1 : \mathscr{M}_{\mathcal{U}}[\mathscr{A}] \longrightarrow \mathscr{A}$ *is left adjoint to the embedding* $\iota : \mathscr{A} \longrightarrow \mathscr{M}_{\mathcal{U}}[\mathscr{A}]$*. In particular* $\mathscr{A}$ *is equivalent to a full reflective subcategory of* $\mathscr{M}_{\mathcal{U}}[\mathscr{A}]$*.*
2. *The functor* $\Pi_1 : \mathscr{M}_{\mathcal{U}}[\mathscr{A}] \longrightarrow \mathscr{A}$ *has also a left adjoint* $L_1 : \mathscr{A} \longrightarrow \mathscr{M}_{\mathcal{U}}[\mathscr{A}]$ *with* $L_1(\mathcal{P}) = [\emptyset, \mathcal{P}, \emptyset \longrightarrow \mathcal{U}(\mathcal{P})]$*, where* $\emptyset \longrightarrow \mathcal{U}(\mathcal{P})$ *is the unique map.*
3. *If furthermore* $\mathcal{U}$ *has a left adjoint* $\mathbf{F} : \mathscr{M} \longrightarrow \mathscr{A}$*, then*
   (a) *The functor* $\Pi_{Arr} : \mathscr{M}_{\mathcal{U}}[\mathscr{A}] \longrightarrow Arr(\mathscr{M})$ *has a left adjoint*
   $$\Gamma : Arr(\mathscr{M}) \longrightarrow \mathscr{M}_{\mathcal{U}}[\mathscr{A}]$$
   (b) *The functor* $\Pi_0 : \mathscr{M}_{\mathcal{U}}[\mathscr{A}] \longrightarrow \mathscr{M}$ *has a left adjoint* $\mathbf{F}^+ : \mathscr{M} \longrightarrow \mathscr{M}_{\mathcal{U}}[\mathscr{A}]$ *such that the composite* $\Pi_1 \circ \mathbf{F}^+$ *is the functor* $\mathbf{F}$*.*
   We have $\mathbf{F}^+ = \Gamma \circ L_0$*.*

Proof. Assertion (1) is the content of Proposition 4.5. Assertion (2) can be easily verified. Assertion (3) is given by Proposition 4.7 and Proposition 4.8. □

Proposition 4.5. *Let* $\mathcal{U} : \mathscr{A} \longrightarrow \mathscr{M}$ *be a functor. Then the functor* $\iota : \mathscr{A} \longrightarrow \mathscr{M}_{\mathcal{U}}[\mathscr{A}]$ *is right adjoint to the functor* $\Pi_1 : \mathscr{M}_{\mathcal{U}}[\mathscr{A}] \longrightarrow \mathscr{A}$*. In particular* $\mathscr{A}$ *is equivalent to a full reflective subcategory of* $\mathscr{M}_{\mathcal{U}}[\mathscr{A}]$*.*

Proof. Let $[\mathcal{F}]$ be an object of $\mathscr{M}_{\mathcal{U}}[\mathscr{A}]$ and $\mathcal{P}$ be an object of $\mathscr{A}$. We wish to show that we have a functorial isomorphism of hom-sets :

$$\mathrm{Hom}_{\mathscr{A}}(\Pi_1([\mathcal{F}]), \mathcal{P}) \cong \mathrm{Hom}_{\mathscr{M}_{\mathcal{U}}[\mathscr{A}]}([\mathcal{F}], \iota(\mathcal{P}))$$

Recall that $\Pi_1([\mathcal{F}]) = \mathcal{F}_1$ and that by definition a map $\sigma = [\sigma_0, \sigma_1] : [\mathcal{F}] \longrightarrow \iota(\mathcal{P})$ consists of two maps $\sigma_0 \in \mathrm{Hom}_{\mathscr{M}}(\mathcal{F}_0, \mathcal{U}(\mathcal{P}))$ and $\sigma_1 \in \mathrm{Hom}_{\mathscr{A}}(\mathcal{F}_1, \mathcal{P})$ such that :

$$\mathrm{Id}_{\mathcal{U}(\mathcal{P})} \circ \sigma_0 = \mathcal{U}(\sigma_1) \circ \pi_{\mathcal{F}} \Longleftrightarrow \sigma_0 = \mathcal{U}(\sigma_1) \circ \pi_{\mathcal{F}} \Longleftrightarrow \sigma = [\mathcal{U}(\sigma_1) \circ \pi_{\mathcal{F}}, \sigma_1].$$

It follows from the above equivalence that the function

$$\varphi : \mathrm{Hom}_{\mathscr{A}}(\mathcal{F}_1, \mathcal{P}) \longrightarrow \mathrm{Hom}_{\mathscr{M}_{\mathcal{U}}[\mathscr{A}]}([\mathcal{F}], \iota(\mathcal{P})),$$

defined by $\varphi(\sigma_1) = [\mathcal{U}(\sigma_1) \circ \pi_{\mathcal{F}}, \sigma_1]$ is surjective. The presence of σ_1 alone in the pair $[\mathcal{U}(\sigma_1) \circ \pi_{\mathcal{F}}, \sigma_1]$ implies that φ is also injective. Therefore φ is an isomorphism as required. The functoriality in both variables $[\mathcal{F}]$ and $\mathcal{P}$ is clear. □

Proposition 4.6. *With the previous notation the following hold.*

1. *For any $[\mathcal{F}] \in \mathscr{M}_{\mathcal{U}}[\mathscr{A}]$, the unit $\eta_1([\mathcal{F}]) : [\mathcal{F}] \longrightarrow \iota(\Pi_1([\mathcal{F}]))$ of the previous adjunction is given by the pair $[\pi_{\mathcal{F}}, \mathrm{Id}_{\mathcal{F}_1}]$.*
2. *For any $\mathcal{P} \in \mathscr{A}$ the counit $\varepsilon_1(\mathcal{P}) : \Pi_1(\iota(\mathcal{P})) \longrightarrow \mathcal{P}$ of the adjunction is the identity.*

Proof. Clear. □

Proposition 4.7. *Let $\mathcal{U} : \mathscr{A} \longrightarrow \mathscr{M}$ be a functor that possesses a left adjoint $\mathbf{F} : \mathscr{M} \longrightarrow \mathscr{A}$. Let $\eta : \mathrm{Id}_{\mathscr{M}} \longrightarrow \mathcal{U}\mathbf{F}$ be the unit of this adjunction. Then we have a functor*

$$\Gamma : Arr(\mathscr{M}) \longrightarrow \mathscr{M}_{\mathcal{U}}[\mathscr{A}]$$

defined as follows.

1. *If $f : X_0 \longrightarrow X_1$, then $\Gamma(f) = [X_0, \mathbf{F}(X_1), \eta_{X_1} \circ f]$ with $\eta_{X_1} : X_1 \longrightarrow \mathcal{U}\mathbf{F}(X_1)$*
2. *If $g : Y_0 \longrightarrow Y_1$ and $\alpha = (\alpha_0, \alpha_1) : f \longrightarrow g$ is a map in $Arr(\mathscr{M})$, then $\Gamma(\alpha) = [\alpha_0, \mathbf{F}(\alpha_1)]$.*

In particular

- $\Pi_0(\Gamma(f)) = X_0$, $\Pi_0(\Gamma(\alpha)) = \alpha_0$
- $\Pi_1(\Gamma(f)) = \mathbf{F}(X_1)$, $\Pi_1(\Gamma(\alpha)) = \mathbf{F}(\alpha_1)$
- $\Pi_{Arr}(\Gamma(f)) = \eta_{X_1} \circ f : X_0 \longrightarrow X_1 \longrightarrow \mathcal{U}\mathbf{F}(X_1)$

Proof. For the second assertion it suffices to show that we have an equality

$$\eta_{Y_1} \circ g \circ \alpha_0 = \mathcal{U}\mathbf{F}(\alpha_1) \circ \eta_{X_1} \circ f.$$

We get this equality by observing that the naturality of $\eta : \mathrm{Id}_{\mathscr{M}} \longrightarrow \mathcal{U}\mathbf{F}$ and part of α being a map in $\mathrm{Arr}(\mathscr{M})$ imply that all three squares below are commutative.

$$\begin{array}{ccc} X_0 & \xrightarrow{f} & X_1 \\ {\scriptstyle \eta_{X_0}}\downarrow & & \downarrow{\scriptstyle \eta_{X_1}} \\ \mathcal{U}\mathbf{F}(X_0) & \xrightarrow{\mathcal{U}\mathbf{F}(f)} & \mathcal{U}\mathbf{F}(X_1) \end{array} \qquad \begin{array}{ccc} X_1 & \xrightarrow{\alpha_1} & Y_1 \\ {\scriptstyle \eta_{X_1}}\downarrow & & \downarrow{\scriptstyle \eta_{Y_1}} \\ \mathcal{U}\mathbf{F}(X_1) & \xrightarrow{\mathcal{U}\mathbf{F}(\alpha_1)} & \mathcal{U}\mathbf{F}(Y_1) \end{array} \qquad \begin{array}{ccc} X_0 & \xrightarrow{f} & X_1 \\ {\scriptstyle \alpha_0}\downarrow & & \downarrow{\scriptstyle \alpha_1} \\ Y_0 & \xrightarrow{g} & Y_1 \end{array}$$

Thus we get

$$\begin{aligned} \eta_{Y_1} \circ g \circ \alpha_0 &= \eta_{Y_1} \circ (g \circ \alpha_0) \\ &= \eta_{Y_1} \circ (\alpha_1 \circ f) \\ &= (\eta_{Y_1} \circ \alpha_1) \circ f \\ &= (\mathcal{U}\mathbf{F}(\alpha_1) \circ \eta_{X_1}) \circ f \end{aligned}$$

□

We are now able to establish the following adjunction.

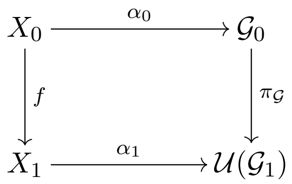
Proposition 4.8. *The functor $\Gamma : Arr(\mathscr{M}) \longrightarrow \mathscr{M}_{\mathcal{U}}[\mathscr{A}]$ is left adjoint to the functor $\Pi_{Arr} : \mathscr{M}_{\mathcal{U}}[\mathscr{A}] \longrightarrow Arr(\mathscr{M})$.*

Proof. The proof is tedious but straightforward so we shall give an outline of it. Let $f : X_0 \longrightarrow X_1$ be an object of $\text{Arr}(\mathscr{M})$ and let $[\mathcal{G}] = [\mathcal{G}_0, \mathcal{G}_1, \pi_{\mathcal{G}}]$ be an object of $\mathscr{M}_{\mathcal{U}}[\mathscr{A}]$. We wish to establish that we have a functorial isomorphism of hom-sets:

$$\text{Hom}_{\mathscr{M}_{\mathcal{U}}[\mathscr{A}]}(\Gamma(f), [\mathcal{G}]) \cong \text{Hom}_{\text{Arr}(\mathscr{M})}(f, \pi_{\mathcal{G}})$$

We define a function $\phi : \text{Hom}_{\text{Arr}(\mathscr{M})}(f, \pi_{\mathcal{G}}) \longrightarrow \text{Hom}_{\mathscr{M}_{\mathcal{U}}[\mathscr{A}]}(\Gamma(f), [\mathcal{G}])$ as follows.

- Let $\alpha : f \longrightarrow \pi_{\mathcal{G}}$ a map in $\text{Arr}(\mathscr{M})$ displayed by the commutative square:



From the adjunction $\mathbf{F} \dashv \mathcal{U}$ there is a functorial isomorphism of hom-sets:

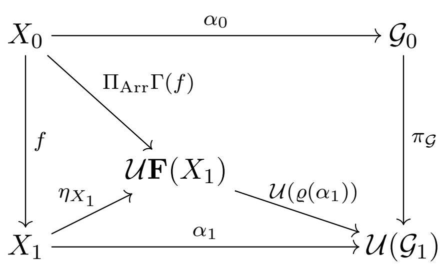
$$\varrho : \text{Hom}_{\mathscr{M}}(X_1, \mathcal{U}(\mathcal{G}_1)) \xrightarrow{\cong} \text{Hom}_{\mathscr{A}}(\mathbf{F}X_1, \mathcal{G}_1)$$

- Consider $\varrho(\alpha_1) : \mathbf{F}(X_1) \longrightarrow \mathcal{G}_1$, the adjoint transpose of $\alpha_1 : X_1 \longrightarrow \mathcal{U}(\mathcal{G}_1)$.
- We define $\phi(\alpha) = [\alpha_0, \varrho(\alpha_1)] \in \text{Hom}_{\mathscr{M}}(X_0, \mathcal{G}_0) \times \text{Hom}_{\mathscr{A}}(\mathbf{F}X_1, \mathcal{G}_1)$.

Let us check that the pair $\phi(\alpha) = [\alpha_0, \varrho(\alpha_1)]$ defines indeed a morphism in $\mathscr{M}_{\mathcal{U}}[\mathscr{A}]$, that is, we must prove that we have an equality: $\mathcal{U}(\varrho(\alpha_1)) \circ \Pi_{\text{Arr}}\Gamma(f) = \pi_{\mathcal{G}} \circ \alpha_1$.
By definition, $\Pi_{\text{Arr}}\Gamma(f) = \eta_{X_1} \circ f$ and from the adjunction $\mathbf{F} \dashv \mathcal{U}$, we have:

$$\alpha_1 = \mathcal{U}(\varrho(\alpha_1)) \circ \eta_{X_1}.$$

Putting all together we have a diagram in which everything commutes:



The inner commutative square gives a map $\Pi_{\text{Arr}}\Gamma(f) \longrightarrow \pi_{\mathcal{G}}$ in $\text{Arr}(\mathscr{M})$. This means that $\phi(\alpha) = [\alpha_0, \varrho(\alpha_1)]$ is a map in $\mathscr{M}_{\mathcal{U}}[\mathscr{A}]$.

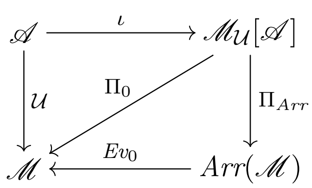
This function is clearly an injection since ϱ is an isomorphism, and we have an inverse function ϕ^{-1} that takes a map $\sigma = [\sigma_0, \sigma_1] : \Gamma(f) \longrightarrow \mathcal{G}$ to the map

$$\phi^{-1}(\sigma) = [\sigma_0, \varrho^{-1}(\sigma_1)] : f \longrightarrow \pi_{\mathcal{G}} .$$ □

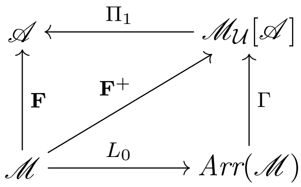
Let us now put together the various adjunctions and see how they interact.

Proposition 4.9. *Let $\mathcal{U} : \mathscr{A} \longrightarrow \mathscr{M}$ be a functor. Then the following hold.*

1. *We have a commutative diagram.*



2. *If moreover $\mathcal{U}$ has a left adjoint $\mathbf{F} : \mathscr{M} \longrightarrow \mathscr{A}$ then we have a commutative diagram of left adjoints functors.*



With $\mathbf{F}^+ = \Gamma \circ L_0$.

4.3.2 Limits and colimits. Given a diagram $D : \mathcal{J} \longrightarrow \mathscr{M}_{\mathcal{U}}[\mathscr{A}]$ we have three induced diagrams:

- $D_0 = \Pi_0(D) : \mathcal{J} \longrightarrow \mathscr{M}$
- $D_1 = \Pi_1(D) : \mathcal{J} \longrightarrow \mathscr{A}$
- $\pi_D = \Pi_{\text{Arr}}(D) : \mathcal{J} \longrightarrow \text{Arr}(\mathscr{M})$.

For every $j \in \mathcal{J}$ we have $D(j) = [D_0(j), D_0(j), \pi_D(j)]$. Our goal here is to see how we compute limits and colimits in the comma category $\mathscr{M}_{\mathcal{U}}[\mathscr{A}] := (\mathscr{M} \downarrow \mathcal{U})$.

Proposition 4.10. *Let $\mathcal{U} : \mathscr{A} \leftrightarrows \mathscr{M} : \mathbf{F}$ be an adjunction between complete and cocomplete categories. Then with the previous notation the following hold.*

1. *There is an induced map*

$$\pi_{\text{colim}} : \operatorname{colim} D_0 \longrightarrow \mathcal{U}(\operatorname{colim} D_1)$$

in $\mathscr{M}$, and the triple $[\operatorname{colim} D_0, \operatorname{colim} D_1, \pi_{\text{colim}}]$ equipped with the obvious maps is the colimit of the diagram $D : \mathcal{J} \longrightarrow \mathscr{M}_{\mathcal{U}}[\mathscr{A}]$.

2. *As the functor $\mathcal{U}$ preserves limits, there is also an induced map*

$$\pi_{\lim} : \lim D_0 \longrightarrow \mathcal{U}(\lim D_1)$$

in $\mathscr{M}$, and the triple $[\lim D_0, \lim D_1, \pi_{\lim}]$ equipped with the obvious maps is the limit of the diagram $D : \mathcal{J} \longrightarrow \mathscr{M}_{\mathcal{U}}[\mathscr{A}]$.

The category $\mathscr{M}_{\mathcal{U}}[\mathscr{A}]$ is also complete and cocomplete.

An immediate consequence is that:

Corollary 4.11. 1. *The functors $\Pi_0 : \mathscr{M}_{\mathcal{U}}[\mathscr{A}] \longrightarrow \mathscr{M}$ and $\Pi_1 : \mathscr{M}_{\mathcal{U}}[\mathscr{A}] \longrightarrow \mathscr{A}$ preserve limits and colimits. In particular they preserve pushouts.*

2. *The functor $\Pi_{Arr} : \mathscr{M}_{\mathcal{U}}[\mathscr{A}] \longrightarrow Arr(\mathscr{M})$ preserves limits.*

In other words the functor $(\Pi_0, \Pi_1) : \mathscr{M}_{\mathcal{U}}[\mathscr{A}] \longrightarrow \mathscr{M} \times \mathscr{A}$ creates limits and colimits (see [64]).

Proof. For every $j \in \mathcal{J}$ denote by $\epsilon_1(j) : D_1(j) \longrightarrow \operatorname{colim} D_1$ the canonical map in $\mathscr{A}$ going to the colimit and consider its image $\mathcal{U}(\epsilon_1(j)) : \mathcal{U}(D_1(j)) \longrightarrow \mathcal{U}(\operatorname{colim} D_1)$. Then the maps

$$\{\mathcal{U}(\epsilon_1(j)) \circ \pi_D(j) : D_0(j) \longrightarrow \mathcal{U}(\operatorname{colim} D_1)\}_{j \in \mathcal{J}}$$

determine a natural transformation from D_0 to the constant diagram of value $\mathcal{U}(\operatorname{colim} D_1)$. Indeed, for every structure map $l : j \longrightarrow j'$ in $\mathcal{J}$ we can apply the functor $\mathcal{U}$ to the equality $\epsilon_1(j) = \epsilon_1(j') \circ D_1(l)$ and get:

$$\mathcal{U}(\epsilon_1(j)) = \mathcal{U}(\epsilon_1(j')) \circ \mathcal{U}(D_1(l)).$$

Moreover, we also have a commutative diagram in $\mathscr{M}$:

$$\begin{array}{ccccc}
D_0(j) & \xrightarrow{D_0(l)} & D_0(j') & & \\
\downarrow{\scriptstyle \pi_D(j)} & & \downarrow{\scriptstyle \pi_D(j')} & & \\
\mathcal{U}(D_1(j)) & \xrightarrow{\mathcal{U}(D_1(l))} & \mathcal{U}(D_1(j')) & \xrightarrow{\mathcal{U}(\epsilon_1(j'))} & \mathcal{U}(\operatorname{colim} D_1)
\end{array}$$

Using the universal property of the colimit of D_0, we find a unique map

$$\pi_{\text{colim}} : \operatorname{colim} D_0 \longrightarrow \mathcal{U}(\operatorname{colim} D_1)$$

that makes everything compatible. If we write $\epsilon_0(j) : D_0(j) \longrightarrow colimD_1$ for the canonical map in $\mathscr{M}$ going to the colimit, then the pair $[\epsilon_0(j), \epsilon_1(j)]$ determines a map in $\mathscr{M}_{\mathcal{U}}[\mathscr{A}]$:

$$[D_0(j), D_0(j), \pi_D(j)] \longrightarrow [\operatorname{colim} D_0, \operatorname{colim} D_1, \pi_{\text{colim}}].$$

It is tedious but straightforward to check that $[\operatorname{colim} D_0, \operatorname{colim} D_1, \pi_{\text{colim}}]$ equipped with these maps satisfies the universal property of the colimit of the diagram D. This proves Assertion (1).

For Assertion (2) we proceed in a dual manner using the fact that $\mathcal{U}$ preserves limits like any right adjoint functor. Using this, we know that $\mathcal{U}(\lim D_1)$ equipped with the obvious maps is the limit of the diagram $\mathcal{U}(D_1)$. Let $p_0(j) : \lim D_0 \longrightarrow D_0(j)$ be the canonical projection so that for every structure map $l : j \longrightarrow j'$ we have $p_0(j') = D_0(l) \circ p_0(j)$. Applying $\mathcal{U}$ to this equality yields:

$$\mathcal{U}(p_0(j')) = \mathcal{U}(D_0(l)) \circ \mathcal{U}(p_0(j)).$$

Then the maps: $\{\pi_D(j) \circ p_0(j) : \lim D_0 \longrightarrow \mathcal{U}(D_1(j))\}_{j \in \mathcal{J}}$, determine a natural transformation from the constant diagram of value $\lim D_0$, to the diagram $\mathcal{U}(D_1)$.
By the universal property of the limit $\mathcal{U}(\lim D_1)$ we find a unique map

$$\pi_{\lim} : \lim D_0 \longrightarrow \mathcal{U}(\lim D_1)$$

that makes everything compatible. In particular for every structure map $l : j \longrightarrow j'$ we have a commutative diagram:

$$\begin{array}{ccccc}
\lim D_0 & \xrightarrow{p_0(j)} & D_0(j) & \xrightarrow{D_0(l)} & D_0(j') \\
\downarrow{\scriptstyle \pi_{\lim}} & & \downarrow{\scriptstyle \pi_D(j)} & & \downarrow{\scriptstyle \pi_D(j')} \\
\mathcal{U}(\lim D_1) & \xrightarrow{\mathcal{U}(p_1(j))} & \mathcal{U}(D_1(j)) & \xrightarrow{\mathcal{U}(D_1(l))} & \mathcal{U}(D_1(j'))
\end{array}$$

It is not hard to check that the object $[\lim D_0, \lim D_1, \pi_{\lim}]$ equipped with the maps defined by the maps

$$[p_0(j), p_1(j)] : [\lim D_0, \lim D_1, \pi_{\lim}] \longrightarrow D(j)$$

satisfies the universal property of the limit for the diagram D. □

4.3.3 Accessibility of comma categories

Note. We list below some technical results on locally presentable categories. Good references on the subject include [1, 26, 64].

Proposition 4.12. *Let $\mathcal{U} : \mathscr{A} \longrightarrow \mathscr{M}$ be an accessible functor between accessible categories. Then the category $\mathscr{M}_{\mathcal{U}}[\mathscr{A}] = (\mathscr{M} \downarrow \mathcal{U})$ is also accessible.*

Proof. This is a special case of Theorem 2.43 in the book of Adámek and Rosický [1] for the comma category $\mathrm{Id}_{\mathscr{M}} \downarrow \mathcal{U} \cong \mathscr{M}_{\mathcal{U}}[\mathscr{A}]$. □

Corollary 4.13. *Let $\mathcal{U} : \mathscr{A} \rightleftarrows \mathscr{M} : \mathbf{F}$ be an adjunction between locally presentable categories. Then the category $\mathscr{M}_{\mathcal{U}}[\mathscr{A}]$ is also locally presentable.*

Proof. With the previous result we know that $\mathscr{M}_{\mathcal{U}}[\mathscr{A}]$ is an accessible category. So it remains to show that $\mathscr{M}_{\mathcal{U}}[\mathscr{A}]$ is also cocomplete. But this is given by Proposition 4.10. □

5. Injective and projective model structures

We are now able to set up the material that is needed to develop a homotopy theory on our comma category $(\mathscr{M} \downarrow \mathcal{U})$. We will work under the following the hypotheses.

Hypotheses

1. $\mathscr{A}$ and $\mathscr{M}$ are two general model categories.
2. $\mathcal{U} : \mathscr{A} \longrightarrow \mathscr{M}$ is a right Quillen whose adjoint left Quillen functor is $\mathbf{F} : \mathscr{M} \longrightarrow \mathscr{A}$.
3. We will mention explicitly when $\mathscr{A}$ and $\mathscr{M}$ are cofibrantly generated (resp. combinatorial) model categories. We assume this in Section 5.5.2 and Section 5.5.4.
4. We will also require $\mathscr{M}$ to be tractable as in [8], when we will localize the projective model structure; but this will be mentioned explicitly as well.

5.1 Detecting the Segal conditions for comma categories.

Consider the adjunction $\Pi_{\mathrm{Arr}} : \mathscr{M}_{\mathcal{U}}[\mathscr{A}] \rightleftarrows \mathrm{Arr}(\mathscr{M}) : \Gamma$ given in Proposition 4.7.

Definition 5.1. Define the **injective localizing set** $\mathbf{K}_{\mathbf{I}_{\mathscr{M}}} = \{\Gamma(\alpha_i)\}_{i \in \mathbf{I}_{\mathscr{M}}} = \Gamma(\alpha_{\mathbf{I}_{\mathscr{M}}})$ as the image of $\alpha_{\mathbf{I}_{\mathscr{M}}}$ under the left adjoint Γ.

An immediate consequence of Lemma 2.22 is:

Proposition 5.2. *In the category $\mathscr{M}_{\mathcal{U}}[\mathscr{A}]$ the following hold.*

1. *An object $[\mathcal{F}] = [\mathcal{F}_0, \mathcal{F}_1, \pi_{\mathcal{F}}]$ is $\mathbf{K}_{\mathbf{I}}$-injective, that is $[\mathcal{F}] \longrightarrow *$ has the RLP with respect to $\mathbf{K}_{\mathbf{I}}$, if and only if, the map $\pi_{\mathcal{F}} : \mathcal{F}_0 \longrightarrow \mathcal{U}(\mathcal{F}_1)$ is a trivial fibration in $\mathscr{M}$. In particular $[\mathcal{F}]$ satisfies the Segal condition.*
2. *For any $\mathcal{P} \in \mathscr{A}$, $\iota(\mathcal{P}) = [\mathcal{U}(\mathcal{P}), \mathcal{P}, \mathrm{Id}_{\mathcal{U}(\mathcal{P})}]$ is $\mathbf{K}_{\mathbf{I}}$-injective.*

Proof. We proceed by adjunction. $[\mathcal{F}] \longrightarrow *$ has the RLP with respect to $\mathbf{K}_{\mathbf{I}}$, if and only if, the map $\pi_{\mathcal{F}} : \mathcal{F}_0 \longrightarrow \mathcal{U}(\mathcal{F}_1)$ has the RLP with respect to $\alpha_{\mathbf{I}}$, if and only if $\pi_{\mathcal{F}}$ is a trivial fibration, thanks to Lemma 2.22. This gives the first assertion. The second assertion is clear since every identity morphism is a trivial fibration. □

Definition 5.3. Let $\mathscr{M}$ be a tractable model category. Define the **projective localizing set** $\mathbf{K}_{\mathbf{I}_{proj}} = \{\Gamma(\zeta_i)\}_{i \in \mathbf{I}_{\mathscr{M}}} = \Gamma(\zeta_{\mathbf{I}_{\mathscr{M}}})$ as the image of $\zeta_{\mathbf{I}_{\mathscr{M}}}$ under the left adjoint Γ.

The following proposition will be used for the projective model structure.

Proposition 5.4. *Let $\mathscr{M}$ be a tractable model category. Let $[\mathcal{F}] = [\mathcal{F}_0, \mathcal{F}_1, \pi_{\mathcal{F}}]$ be an object in the category $\mathscr{M}_{\mathcal{U}}[\mathscr{A}]$ and assume that:*

1. *$[\mathcal{F}]$ is $\mathbf{K}_{\mathbf{I}_{proj}}$-injective, that is $[\mathcal{F}] \longrightarrow *$ has the RLP with respect to $\mathbf{K}_{\mathbf{I}_{proj}}$;*
2. *$\pi_{\mathcal{F}}$ is a map between fibrant objects i.e, the unique map $[\mathcal{F}] \longrightarrow *$ is a level-wise fibration.*

Then $[\mathcal{F}]$ satisfies the Segal conditions, that is, $\pi_{\mathcal{F}}$ is a weak equivalence.

Proof. Just proceed by adjunction with Lemma 2.24. □

5.2 Injective and projective model structures on $\mathscr{M}_{\mathcal{U}}[\mathscr{A}]$. Let us consider a right Quillen functor $\mathcal{U} : \mathscr{A} \longrightarrow \mathscr{M}$ that possesses a left adjoint $\mathbf{F} : \mathscr{M} \longrightarrow \mathscr{A}$. We are going to mimic Hovey's result and have an **injective model structure** on $\mathscr{M}_{\mathcal{U}}[\mathscr{A}] := (\mathscr{M} \downarrow \mathcal{U})$, such that when $\mathcal{U} = \mathrm{Id}_{\mathscr{M}}$, we recover Hovey's theorem. Stanculescu [88] and Toën [96] considered similar model structures on the (equivalent) category $(\mathbf{F} \downarrow \mathscr{A})$.

Recall that from the adjunction $\mathbf{F} \dashv \mathcal{U}$ there is a functorial isomorphism of hom-sets

$$\varrho : \mathrm{Hom}_{\mathscr{M}}(\mathcal{F}_0, \mathcal{U}(\mathcal{F}_1)) \xrightarrow{\cong} \mathrm{Hom}_{\mathscr{A}}(\mathbf{F}(\mathcal{F}_0), \mathcal{F}_1).$$

Then if $\sigma = [\sigma_0, \sigma_1] : [\mathcal{F}] \longrightarrow [\mathcal{G}]$ is a map in $\mathscr{M}_{\mathcal{U}}[\mathscr{A}]$ we have two commutative squares that are mutually adjoint:

$$\begin{array}{ccc} \mathcal{F}_0 & \xrightarrow{\sigma_0} & \mathcal{G}_0 \\ \downarrow{\scriptstyle \pi_{\mathcal{F}}} & & \downarrow{\scriptstyle \pi_{\mathcal{G}}} \\ \mathcal{U}(\mathcal{F}_1) & \xrightarrow{\mathcal{U}(\sigma_1)} & \mathcal{U}(\mathcal{G}_1) \end{array} \iff \begin{array}{ccc} \mathbf{F}(\mathcal{F}_0) & \xrightarrow{\mathbf{F}(\sigma_0)} & \mathbf{F}(\mathcal{G}_0) \\ \downarrow{\scriptstyle \varrho(\pi_{\mathcal{F}})} & & \downarrow{\scriptstyle \varrho(\pi_{\mathcal{G}})} \\ \mathcal{F}_1 & \xrightarrow{\sigma_1} & \mathcal{G}_1 \end{array}$$

Definition 5.5. Let $\sigma : [\mathcal{F}] \longrightarrow [\mathcal{G}]$ be a map in $\mathscr{M}_{\mathcal{U}}[\mathscr{A}]$. With the previous notation we will say that:

1. σ is a **injective (trivial) cofibration** if σ_0 is a (trivial) cofibration in $\mathscr{M}$ and σ_1 is a (trivial) cofibration in $\mathscr{A}$.
2. σ is a **level-wise weak equivalence** (resp. **level-wise fibration**) if:
   - σ_0 is a weak equivalence (resp. fibration) in $\mathscr{M}$ and
   - σ_1 is a weak equivalence (resp. fibration) in $\mathscr{A}$.
3. σ is an **injective (trivial) fibration** if:
   - $\sigma_1 : \mathcal{F}_1 \longrightarrow \mathcal{G}_1$ is a (trivial) fibration in $\mathscr{A}$ and
   - the induced map $\mathcal{F}_0 \longrightarrow \mathcal{U}(\mathcal{F}_1) \times_{\mathcal{U}(\mathcal{G}_1)} \mathcal{G}_0$ is a (trivial) fibration in $\mathscr{M}$.
4. σ is a **projective (trivial) cofibration** if:
   - $\sigma_0 : \mathcal{F}_0 \longrightarrow \mathcal{G}_0$ is a (trivial) cofibration in $\mathscr{M}$ and
   - the induced map $\mathcal{F}_1 \cup^{\mathbf{F}(\mathcal{F}_0)} \mathbf{F}(\mathcal{G}_0) \longrightarrow \mathcal{G}_1$ is a (trivial) cofibration in $\mathscr{A}$.

We need the following result to establish the Reedy model structure. The proof is straightforward but we include it for completeness.

Lemma 5.6. *Let $\mathcal{U} : \mathscr{A} \longrightarrow \mathscr{M}$ be a right Quillen functor. Then a map $\sigma : [\mathcal{F}] \longrightarrow [\mathcal{G}]$ is an injective trivial fibration if and only if it is an injective fibration and a level-wise weak equivalence.*

Proof. To prove the only if part, it suffices to show that σ is a level-wise weak equivalence. If $\sigma = [\sigma_0, \sigma_1]$ is an injective trivial fibration, then σ_1 is a trivial fibration in $\mathscr{A}$ by definition, and therefore $\mathcal{U}(\sigma_1)$ is also trivial fibration in $\mathscr{M}$, since $\mathcal{U}$ preserves trivial fibrations. Now in any model category, trivial fibrations are closed under pullback, and we see that the canonical map $p : \mathcal{U}(\mathcal{F}_1) \times_{\mathcal{U}(\mathcal{G}_1)} \mathcal{G}_0 \longrightarrow \mathcal{G}_0$ is also a trivial fibration. The other part of being an injective trivial fibration means that the map $\mathcal{F}_0 \longrightarrow \mathcal{U}(\mathcal{F}_1) \times_{\mathcal{U}(\mathcal{G}_1)} \mathcal{G}_0$ is a trivial fibration. The map σ_0 is the composite the previous map followed by the map p, and both are trivial fibrations, thus σ_0 is a trivial fibration as well. In the end σ_1 and σ_0 are trivial fibrations in the respective model categories, in particular each of them is a weak equivalence.

For the if part, we simply need to show that the map $\mathcal{F}_0 \longrightarrow \mathcal{U}(\mathcal{F}_1) \times_{\mathcal{U}(\mathcal{G}_1)} \mathcal{G}_0$ is a weak equivalence since it is already a fibration. The argument is based on the 3-for-2 property of weak equivalences in $\mathscr{M}$. Indeed, assume that $\sigma = [\sigma_0, \sigma_1]$ is an injective fibration and a level-wise weak equivalence. Then σ_1 is a trivial fibration in $\mathscr{A}$ by definition, and therefore $\mathcal{U}(\sigma_1)$ and then its pullback p are also a trivial fibrations. As mentioned above, we have a factorization σ_0 as the map $\mathcal{F}_0 \longrightarrow \mathcal{U}(\mathcal{F}_1) \times_{\mathcal{U}(\mathcal{G}_1)} \mathcal{G}_0$ followed by the map $p : \mathcal{U}(\mathcal{F}_1) \times_{\mathcal{U}(\mathcal{G}_1)} \mathcal{G}_0 \longrightarrow \mathcal{G}_0$. Since σ_0 and p are weak equivalences, then by 3-for-2, the map $\mathcal{F}_0 \longrightarrow \mathcal{U}(\mathcal{F}_1) \times_{\mathcal{U}(\mathcal{G}_1)} \mathcal{G}_0$ is also a weak equivalence as desired. □

The dual statement is:

Lemma 5.7. *A map $\sigma : [\mathcal{F}] \longrightarrow [\mathcal{G}]$ is a projective trivial cofibration if and only if it is a level-wise weak equivalence and a projective cofibration.*

The proof is dual to the previous one so we just need to adapt it.

Proof. To prove the only if part, it suffices to show that σ is a level-wise weak equivalence. If $\sigma = [\sigma_0, \sigma_1]$ is a projective trivial cofibration, then σ_0 is a trivial cofibration in $\mathscr{M}$ by definition, and therefore $\mathbf{F}(\sigma_0)$ is also trivial cofibration in $\mathscr{A}$, since $\mathbf{F}$ preserves trivial cofibrations. Now in any model category, trivial cofibrations are closed under cobase change. It follows that the canonical map $q : \mathcal{F}_1 \longrightarrow \mathcal{F}_1 \cup^{\mathbf{F}(\mathcal{F}_0)} \mathbf{F}(\mathcal{G}_0)$ is a trivial cofibration in $\mathscr{A}$. The other part of being a projective trivial cofibration means that the map $\mathcal{F}_1 \cup^{\mathbf{F}(\mathcal{F}_0)} \mathbf{F}(\mathcal{G}_0) \longrightarrow \mathcal{G}_1$ is a trivial cofibration. The map σ_1 is the composite of the map q followed by the previous map, and both are trivial cofibrations. Then σ_0 is a trivial cofibrations as well. In the end σ_0 and σ_1 are trivial cofibrations in the respective model category, in particular each of them is a weak equivalence.

For the if part, we simply need to show that the map $\mathcal{F}_1 \cup^{\mathbf{F}(\mathcal{F}_0)} \mathbf{F}(\mathcal{G}_0) \longrightarrow \mathcal{G}_1$ is a weak equivalence since it is already a cofibration. The argument is also based on the 3-for-2 property of weak equivalences in $\mathscr{A}$. Indeed, assume that $\sigma = [\sigma_0, \sigma_1]$ is a projective cofibration and a level-wise weak equivalence. Then σ_0 is a trivial cofibration in $\mathscr{M}$ by definition, therefore $\mathbf{F}(\sigma_0)$ and its cobase change q are also a trivial cofibrations in $\mathscr{A}$. As mentioned above, we have a factorization σ_1 as the map q followed by the map $\mathcal{F}_1 \cup^{\mathbf{F}(\mathcal{F}_0)} \mathbf{F}(\mathcal{G}_0) \longrightarrow \mathcal{G}_1$. Since σ_1 and q are weak equivalences, then by 3-for-2, the map

$$\mathcal{F}_1 \cup^{\mathbf{F}(\mathcal{F}_0)} \mathbf{F}(\mathcal{G}_0) \longrightarrow \mathcal{G}_1$$

is also a weak equivalence as desired. □

5.3 Factorizations. We generalize here to comma categories the necessary factorizations that are required to get the Reedy model structure. We simply follow the classical method.

5.3.1 Injective factorizations

Proposition 5.8. *Let $\mathcal{U} : \mathscr{A} \longrightarrow \mathscr{M}$ be a right Quillen functor. Then with the previous definition the following hold.*

1. *Any map $\sigma : [\mathcal{F}] \longrightarrow [\mathcal{G}]$ can be factored as an injective cofibration followed by an injective trivial fibration.*
2. *Any map $\sigma : [\mathcal{F}] \longrightarrow [\mathcal{G}]$ can be factored as an injective trivial cofibration followed by an injective fibration.*

If we decide to prove each assertion separately, this will be very long. So we will use a generic language of *(weak) factorization systems* (see for example [81]). In any model category we have two weak factorizations systems $(\mathbf{cof}, \mathbf{fib} \cap \mathscr{W})$ and $(\mathbf{cof} \cap \mathscr{W}, \mathbf{fib})$, where **cof**, **fib** and $\mathscr{W}$ are respectively the classes of cofibrations, fibrations and weak equivalences. Then we have two different factorization systems $(\mathscr{L}_{\mathscr{A}}, \mathscr{R}_{\mathscr{A}})$ on $\mathscr{A}$ and $(\mathscr{L}_{\mathscr{M}}, \mathscr{R}_{\mathscr{M}})$ on $\mathscr{M}$ that determine the model structure on each model category. The functor $\mathcal{U}$ is such that $\mathcal{U}(\mathscr{R}_{\mathscr{A}}) \subseteq \mathscr{R}_{\mathscr{M}}$ for each of the corresponding factorization systems.

Proof of Proposition 5.8. Let $\sigma = [\sigma_0, \sigma_1] : [\mathcal{F}] \longrightarrow [\mathcal{G}]$ be a map in $\mathscr{M}_{\mathcal{U}}[\mathscr{A}]$. Use the axiom of the model category $\mathscr{A}$ with respect to the factorization system $(\mathscr{L}_{\mathscr{A}}, \mathscr{R}_{\mathscr{A}})$ to write σ_1 as $\sigma_1 = r(\sigma_1) \circ l(\sigma_1)$:

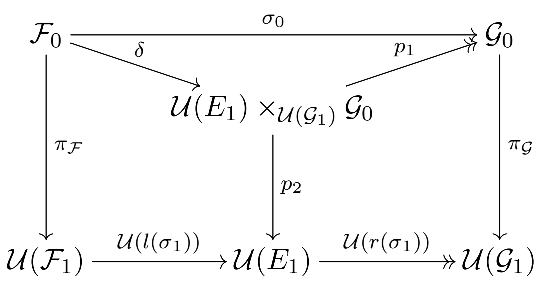
$$\mathcal{F}_1 \xrightarrow{\sigma_1} \mathcal{G}_1 = \mathcal{F}_1 \xhookrightarrow{l(\sigma_1)} E_1 \xrightarrow{r(\sigma_1)} \mathcal{G}_1,$$

with $r(\sigma_1) \in \mathscr{R}_{\mathscr{A}}$ and $l(\sigma_1) \in \mathscr{L}_{\mathscr{A}}$. The image under $\mathcal{U}$ of this factorization, gives a factorization $\mathcal{U}(\sigma_1) = \mathcal{U}(r(\sigma_1)) \circ \mathcal{U}(l(\sigma_1))$, with $\mathcal{U}(r(\sigma_1)) \in \mathscr{R}_{\mathscr{M}}$ since $\mathcal{U}(\mathscr{R}_{\mathscr{A}}) \subseteq \mathscr{R}_{\mathscr{M}}$.

Form the pullback square in $\mathscr{M}$ defined by the pullback data:

$$\mathcal{U}(E_1) \xrightarrow{\mathcal{U}(r(\sigma_1))} \mathcal{U}(\mathcal{G}_1) \xleftarrow{\pi_{\mathcal{G}}} \mathcal{G}_0,$$

and let $p_1 : \mathcal{U}(E_1) \times_{\mathcal{U}(\mathcal{G}_1)} \mathcal{G}_0 \longrightarrow \mathcal{G}_0$ and $p_2 : \mathcal{U}(E_1) \times_{\mathcal{U}(\mathcal{G}_1)} \mathcal{G}_0 \longrightarrow \mathcal{U}(\mathcal{E}_1)$ be the canonical maps. Then $p_1 \in \mathscr{R}_{\mathscr{M}}$ because $\mathscr{R}_{\mathscr{M}}$ is closed under pullbacks. The universal property of the pullback square gives a unique map $\delta : \mathcal{F}_0 \longrightarrow \mathcal{U}(E_1) \times_{\mathcal{U}(\mathcal{G}_1)} \mathcal{G}_0$, such that everything below commutes.



Now we use the factorization system $(\mathscr{L}_{\mathscr{M}}, \mathscr{R}_{\mathscr{M}})$ to factor the map δ:

$$\delta : \mathcal{F}_0 \longrightarrow \mathcal{U}(E_1) \times_{\mathcal{U}(\mathcal{G}_1)} \mathcal{G}_0 = \mathcal{F}_0 \xhookrightarrow{l(\delta)} m_0 \xrightarrow{r(\delta)} \mathcal{U}(E_1) \times_{\mathcal{U}(\mathcal{G}_1)} \mathcal{G}_0,$$

with $r(\delta) \in \mathscr{R}_{\mathscr{M}}$ and $l(\delta) \in \mathscr{L}_{\mathscr{M}}$. Let $[\mathcal{E}] = [\mathcal{E}_0, \mathcal{E}_1, \pi_{\mathcal{E}}]$ be the object of $\mathscr{M}_{\mathcal{U}}[\mathscr{A}]$ defined by

$$\mathcal{E}_0 = m_0, \quad \mathcal{E}_1 = E_1, \quad \pi_{\mathcal{E}} = p_2 \circ r(\delta).$$

We have a map $l(\sigma) : [\mathcal{F}] \longrightarrow [\mathcal{E}]$ given by the pair $[l(\delta), l(\sigma_1)] \in \mathscr{L}_{\mathscr{M}} \times \mathscr{L}_{\mathscr{A}}$, and a map $r(\sigma) : [\mathcal{E}] \longrightarrow [\mathcal{G}]$ given by the pair $[p_1 \circ r(\delta), r(\sigma_1)]$, with $[r(\delta), r(\sigma_1)] \in \mathscr{R}_{\mathscr{M}} \times \mathscr{R}_{\mathscr{A}}$ such that $\sigma = r(\sigma) \circ l(\sigma)$. This gives the assertions. □

5.3.2 Projective factorizations

Proposition 5.9. *Let $\mathcal{U} : \mathscr{A} \longrightarrow \mathscr{M}$ be a right Quillen functor. Then with the previous definition the following hold.*

1. *Any map $\sigma : [\mathcal{F}] \longrightarrow [\mathcal{G}]$ can be factored as a projective cofibration followed by a projective trivial fibration.*
2. *Any map $\sigma : [\mathcal{F}] \longrightarrow [\mathcal{G}]$ can be factored as a projective trivial cofibration followed by a projective fibration.*

Proof. Consider a factorization system $(\mathscr{L}_{\mathscr{A}}, \mathscr{R}_{\mathscr{A}})$ on $\mathscr{A}$ and a factorization system $(\mathscr{L}_{\mathscr{M}}, \mathscr{R}_{\mathscr{M}})$ on $\mathscr{M}$ such that $\mathcal{U}(\mathscr{R}_{\mathscr{A}}) \subseteq \mathscr{R}_{\mathscr{M}}$ and $\mathbf{F}(\mathscr{L}_{\mathscr{M}}) \subseteq \mathscr{L}_{\mathscr{A}}$. Let $\sigma = [\sigma_0, \sigma_1] : [\mathcal{F}] \longrightarrow [\mathcal{G}]$ be a map in $\mathscr{M}_{\mathcal{U}}[\mathscr{A}]$. Use the axiom of the model category $\mathscr{A}$ with respect to the factorization system $(\mathscr{L}_{\mathscr{A}}, \mathscr{R}_{\mathscr{A}})$ to write σ_0 as $\sigma_0 = r(\sigma_0) \circ l(\sigma_0)$:

$$\mathcal{F}_0 \xrightarrow{\sigma_0} \mathcal{G}_0 = \mathcal{F}_1 \xhookrightarrow{l(\sigma_0)} m_0 \xrightarrow{r(\sigma_0)} \mathcal{G}_0,$$

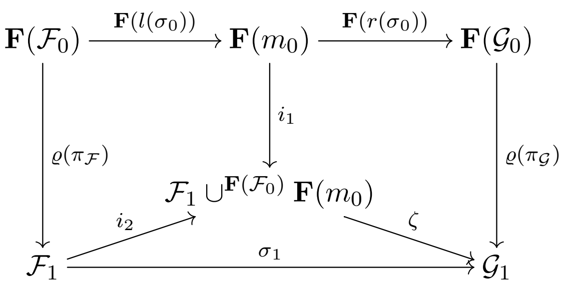
with $r(\sigma_0) \in \mathscr{R}_{\mathscr{M}}$ and $l(\sigma_0) \in \mathscr{L}_{\mathscr{M}}$. The image under $\mathbf{F}$ of this factorization, gives a factorization $\mathbf{F}(\sigma_0) = \mathbf{F}(r(\sigma_0)) \circ \mathbf{F}(l(\sigma_0))$, with $\mathbf{F}(l(\sigma_0)) \in \mathscr{L}_{\mathscr{A}}$ since $\mathbf{F}(\mathscr{L}_{\mathscr{M}}) \subseteq \mathscr{L}_{\mathscr{A}}$. Form the pushout square in $\mathscr{A}$ defined by the pushout data

$$\mathcal{F}_1 \xleftarrow{\varrho(\pi_{\mathcal{F}})} \mathbf{F}(\mathcal{F}_0) \xrightarrow{\mathbf{F}(l(\sigma_0))} \mathbf{F}(m_0),$$

and let $i_2 : \mathcal{F}_1 \longrightarrow \mathcal{F}_1 \cup^{\mathbf{F}(\mathcal{F}_0)} \mathbf{F}(\mathcal{G}_0)$ and $i_1 : \mathbf{F}(m_0) \longrightarrow \mathcal{F}_1 \cup^{\mathbf{F}(\mathcal{F}_0)} \mathbf{F}(\mathcal{G}_0)$ be the canonical maps. Then $i_2 \in \mathscr{L}_{\mathscr{A}}$ because $\mathscr{L}_{\mathscr{A}}$ is closed under pushouts. The universal property of the pushout square gives a unique map

$$\zeta : \mathcal{F}_1 \cup^{\mathbf{F}(\mathcal{F}_0)} \mathbf{F}(m_0) \longrightarrow \mathcal{G}_1,$$

such that everything below commutes.



Now we use the factorization system $(\mathscr{L}_{\mathscr{A}}, \mathscr{R}_{\mathscr{A}})$ to factor the map ζ:

$$\delta : \mathcal{F}_1 \cup^{\mathbf{F}(\mathcal{F}_0)} \mathbf{F}(m_0) \longrightarrow \mathcal{G}_1 = \mathcal{F}_1 \cup^{\mathbf{F}(\mathcal{F}_0)} \mathbf{F}(m_0) \xhookrightarrow{l(\zeta)} E_1 \xrightarrow{r(\zeta)} \mathcal{G}_1,$$

with $l(\zeta) \in \mathscr{L}_{\mathscr{A}}$ and $r(\zeta) \in \mathscr{R}_{\mathscr{A}}$. Let $[\mathcal{E}] = [\mathcal{E}_0, \mathcal{E}_1, \pi_{\mathcal{E}}]$ be the object of $\mathscr{M}_{\mathcal{U}}[\mathscr{A}]$ defined by

$$\mathcal{E}_0 = m_0, \quad \mathcal{E}_1 = E_1, \quad \pi_{\mathcal{E}} = \varrho^{-1}(l(\zeta) \circ i_1) \in \operatorname{Hom}_{\mathscr{M}}(m_0, \mathcal{U}(E_1)).$$

We have a map $l(\sigma) : [\mathcal{F}] \longrightarrow [\mathcal{E}]$ given by the pair $[l(\sigma_0), l(\zeta) \circ i_2]$, with $[l(\sigma_0), l(\zeta)] \in \mathscr{L}_{\mathscr{M}} \times \mathscr{L}_{\mathscr{A}}$, and a map $r(\sigma) : [\mathcal{E}] \longrightarrow [\mathcal{G}]$ given by the pair $[r(\sigma_0), r(\zeta)] \in \mathscr{R}_{\mathscr{M}} \times \mathscr{R}_{\mathscr{A}}$; such that $\sigma = r(\sigma) \circ l(\sigma)$. This gives the assertions. □

5.4 Lifting properties. On the model category $\mathscr{A}$ we will use here again the generic notation $(\mathscr{L}_{\mathscr{A}}, \mathscr{R}_{\mathscr{A}})$ for both factorization systems. Similarly we will denote by $(\mathscr{L}_{\mathscr{M}}, \mathscr{R}_{\mathscr{M}})$ both factorization systems on $\mathscr{M}$. In both cases we have $\mathcal{U}(\mathscr{R}_{\mathscr{A}}) \subseteq \mathscr{R}_{\mathscr{M}}$ and $\mathbf{F}(\mathscr{L}_{\mathscr{M}}) \subseteq \mathscr{L}_{\mathscr{A}}$. Recall that $\sigma = [\sigma_0, \sigma_1] : [\mathcal{F}] \longrightarrow [\mathcal{G}]$ is an injective (trivial) cofibration in $\mathscr{M}_{\mathcal{U}}[\mathscr{A}]$ if $[\sigma_0, \sigma_1] \in \mathscr{L}_{\mathscr{M}} \times \mathscr{L}_{\mathscr{A}}$. Moreover, $\beta = [\beta_0, \beta_1] : [\mathcal{P}] \longrightarrow [\mathcal{Q}]$ is an injective (trivial) fibration if $[\delta, \beta_1] \in \mathscr{L}_{\mathscr{M}} \times \mathscr{L}_{\mathscr{A}}$, where $\delta : \mathcal{P}_0 \longrightarrow \mathcal{U}(\mathcal{P}_1) \times_{\mathcal{U}(\mathcal{Q}_1)} \mathcal{Q}_0$ is the induced map.

5.4.1 *Injective lifting properties*

Proposition 5.10. *Let $\mathcal{U} : \mathscr{A} \longrightarrow \mathscr{M}$ be a right Quillen functor. Then with the previous definitions the following hold.*

1. *Any lifting problem defined by an injective cofibration and an injective trivial fibration has a solution.*
2. *Any lifting problem defined by an injective trivial cofibration and an injective fibration has a solution.*

Proof. The proof is the same for both assertions by considering the appropriate factorization system on $\mathscr{A}$ and $\mathscr{M}$. Consider a lifting problem in $\mathscr{M}_{\mathcal{U}}[\mathscr{A}]$ defined by $\sigma : [\mathcal{F}] \xrightarrow{[\sigma_0,\sigma_1]} [\mathcal{G}]$ and $\beta : [\mathcal{P}] \xrightarrow{[\beta_0,\beta_1]} [\mathcal{Q}]$ as follows.

$$\begin{array}{ccc} [\mathcal{F}] & \xrightarrow{(\theta_0,\theta_1)} & [\mathcal{P}] \\ {\scriptstyle (\sigma_0,\sigma_1)}\downarrow & & \downarrow{\scriptstyle (\beta_0,\beta_1)} \\ [\mathcal{G}] & \xrightarrow{(\gamma_0,\gamma_1)} & [\mathcal{Q}] \end{array}$$

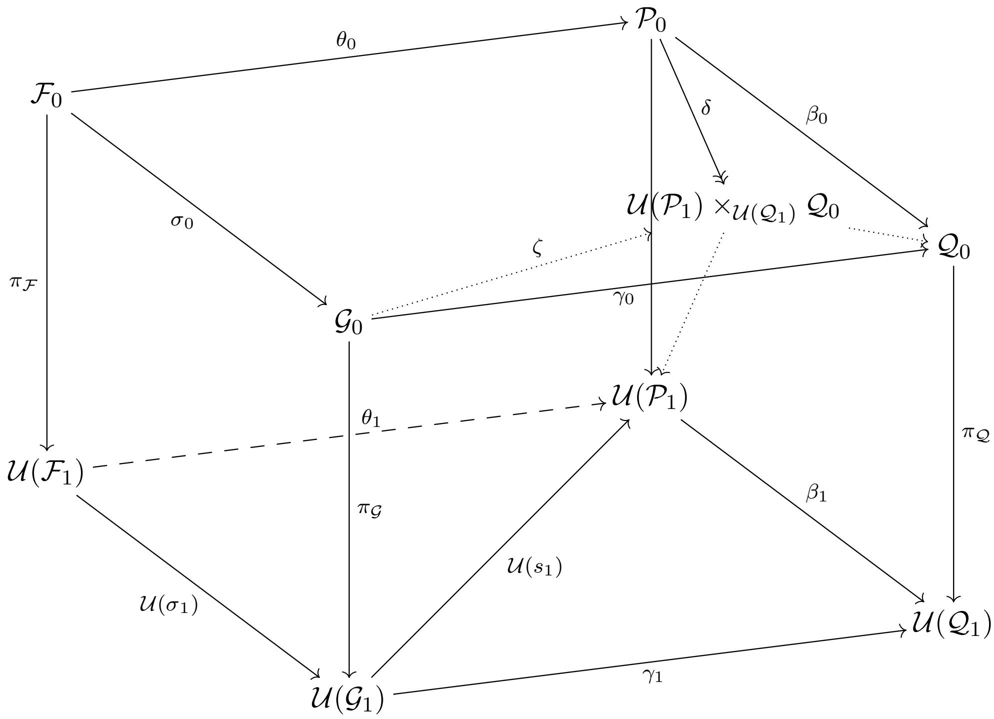
The image of this lifting problem under $\Pi_1 : \mathscr{M}_{\mathcal{U}}[\mathscr{A}] \longrightarrow \mathscr{A}$ is a lifting problem defined by σ_1 and β_1. Therefore if $[\sigma_1, \beta_1] \in \mathscr{L}_{\mathscr{A}} \times \mathscr{R}_{\mathscr{A}}$, then there is a solution $s_1 : \mathcal{G}_1 \longrightarrow \mathcal{P}_1$ of this lifting problem in $\mathscr{A}$. Then by functoriality of $\mathcal{U}$, the map $\mathcal{U}(s_1)$ is a solution to the induced lifting problem defined by $\mathcal{U}(\sigma_1)$ and $\mathcal{U}(\beta_1)$ in $\mathscr{M}$. Part of $\mathcal{U}(s_1)$ being a solution gives an equality $\mathcal{U}(\gamma_1) = \mathcal{U}(\beta_1) \circ \mathcal{U}(s_1)$. Moreover, $[\gamma] = [\gamma_0, \gamma_1]$ being a morphism in $\mathscr{M}_{\mathcal{U}}[\mathscr{A}]$ gives the equality $\pi_{\mathcal{Q}} \circ \gamma_0 = \mathcal{U}(\gamma_1) \circ \pi_{\mathcal{G}}$.

Now consider the map $\mathcal{U}(s_1) \circ \pi_{\mathcal{G}} \in \mathrm{Hom}_{\mathscr{M}}(\mathcal{G}_0, \mathcal{U}(\mathcal{P}_1))$ and the map $\gamma_0 \in \mathrm{Hom}_{\mathscr{M}}(\mathcal{G}_0, \mathcal{Q}_0)$. Then by the above, it is not hard to see that these maps complete the pullback data

$$\mathcal{U}(\mathcal{P}_1) \xrightarrow{\mathcal{U}(\beta_1)} \mathcal{U}(\mathcal{Q}_1) \xleftarrow{\pi_{\mathcal{Q}}} \mathcal{Q}_0$$

into a commutative square ($\pi_{\mathcal{Q}} \circ \gamma_0 = \mathcal{U}(\beta_1) \circ \mathcal{U}(s_1) \circ \pi_{\mathcal{G}}$). Therefore, by the universal property of the pullback square there is a unique map: $\zeta : \mathcal{G}_0 \longrightarrow \mathcal{U}(\mathcal{P}_1) \times_{\mathcal{U}(\mathcal{Q}_1)} \mathcal{Q}_0$, making everything compatible. In particular γ_0 and $\mathcal{U}(s_1) \circ \pi_{\mathcal{G}}$ factor through ζ.

Our original lifting problem in $\mathscr{M}_{\mathcal{U}}[\mathscr{A}]$ defined by $[\sigma]$ and $[\beta]$ is represented by a commutative cube in $\mathscr{M}$. If we unfold it, we find that everything commutes in the diagram hereafter:



Thus we get a commutative square which corresponds to a lifting problem defined by the map $\sigma_0 : \mathcal{F}_0 \longrightarrow \mathcal{G}_0$ and the map $\delta : \mathcal{P}_0 \longrightarrow \mathcal{U}(\mathcal{P}_1) \times_{\mathcal{U}(\mathcal{Q}_1)} \mathcal{Q}_0$:

$$\begin{array}{ccc} \mathcal{F}_0 & \xrightarrow{\theta_0} & \mathcal{P}_0 \\ {\scriptstyle \sigma_0}\downarrow & & \downarrow{\scriptstyle \delta} \\ \mathcal{G}_0 & \xrightarrow{\zeta} & \mathcal{U}(\mathcal{P}_1) \times_{\mathcal{U}(\mathcal{Q}_1)} \mathcal{Q}_0 \end{array}$$

Now it suffices to observe that this lifting problem has a solution if and only if our original lifting problem has a solution. Indeed, if $s_0 : \mathcal{G}_0 \longrightarrow \mathcal{P}_0$ is a solution to the previous lifting problem, then we have map $[s] = [s_0, s_1] : [\mathcal{G}] \longrightarrow [\mathcal{P}]$ that is a solution to the original problem. Conversely given a solution $[s] = [s_0, s_1] : [\mathcal{G}] \longrightarrow [\mathcal{P}]$ to the original lifting problem , then the component $s_0 : \mathcal{G}_0 \longrightarrow \mathcal{P}_0$ is a solution to the lifting problem defined by σ_0 and $\delta : \mathcal{P}_0 \longrightarrow \mathcal{U}(\mathcal{P}_1) \times_{\mathcal{U}(\mathcal{Q}_1)} \mathcal{Q}_0$. Finally, it is clear that the lifting problem defined by σ_0 and δ has a solution $s_0 \in \mathrm{Hom}_{\mathscr{M}}(\mathcal{G}_0, \mathcal{P}_0)$, since $[\sigma_0, \delta] \in \mathscr{L}_{\mathscr{M}} \times \mathscr{R}_{\mathscr{M}}$. □

5.4.2 Projective lifting properties

Proposition 5.11. *Let $\mathcal{U} : \mathscr{A} \longrightarrow \mathscr{M}$ be a right Quillen functor. Then with the previous definitions the following hold.*

1. *Any lifting problem defined by a projective cofibration and a projective trivial fibration has a solution.*
2. *Any lifting problem defined by a projective trivial cofibration and projective fibration has a solution.*

Proof. We proceed in a dual manner to the proof of Proposition 5.10 with the same notation for both factorizations systems on $\mathscr{A}$ and $\mathscr{M}$. Consider a lifting problem in $\mathscr{M}_{\mathcal{U}}[\mathscr{A}]$ defined by

$\sigma : [\mathcal{F}] \xrightarrow{[\sigma_0,\sigma_1]} [\mathcal{G}]$ and $\beta : [\mathcal{P}] \xrightarrow{[\beta_0,\beta_1]} [\mathcal{Q}]$ as follows.

$$\begin{array}{ccc} [\mathcal{F}] & \xrightarrow{(\theta_0,\theta_1)} & [\mathcal{P}] \\ {\scriptstyle (\sigma_0,\sigma_1)}\downarrow & & \downarrow{\scriptstyle (\beta_0,\beta_1)} \\ [\mathcal{G}] & \xrightarrow{(\gamma_0,\gamma_1)} & [\mathcal{Q}] \end{array}$$

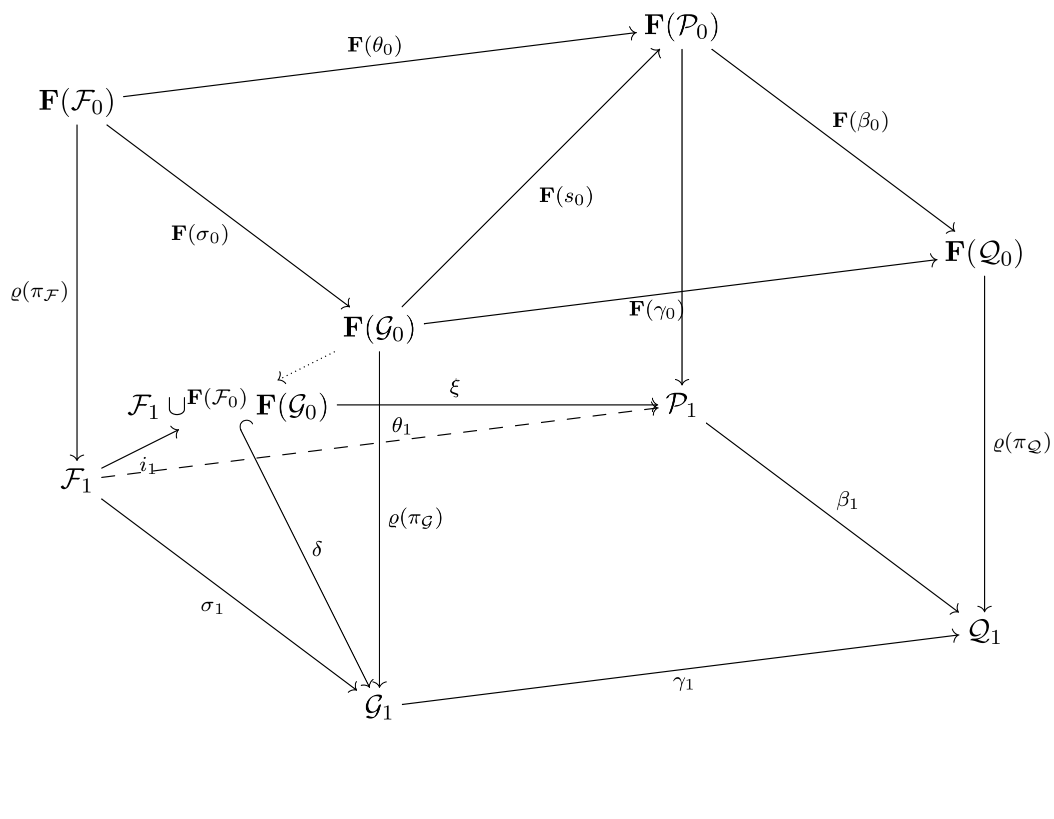
The image of this lifting problem under $\Pi_0 : \mathscr{M}_{\mathcal{U}}[\mathscr{A}] \longrightarrow \mathscr{M}$ is a lifting problem defined by σ_0 and β_0. Therefore if $[\sigma_0, \beta_0] \in \mathscr{L}_{\mathscr{M}} \times \mathscr{R}_{\mathscr{M}}$, then there is a solution $s_0 : \mathcal{G}_0 \longrightarrow \mathcal{P}_0$ of this lifting problem in $\mathscr{M}$. Then by functoriality of $\mathbf{F}$, the map $\mathbf{F}(s_0)$ is a solution to the induced lifting problem defined by $\mathbf{F}(\sigma_0)$ and $\mathbf{F}(\beta_0)$ in $\mathscr{A}$. Part of $\mathbf{F}(s_0)$ being a solution gives an equality $\mathbf{F}(\theta_0) = \mathbf{F}(s_0) \circ \mathbf{F}(\sigma_0)$. And $[\theta] = [\theta_0, \theta_1]$ being a morphism in $\mathscr{M}_{\mathcal{U}}[\mathscr{A}]$ gives by adjunction a morphism $\varrho(\pi_{\mathcal{F}}) \xrightarrow{[\mathbf{F}(\sigma_0),\theta_1]} \varrho(\pi_{\mathcal{Q}})$ in $\mathrm{Arr}(\mathscr{A})$. In particular we have an the equality $\varrho(\pi_{\mathcal{P}}) \circ \mathbf{F}(\theta_0) = \theta_1 \circ \varrho(\pi_{\mathcal{F}})$.

Now consider the map $\mathbf{F}(s_0) \circ \pi_{\mathcal{P}} \in \mathrm{Hom}_{\mathscr{A}}(\mathbf{F}(\mathcal{G}_0), \mathcal{P}_1)$ and the map $\theta_1 \in \mathrm{Hom}_{\mathscr{A}}(\mathcal{F}_1, \mathcal{P}_1)$. Then using the previous equalities, it is not hard to see that these maps complete the pushout data

$$\mathbf{F}(\mathcal{G}_0) \xleftarrow{\mathbf{F}(\sigma_0)} \mathbf{F}(\mathcal{F}_0) \xrightarrow{\varrho(\pi_{\mathcal{F}})} \mathcal{F}_1$$

into a commutative square ($\theta_1 \circ \varrho(\pi_{\mathcal{F}}) = \varrho(\pi_{\mathcal{P}}) \circ \mathbf{F}(s_0) \circ \mathbf{F}(\sigma_0)$). Therefore, by the universal property of the pushout square, there is a unique map: $\xi : \mathcal{F}_1 \cup^{\mathbf{F}(\mathcal{F}_0)} \mathbf{F}(\mathcal{G}_0) \longrightarrow \mathcal{P}_1$, making everything compatible. In particular $\varrho(\pi_{\mathcal{Q}}) \circ \mathbf{F}(s_0)$ and θ_1 factor through ξ.

The original lifting problem in $\mathscr{M}_{\mathcal{U}}[\mathscr{A}]$ defined by $[\sigma]$ and $[\beta]$ is represented by adjunction by a commutative cube in $\mathscr{A}$. If we unfold it, we find that everything commutes in the diagram hereafter:



Thus we get a commutative square that corresponds to a lifting problem defined by the maps $\delta : \mathcal{F}_1 \cup^{\mathbf{F}(\mathcal{F}_0)} \mathbf{F}(\mathcal{G}_0) \longrightarrow \mathcal{G}_1$ and $\beta_1 : \mathcal{P}_1 \longrightarrow \mathcal{Q}_1$:

$$\begin{array}{ccc} \mathcal{F}_1 \cup^{\mathbf{F}(\mathcal{F}_0)} \mathbf{F}(\mathcal{G}_0) & \xrightarrow{\xi} & \mathcal{P}_1 \\ {\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle\beta_1} \\ \mathcal{G}_1 & \xrightarrow{\gamma_1} & \mathcal{Q}_1 \end{array}$$

As in the injective case, it suffices to observe that this lifting problem has a solution if and only if our original lifting problem has a solution. Indeed, if $s_1 : \mathcal{G}_1 \longrightarrow \mathcal{P}_1$ is a solution to the previous lifting problem, then we have map $[s] = [s_0, s_1] : [\mathcal{G}] \longrightarrow [\mathcal{P}]$ that is a solution to the original problem. Conversely given a solution $[s] = [s_0, s_1] : [\mathcal{G}] \longrightarrow [\mathcal{P}]$ to the original lifting problem , then the component $s_1 : \mathcal{G}_1 \longrightarrow \mathcal{P}_1$ is a solution to the lifting problem defined by $\delta : \mathcal{F}_1 \cup^{\mathbf{F}(\mathcal{F}_0)} \mathbf{F}(\mathcal{G}_0) \longrightarrow \mathcal{G}_1$ and β_1. Finally, it is clear that the lifting problem defined by δ and β_1 has a solution $s_1 \in \mathrm{Hom}_{\mathscr{A}}(\mathcal{G}_1, \mathcal{P}_1)$, since $[\sigma_0, \delta] \in \mathscr{L}_{\mathscr{A}} \times \mathscr{R}_{\mathscr{A}}$. □

5.5 The model structures. We are able to state our theorems.

5.5.1 Injective model structure

Theorem 5.12. *Let $\mathcal{U} : \mathscr{A} \longrightarrow \mathscr{M}$ be a right Quillen functor. Then with the previous definitions, the following hold. There is a model structure on the category $(\mathscr{M} \downarrow \mathcal{U}) = \mathscr{M}_{\mathcal{U}}[\mathscr{A}]$ which may be described as follows.*

- *A map $\sigma : [\mathcal{F}] \longrightarrow [\mathcal{G}]$ is a weak equivalence if and only if it is a level-wise weak equivalence.*
- *A map $\sigma : [\mathcal{F}] \longrightarrow [\mathcal{G}]$ is cofibration if it is an injective cofibration.*
- *A map $\sigma : [\mathcal{F}] \longrightarrow [\mathcal{G}]$ is fibration if it is an injective fibration.*

We will denote this model category by $\mathscr{M}_{\mathcal{U}}[\mathscr{A}]_{\mathrm{inj}}$.

Proof. The class of level-wise weak equivalences clearly satisfies the 2-out-of-3 property. The three classes of cofibrations, fibrations and weak equivalences are closed under composition and retracts. With Proposition 5.10, Proposition 5.8 and Lemma 5.6, one can easily verify that the axioms of a model structure hold (see for example [44, Definition 1.1.3]). □

Corollary 5.13. *Let $\mathcal{U} : \mathscr{A} \longrightarrow \mathscr{M}$ be a right Quillen functor.*

1. *We have a Quillen adjunction $\Pi_1 : \mathscr{M}_{\mathcal{U}}[\mathscr{A}]_{\mathrm{inj}} \rightleftarrows \mathscr{A} : \iota$, where Π_1 is left Quillen.*
2. *We have a Quillen adjunction $L_1 : \mathscr{A} \rightleftarrows \mathscr{M}_{\mathcal{U}}[\mathscr{A}]_{\mathrm{inj}} : \Pi_1$, where L_1 is left Quillen.*
3. *We have a Quillen adjunction $\Gamma : Arr(\mathscr{M})_{\mathrm{inj}} \rightleftarrows \mathscr{M}_{\mathcal{U}}[\mathscr{A}]_{\mathrm{inj}} : \Pi_{Arr}$, where Γ is left Quillen.*
4. *We also have a Quillen adjunction $\Pi_0 : \mathscr{M}_{\mathcal{U}}[\mathscr{A}]_{\mathrm{inj}} \rightleftarrows \mathscr{M} : \mathbf{F}^+$, where $\mathbf{F}^+ = \Gamma \circ L_0$ is left Quillen.*
5. *The functors Π_1 and Π_0 preserve the weak equivalences.*

Proof. The functor $\Pi_1 : \mathscr{M}_{\mathcal{U}}[\mathscr{A}]_{\mathrm{inj}} \longrightarrow \mathscr{A}$ preserves (trivial) cofibrations and (trivial) fibrations. So clearly it is simultaneously a left Quillen functor and a right Quillen functor. This gives the first two assertions.

For Assertion (3), it suffices to observe that if $\sigma = [\sigma_0, \sigma_1]$ is an injective trivial fibration then $\Pi_{\mathrm{Arr}}(\sigma) = [\sigma_0, \mathcal{U}(\sigma_1)]$ is an injective (trivial) fibration in $\mathrm{Arr}(\mathscr{M})_{\mathrm{inj}}$ by definition ($\mathcal{U}$ being a right Quillen functor). Therefore Π_{Arr} is right Quillen which means automatically that Γ is left Quillen.

Assertion (4) is clear because any injective (trivial) fibration is also projective (trivial) fibration; that is a level-wise (trivial) fibration. Therefore the functor Π_0 preserves the fibrations and the trivial fibrations; which means that Π_0 is a right Quillen functor. The last assertion follows from the definition of a level-wise weak equivalence. □

Cofibrantly generated. Let us now assume that $\mathscr{M}$ and $\mathscr{A}$ are cofibrantly generated in the sense of [44, Definition 2.1.17]. We will denote by $\mathbf{I}_{\mathscr{M}}$ and $\mathbf{J}_{\mathscr{M}}$ the respective generating sets of cofibrations and trivial cofibrations for $\mathscr{M}$. Similarly let $\mathbf{I}_{\mathscr{A}}$ and $\mathbf{J}_{\mathscr{A}}$ be the respective generating sets of cofibrations and trivial cofibrations of $\mathscr{A}$. Given $s : A \longrightarrow B \in \operatorname{Arr}(\mathscr{M})$, we have introduced in Notation 2.12 a map $\alpha_s : s \xrightarrow{(s,\mathrm{Id})} \mathrm{Id}_B$ in $\operatorname{Arr}(\mathscr{M})$.

Theorem 5.14. *If $\mathscr{A}$ and $\mathscr{M}$ are cofibrantly generated (resp. combinatorial), then $\mathscr{M}_{\mathcal{U}}[\mathscr{A}]_{\mathrm{inj}}$ is cofibrantly generated (resp. combinatorial). Moreover,*

1. *The set $\mathbf{I}_{\mathscr{M}_{\mathcal{U}}[\mathscr{A}]_{\mathrm{inj}}} = L_1(\mathbf{I}_{\mathscr{A}}) \coprod \Gamma(\alpha_{\mathbf{I}_{\mathscr{M}}})$ is a generating set of cofibrations in $\mathscr{M}_{\mathcal{U}}[\mathscr{A}]_{\mathrm{inj}}$.*
2. *The set $\mathbf{J}_{\mathscr{M}_{\mathcal{U}}[\mathscr{A}]_{\mathrm{inj}}} = L_1(\mathbf{J}_{\mathscr{A}}) \coprod \Gamma(\alpha_{\mathbf{J}_{\mathscr{M}}})$ is a generating set of trivial cofibrations in $\mathscr{M}_{\mathcal{U}}[\mathscr{A}]_{\mathrm{inj}}$.*

Proof. Corollary 4.13 says that the category $\mathscr{M}_{\mathcal{U}}[\mathscr{A}]$ is locally presentable. So we just need to prove that it is cofibrantly generated if we want to prove that it is a combinatorial model category.

A map $\sigma = [\sigma_0, \sigma_1]$ has the RLP with respect to all maps in $L_1(\mathbf{I}_{\mathscr{A}}) \coprod \Gamma(\alpha_{\mathbf{I}_{\mathscr{M}}})$ if and only if it is simultaneously $L_1(\mathbf{I}_{\mathscr{A}})$-injective and $\Gamma(\alpha_{\mathbf{I}_{\mathscr{M}}})$-injective. One the one hand, using the adjunction $L_1 \dashv \Pi_1$, $\sigma = [\sigma_0, \sigma_1]$ is $L_1(\mathbf{I}_{\mathscr{A}})$-injective if and only if σ_1 is $\mathbf{I}_{\mathscr{A}}$-injective, if and only if, σ_1 is a trivial fibration.

On the other hand, using the adjunction $\Gamma \dashv \Pi_1$, $\sigma : [\mathcal{P}] \xrightarrow{[\sigma_0,\sigma_1]} [\mathcal{Q}]$ is $\Gamma(\alpha_{\mathbf{I}_{\mathscr{M}}})$-injective if and only if $\Pi_{\mathrm{Arr}}(\sigma) = [\sigma_0, \mathcal{U}(\sigma_1)]$ is $\alpha_{\mathbf{I}_{\mathscr{M}}}$-injective. By Lemma 2.22, $\Pi_{\mathrm{Arr}}(\sigma) = [\sigma_0, \mathcal{U}(\sigma_1)]$ is $\alpha_{\mathbf{I}_{\mathscr{M}}}$-injective if and only if $\delta : \mathcal{P}_0 \longrightarrow \mathcal{U}(\mathcal{P}_1) \times_{\mathcal{U}(\mathcal{Q}_1)} \mathcal{Q}_0$ is $\mathbf{I}_{\mathscr{M}}$-injective, if and only if δ is a trivial fibration. This gives Assertion (1). The second assertion is proved the same way. □

5.5.2 Localizing the injective model structure. Recall that the functor $\Pi_1 : \mathscr{M}_{\mathcal{U}}[\mathscr{A}] \longrightarrow \mathscr{A}$ preserves the weak equivalences. Let $\mathscr{W}_{\mathscr{M}_{\mathcal{U}}[\mathscr{A}]}$ be the class of level-wise weak equivalences and let $\mathscr{W}_{\mathscr{A}}$ be the class of weak equivalences in $\mathscr{A}$. Denote by $\mathscr{W}_L = {\Pi_1}^{-1}(\mathscr{W}_{\mathscr{A}})$. Clearly $\mathscr{W}_{\mathscr{M}_{\mathcal{U}}[\mathscr{A}]} \subseteq \mathscr{W}_L$.

Definition 5.15. Elements of $\mathscr{W}_L$ will be called *new weak equivalences.*

Proposition 5.16. *For any $i \in \mathbf{I}_{\mathscr{M}}$, $\Gamma(\alpha_i)$ is an injective cofibration and a new weak equivalence.*

Proof. If $i : U \longrightarrow V$, then $\alpha_i : i \xrightarrow{(i,\mathrm{Id}_V)} \mathrm{Id}_V$ is an injective cofibration in $\operatorname{Arr}(\mathscr{M})$ whose components are i and Id_V. Then $\Gamma(\alpha_i) : \Gamma(i) \xrightarrow{[i,\mathbf{F}(\mathrm{Id}_V)]} \Gamma(\mathrm{Id}_V)$ is an injective cofibration in $\mathscr{M}_{\mathcal{U}}[\mathscr{A}]$ whose components are i and $\mathbf{F}(\mathrm{Id}_V)$, by definition of Γ (see Proposition 4.7). $\Pi_1(\Gamma(\alpha_i)) = \mathbf{F}(\mathrm{Id}_V)$ is an isomorphism, thus a weak equivalence, and

$$\Pi_1(\Gamma(\alpha_i)) \in \mathscr{W}_{\mathscr{A}} \Leftrightarrow \Gamma(\alpha_i) \in {\Pi_1}^{-1}(\mathscr{W}_{\mathscr{A}}) = \mathscr{W}_L.$$ □

We are now in the situation of Smith's theorem to localize the injective model structure to get a model structure on $\mathscr{M}_{\mathcal{U}}[\mathscr{A}]$, with the class of new weak equivalences $\mathscr{W}_L$ and the same generating set of cofibrations $\mathbf{I}_{\mathscr{M}_{\mathcal{U}}[\mathscr{A}]_{\mathrm{inj}}}$. We will use Smith's theorem and its consequences that can be found in Beke [12, Theorem 4.1, Proposition 4.2, Proposition 4.4].

Theorem 5.17. *Let $\mathcal{U} : \mathscr{A} \longrightarrow \mathscr{M}$ be a right Quillen functor between combinatorial model categories.*

1. *The data $(\mathbf{I}_{\mathscr{M}_{\mathcal{U}}[\mathscr{A}]_{\text{inj}}}, \mathscr{W}_L)$ define a combinatorial model structure on $\mathscr{M}_{\mathcal{U}}[\mathscr{A}]$ that will be denoted by $\mathscr{M}_{\mathcal{U}}[\mathscr{A}]_{\text{inj}}{}^+$ and which may be described as follows.*
   - *A map $\sigma = [\sigma_0, \sigma_1]$ is a weak equivalence if it is in $\mathscr{W}_L$, that is if σ_1 is a weak equivalence in $\mathscr{A}$.*
   - *The cofibrations are the injective cofibrations.*
   - *The fibrations are the maps satisfying the RLP with respect to every map that is simultaneously a cofibration and a weak equivalence.*
2. *We have a Quillen equivalence $\Pi_1 : \mathscr{M}_{\mathcal{U}}[\mathscr{A}]_{\text{inj}}{}^+ \rightleftarrows \mathscr{A} : \iota$, where Π_1 is left Quillen.*
3. *Any fibrant object $[\mathcal{F}] = [\mathcal{F}_0, \mathcal{F}_1, \pi_{\mathcal{F}}]$ in $\mathscr{M}_{\mathcal{U}}[\mathscr{A}]_{\text{inj}}{}^+$ satisfies the Segal condition, that is $\pi_{\mathcal{F}} : \mathcal{F}_0 \longrightarrow \mathcal{U}(\mathcal{F}_1)$ is a weak equivalence.*

Proof. Assertion (1) is a direct consequence of a result of Beke [12, Proposition 4.2], which is itself a consequence of Smith recognition theorem. The Quillen equivalence is given by Proposition 4.4 in Beke [12] since $\iota : \mathscr{A} \longrightarrow \mathscr{M}_{\mathcal{U}}[\mathscr{A}]$ exhibits $\mathscr{A}$ as a full reflective subcategory and the functor Π_1 preserves the weak equivalences. This gives Assertion (2).

For Assertion (3) we proceed by adjunction. If $[\mathcal{F}]$ is fibrant in $\mathscr{M}_{\mathcal{U}}[\mathscr{A}]_{\text{inj}}{}^+$, then the unique map $[\mathcal{F}] \longrightarrow *$ has the RLP with respect to all trivial cofibrations. And by Proposition 5.16, all elements of $\Gamma(\alpha_{\mathbf{I}_{\mathscr{M}}})$ are trivial cofibrations in $\mathscr{M}_{\mathcal{U}}[\mathscr{A}]_{\text{inj}}{}^+$, therefore $[\mathcal{F}] \longrightarrow *$ has the RLP with respect to every element in $\Gamma(\alpha_{\mathbf{I}_{\mathscr{M}}})$ i.e, $[\mathcal{F}]$ is $\Gamma(\alpha_{\mathbf{I}_{\mathscr{M}}})$-injective. Now by Proposition 5.2, we know that $[\mathcal{F}]$ satisfies the Segal condition ($\pi_{\mathcal{F}}$ is a trivial fibration) and the assertion follows. □

The following gives an explicit description of the fibrations.

Theorem 5.18. *With the previous definitions we have the following.*

1. *The set $\mathbf{J}^+_{\mathscr{M}_{\mathcal{U}}[\mathscr{A}]_{\text{inj}}} = \mathbf{J}_{\mathscr{M}_{\mathcal{U}}[\mathscr{A}]_{\text{inj}}} \coprod \Gamma(\alpha_{\mathbf{I}_{\mathscr{M}}})$ is a generating set of trivial cofibrations.*
2. *A map $\theta : [\mathcal{P}] \xrightarrow{[\sigma_0,\sigma_1]} [\mathcal{Q}]$ is fibration in $\mathscr{M}_{\mathcal{U}}[\mathscr{A}]_{\text{inj}}{}^+$, if it is an injective fibration such that the induced map $\delta : \mathcal{P}_0 \longrightarrow \mathcal{U}(\mathcal{P}_1) \times_{\mathcal{U}(\mathcal{Q}_1)} \mathcal{Q}_0$ is a trivial fibration in $\mathscr{M}$.*

Proof. The second assertion is a consequence of the first considering the fact that a map θ is $\Gamma(\alpha_{\mathbf{I}_{\mathscr{M}}})$-injective if and only if the map δ is $\mathbf{I}_{\mathscr{M}}$-injective, thus a trivial fibration in $\mathscr{M}$ (see Lemma 2.22).

To get the first assertion, we shall prove that $\mathbf{cof}(\mathbf{I}_{\mathscr{M}_{\mathcal{U}}[\mathscr{A}]_{\text{inj}}}) \cap \mathscr{W}_L \subseteq \mathbf{cof}(\mathbf{J}^+_{\mathscr{M}_{\mathcal{U}}[\mathscr{A}]_{\text{inj}}})$.
Let $\sigma : [\mathcal{F}] \xrightarrow{[\sigma_0,\sigma_1]} [\mathcal{G}]$ be an element in $\mathbf{cof}(\mathbf{I}_{\mathscr{M}_{\mathcal{U}}[\mathscr{A}]_{\text{inj}}}) \cap \mathscr{W}_L$. Apply the small object argument to factor σ as a relative $\mathbf{J}^+_{\mathscr{M}_{\mathcal{U}}[\mathscr{A}]_{\text{inj}}}$-cell complex followed by a $\mathbf{J}^+_{\mathscr{M}_{\mathcal{U}}[\mathscr{A}]_{\text{inj}}}$-injective map, $\sigma = r(\sigma) \circ l(\sigma)$, where $r(\sigma) = [r(\sigma)_0, r(\sigma)_1]$ and $l(\sigma) = [l(\sigma)_0, l(\sigma)_1]$. If we look at this factorization in $\mathscr{A}$ it gives a factorization of $\sigma_1 = r(\sigma)_1 \circ l(\sigma)_1$ with $l(\sigma)_1 \in \mathbf{cell}(\mathbf{J}_{\mathscr{A}})$ and $r(\sigma)_1$ is $\mathbf{J}_{\mathscr{A}}$-injective map, thus a fibration. By assumption $\sigma \in \mathscr{W}_L$ which means that σ_1 is a weak equivalence in $\mathscr{A}$ and so is $l(\sigma)_1$; therefore by 3-for-2, $r(\sigma)_1$ is also a weak equivalence in $\mathscr{A}$. So we find that $r(\sigma)_1$ is a trivial fibration i.e, $r(\sigma)$ is $L_1(\mathbf{I}_{\mathscr{A}})$-injective.

On the other hand, part of $r(\sigma)$ being $\mathbf{J}^+_{\mathscr{M}_{\mathcal{U}}[\mathscr{A}]_{\text{inj}}}$-injective implies that $r(\sigma)$ is in particular $\Gamma(\alpha_{\mathbf{I}_{\mathscr{M}}})$-injective since $\Gamma(\alpha_{\mathbf{I}_{\mathscr{M}}}) \subset \mathbf{J}^+_{\mathscr{M}_{\mathcal{U}}[\mathscr{A}]_{\text{inj}}}$. So in the end $r(\sigma)$ is $\mathbf{I}_{\mathscr{M}_{\mathcal{U}}[\mathscr{A}]_{\text{inj}}}$-injective, which means that $r(\sigma)$ is an injective trivial fibration. The map σ is an injective cofibration so it possesses the LLP with respect to $r(\sigma)$. Putting this together with the factorization $\sigma = r(\sigma) \circ l(\sigma)$, the retract argument as in [44, Lemme 1.1.9] implies that σ is a retract of $l(\sigma) \in \mathbf{cell}(\mathbf{J}^+_{\mathscr{M}_{\mathcal{U}}[\mathscr{A}]_{\text{inj}}})$, thus $\sigma \in \mathbf{cof}(\mathbf{J}^+_{\mathscr{M}_{\mathcal{U}}[\mathscr{A}]_{\text{inj}}})$. □

5.5.3 Projective model structure

Theorem 5.19. *Let $\mathcal{U} : \mathscr{A} \longrightarrow \mathscr{M}$ be a right Quillen functor. Then with the previous definitions, the following hold. There is a model structure on the category $(\mathscr{M} \downarrow \mathcal{U}) = \mathscr{M}_{\mathcal{U}}[\mathscr{A}]$ which may be described as follows.*

- *A map $\sigma : [\mathcal{F}] \longrightarrow [\mathcal{G}]$ is a weak equivalence if and only if it is a level-wise weak equivalence.*
- *A map $\sigma : [\mathcal{F}] \longrightarrow [\mathcal{G}]$ is cofibration if it is a projective cofibration.*
- *A map $\sigma : [\mathcal{F}] \longrightarrow [\mathcal{G}]$ is fibration if it is a projective (= level) fibration.*

We will denote this model category by $\mathscr{M}_{\mathcal{U}}[\mathscr{A}]_{\text{proj}}$.

Proof. The class of level-wise weak equivalences clearly satisfies the 2-out-of-3 property. The three classes of cofibrations, fibrations and weak equivalences are closed under composition and retracts. With Proposition 5.11, Proposition 5.9 and Lemma 5.7, one can easily verify that the axioms of a model structure hold. □

Corollary 5.20. *Let $\mathcal{U} : \mathscr{A} \longrightarrow \mathscr{M}$ be a right Quillen functor.*

1. *We have a Quillen adjunction $\Pi_1 : \mathscr{M}_{\mathcal{U}}[\mathscr{A}]_{\text{proj}} \rightleftarrows \mathscr{A} : \iota$, where Π_1 is left Quillen.*
2. *We have a Quillen adjunction $L_1 : \mathscr{A} \rightleftarrows \mathscr{M}_{\mathcal{U}}[\mathscr{A}]_{\text{proj}} : \Pi_1$, where L_1 is left Quillen.*
3. *We have a Quillen adjunction $\Gamma : Arr(\mathscr{M})_{\text{proj}} \rightleftarrows \mathscr{M}_{\mathcal{U}}[\mathscr{A}]_{\text{proj}} : \Pi_{Arr}$, where Γ is left Quillen.*
4. *We also have a Quillen adjunction $\Pi_0 : \mathscr{M}_{\mathcal{U}}[\mathscr{A}]_{\text{proj}} \rightleftarrows : \mathbf{F}^+$, where $\mathbf{F}^+ = \Gamma \circ L_0$ is left Quillen.*
5. *The functors Π_1 and Π_0 preserve the weak equivalences.*

Proof. The functor $\Pi_1 : \mathscr{M}_{\mathcal{U}}[\mathscr{A}]_{\text{proj}} \longrightarrow \mathscr{A}$ preserves (trivial) cofibrations and (trivial) fibrations. So clearly it is simultaneously a left Quillen functor and a right Quillen functor. This gives the first two assertions.

For Assertion (3), it suffices to observe that if $\sigma = [\sigma_0, \sigma_1]$ is a projective trivial fibration then $\Pi_{\text{Arr}}(\sigma) = [\sigma_0, \mathcal{U}(\sigma_1)]$ is a projective (trivial) fibration in $\text{Arr}(\mathscr{M})_{\text{proj}}$ by definition ($\mathcal{U}$ being a right Quillen functor). Therefore Π_{Arr} is right Quillen which means automatically that Γ is left Quillen.

Assertion (4) is straightforward because by definition the projective (trivial) fibrations are the level-wise (trivial) fibrations so clearly Π_0 is right Quillen. The last assertion follows from the definition of a level-wise weak equivalences. □

Cofibrantly generated. Let us now assume that $\mathscr{M}$ and $\mathscr{A}$ are cofibrantly generated as before. We have a similar theorem as in the injective model structure.

Theorem 5.21. *If $\mathscr{A}$ and $\mathscr{M}$ are cofibrantly generated (resp. combinatorial), then $\mathscr{M}_{\mathcal{U}}[\mathscr{A}]_{\text{proj}}$ is cofibrantly generated (resp. combinatorial).*

1. *The set $\mathbf{I}_{\mathscr{M}_{\mathcal{U}}[\mathscr{A}]_{\text{proj}}} = L_1(\mathbf{I}_{\mathscr{A}}) \coprod \Gamma(L_0(\mathbf{I}_{\mathscr{M}})) = L_1(\mathbf{I}_{\mathscr{A}}) \coprod \mathbf{F}^+(\mathbf{I}_{\mathscr{M}})$ is a generating set of cofibrations in $\mathscr{M}_{\mathcal{U}}[\mathscr{A}]_{\text{proj}}$.*
2. *The set $\mathbf{J}_{\mathscr{M}_{\mathcal{U}}[\mathscr{A}]_{\text{proj}}} = L_1(\mathbf{J}_{\mathscr{A}}) \coprod \Gamma(L_0(\mathbf{J}_{\mathscr{M}})) = L_1(\mathbf{J}_{\mathscr{A}}) \coprod \mathbf{F}^+(\mathbf{J}_{\mathscr{M}})$ is a generating set of trivial cofibrations in $\mathscr{M}_{\mathcal{U}}[\mathscr{A}]_{\text{proj}}$.*

Proof. As with the injective case, Corollary 4.13 says that the category $\mathscr{M}_{\mathcal{U}}[\mathscr{A}]$ is locally presentable. So it remains to prove that it is cofibrantly generated to prove that it is a combinatorial model category.

A map $\sigma = [\sigma_0, \sigma_1]$ has the RLP with respect to all maps in $L_1(\mathbf{I}_{\mathscr{A}}) \coprod \Gamma(L_0(\mathbf{I}_{\mathscr{M}}))$ if and only if it is simultaneously $L_1(\mathbf{I}_{\mathscr{A}})$-injective and $\Gamma(L_0(\mathbf{I}_{\mathscr{M}}))$-injective. One the one hand, using the adjunction $L_1 \dashv \Pi_1$, $\sigma = [\sigma_0, \sigma_1]$ is $L_1(\mathbf{I}_{\mathscr{A}})$-injective if and only if σ_1 is $\mathbf{I}_{\mathscr{A}}$-injective, if and only if, σ_1 is a trivial fibration.

On the other hand, using the adjunction $\mathbf{F}^+ \dashv (\mathrm{Ev}_0 \circ \Pi_{\mathrm{Arr}})$, $\sigma = [\sigma_0, \sigma_1]$ is $\mathbf{F}^+(\mathbf{I}_{\mathscr{M}})$-injective if and only if $\mathrm{Ev}_0\, \Pi_{\mathrm{Arr}}(\sigma) = \sigma_0$ is $\mathbf{I}_{\mathscr{M}}$-injective, if and only if σ_0 is a trivial fibration. This gives Assertion (1). The second assertion is proved the same way. □

5.5.4 Localizing the projective model structure. Given $s : A \longrightarrow B \in \mathrm{Arr}(\mathscr{M})$, we have introduced in Notation 2.15 a map $\zeta_s : s \xrightarrow{(s,\mathrm{Id})} \mathrm{Id}_B$ in $\mathrm{Arr}(\mathscr{M})$ and we have defined the universal projective localizing set $\zeta_{\mathbf{I}}$. Let $\mathbf{K}_{\mathbf{I}_{proj}} = \Gamma(\zeta_{\mathbf{I}_{\mathscr{M}}}) = \{\Gamma(\zeta_i)\}_{i \in \mathbf{I}_{\mathscr{M}}}$.

As with the injective model structure, we consider the same class $\mathscr{W}_L = {\Pi_1}^{-1}(\mathscr{W}_{\mathscr{A}})$ of *new weak equivalences.*

Proposition 5.22. *For any $i \in \mathbf{I}_{\mathscr{M}}$, $\Gamma(\zeta_i)$ is a projective cofibration and a new weak equivalence.*

Proof. If $i : U \longrightarrow V$, then $\zeta_i : i \xrightarrow{(i,j_0)} j_1$ is an projective cofibration in $\mathrm{Arr}(\mathscr{M})$ by construction (see Proposition 2.17). The components of ζ_i are i and j_0. With the notation introduced in Notation 2.15, $\Gamma(\zeta_i) : \Gamma(i) \xrightarrow{[i,\mathbf{F}(j_0)]} \Gamma(j_1)$ is therefore a projective cofibration in $\mathscr{M}_{\mathcal{U}}[\mathscr{A}]$ whose components are i and $\mathbf{F}(j_0)$, since Γ is left Quillen (see Proposition 4.7). $\Pi_1(\Gamma(\zeta_i)) = \mathbf{F}(j_0)$ is a trivial cofibration because j_0 is trivial cofibration and $\mathbf{F}$ is left Quillen. In particular, $\mathbf{F}(j_1) = \Pi_1(\Gamma(\zeta_i))$ is a weak equivalence, that is

$$\Pi_1(\Gamma(\zeta_i)) \in \mathscr{W}_{\mathscr{A}} \Leftrightarrow \Gamma(\zeta_i) \in {\Pi_1}^{-1}(\mathscr{W}_{\mathscr{A}}) = \mathscr{W}_L.$$ □

As with the injective case, we are in the situation of Smith's theorem to localize the projective model structure, to obtain a model structure on $\mathscr{M}_{\mathcal{U}}[\mathscr{A}]$ with the class of new weak equivalences $\mathscr{W}_L$, and the same generating set of cofibrations $\mathbf{I}_{\mathscr{M}_{\mathcal{U}}[\mathscr{A}]_{\mathrm{proj}}}$.

Theorem 5.23. *Let $\mathscr{M}$ be a tractable model category and let $\mathcal{U} : \mathscr{A} \longrightarrow \mathscr{M}$ be a right Quillen functor between combinatorial model categories.*

1. *The data $(\mathbf{I}_{\mathscr{M}_{\mathcal{U}}[\mathscr{A}]_{\mathrm{proj}}}, \mathscr{W}_L)$ define a combinatorial model structure on $\mathscr{M}_{\mathcal{U}}[\mathscr{A}]$ that will be denoted by ${\mathscr{M}_{\mathcal{U}}[\mathscr{A}]_{\mathrm{proj}}}^+$ and which may be described as follows.*
    - *A map $\sigma = [\sigma_0, \sigma_1]$ is a weak equivalence if it is in $\mathscr{W}_L$, that is if σ_1 is a weak equivalence in $\mathscr{A}$.*
    - *The cofibrations are the projective cofibrations.*
    - *The fibrations are the maps satisfying the RLP with respect to every map that is simultaneously a cofibration and a weak equivalence.*
2. *We have a Quillen equivalence $\Pi_1 : {\mathscr{M}_{\mathcal{U}}[\mathscr{A}]_{\mathrm{proj}}}^+ \rightleftarrows \mathscr{A} : \iota$, where Π_1 is left Quillen.*
3. *Any fibrant object $[\mathcal{F}] = [\mathcal{F}_0, \mathcal{F}_1, \pi_{\mathcal{F}}]$ in ${\mathscr{M}_{\mathcal{U}}[\mathscr{A}]_{\mathrm{proj}}}^+$ satisfies the Segal condition, that is $\pi_{\mathcal{F}} : \mathcal{F}_0 \longrightarrow \mathcal{U}(\mathcal{F}_1)$ is a weak equivalence.*

Proof. Assertion (1) is a direct consequence of Proposition 4.2 in Beke [12], which is itself a consequence of Smith recognition theorem. The Quillen equivalence is given by Proposition 4.4 in Beke [12] since $\iota : \mathscr{A} \longrightarrow \mathscr{M}_{\mathcal{U}}[\mathscr{A}]$ exhibits $\mathscr{A}$ as a full reflective subcategory and the functor Π_1 preserves the weak equivalences. This gives Assertion (2).

For Assertion (3) we proceed by adjunction. If $[\mathcal{F}]$ is fibrant in ${\mathscr{M}_{\mathcal{U}}[\mathscr{A}]_{\mathrm{proj}}}^+$, then the unique map $[\mathcal{F}] \longrightarrow *$ has the RLP with respect to all trivial cofibrations. And by Proposition 5.22, all elements of $\Gamma(\zeta_{\mathbf{I}_\mathscr{M}})$ are trivial cofibrations in $\mathscr{M}_\mathcal{U}[\mathscr{A}]_{\text{proj}}{}^+$, therefore $[\mathcal{F}] \longrightarrow *$ has the RLP with respect to every element in $\Gamma(\zeta_{\mathbf{I}_\mathscr{M}})$ i.e, $[\mathcal{F}]$ is $\Gamma(\zeta_{\mathbf{I}_\mathscr{M}})$-injective. Now by Proposition 5.4, we know that $[\mathcal{F}]$ satisfies the Segal condition ($\pi_\mathcal{F}$ is a weak equivalence between fibrant objects) and the assertion follows. □

Corollary 5.24. *The identity functor* $\text{Id} : \mathscr{M}_\mathcal{U}[\mathscr{A}]_{\text{proj}}{}^+ \longrightarrow \mathscr{M}_\mathcal{U}[\mathscr{A}]_{\text{inj}}{}^+$ *is a left Quillen functor which is a Quillen equivalence.*

Proof. It is the identity functor and we have the same class of new weak equivalences. Moreover any projective cofibration is an injective cofibration. □

Unlike the injective model structure, it is difficult to give an explicit description of the fibrations in general without any assumption of (left) properness on $\mathscr{M}$. We will deal with this in a subsequent paper.

Remark 5.25. We will close this section with some observations.

1. If $\mathscr{M}$ is tractable, we can show that $\mathscr{M}_\mathcal{U}[\mathscr{A}]_{\text{inj}}{}^+$ is the left Bousfield localization with respect $\Gamma(\alpha_{\mathbf{I}_\mathscr{M}})$. Indeed, elements in $\Gamma(\alpha_{\mathbf{I}_\mathscr{M}}))$ are maps between cofibrant objects so there is no need to take a cofibrant approximation of these maps to define their image under a left derived functor.
2. There are other model structures on $(\mathscr{M} \downarrow \mathcal{U})$ that one can get from the injective and the projective model structures on $\text{Arr}(\mathscr{M})$ with the adjunction $\Gamma \dashv \Pi_{\text{Arr}}$. They will be considered later.

6. Homotopy of Quillen-Segal algebras

In this section $\mathscr{M}$ is a combinatorial (monoidal) model category and $\mathscr{A} = \mathcal{O}\text{-}Alg(\mathscr{M})$. We apply the previous material to the forgetful functor $\mathcal{U} : \mathcal{O}\text{-}Alg(\mathscr{M}) \longrightarrow \mathscr{M}$ for an operad or monad $\mathcal{O}$. If $\mathcal{O}$ is a properad or PROP, then we have to consider another model category $\mathscr{M}'$ and a monad $\mathcal{O}'$. For example $\mathscr{M}'$ is a category of C-colored objects in $\mathscr{M}$, where C is a set of colors (see [39]).

Theorem 6.1. *Let $\mathscr{M}$ be a combinatorial model category and let $\mathcal{O}$ be an operad enriched over $\mathscr{M}$ or a monad on $\mathscr{M}$. Let $\mathcal{U} : \mathcal{O}\text{-}Alg(\mathscr{M}) \longrightarrow \mathscr{M}$ be the forgetful functor. Then the following hold.*

1. *The transferred model structure on $\mathcal{O}\text{-}Alg(\mathscr{M})$ exists if and only if the projective and the injective model structure on $\mathscr{M}_\mathcal{U}[\mathcal{O}\text{-}Alg(\mathscr{M})] := (\mathscr{M} \downarrow \mathcal{U})$ exist.*
2. *In the latter case there is a model structure on $\mathscr{M}_\mathcal{U}[\mathcal{O}\text{-}Alg(\mathscr{M})]$ such that:*
   - *(a) fibrant objects are Quillen-Segal $\mathcal{O}$-algebras*
   - *(b) the adjunction $\iota : \mathcal{O}\text{-}Alg(\mathscr{M}) \leftrightarrows \mathscr{M}_\mathcal{U}[\mathcal{O}\text{-}Alg(\mathscr{M})] : \Pi_1$, is a Quillen equivalence where ι is right Quillen.*

The model structure in Assertion (2) is the localization of injective model structure that does not require $\mathscr{M}$ to be tractable to impose the Segal conditions on the fibrant objects.

Proof. It suffices to prove Assertion (1) because the second assertion is a direct application of Theorem 5.17 above. For this assertion, the if part is given by Theorem 5.12 and Theorem 5.19. So it remains to prove the only if part, namely that if the projective (or injective) model structure exists on $\mathscr{M}_\mathcal{U}[\mathcal{O}\text{-}Alg(\mathscr{M})]$, then the transferred model structure on $\mathcal{O}\text{-}Alg(\mathscr{M})$ exists.

This is easy but we include the proof for the reader's convenience. In fact, in this case, these model structures are directly right-induced from the injective and projective model structures on $\text{Arr}(\mathscr{M})$ through the adjunction $\Gamma : \text{Arr}(\mathscr{M}) \rightleftarrows \mathscr{M}_{\mathcal{U}}[\mathcal{O}\text{-}Alg(\mathscr{M})] : \Pi_{\text{Arr}}$.

By the classical argument of right-induced model structures, it suffices to show that if $f : X_0 \longrightarrow X_1$ is an element of $\mathbf{J}_{\mathscr{M}}$ then any cobase of $\mathbf{F}(f)$ in $\mathcal{O}\text{-}Alg(\mathscr{M})$ is a weak equivalence of $\mathcal{O}$-algebras. Consider the left adjoint $L_1 : \mathcal{O}\text{-}Alg(\mathscr{M}) \longrightarrow \mathscr{M}_{\mathcal{U}}[\mathcal{O}\text{-}Alg(\mathscr{M})]$. One can easily check that $L_1(\mathbf{F}(f))$ is a projective trivial cofibration (it has the LLP with respect to every projective fibrations). Given a pushout diagram in $\mathcal{O}\text{-}Alg(\mathscr{M})$, its image under L_1 is also a pushout diagram in $\mathscr{M}_{\mathcal{U}}[\mathcal{O}\text{-}Alg(\mathscr{M})]$ because L_1 preserves colimits (like any left adjoint). Moreover, colimits in $\mathscr{M}_{\mathcal{U}}[\mathcal{O}\text{-}Alg(\mathscr{M})]$ are computed level-wise, and indeed we have $\Pi_1 \circ L_1 = \text{Id}_{\mathcal{O}\text{-}Alg(\mathscr{M})}$. Since $L_1(\mathbf{F}(f))$ is a trivial cofibration in $\mathscr{M}_{\mathcal{U}}[\mathcal{O}\text{-}Alg(\mathscr{M})]$, then any cobase change of it is a level-wise weak equivalence, therefore any cobase change of $\mathbf{F}(f)$ in $\mathcal{O}\text{-}Alg(\mathscr{M})$ is a weak equivalence as desired. □

For the projective case we have the following result.

Theorem 6.2. *Let $\mathscr{M}$ be a tractable model category and let $\mathcal{O}$ be an operad enriched over $\mathscr{M}$ or a monad on $\mathscr{M}$. Let $\mathcal{U} : \mathcal{O}\text{-}Alg(\mathscr{M}) \longrightarrow \mathscr{M}$ be the forgetful functor. Then there is a model structure on $\mathscr{M}_{\mathcal{U}}[\mathcal{O}\text{-}Alg(\mathscr{M})]$ such that:*

1. *Fibrant objects are Quillen-Segal $\mathcal{O}$-algebras.*
2. *The adjunction $\iota : \mathcal{O}\text{-}Alg(\mathscr{M}) \leftrightarrows \mathscr{M}_{\mathcal{U}}[\mathcal{O}\text{-}Alg(\mathscr{M})] : \Pi_1$, is a Quillen equivalence where ι is right Quillen.*

Remark 6.3. Let $\Phi : \mathcal{O}' \longrightarrow \mathcal{O}$ be a map of $\mathscr{M}$-operads. Then there is a functor $\Phi^\star : \mathcal{O}\text{-}Alg(\mathscr{M}) \longrightarrow \mathcal{O}'\text{-}Alg(\mathscr{M})$ that "restricts the action". Therefore if the transfer model structures on $\mathcal{O}\text{-}Alg(\mathscr{M})$ and $\mathcal{O}'\text{-}\text{Alg}(\mathscr{M})$ exist, then $\Phi^\star$ is right Quillen and according to Theorem 5.17 and Theorem 5.23 we have that:

1. The inclusion $\iota : \mathcal{O}\text{-}Alg(\mathscr{M}) \longrightarrow (\mathcal{O}'\text{-}Alg(\mathscr{M}) \downarrow \Phi^\star)$ is a Quillen equivalence.
2. The inclusion $\iota : \mathcal{O}\text{-}Alg(\mathscr{M}) \longrightarrow (\mathscr{M} \downarrow \mathcal{U})$ is a Quillen equivalence.
3. By 3-for-2 of Quillen-equivalences, the functor $(\mathcal{O}'\text{-}Alg(\mathscr{M}) \downarrow \Phi^\star) \longrightarrow (\mathscr{M} \downarrow \mathcal{U})$ is also a Quillen equivalence.

6.1 Future applications Let us take a moment to outline some possible directions for future applications of the theory of Quillen-Segal algebras and of Quillen-Segal structures in general.

As mentioned before, the theory of operads has its roots in stable homotopy theory with the work of May [71] and Stasheff [90]; but nowadays it has found significant applications in several fields of mathematics. For example commutative Frobenius algebras appear in *Topological Quantum Field Theory* (see Kock [56]). In Symplectic Topology, the Fukaya category of symplectic manifold carries an A_∞-structure (see [35]). Batanin [9] and Trimble (see [61]) used (higher) operads to define higher categories. We shall refer the reader to Loday-Vallette [63], Markl-Shnider-Stasheff [70], Vallette [101] and the many references therein for a detailed account on the subject.

Quillen-Segal algebras generalize usual algebras, so we can obviously adapt the above applications. But our motivation was not to simply generalize these existing algebraic structures; but rather it is the theory of co-Segal categories initiated in [6] and the related applications that motivated their definition. Before going further, it is important to observe that in general, given an object $A \in \mathscr{M}$, it is hard to determine that an operad $\mathcal{O}$ acts directly on it. The most natural thing that happens is that "A has the homotopy type of an $\mathcal{O}$-algebra B", in the sense that A is isomorphic to the underlying object $\mathcal{U}(B)$ of B in the homotopy category $\mathbf{ho}(\mathscr{M})$ of $\mathscr{M}$. This means that either we have a weak equivalence $\pi : A \longrightarrow \mathcal{U}(B)$ in $\mathscr{M}$, or that there is a zig-zag of weak equivalences in $\mathscr{M}$ that connects A and $\mathcal{U}(B)$. In the latter case under suitable hypotheses, a version of Whitehead's theorem applies (see [44]) and we can reduce to the first case, that is, we have a weak equivalence $\pi : A \longrightarrow \mathcal{U}(B)$, and thus a Quillen-Segal algebra structure on A. Trying to lift the algebra structure to A is the *Homotopy Transfer Problem* and this is a hard problem in general. So we hope that when we consider it as Quillen-Segal algebra, we can simplify some constructions in Homotopical Algebra, Rational Homotopy Theory, Formality, etc.

- ⋄ We are particularly interested in the Quillen-Segal algebras arising from Geometry as in the paper of Deligne, Griffiths, Morgan and Sullivan [29].
- ⋄ We also want to understand the concept of *infinity morphism* between $\mathcal{O}$-algebras considered by Vallette in [99].

Quillen-Segal algebras and higher linear categories. A particular case of interest that leads to an example of co-Segal category goes as follows. Recall that given the symmetric monoidal model category $\mathscr{M} = (\underline{M}, \otimes, I)$ and a set X, there is a multisorted (or colored) operad $\mathcal{O}_X$ such that the category of $\mathcal{O}_X$-algebras with coefficients in $\mathscr{M}$, $\mathcal{O}_X\text{-Alg}(\mathscr{M})$, is equivalent to the category $\mathscr{M}\text{-}\mathbf{Cat}(X)$ of enriched $\mathscr{M}$-categories with X as the set of objects (see Berger-Moerdijk [14], [15]). If we denote by $\mathscr{M}\text{-}\mathbf{Graph}(X) := \prod_{X\times X} \mathscr{M}$ the category of enriched $\mathscr{M}$-graphs over X then there is a forgetful functor

$$\mathcal{U} : \mathscr{M}\text{-}\mathbf{Cat}(X) \longrightarrow \mathscr{M}\text{-}\mathbf{Graph}(X).$$

Then a Quillen-Segal $\mathcal{O}_X$-algebra gives rise to a co-Segal category. More precisely if $\mathcal{C}$ is an $\mathscr{M}$-category and we are given weak equivalences $\varepsilon_{ab} : \widetilde{\mathcal{C}}(a,b) \longrightarrow \mathcal{C}(a,b)$ for every $(a,b) \in X^2$ (perhaps obtained via projective resolution), one would like to define a new category where the hom-object is $\widetilde{\mathcal{C}}(a,b)$. If we put these maps together we get the following zig-zag which is the main feature of a co-Segal category as in [6]:

$$\widetilde{\mathcal{C}}(a,b) \otimes \widetilde{\mathcal{C}}(b,c) \xrightarrow{c_{abc}\circ(\varepsilon_{ab}\otimes\varepsilon_{bc})} \widetilde{\mathcal{C}}(a,b,c) \xleftarrow{\sim} \widetilde{\mathcal{C}}(a,c).$$

Here $\widetilde{\mathcal{C}}(a,b,c) := \mathcal{C}(a,c)$ and the map $c_{abc} : \mathcal{C}(a,b) \otimes \mathcal{C}(b,c) \longrightarrow \mathcal{C}(a,c)$ is the composition in $\mathcal{C}$. The co-Segal category $\widetilde{\mathcal{C}}$ is a weakly enriched $\mathscr{M}$-category.

- ⋄ The main motivation to develop such theory is to get good notion of higher linear categories; that is when $\mathscr{M}$ is the category of chain complexes over a commutative ring. Higher linear categories play an important role in Modern Algebraic Geometry e.g., Higher Tanaka duality, non commutative motives (see [58, 67, 97, 103]).
- ⋄ Another important application of the theory of higher linear categories was to give another proof of Deligne's conjecture based on the previous work of Kock-Toën [57].
- ⋄ In [93], Tamarkin showed that the collection of *dg*-categories form a homotopy 2-category in a very specific sense. We would like to understand his constructions through Quillen-Segal algebras. Theses ideas suggest to further study Quillen-Segal algebras when $\mathscr{M}$ is the category of dg-categories with the Tabuada model structure [92].

Obstruction to left properness for algebras The model structure on $\mathcal{O}\text{-}Alg(\mathscr{M})$ is *created* along the adjunction $\mathcal{U} : \mathcal{O}\text{-}Alg(\mathscr{M}) \rightleftarrows \mathscr{M} : \mathbf{F}$ by a classical argument of Quillen [78]. In many cases, the key ingredient is a careful analysis of the cobase change (pushout) of $\mathbf{F}(j)$ for any element j of the generating set of trivial cofibrations in $\mathscr{M}$ (see for example Schwede-Shipley [85]). Obviously such a model structure does not always exist and it is important to figure out why precisely. When the model structure exists, there is a growing interest in the preservation of *left properness* or *relative left properness* from $\mathscr{M}$ to $\mathcal{O}\text{-}Alg(\mathscr{M})$. There are many contributions in the literature on the subject (see [10, 32, 39, 75, 76, 80]).

But it now appears that it is important to analyze these pushouts in the comma category $\mathscr{M}_{\mathcal{U}}[\mathcal{O}\text{-}Alg(\mathscr{M})]$ of pre-Quillen-Segal algebras through the embedding:

$$\iota : \mathcal{O}\text{-}Alg(\mathscr{M}) \longrightarrow \mathscr{M}_{\mathcal{U}}[\mathcal{O}\text{-}Alg(\mathscr{M})]\,.$$

The reason is that we keep track of *everything* about the algebras and their respective underlying objects when we work in this comma category. We will study in a future work the obstruction of $\mathcal{O}\text{-}Alg(\mathscr{M})$ to be left proper in the same spirit as Batanin and Berger [10]. But for now we outline the fact the existence of the model structure boils down to a structure of Quillen-Segal algebra on some specific objects. The result is about the pushouts that are needed to get the right-induced model structure.

Proposition 6.4. *Let $j : a \longrightarrow b$ be a trivial cofibration in $\mathscr{M}$, and let $\mathcal{E}$ be an $\mathcal{O}$-algebra. Consider pushout data in $\mathcal{O}\text{-}Alg(\mathscr{M})$: $\mathbf{F}(b) \xleftarrow{\mathbf{F}(j)} \mathbf{F}(a) \xrightarrow{f} \mathcal{E}$, with its adjoint transpose pushout data in $\mathscr{M}$: $b \xleftarrow{j} a \xrightarrow{\rho(f)} \mathcal{U}(\mathcal{E})$.*

- *Let $\mathcal{G} := \mathbf{F}(b) \cup^{\mathbf{F}(a)} \mathcal{E}$ be the colimit of the pushout data $\mathbf{F}(b) \xleftarrow{\mathbf{F}(j)} \mathbf{F}(a) \xrightarrow{f} \mathcal{E}$ and let $\delta(j) : \mathcal{E} \longrightarrow \mathcal{G}$ be the canonical map.*
- *Let $V := b \cup^{a} \mathcal{U}(\mathcal{E})$ be the colimit of the pushout data $b \xleftarrow{j} a \xrightarrow{\rho(f)} \mathcal{U}(\mathcal{E})$ and let $\xi(j) : \mathcal{U}(\mathcal{E}) \longrightarrow V$ the canonical map going to the colimit.*

1. *Then there is a universal map in $\mathscr{M}$: $\pi : V \longrightarrow \mathcal{U}(\mathcal{G})$ and we have an object $[V, \mathcal{G}, \pi] \in \mathscr{M}_{\mathcal{U}}[\mathcal{O}\text{-}Alg(\mathscr{M})]$.*
2. *The algebra map $\delta(j) : \mathcal{E} \longrightarrow \mathcal{G}$ is weak equivalence if and only if the map $\pi : V \longrightarrow \mathcal{U}(\mathcal{G})$ is weak equivalence, that is, if and only if $[V, \mathcal{G}, \pi] \in \mathscr{M}_{\mathcal{U}}[\mathcal{O}\text{-}Alg(\mathscr{M})]$ is a Quillen-Segal algebra.*

We will give a similar statement about left properness later in a future work. This can be helpful to understand why the category $\mathcal{O}\text{-}Alg(\mathscr{M})$ fails to be left proper and in particular to understand some examples such as the one recently given by Dwyer as explained in [39].

Proof. The idea is to observe that the two pushouts are related in $\mathscr{M}_{\mathcal{U}}[\mathcal{O}\text{-}Alg(\mathscr{M})]$ when we project everything in $\mathscr{M}$. With the embedding $\iota : \mathcal{O}\text{-}Alg(\mathscr{M}) \hookrightarrow \mathscr{M}_{\mathcal{U}}[\mathcal{O}\text{-}Alg(\mathscr{M})]$ and the adjunction $\mathbf{F}^+ \dashv \Pi_0$ of Proposition 4.9, the pushout data $b \xleftarrow{j} a \xrightarrow{\rho(f)} \mathcal{U}(\mathcal{E})$ is uniquely equivalent to the following pushout data in $\mathscr{M}_{\mathcal{U}}[\mathcal{O}\text{-}Alg(\mathscr{M})]$:

$$\mathbf{F}^+(b) \xleftarrow{\mathbf{F}^+(j)} \mathbf{F}^+(a) \xrightarrow{\psi(f)} \iota(\mathcal{E}).$$

Let $[\mathcal{H}] = [\mathcal{H}_0, \mathcal{H}_1, \pi_{\mathcal{H}}]$ be the colimit of this pushout in $\mathscr{M}_{\mathcal{U}}[\mathcal{O}\text{-}Alg(\mathscr{M})]$. If we look at the way we compute colimits in the comma category $\mathscr{M}_{\mathcal{U}}[\mathcal{O}\text{-}Alg(\mathscr{M})]$, one gets the following:

- $\mathcal{H}_0 \cong V$ in $\mathscr{M}$;
- $\mathcal{H}_1 \cong \mathcal{G}$ in $\mathcal{O}\text{-}Alg(\mathscr{M})$;
- the map $\pi_{\mathcal{H}} : \mathcal{H}_0 \longrightarrow \mathcal{U}(\mathcal{H}_1)$ gives our map $\pi : V \longrightarrow \mathcal{U}(\mathcal{G})$.

- More importantly, the image under $\mathcal{U}$ of the algebra map $\delta(j) : \mathcal{E} \longrightarrow \mathcal{G}$ factors through $\xi(j) : \mathcal{U}(\mathcal{E}) \longrightarrow V$ as:

$$\mathcal{U}(\delta(j)) = \pi \circ \xi(j).$$

The map $\xi(j)$ is a trivial cofibration as the cobase change in $\mathscr{M}$ of the trivial cofibration j; in particular it is a weak equivalence in $\mathscr{M}$. Therefore by 3-for-2, $\mathcal{U}(\delta(j))$ is a weak equivalence in $\mathscr{M}$ if and only if π is a weak equivalence in $\mathscr{M}$. □

The previous result might seem obvious but we can use it to understand and to produce many examples where we do not have a model structure, especially in equivariant settings. We shall refer the reader to Bergner [20], Stephan [91] and the many references therein for more details on the subject.

Finally let us remind the reader that given a right Quillen functor $\mathcal{U} : \mathscr{A} \longrightarrow \mathscr{M}$ between combinatorial model categories, even before knowing that $\mathcal{U}$ is a Quillen equivalence, our theorems say that the comma category $\mathscr{M}_{\mathcal{U}}[\mathscr{A}]$ is *already* Quillen-equivalent to $\mathscr{A}$. And this category $\mathscr{M}_{\mathcal{U}}[\mathscr{A}]$ is partially controlled by the target category $\mathscr{M}$.

7. Quillen-Segal theories and Stable homotopy

In this section we assume that $\mathscr{A} = \mathscr{M}$ and that $\mathcal{U} : \mathscr{M} \longrightarrow \mathscr{M}$ is an endofunctor which is right Quillen. The example that we shall keep in mind is $\mathcal{U} = \Omega : \mathbf{sSet}_* \longrightarrow \mathbf{sSet}_*$. The left adjoint $\mathbf{F} : \mathscr{M} \longrightarrow \mathscr{M}$ is to be thought of as the suspension functor $\Sigma = S^1 \wedge -$. We will write $\mathcal{U}^n$ for the composite of $\mathcal{U}$ with itself n times and similarly we have $\mathbf{F}^n$; with the convention $\mathbf{F}^0 = \mathrm{Id}_{\mathscr{M}} = \mathcal{U}^0$.

There are many references in the literature on stable homotopy theory. Classical ones include Adams [2], Bousfield-Friedlander [24], Goerss-Jardine [37], Lima [62]. Modern foundations are to be found in Elmendorf-Kriz-Mandell-May [34], Hovey [45], Schwede [84] and the many references therein. In the world of ∞-categories we shall refer the reader to Lurie [66].

Definition 7.1. A spectrum X is a sequence of pointed simplicial sets $(X_n)_{n\in\mathbb{N}}$ together with basepoint preserving maps $\gamma : S^1 \wedge X_n \longrightarrow X_{n+1}$. A map $f : X \longrightarrow Y$ of spectra is a sequence (f_n) of maps $f_n : X_n \longrightarrow Y_n$ in $\mathbf{sSet}_*$ such that the following commutes for every $n \geq 0$.

$$\begin{array}{ccc} S^1 \wedge X_n & \xrightarrow{\mathrm{Id} \wedge f_n} & S^1 \wedge Y_n \\ \downarrow{\scriptstyle \gamma_X} & & \downarrow{\scriptstyle \gamma_Y} \\ X_{n+1} & \xrightarrow{f_{n+1}} & Y_{n+1} \end{array}$$

If we take a general left adjoint T that generalizes the suspension functor, we find the definition of spectrum given by Hovey [45, Definition 1.1]. With the adjunction $(S^1 \wedge -) \dashv \Omega$, the previous definition is equivalent to the one below that we shall work with (see Adams [2, Part III]).

Definition 7.2. A spectrum X is a sequence of pointed simplicial sets $(X_n)_{n\in\mathbb{N}}$ together with basepoint preserving maps $\varepsilon : X_n \longrightarrow \Omega(X_{n+1})$. A map $f : X \longrightarrow Y$ of spectrum is a sequence (f_n) of maps $f_n : X_n \longrightarrow Y_n$ in $\mathbf{sSet}_*$ such that the following commutes for every $n \geq 0$.

$$\begin{array}{ccc} X_n & \xrightarrow{f_n} & Y_n \\ \Big\downarrow{\scriptstyle \varepsilon_X} & & \Big\downarrow{\scriptstyle \varepsilon_Y} \\ \Omega(X_{n+1}) & \xrightarrow{\Omega(f_{n+1})} & \Omega(Y_{n+1}) \end{array}$$

The category of (pre)spectra will be denoted by $\mathcal{S}\mathrm{p}_\Omega(\mathbb{N}, \mathbf{sSet}_*)$.

Remark 7.3. For every n we have an object $[X]_n \in (\mathbf{sSet}_* \downarrow \Omega)$ given by $[X]_n = [X_n, X_{n+1}, \varepsilon]$. This gives a sequence of objects in $(\mathbf{sSet}_* \downarrow \Omega)$. In virtue of this, we will study spectra through sequences of objects in $\mathbf{sSet}_{*\Omega}[\mathbf{sSet}_*] := (\mathbf{sSet}_* \downarrow \Omega)$.

From the definition, we have the following observations.

1. There is a functor $P : \mathcal{S}\mathrm{p}_\Omega(\mathbb{N}, \mathbf{sSet}_*) \longrightarrow \prod_{n\in\mathbb{N}} \mathbf{sSet}_{*\Omega}[\mathbf{sSet}_*]$ that maps X to $([X]_n)_{n\in\mathbb{N}}$.
2. Projecting the nth factor yields a functor $\mathcal{T}_n : \mathcal{S}\mathrm{p}_\Omega(\mathbb{N}, \mathbf{sSet}_*) \longrightarrow \mathrm{Arr}(\mathbf{sSet}_*)$ that maps $X \mapsto \varepsilon_n$.
3. As n runs through $\mathbb{N}$, we have a Quillen-Segal theory on the category $\mathcal{S}\mathrm{p}_\Omega(\mathbb{N}, \mathbf{sSet}_*)$ given by the family of functor $(\mathcal{T}_n)_{n\in\mathbb{N}}$.

Our goal here is to show that we can transfer the model structure from $\prod_{n\in\mathbb{N}} \mathbf{sSet}_{*\Omega}[\mathbf{sSet}_*]$ to the category $\mathcal{S}\mathrm{p}_\Omega(\mathbb{N}, \mathbf{sSet}_*)$ with the classical argument of right-induced model structures (see for example [12, 38, 78]). On $\mathbf{sSet}_{*\Omega}[\mathbf{sSet}_*]$ we have the injective model structure $\mathbf{sSet}_{*\Omega}[\mathbf{sSet}_*]_{inj}$ (Theorem 5.12) and the projective model structure $\mathbf{sSet}_{*\Omega}[\mathbf{sSet}_*]_{proj}$ (Theorem 5.19). Each model structure will induce a model structure on $\mathcal{S}\mathrm{p}_\Omega(\mathbb{N}, \mathbf{sSet}_*)$ called the projective and injective model structures, just as for (unbounded) chain complexes (see [44]). We do this in the general setting of a model category $\mathscr{M}$. But first we need to set up some definitions and properties.

7.1 $\mathbb{Z}$-sequences and spectra in general model categories.

As pointed out by Adams, the indexing set for spectra can be either $\mathbb{Z}$ or $\mathbb{N}$. We will index our sequences over $\mathbb{Z}$ and mention explicitly when we index over $\mathbb{N}$. Many of the constructions that will follow hold for any subset $\mathbb{O} \subseteq \mathbb{Z}$ which is an *integer interval* in the sense that: if $m \in \mathbb{O}$ and if $n \in \mathbb{O}$ with $m < n$, then for any $p \in \mathbb{Z}$ such that $m < p < n$ we have $p \in \mathbb{O}$. In particular $\mathbb{O}$ can be isomorphic as an ordered set to an *ordinal*, possibly finite. Recall that an *ordinal* is a set A such that if $x \in y$ and $y \in A$ then $x \in A$; and such that A is well ordered by the strict relation: $x < y \iff x \in y$ for $x, y \in A$. The reader may refer to Krivine [59] for the theory of ordinals and cardinals.

Definition 7.4. Let $\mathbb{O}$ be a subset of $\mathbb{Z}$.

1. Say that $\mathbb{O}$ is an *indexing set* if $\mathbb{O}$ is an integer interval.
2. Denote by $\mathbb{Z}_{\mathrm{disc}}$, $\mathbb{N}_{\mathrm{disc}}$ and $\mathbb{O}_{\mathrm{disc}}$ the respective discrete categories associated to $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{O}$.

Unless otherwise specified the letter $\mathbb{O}$ will always refer to an indexing set. Let us now generalize Definition 7.2 for a general model category $\mathscr{M}$.

Definition 7.5. Let $\mathcal{U} : \mathscr{M} \longrightarrow \mathscr{M}$ be a right Quillen functor.

1. A $\mathcal{U}$-prespectrum indexed by $\mathbb{Z}$ is a sequence of objects of $\mathscr{M}$, $X = (X_n)_{n\in\mathbb{Z}}$ together with maps $\varepsilon : X_n \longrightarrow \mathcal{U}(X_{n+1})$. A map $f : X \longrightarrow Y$ of $\mathcal{U}$-prespectra is a sequence of maps $f_n : X_n \longrightarrow Y_n$ such that the obvious diagrams commute as in Definition 7.2.

2. More generally for $\mathbb{O} \subseteq \mathbb{Z}$, a $\mathcal{U}$-prespectrum indexed by $\mathbb{O}$ is a sequence of objects of $\mathscr{M}$, $X = (X_n)_{n\in\mathbb{O}}$ together with maps $\varepsilon : X_n \longrightarrow \mathcal{U}(X_{n+1})$. Maps are defined the same way as for the case $\mathbb{O} = \mathbb{Z}$.

Warning. 1. So far if $[\mathcal{X}]$ is an object of $(\mathscr{M} \downarrow \mathcal{U})$, we have written $[\mathcal{X}] = [\mathcal{X}_0, \mathcal{X}_1, \pi_{\mathcal{X}}]$. But when we consider sequences $([\mathcal{X}_n])_n$ there will be multiple indices, so we will write instead $[\mathcal{X}] = [\mathcal{X}^0, \mathcal{X}^1, \varepsilon : \mathcal{X}^0 \longrightarrow \mathcal{U}(\mathcal{X}^1)]$ to put the indexing integer as $\mathcal{X}_n^0$. We have changed the letter π to ε because the letter π refers to homotopy groups.

2. In the upcoming definitions we will be talking about $\mathbb{Z}$-sequences but this simply means $\mathbb{Z}$-indexed sequences and **not** a λ-sequence as in [44, Definition 2.1.1].

Definition 7.6. Let $\mathcal{U} : \mathscr{M} \longrightarrow \mathscr{M}$ be a right Quillen functor.

1. A $\mathbb{Z}$-sequence with coefficient in $\mathscr{M}_{\mathcal{U}}[\mathscr{M}]$ is an object $[\mathcal{X}_\bullet] = ([\mathcal{X}_n])_{n\in\mathbb{Z}} \in \prod_{n\in\mathbb{Z}} \mathscr{M}_{\mathcal{U}}[\mathscr{M}]$, in that for every $n \in \mathbb{Z}$,
$$[\mathcal{X}_n] = [\mathcal{X}_n^0, \mathcal{X}_n^1, \varepsilon_n : \mathcal{X}_n^0 \longrightarrow \mathcal{U}(\mathcal{X}_n^1)].$$
Equivalently $[\mathcal{X}_\bullet]$ determines a functor $[\mathcal{X}_\bullet] : \mathbb{Z}_{disc} \longrightarrow \mathscr{M}_{\mathcal{U}}[\mathscr{M}]$, that maps $n \mapsto [\mathcal{X}_n]$.
2. Say that a $\mathbb{Z}$-sequence $[\mathcal{X}_\bullet]$ is **linked** if for every n there is an isomorphism:
$$\tau_n : \mathcal{X}_n^1 \xrightarrow{\cong} \mathcal{X}_{n+1}^0.$$
We will denote by $[\mathcal{X}_\bullet, \tau]$ the linked sequence.
3. Say that a morphism $\sigma : [\mathcal{X}_\bullet, \tau] \xrightarrow{(\sigma_n)} [\mathcal{Y}_\bullet, \tau']$ is a **linked morphism**, if for every n, we have $\tau_n' \circ \sigma_n^1 = \sigma_{n+1}^0 \circ \tau_n$, or equivalently $\sigma_{n+1}^0 = \tau_n' \circ \sigma_n^1 \circ \tau_n^{-1}$. In other words the following commutes.
$$\begin{array}{ccc} \mathcal{X}_n^1 & \xrightarrow{\sigma_n^1} & \mathcal{Y}_n^1 \\ \cong\downarrow\tau_n & & \cong\downarrow\tau_n' \\ \mathcal{X}_{n+1}^0 & \xrightarrow{\sigma_{n+1}^0} & \mathcal{Y}_{n+1}^0 \end{array}$$
4. Say that a linked sequence $[\mathcal{X}_\bullet, \tau]$ is **strictly linked** if for every n, τ_n is the identity morphism.
5. Similarly for $\mathbb{O} \subseteq \mathbb{Z}$, an $\mathbb{O}$-sequence $[\mathcal{X}_\bullet]$ is a functor $[\mathcal{X}_\bullet] : \mathbb{O}_{\text{disc}} \longrightarrow \mathscr{M}_{\mathcal{U}}[\mathscr{M}]$.
6. An $\mathbb{O}$-sequence $[\mathcal{X}_\bullet]$ is linked if for every $n \in \mathbb{O}$ such that $n+1 \in \mathbb{O}$, there is an isomorphism:
$$\tau_n : \mathcal{X}_n^1 \xrightarrow{\cong} \mathcal{X}_{n+1}^0.$$
Strictly linked $\mathbb{O}$-sequences and linked morphisms are defined the same way as for the case $\mathbb{O} = \mathbb{Z}$.

Pictorially, we can represent an unlinked $\mathbb{Z}$-sequence by a 'stair diagram':

$$\begin{array}{cccc} \mathcal{X}_n^0 & \xrightarrow{\varepsilon_n} & \mathcal{U}(\underbrace{\mathcal{X}_n^1}) & \\ & & \vdots\ ? & \\ & & \mathcal{X}_{n+1}^0 & \xrightarrow{\varepsilon_{n+1}} \mathcal{U}(\mathcal{X}_{n+1}^1) \end{array}$$

When the sequence is linked we can move up the lower stair and get a map $\mathcal{U}(\tau_n) \circ \varepsilon_n : \mathcal{X}_n^0 \longrightarrow \mathcal{U}(\mathcal{X}_{n+1}^0)$. Each of these maps has an adjoint $\mathbf{F}(\mathcal{X}_n^0) \longrightarrow \mathcal{X}_{n+1}^0$. If $\mathscr{M} = \mathbf{sSet}_*$ and if $\mathbf{F}$ is the suspension functor given by $\mathbf{F} = S^1 \wedge -$, then the previous map takes the form $S^1 \wedge \mathcal{X}_n^0 \longrightarrow \mathcal{X}_{n+1}^0$, and we see that we have a spectrum as in Definition 7.1 above, when the sequence is indexed by $\mathbb{N}$. If we write $X_n := \mathcal{X}_n^0$, then the sequence $(X_n)_{n\in\mathbb{N}}$ with the map $\mathcal{U}(\tau_n) \circ \varepsilon_n$ defines an object of $\mathcal{S}\mathrm{p}_{\Omega}(\mathbb{N}, \mathbf{sSet}_*)$ as in Definition 7.2.

Proposition-Definition 7.7. *With the previous notation we have:*

1. *The category of $\mathbb{Z}$-sequences is equivalent to the functor category*

$$\mathrm{Hom}(\mathbb{Z}_{\mathrm{disc}}, \mathscr{M}_{\mathcal{U}}[\mathscr{M}]) = (\mathscr{M}_{\mathcal{U}}[\mathscr{M}])^{\mathbb{Z}_{\mathrm{disc}}}.$$

2. *The category of $\mathbb{O}$-sequences is equivalent to the functor category*

$$\mathrm{Hom}(\mathbb{O}_{\mathrm{disc}}, \mathscr{M}_{\mathcal{U}}[\mathscr{M}]) = (\mathscr{M}_{\mathcal{U}}[\mathscr{M}])^{\mathbb{O}_{\mathrm{disc}}}.$$

3. *Linked $\mathbb{Z}$-sequences (resp. $\mathbb{O}$-sequences) and linked morphisms form a category.*
4. *Strictly linked $\mathbb{Z}$-sequences (resp. $\mathbb{O}$-sequences) form a subcategory of the category of linked $\mathbb{Z}$-sequences (resp. $\mathbb{O}$-sequences).*

Notation 7.8. Let us take a moment to set up some notation that will simplify our discussion. Let $\mathbb{O} \subseteq \mathbb{Z}$ be an indexing set that will be in general $\mathbb{N}$ or $\mathbb{Z}$.

1. $[\mathcal{X}_\bullet], [\mathcal{Y}_\bullet], \ldots$, are sequences objects of $\mathscr{M}_{\mathcal{U}}[\mathscr{M}]$.
2. $[\mathcal{X}_\bullet, \tau], [\mathcal{Y}_\bullet, \tau'], \ldots$, are linked sequences of objects of $\mathscr{M}_{\mathcal{U}}[\mathscr{M}]$.
3. $[\mathcal{X}_\bullet, 1], [\mathcal{Y}_\bullet, 1], \ldots$, are **strictly linked sequences** of objects of $\mathscr{M}_{\mathcal{U}}[\mathscr{M}]$.
4. $\mathcal{S}\mathrm{p}_{\mathcal{U}}(\mathbb{Z}, \mathscr{M})$ = the category of $\mathcal{U}$-prespectra indexed by $\mathbb{Z}$ (Definition 7.5).
5. $\mathcal{S}\mathrm{p}_{\mathcal{U}}(\mathbb{O}, \mathscr{M})$ = the category of $\mathcal{U}$-prespectra indexed by $\mathbb{O}$.
6. $\mathcal{LS}_{\mathcal{U}}^{+}(\mathbb{O}, \mathscr{M})$ = the category of linked sequences indexed by $\mathbb{O}$ as in Definition 7.6.
7. $\mathcal{LS}_{\mathcal{U}}(\mathbb{O}, \mathscr{M})$ = the category of **strictly linked** sequences indexed by $\mathbb{O}$.
8. $P : \mathcal{LS}_{\mathcal{U}}(\mathbb{O}, \mathscr{M}) \longrightarrow (\mathscr{M}_{\mathcal{U}}[\mathscr{M}])^{\mathbb{O}_{\mathrm{disc}}}$ is the forgetful functor (it forgets the links).
9. $P_n : \mathcal{LS}_{\mathcal{U}}(\mathbb{O}, \mathscr{M}) \longrightarrow \mathscr{M}_{\mathcal{U}}[\mathscr{M}]$ is the composite of P followed by the nth projection:

$$P_n([\mathcal{X}_\bullet, 1]) = [\mathcal{X}_n] \,.$$

Proposition-Definition 7.9. *Let $[\mathcal{X}_\bullet, \tau]$ be an object of $\mathcal{LS}_{\mathcal{U}}^{+}(\mathbb{Z}, \mathscr{M})$.*

1. *Define the associated $\mathcal{U}$-prespectrum as the sequence $(X_n)_{n\in\mathbb{Z}}$ with the maps $f_n : X_n \longrightarrow \mathcal{U}(X_{n+1})$ given by:*

$$X_n := \mathcal{X}_n^0, \qquad f_n := \mathcal{U}(\tau_n) \circ \varepsilon_n \,.$$

This construction defines a functor $Q : \mathcal{LS}_{\mathcal{U}}^{+}(\mathbb{Z}, \mathscr{M}) \longrightarrow \mathcal{S}\mathrm{p}_{\mathcal{U}}(\mathbb{Z}, \mathscr{M})$

2. *For any indexing set $\mathbb{O} \subseteq \mathbb{Z}$, the functor Q restricts to an isomorphism of categories: $\mathcal{LS}_{\mathcal{U}}(\mathbb{O}, \mathscr{M}) \cong \mathcal{S}\mathrm{p}_{\mathcal{U}}(\mathbb{O}, \mathscr{M})$*
3. *In particular if we take $\mathbb{O} = \mathbb{N}$ and $\mathscr{M} = \mathbf{sSet}_*$, we get an isomorphism of categories: $\mathcal{S}\mathrm{p}_{\Omega}(\mathbb{N}, \mathbf{sSet}_*) \cong \mathcal{LS}_{\Omega}(\mathbb{N}, \mathbf{sSet}_*)$.*

Proof. Clear. ∎

With the following result we also deduce that the functor $Q : \mathcal{LS}_{\mathcal{U}}^{+}(\mathbb{Z}, \mathscr{M}) \longrightarrow \mathcal{S}\mathrm{p}_{\mathcal{U}}(\mathbb{Z}, \mathscr{M})$ is an equivalence of categories.

Proposition 7.10. *For any indexing set* $\mathbb{O} \subseteq \mathbb{Z}$*, the inclusion* $\mathcal{LS}_{\mathcal{U}}(\mathbb{O}, \mathscr{M}) \hookrightarrow \mathcal{LS}^{+}_{\mathcal{U}}(\mathbb{O}, \mathscr{M})$ *is an equivalence of categories. A quasi-inverse is the functor* $\Theta : \mathcal{LS}^{+}_{\mathcal{U}}(\mathbb{O}, \mathscr{M}) \longrightarrow \mathcal{LS}_{\mathcal{U}}(\mathbb{O}, \mathscr{M})$ *defined as follows.*

1. Θ *maps* $[\mathcal{X}_\bullet, \tau]$ *to the strictly linked sequence* $[\widetilde{\mathcal{X}}_\bullet, 1]$ *where* $[\widetilde{\mathcal{X}}_\bullet] = ([\widetilde{\mathcal{X}}_n])_{n\in\mathbb{O}}$ *is given by*
$$[\widetilde{\mathcal{X}}_n] = [\mathcal{X}^0_n, \mathcal{X}^0_{n+1}, \mathcal{U}(\tau_n) \circ \varepsilon_n].$$
2. Θ *maps* $\sigma = (\sigma_n)$ *to* $\widetilde{\sigma} = (\widetilde{\sigma}_n)$ *given by:*
$$\widetilde{\sigma}_n = (\widetilde{\sigma}^0_n, \widetilde{\sigma}^1_n) := (\sigma^0_n, \tau'_n \circ \sigma^1_n \circ \tau_n^{-1}).$$

Proof. We give the proof for the case $\mathbb{O} = \mathbb{Z}$, the general case is treated the same way. Clearly for every n we have $\widetilde{\mathcal{X}}^1_n = \widetilde{\mathcal{X}}^0_{n+1}$ by construction. This means that $\Theta([\mathcal{X}_\bullet, \tau])$ is strictly linked. By the same reasoning $\Theta(\sigma)$ is a linked morphism if σ is. These constructions are clearly functorial and Θ is well defined.

The proposition will be proved as soon as we establish that $\sigma \mapsto \widetilde{\sigma}$ is an isomorphism of hom-sets:
$$\Theta : \mathrm{Hom}_{\mathcal{LS}^{+}_{\mathcal{U}}(\mathbb{O},\mathscr{M})}([\mathcal{X}_\bullet, \tau], [\mathcal{Y}_\bullet, \tau']) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{LS}_{\mathcal{U}}(\mathbb{O},\mathscr{M})}([\widetilde{\mathcal{X}}_\bullet, 1], [\widetilde{\mathcal{Y}}_\bullet, 1]).$$
And this is clear since we have an inverse that takes $(\widetilde{\sigma}^0_n, \widetilde{\sigma}^1_n) \mapsto (\sigma^0_n, {\tau'_n}^{-1} \circ \widetilde{\sigma}^1_n \circ \tau_n) = (\sigma^0_n, {\tau'_n}^{-1} \circ \widetilde{\sigma}^0_{n+1} \circ \tau_n)$. □

With the previous proposition we will focus on the strictly linked sequences to simplify the constructions.

Remark 7.11. In the standard situation of spectra, $\mathscr{M}$ is a category of pointed objects, and therefore the initial object $\emptyset$ and the terminal object $*$ are uniquely isomorphic to form a zero object in $\mathscr{M}$. In the upcoming results we will only use this property when really needed. So we shall write distinctly $\emptyset$ for the initial object and $*$ for the terminal object.

Note. From now on our results and our constructions will be given for $\mathbb{Z}$-sequences but they also hold also for any indexing set $\mathbb{O} \subseteq \mathbb{Z}$. We do this to simplify the constructions and to avoid long proofs that involve cases such as "for every $n \in \mathbb{O}$ such that $n + 1 \in \mathbb{O}$".

Proposition 7.12. *The category* $\mathcal{LS}_{\mathcal{U}}(\mathbb{Z}, \mathscr{M})$ *is complete and cocomplete and the following hold.*

1. *The functor* $P : \mathcal{LS}_{\mathcal{U}}(\mathbb{Z}, \mathscr{M}) \longrightarrow (\mathscr{M}_{\mathcal{U}}[\mathscr{M}])^{\mathbb{Z}_{\mathrm{disc}}}$ *creates limits and colimits which are computed level-wise.*
2. *The functor* P *has a left adjoint and this adjunction is monadic.*

We defer the proof for the moment because we need the following intermediate result that will simplify the proof of the proposition.

Lemma 7.13. *The projection functor* P_n *has a left adjoint* $\Upsilon_n : \mathscr{M}_{\mathcal{U}}[\mathscr{M}] \longrightarrow \mathcal{LS}_{\mathcal{U}}(\mathbb{Z}, \mathscr{M})$ *defined as follows.*

1. *For* $[\mathcal{A}] = [\mathcal{A}^0, \mathcal{A}^1, \varepsilon] \in \mathscr{M}_{\mathcal{U}}[\mathscr{M}]$ *we define* $\Upsilon_n([\mathcal{A}]) = ([\widetilde{\mathcal{A}}_k])_{k\in\mathbb{Z}} \in \prod_{k\in\mathbb{Z}} \mathscr{M}_{\mathcal{U}}[\mathscr{M}]$*, by the formulas:*
   - $[\widetilde{\mathcal{A}}_k] = L_1(\emptyset) = [\emptyset, \emptyset, \emptyset \xrightarrow{!} \mathcal{U}(\emptyset)]$ *if* $k < n - 1$
   - $[\widetilde{\mathcal{A}}_k] = L_1(\mathcal{A}^0) = [\emptyset, \mathcal{A}^0, \emptyset \xrightarrow{!} \mathcal{U}(\mathcal{A}^0)]$ *if* $k = n - 1$
   - $[\widetilde{\mathcal{A}}_k] = [\mathcal{A}] = [\mathcal{A}^0, \mathcal{A}^1, \mathcal{A}^0 \xrightarrow{\varepsilon} \mathcal{U}(\mathcal{A}^1)]$ *if* $k = n$

- $[\widetilde{\mathcal{A}}_k] = \mathbf{F}^+(\mathbf{F}^{k-(n+1)}(\mathcal{A}^1)) = [\mathbf{F}^{k-(n+1)}(\mathcal{A}^1), \mathbf{F}(\mathbf{F}^{k-(n+1)}(\mathcal{A}^1)), \eta_{\mathbf{F}^{k-(n+1)}(\mathcal{A}^1)}]$ *if* $k \geq n+1$*; where* η *is the unit of the adjunction* $\mathbf{F} \dashv \mathcal{U}$*. Equivalently we have the inductive formula:*
$$[\widetilde{\mathcal{A}}_k] = \mathbf{F}^+(\widetilde{\mathcal{A}}^1_{k-1}) = [\widetilde{\mathcal{A}}^1_{k-1}, \mathbf{F}(\widetilde{\mathcal{A}}^1_{k-1}), \eta_{\widetilde{\mathcal{A}}^1_{k-1}}], \quad k \geq n+1.$$

2. *If* $\sigma = [\sigma_0; \sigma_1] \in \mathrm{Hom}_{\mathscr{M}_{\mathcal{U}}[\mathscr{M}]}([\mathcal{A}], [\mathcal{B}])$ *we define* $\Upsilon_n(\sigma) = (\widetilde{\sigma}_k)_{k \in \mathbb{Z}}$ *by the formulas:*
   - $\widetilde{\sigma}_k = [\mathrm{Id}_\emptyset, \mathrm{Id}_\emptyset]$ *if* $k < n-1$
   - $\widetilde{\sigma}_k = [\mathrm{Id}_\emptyset, \sigma_0]$ *if* $k = n-1$
   - $\widetilde{\sigma}_k = \sigma = [\sigma_0, \sigma_1]$ *if* $k = n$
   - $\widetilde{\sigma}_k = \mathbf{F}^+(\mathbf{F}^{k-(n+1)}(\sigma_1)) = [\mathbf{F}^{k-(n+1)}(\sigma_1), \mathbf{F}(\mathbf{F}^{k-(n+1)}(\sigma_1))]$ *if* $k \geq n+1$*. In a short form we have the inductive formula:*
$$\widetilde{\sigma}_k = \mathbf{F}^+(\widetilde{\sigma}^1_{k-1}) = [\widetilde{\sigma}^1_{k-1}, \mathbf{F}(\widetilde{\sigma}^1_{k-1})], \quad k \geq n+1.$$

Before we give the proof, let us take a moment to see what Υ_n really does. If we are given an object $[\mathcal{A}] = [\mathcal{A}^0, \mathcal{A}^1, \mathcal{A}^0 \xrightarrow{\varepsilon} \mathcal{U}(\mathcal{A}^1)]$, by adjunction the map ε corresponds to a unique map $\mathbf{F}(\mathcal{A}^0) \xrightarrow{\varrho(\varepsilon)} \mathcal{A}^1$. Then $\Upsilon_n([\mathcal{A}])$ is the prespectrum given by the sequence:

$$\emptyset, \cdots, \emptyset, \underbrace{\mathbf{F}(\mathcal{A}^0)}_{n\text{th}}, \underbrace{\mathcal{A}^1}_{(n+1)\text{th}}, \underbrace{\mathbf{F}(\mathcal{A}^1)}_{(n+2)\text{th}}, \mathbf{F}^2(\mathcal{A}^1), \cdots, \underbrace{\mathbf{F}^{k-(n+1)}(\mathcal{A}^1) \cdots}_{k \geq (n+3)},$$

with the connecting morphisms:

$$\cdots, \emptyset \xrightarrow{\mathrm{Id}_\emptyset} \emptyset, \mathbf{F}(\mathcal{A}^0) \xrightarrow{\varrho(\varepsilon)} \mathcal{A}^1, \mathbf{F}(\mathcal{A}^1) \xrightarrow{\mathrm{Id}} \mathbf{F}(\mathcal{A}^1), \mathbf{F}^2(\mathcal{A}^1) \xrightarrow{\mathrm{Id}} \mathbf{F}^2(\mathcal{A}^1), \cdots \mathrm{Id}_{\mathbf{F}^{k-(n+1)}(\mathcal{A}^1)}, \cdots.$$

In our formalism, we consider the adjoint definition of the spectrum given by the following sequence and its connecting morphisms:

$$\emptyset, \cdots, \emptyset, \underbrace{\mathcal{A}^0}_{n\text{th}}, \underbrace{\mathcal{A}^1}_{(n+1)\text{th}}, \underbrace{\mathbf{F}(\mathcal{A}^1)}_{(n+2)\text{th}}, \mathbf{F}^2(\mathcal{A}^1), \cdots, \underbrace{\mathbf{F}^{k-(n+1)}(\mathcal{A}^1) \cdots}_{k \geq (n+3)}.$$

$$\eta_\emptyset, \cdots, \emptyset \xrightarrow{!} \mathcal{U}\mathcal{F}(\mathcal{A}^0), \mathcal{A}^0 \xrightarrow{\varepsilon} \mathcal{U}(\mathcal{A}^1), \mathcal{A}^1 \xrightarrow{\eta_{\mathcal{A}^1}} \mathcal{U}\mathbf{F}(\mathcal{A}^1), \mathbf{F}(\mathcal{A}^1) \xrightarrow{\eta} \mathcal{U}\mathbf{F}^2(\mathcal{A}^1), \cdots \eta_{\mathbf{F}^{k-(n+1)}(\mathcal{A}^1)}, \cdots.$$

Proof of Lemma 7.13. It is clear that $\Upsilon_n([\mathcal{A}])$ is a linked sequence and that $\Upsilon_n(\sigma)$ is a linked morphism. The constructions are clearly functorial so the previous data define a functor. The lemma will follow as soon as we show that for every $[\mathcal{A}] \in \mathscr{M}_{\mathcal{U}}[\mathscr{M}]$ and for every $[\mathcal{X}_\bullet, 1] \in \mathcal{LS}_{\mathcal{U}}(\mathbb{Z}, \mathscr{M})$, there is a functorial isomorphism of hom-sets:

$$\varphi : \mathrm{Hom}_{\mathscr{M}_{\mathcal{U}}[\mathscr{M}]}([\mathcal{A}], [\mathcal{X}_n]) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{LS}_{\mathcal{U}}(\mathbb{Z}, \mathscr{M})}(\Upsilon_n([\mathcal{A}]), [\mathcal{X}_\bullet, 1]).$$

This is straightforward but we give the proof for the reader's convenience. If σ is an element of $\mathrm{Hom}_{\mathscr{M}_{\mathcal{U}}[\mathscr{M}]}([\mathcal{A}], [\mathcal{X}_n])$ we define $\varphi(\sigma) = (\widetilde{\sigma}_k) \in \mathrm{Hom}_{\mathcal{LS}_{\mathcal{U}}(\mathbb{Z}, \mathscr{M})}(\Upsilon_n([\mathcal{A}]), [\mathcal{X}_\bullet, 1])$ as the linked morphism given by the formulas:

- if $k < n-1$, $\widetilde{\sigma}_k : L_1(\emptyset) \longrightarrow [\mathcal{X}_k]$ is the adjoint map to the unique morphism $\emptyset \xrightarrow{!} \mathcal{X}^1_k$, in the adjunction $L_1 \dashv \Pi_1$ of Theorem 4.4. The components of $\widetilde{\sigma}_k$ are $\widetilde{\sigma}^0_k = \emptyset \xrightarrow{!} \mathcal{X}^0_k$ and $\widetilde{\sigma}^1_k = \emptyset \xrightarrow{!} \mathcal{X}^1_k$ the unique morphisms from the initial object.
- if $k = n-1$, $\widetilde{\sigma}_k = [\mathrm{Id}_\emptyset, \sigma_0] : L_1(\mathcal{A}^0) \longrightarrow [\mathcal{X}_{n-1}]$ is the adjoint map to $\sigma_0 : \mathcal{A}^0 \longrightarrow \underbrace{\mathcal{X}^0_n}_{=\mathcal{X}^1_{n-1}}$.

  Thus this morphism is uniquely determined by σ_0.

- if $k = n$, $\widetilde{\sigma}_k = \sigma : [\mathcal{A}] \longrightarrow [\mathcal{X}_n]$.
- Inductively for $k \geq n+1$, $\widetilde{\sigma}_k : [\widetilde{\mathcal{A}}_k] = \mathbf{F}^+(\widetilde{\mathcal{A}}^1_{k-1}) \longrightarrow [\mathcal{X}_k]$ is the adjoint map to

$$\widetilde{\sigma}^1_{k-1} : \widetilde{\mathcal{A}}^1_{k-1} \longrightarrow \underbrace{\mathcal{X}^1_{k-1}}_{=\mathcal{X}^0_k}.$$

The function φ is clearly 1-1 because the nth component is σ. And we have an inverse function $\varphi^{-1} : \operatorname{Hom}_{\mathcal{LS}_{\mathcal{U}}(\mathbb{Z},\mathscr{M})}(\Upsilon_n([\mathcal{A}]), [\mathcal{X}_\bullet, 1]) \longrightarrow \operatorname{Hom}_{\mathscr{M}_{\mathcal{U}}[\mathscr{M}]}([\mathcal{A}], [\mathcal{X}_n])$ that projects the nth component. And the lemma follows. □

Proposition-Definition 7.14. *For $n \in \mathbb{Z}$, define the nth* Dirac mass *functor*

$$\delta_n : \mathscr{M}_{\mathcal{U}}[\mathscr{M}] \longrightarrow (\mathscr{M}_{\mathcal{U}}[\mathscr{M}])^{\mathbb{Z}_{\text{disc}}}$$

as the left adjoint to the nth evaluation $Ev_n : (\mathscr{M}_{\mathcal{U}}[\mathscr{M}])^{\mathbb{Z}_{\text{disc}}} \longrightarrow \mathscr{M}_{\mathcal{U}}[\mathscr{M}]$ defined by $Ev_n([\mathcal{X}_\bullet, 1]) = [\mathcal{X}_n]$. For an object $[\mathcal{A}] \in \mathscr{M}_{\mathcal{U}}[\mathscr{M}]$, $\delta_n([\mathcal{A}]) = (\delta_n([\mathcal{A}])_k)_{k\in\mathbb{Z}}$ is given by:

- $\delta_n([\mathcal{A}])_k = [\mathcal{A}]$ *if $k = n$.*
- $\delta_n([\mathcal{A}])_k = L_1(\emptyset) = [\emptyset, \emptyset, \emptyset \xrightarrow{!} \mathcal{U}(\emptyset)]$ *if $k \neq n$.*

1. *We have a functorial isomorphism of hom-sets:*

$$\operatorname{Hom}_{\mathscr{M}_{\mathcal{U}}[\mathscr{M}]}([\mathcal{A}], [\mathcal{X}_n]) \cong \operatorname{Hom}_{(\mathscr{M}_{\mathcal{U}}[\mathscr{M}])^{\mathbb{Z}_{\text{disc}}}}(\delta_n([\mathcal{A}]), [\mathcal{X}_\bullet]).$$

2. *For every $[\mathcal{X}_\bullet] = ([\mathcal{X}_n])_{n\in\mathbb{Z}} \in (\mathscr{M}_{\mathcal{U}}[\mathscr{M}])^{\mathbb{Z}_{\text{disc}}}$ we have $[\mathcal{X}_\bullet] \cong \sum_n \delta_n([\mathcal{X}_n]) = \coprod_n \delta_n([\mathcal{X}_n])$.*

Proof of Proposition 7.12 We can give the proof of Proposition 7.12 as follows.

Proof. That $P : \mathcal{LS}_{\mathcal{U}}(\mathbb{Z}, \mathscr{M}) \longrightarrow (\mathscr{M}_{\mathcal{U}}[\mathscr{M}])^{\mathbb{Z}_{\text{disc}}}$ creates limits and colimits is obvious and we leave it as an exercise for the reader. So Assertion (1) is clear. Since $\mathcal{LS}_{\mathcal{U}}(\mathbb{Z}, \mathscr{M})$ is complete, coproducts exist and by a classical argument we can find a left adjoint to P as follows. For $[\mathcal{A}_\bullet] = ([\mathcal{A}_n])_{n\in\mathbb{Z}} \in (\mathscr{M}_{\mathcal{U}}[\mathscr{M}])^{\mathbb{Z}_{\text{disc}}}$, let $\Upsilon : (\mathscr{M}_{\mathcal{U}}[\mathscr{M}])^{\mathbb{Z}_{\text{disc}}} \longrightarrow \mathcal{LS}_{\mathcal{U}}(\mathbb{Z}, \mathscr{M})$ be the functor defined by

$$\Upsilon([\mathcal{A}_\bullet]) := \sum_n \Upsilon_n([\mathcal{A}_n]) = \coprod_n \Upsilon_n([\mathcal{A}_n]).$$

For every $[\mathcal{X}_\bullet, 1] \in \mathcal{LS}_{\mathcal{U}}(\mathbb{Z}, \mathscr{M})$ we have the following isomorphism of hom-sets:

$$\begin{aligned}
\operatorname{Hom}_{\mathcal{LS}_{\mathcal{U}}(\mathbb{Z},\mathscr{M})}(\Upsilon([\mathcal{A}_\bullet]), [\mathcal{X}_\bullet, 1]) &= \operatorname{Hom}_{\mathcal{LS}_{\mathcal{U}}(\mathbb{Z},\mathscr{M})}(\coprod_n \Upsilon_n([\mathcal{A}_n]), [\mathcal{X}_\bullet, 1]) \\
&\cong \prod_n \operatorname{Hom}_{\mathscr{M}_{\mathcal{U}}[\mathscr{M}]}([\mathcal{A}_n], P_n([\mathcal{X}_\bullet, 1])) \\
&\cong \prod_n \operatorname{Hom}_{\mathscr{M}_{\mathcal{U}}[\mathscr{M}]}([\mathcal{A}_n], [\mathcal{X}_n]) = \prod_n \operatorname{Hom}_{\mathscr{M}_{\mathcal{U}}[\mathscr{M}]}(\mathcal{A}_n, \operatorname{Ev}_n([\mathcal{X}_\bullet, 1])) \\
&\cong \prod_n \operatorname{Hom}_{(\mathscr{M}_{\mathcal{U}}[\mathscr{M}])^{\mathbb{Z}_{\text{disc}}}}(\delta_n([\mathcal{A}_n]), P([\mathcal{X}_\bullet, 1])) \\
&\cong \operatorname{Hom}_{(\mathscr{M}_{\mathcal{U}}[\mathscr{M}])^{\mathbb{Z}_{\text{disc}}}}(\coprod_n \delta_n([\mathcal{A}_n]), P([\mathcal{X}_\bullet, 1])) \\
&\cong \operatorname{Hom}_{(\mathscr{M}_{\mathcal{U}}[\mathscr{M}])^{\mathbb{Z}_{\text{disc}}}}([\mathcal{A}_\bullet], \underbrace{P([\mathcal{X}_\bullet, 1])}_{=[\mathcal{X}_\bullet]})
\end{aligned}$$

Finally this adjunction is monadic by Beck monadicity theorem (see [7]). Indeed, P reflects isomorphisms because if $\sigma = (\sigma_k)$ is a linked morphism such that each σ_k has an inverse σ_k^{-1} in $\mathscr{M}_{\mathcal{U}}[\mathscr{M}]$, then the sequence (σ_k^{-1}) is a linked morphism which is the inverse of σ. Moreover as mentioned before the functor P creates colimits in particular it creates coequalizers of P-split pairs. And with the left adjoint Υ we are in the hypotheses of Beck's theorem. This ends the proof of the Proposition. □

7.2 Homotopy of $\mathbb{Z}$-sequences. The category of $\mathbb{Z}$-sequences is the diagram category:

$$(\mathscr{M}_{\mathcal{U}}[\mathscr{M}])^{\mathbb{Z}_{\text{disc}}} = \text{Hom}(\mathbb{Z}_{\text{disc}}, \mathscr{M}_{\mathcal{U}}[\mathscr{M}])\,.$$

With Theorem 5.12 and Theorem 5.19, there is an injective and projective model structure on $\mathscr{M}_{\mathcal{U}}[\mathscr{M}]$. We will consider the homotopy theory of the diagram categories $\text{Hom}(\mathbb{Z}_{\text{disc}}, \mathscr{M}_{\mathcal{U}}[\mathscr{M}]_{inj})$ and $\text{Hom}(\mathbb{Z}, \mathscr{M}_{\mathcal{U}}[\mathscr{M}]_{proj})$. Model structures on diagram categories have been studied for decades and they are well known in the literature (see for example [31, 40, 87]). But in our case things are even simpler because this is just a product of model categories.

Hypotheses and Notation

1. In this section we assume that $\mathscr{M}$ is combinatorial for simplicity. The general case of a cofibrantly generated model category will be studied in a different paper.
2. We will also assume that $\mathscr{M}$ is left proper (see [40]) and we will mention when we use this hypothesis.
3. We will use the generic notation $\mathscr{M}_{\mathcal{U}}[\mathscr{M}]_{ms}$ with $ms \in \{inj, proj\}$ to represent both model categories $\mathscr{M}_{\mathcal{U}}[\mathscr{M}]_{inj}$ and $\mathscr{M}_{\mathcal{U}}[\mathscr{M}]_{proj}$. The subscript ms refers to "model structure".
4. Similarly $\text{Arr}(\mathscr{M})_{ms}$ represents both $\text{Arr}(\mathscr{M})_{\text{inj}}$ and $\text{Arr}(\mathscr{M})_{\text{proj}}$.
5. The functor $\Pi_{\text{Arr}} : \mathscr{M}_{\mathcal{U}}[\mathscr{M}]_{ms} \longrightarrow \text{Arr}(\mathscr{M})_{ms}$ is right Quillen for $ms \in \{inj, proj\}$

Note. If $\mathbb{O} \subseteq \mathbb{Z}$ we have a restriction $\text{Hom}(\mathbb{Z}_{\text{disc}}, \mathscr{M}_{\mathcal{U}}[\mathscr{M}]) \longrightarrow \text{Hom}(\mathbb{O}_{\text{disc}}, \mathscr{M}_{\mathcal{U}}[\mathscr{M}])$ and the constructions that will follow hold for the diagram category $\text{Hom}(\mathbb{O}_{\text{disc}}, \mathscr{M}_{\mathcal{U}}[\mathscr{M}])$.

Definition 7.15. Let $\sigma : [\mathcal{X}_\centerdot] \longrightarrow [\mathcal{Y}_\centerdot]$ be a morphism in $\text{Hom}(\mathbb{Z}_{\text{disc}}, \mathscr{M}_{\mathcal{U}}[\mathscr{M}]_{ms})$.

- Say that σ is a level-wise weak equivalence if for every n, $\sigma_n : [\mathcal{X}_n] \longrightarrow [\mathcal{Y}_n]$ is a weak equivalence in $\mathscr{M}_{\mathcal{U}}[\mathscr{M}]_{ms}$.
- Say that σ is a level-wise (trivial) cofibration if for every n, $\sigma_n : [\mathcal{X}_n] \longrightarrow [\mathcal{Y}_n]$ is a (trivial) cofibration in $\mathscr{M}_{\mathcal{U}}[\mathscr{M}]_{ms}$.
- Say that σ is a level-wise (trivial) fibration if for every n, $\sigma_n : [\mathcal{X}_n] \longrightarrow [\mathcal{Y}_n]$ is a (trivial) fibration in $\mathscr{M}_{\mathcal{U}}[\mathscr{M}]_{ms}$.

Theorem 7.16. *Let $\mathscr{M}$ be a combinatorial model category and let $\mathcal{U} : \mathscr{M} \longrightarrow \mathscr{M}$ be a right Quillen functor. With the previous definitions the following hold.*

1. *There is an injective model structure on the category $\text{Hom}(\mathbb{Z}_{\text{disc}}, \mathscr{M}_{\mathcal{U}}[\mathscr{M}]_{ms})$ where the cofibrations and the weak equivalences are level-wise.*
2. *There is a projective model structure on the category $\text{Hom}(\mathbb{Z}_{\text{disc}}, \mathscr{M}_{\mathcal{U}}[\mathscr{M}]_{ms})$ where the cofibrations and the weak equivalences are level-wise.*

These model categories are combinatorial. They are left proper if $\mathscr{M}$ is in addition left proper.

Proof. The existence of these model structures can be found for example in [87]. The left properness follows from the fact that $\mathscr{M}_{\mathcal{U}}[\mathscr{M}]_{inj}$ and $\mathscr{M}_{\mathcal{U}}[\mathscr{M}]_{proj}$ are left proper since pushouts are computed object wise. We have seen that $\mathscr{M}_{\mathcal{U}}[\mathscr{M}]$ is locally presentable, therefore $\text{Hom}(\mathbb{Z}_{\text{disc}}, \mathscr{M}_{\mathcal{U}}[\mathscr{M}])$ is also locally presentable. Finally both $\mathscr{M}_{\mathcal{U}}[\mathscr{M}]_{inj}$ and $\mathscr{M}_{\mathcal{U}}[\mathscr{M}]_{inj}$ are combinatorial. □

7.2.1 Strict model structures for prespectra. We wish to transfer the model structure from $\text{Hom}(\mathbb{Z}_{\text{disc}}, \mathscr{M}_{\mathcal{U}}[\mathscr{M}]_{ms})$ to $\mathcal{LS}_{\mathcal{U}}(\mathbb{Z}, \mathscr{M})$. Since the identity functor $\text{Id} : \text{Arr}(\mathscr{M})_{\text{proj}} \longrightarrow \text{Arr}(\mathscr{M})_{\text{inj}}$ is a left Quillen functor i.e, any projective (trivial) cofibration is an injective (trivial) cofibration, it suffices to have a transfer from the model category $\text{Hom}(\mathbb{Z}_{\text{disc}}, \mathscr{M}_{\mathcal{U}}[\mathscr{M}]_{inj})$. The ingredient that is needed is the following lemma.

Lemma 7.17. *Let $\sigma : [\mathcal{A}] \xrightarrow{[\sigma_0, \sigma_1]} [\mathcal{B}]$ be a map in $\mathscr{M}_{\mathcal{U}}[\mathscr{M}]$. Then the following hold.*

1. *For every n, $P(\Upsilon_n(\sigma)) : P(\Upsilon_n([\mathcal{A}])) \longrightarrow P(\Upsilon_n([\mathcal{B}]))$ is a level-wise (trivial) cofibration in $\text{Hom}(\mathbb{Z}_{\text{disc}}, \mathscr{M}_{\mathcal{U}}[\mathscr{M}]_{inj})$ if σ is a (trivial) cofibration in $\mathscr{M}_{\mathcal{U}}[\mathscr{M}]_{inj}$.*
2. *For every n and for any pushout square in $\mathcal{LS}_{\mathcal{U}}(\mathbb{Z}, \mathscr{M})$ as follows*

$$\begin{array}{ccc} \Upsilon_n([\mathcal{A}]) & \longrightarrow & [\mathcal{X}_{\bullet}, 1] \\ \downarrow{\scriptstyle \Upsilon_n(\sigma)} & & \downarrow{\scriptstyle u} \\ \Upsilon_n([\mathcal{B}]) & \longrightarrow & \Upsilon_n([\mathcal{B}]) \cup^{\Upsilon_n([\mathcal{A}])} [\mathcal{X}_{\bullet}, 1] \end{array}$$

the map $P(u) : P([\mathcal{X}_{\bullet}, 1]) \longrightarrow P(\Upsilon_n([\mathcal{B}]) \cup^{\Upsilon_n([\mathcal{A}])} [\mathcal{X}_{\bullet}, 1])$ is a level-wise trivial cofibration in $\text{Hom}(\mathbb{Z}_{\text{disc}}, \mathscr{M}_{\mathcal{U}}[\mathscr{M}]_{inj})$ if σ is trivial cofibration in $\mathscr{M}_{\mathcal{U}}[\mathscr{M}]_{inj}$. In particular $P(u)$ is a level-wise weak equivalence in both $\text{Hom}(\mathbb{Z}_{\text{disc}}, \mathscr{M}_{\mathcal{U}}[\mathscr{M}]_{inj})$ and $\text{Hom}(\mathbb{Z}_{\text{disc}}, \mathscr{M}_{\mathcal{U}}[\mathscr{M}]_{proj})$.

Proof. By definition given $\sigma : [\mathcal{A}] \xrightarrow{[\sigma_0, \sigma_1]} [\mathcal{B}]$, $\Upsilon_n(\sigma) = (\widetilde{\sigma}_k)_{k \in \mathbb{N}}$ is given by the formulas:

- $\widetilde{\sigma}_k = [\text{Id}_{\emptyset}, \text{Id}_{\emptyset}]$ if $k < n-1$
- $\widetilde{\sigma}_k = [\text{Id}_{\emptyset}, \sigma_0]$ if $k = n-1$
- $\widetilde{\sigma}_k = \sigma = [\sigma_0, \sigma_1]$ if $k = n$
- $\widetilde{\sigma}_k = \mathbf{F}^+(\mathbf{F}^{k-(n+1)}(\sigma_1)) = [\mathbf{F}^{k-(n+1)}(\sigma_1), \mathbf{F}(\mathbf{F}^{k-(n+1)}(\sigma_1))]$ if $k \geq n+1$.

Since $\mathbf{F}$ is a left Quillen functor, it preserves the cofibrations and the trivial cofibrations. Therefore for every $k \geq n+1$, $\mathbf{F}^{k-(n+1)}(\sigma_1)$ and $\mathbf{F}(\mathbf{F}^{k-(n+1)}(\sigma_1))$ are (trivial) cofibrations if σ_1 is a (trivial) cofibration. This means that $\widetilde{\sigma}_k$ is a level-wise (trivial) cofibration if σ_1 is for $k \geq n+1$.

For $k \leq n$, the components of $\widetilde{\sigma}_k$ are one of the maps $\text{Id}_{\emptyset}$, σ_0 or σ_1. So clearly $\widetilde{\sigma}_k$ is a level-wise (trivial) cofibration if σ_0 and σ_1 are (trivial) cofibrations. This gives Assertion (1).

Assertion (2) is clear because colimits and in particular pushouts in $\mathcal{LS}_{\mathcal{U}}(\mathbb{Z}, \mathscr{M})$ are computed in $\text{Hom}(\mathbb{Z}_{\text{disc}}, \mathscr{M}_{\mathcal{U}}[\mathscr{M}])$. Finally since $\mathscr{M}_{\mathcal{U}}[\mathscr{M}]_{inj}$ is a model category, the cobase change of any trivial cofibration is again a trivial cofibration. □

Projective model structures Recall that for simplicity we have assumed that $\mathscr{M}$ is combinatorial. But the following theorem holds word-for-word if $\mathscr{M}$ is a cofibrantly generated model category.

Theorem 7.18. *Let $\mathscr{M}$ be a combinatorial model category and consider the projective model structure on $\text{Hom}(\mathbb{Z}_{\text{disc}}, \mathscr{M}_{\mathcal{U}}[\mathscr{M}]_{ms})$. Then there is a right-induced model structure on the category $\mathcal{LS}_{\mathcal{U}}(\mathbb{Z}, \mathscr{M}) \cong \mathcal{S}p_{\mathcal{U}}(\mathbb{Z}, \mathscr{M})$ which is combinatorial and which may be described as follows.*

1. *A map σ is a weak equivalence if $P(\sigma)$ is a weak equivalence in $\text{Hom}(\mathbb{Z}_{\text{disc}}, \mathscr{M}_{\mathcal{U}}[\mathscr{M}])_{proj}$.*
2. *A map σ is a fibration if $P(\sigma)$ is a fibration in $\text{Hom}(\mathbb{Z}_{\text{disc}}, \mathscr{M}_{\mathcal{U}}[\mathscr{M}])_{proj}$.*
3. *A map σ is a cofibration if it possesses the LLP with respect to any map that is simultaneously a fibration and a weak equivalence.*

The adjunction $P : \mathcal{LS}_{\mathcal{U}}(\mathbb{Z}, \mathscr{M}) \rightleftarrows \text{Hom}(\mathbb{Z}_{\text{disc}}, \mathscr{M}_{\mathcal{U}}[\mathscr{M}]_{ms})_{proj} : \Upsilon$ is a Quillen adjunction and for every n, $\mathcal{T}_n = \Pi_{Arr} \circ Ev_n \circ P_n : \mathcal{LS}_{\mathcal{U}}(\mathbb{Z}, \mathscr{M}) \longrightarrow Arr(\mathscr{M})_{ms}$ is right Quillen.

Proof. This is a classical argument of Quillen [78], see Beke [11, 12]. The main ingredient is precisely Lemma 7.17 above. Indeed, any projective trivial cofibration is an injective cofibration.

To prove that $\mathcal{LS}_{\mathcal{U}}(\mathbb{Z},\mathscr{M})$ is combinatorial boils down to show that it is locally presentable. This can be proved directly but we can take a shortcut with a well known method that appears for example in Kelly-Lack [55]. One uses the fact that the adjunction $\Upsilon \dashv P$ is monadic and the induced monad is clearly finitary, that is, it preserves filtered colimits. □

Corollary 7.19. *If in addition $\mathscr{M}$ is left proper then $\mathcal{LS}_{\mathcal{U}}(\mathbb{Z},\mathscr{M})$ is left proper.*

Proof. Left properness follows from the fact that pushouts in $\mathcal{LS}_{\mathcal{U}}(\mathbb{Z},\mathscr{M})$ are computed object wise in $\mathscr{M}_{\mathcal{U}}[\mathscr{M}]$, and clearly $\mathscr{M}_{\mathcal{U}}[\mathscr{M}]_{inj}$ and $\mathscr{M}_{\mathcal{U}}[\mathscr{M}]_{proj}$ are left proper if $\mathscr{M}$ is. □

Remark 7.20. We have a similar statement when we consider the injective model structure on $\text{Hom}(\mathbb{Z}_{\text{disc}},\mathscr{M}_{\mathcal{U}}[\mathscr{M}]_{ms})$. The previous theorem holds for any indexing set $\mathbb{O}\subseteq\mathbb{Z}$, in particular it holds for $\mathbb{N}$-indexed sequences that we study in the next subsection.

7.3 Bousfield-Friedlander strict model structure.

In Bousfield-Friedlander [24], Hovey [45], spectra are indexed over $\mathbb{N}$ so in this section we will focus on the category $\mathcal{LS}_{\mathcal{U}}(\mathbb{N},\mathscr{M})\cong\mathcal{S}\text{p}_{\mathcal{U}}(\mathbb{N},\mathscr{M})$ of strictly linked $\mathbb{N}$-sequences. All previous results hold, including Theorem 7.18.

What is important here is that we can be more specific when we consider the "projective-projective" model category $\text{Hom}(\mathbb{N}_{\text{disc}},\mathscr{M}_{\mathcal{U}}[\mathscr{M}]_{proj})_{proj}$. We do not require $\mathscr{M}$ to be combinatorial (or cofibrantly generated). And the argument also holds for a bounded below indexing set $\mathbb{O}\subset\mathbb{Z}$.

Theorem 7.21. *Let $\mathcal{U}:\mathscr{M}\longrightarrow\mathscr{M}$ be a right Quillen functor and let $\mathscr{M}_{\mathcal{U}}[\mathscr{M}]_{proj}$ be the projective model structure on $\mathscr{M}_{\mathcal{U}}[\mathscr{M}]$ as in Theorem 5.19. Then there is a model structure on $\mathcal{LS}_{\mathcal{U}}(\mathbb{N},\mathscr{M})\cong\mathcal{S}p_{\mathcal{U}}(\mathbb{N},\mathscr{M})$ called the **projective strict model structure** which may be described as follows.*

- *A map $\sigma:[\mathcal{X}_\bullet,1]\longrightarrow[\mathcal{Y}_\bullet,1]$ is a weak equivalence if for every n, $\sigma_n:[\mathcal{X}_n]\longrightarrow[\mathcal{Y}_n]$ is a weak equivalence in $\mathscr{M}_{\mathcal{U}}[\mathscr{M}]_{proj}$, that is a level-wise weak equivalence.*
- *A map $\sigma:[\mathcal{X}_\bullet,1]\longrightarrow[\mathcal{Y}_\bullet,1]$ is a fibration if for every n, $\sigma_n:[\mathcal{X}_n]\longrightarrow[\mathcal{Y}_n]$ is a fibration in $\mathscr{M}_{\mathcal{U}}[\mathscr{M}]_{proj}$, that is a level-wise fibration.*
- *A map $\sigma:[\mathcal{X}_\bullet,1]\longrightarrow[\mathcal{Y}_\bullet,1]$ is a cofibration if for every n, $\sigma_n:[\mathcal{X}_n]\xrightarrow{[\sigma_n^0,\sigma_n^1]}[\mathcal{Y}_n]$ n is a cofibration in $\mathscr{M}_{\mathcal{U}}[\mathscr{M}]_{proj}$, that is for every n , σ_n^0 is a cofibration and the canonical map $\delta:\mathcal{X}_{n+1}^1\cup^{\mathbf{F}(\mathcal{X}_n^0)}\mathbf{F}(\mathcal{Y}_n^0)\longrightarrow\mathcal{Y}_{n+1}^1$ is a cofibration in $\mathscr{M}$.*
- *If $\mathscr{M}$ is combinatorial (resp. left proper) then $\mathcal{LS}_{\mathcal{U}}(\mathbb{N},\mathscr{M})$ is also combinatorial (resp. left proper).*

We will denote this model category by $\mathcal{LS}_{\mathcal{U}}(\mathbb{N},\mathscr{M})_{proj}$.

Proof. One gets the factorization axioms and the lifting properties inductively because $\mathbb{N}$ is bounded below and is a direct Reedy category. The main reason is that given a map $\sigma_n:[\mathcal{X}_n]\xrightarrow{[\sigma_n^0,\sigma_n^1]}[\mathcal{Y}_n]$ in $\mathscr{M}_{\mathcal{U}}[\mathscr{M}]$, when we factor it in the projective model structure we always start with a factorization of σ_n^0. We also do the same thing when we construct a solution for lifting properties. Proceeding this way we can factor any linked morphism σ being careful to take the factorization of σ_n^1 and apply them to $\sigma_{n+1}^0=\sigma_n^1$ to build in $\mathscr{M}_{\mathcal{U}}[\mathscr{M}]$ a factorization of $\sigma_{n+1}=[\sigma_{n+1}^0,\sigma_{n+1}^1]$. One gets inductively the required factorizations (resp liftings) at each step by linked morphisms. The same argument is used by Rosický [82].

If $\mathscr{M}$ is combinatorial (resp left proper) then this model structure coincides with the right-induced model structure obtained with the analogue of Theorem 7.18. This model structure is transferred from $\text{Hom}(\mathbb{N}_{\text{disc}}, \mathscr{M}_{\mathcal{U}}[\mathscr{M}]_{proj})_{proj}$. Indeed we have the same fibrations and the same weak equivalences. □

Once again the previous result holds when $\mathscr{M}$ is cofibrantly generated.

Corollary 7.22. *If* $\mathscr{M} = \mathbf{sSet}_*$ *with the standard Kan-Quillen model structure, then the "projective-projective" model structure on* $\mathcal{S}p_{\Omega}(\mathbb{N}, \mathbf{sSet}_*) \cong \mathcal{LS}_{\Omega}(\mathbb{N}, \mathbf{sSet}_*)$ *of Theorem 7.18 and Theorem 7.21 coincides with the strict model structure of Bousfield-Friedlander that is also obtained by Hovey [45], Schwede [84].*

Proof. Clear. □

7.3.1 Stable homotopy category. We have a projective Quillen-Segal theory $\mathcal{T} = (\mathcal{T}_n)_{n\in\mathbb{N}}$ on $\mathcal{LS}_{\mathcal{U}}(\mathbb{N}, \mathscr{M})_{proj}$ where $\mathcal{T}_n : \mathcal{LS}_{\mathcal{U}}(\mathbb{N}, \mathscr{M})_{proj} \longrightarrow \text{Arr}(\mathscr{M})_{proj}$ is given by $\mathcal{T}_n([\mathcal{X}_\bullet, 1]) = [\varepsilon : \mathcal{X}_n^0 \longrightarrow \mathcal{U}(\mathcal{X}_{n+1}^1)]$. Each $\mathcal{T}_n$ is right Quillen and its the left adjoint is $\Upsilon_n\Gamma$. In virtue of Theorem 2.9 we can localize the theory. We assume that $\mathscr{M}$ is a tractable and left proper model category such as $\mathbf{sSet}_*$. We remind the reader that the categories $\mathcal{S}p_{\mathcal{U}}(\mathbb{N}, \mathscr{M})$ and $\mathcal{LS}_{\mathcal{U}}(\mathbb{N}, \mathscr{M})$ are isomorphic so we will freely identify them.

Theorem 7.23. *Let* $\mathcal{U} : \mathscr{M} \longrightarrow \mathscr{M}$ *be a right Quillen functor for a tractable and left proper model category* $\mathscr{M}$*. Then the following hold.*

1. *There is a left proper model structure on the category* $\mathcal{S}p_{\mathcal{U}}(\mathbb{N}, \mathscr{M}) \cong \mathcal{LS}_{\mathcal{U}}(\mathbb{N}, \mathscr{M})$ *such that every fibrant object* $[\mathcal{X}_\bullet, 1]$ *is a* $\mathcal{U}$*-spectrum.*
2. *If* $\mathscr{M} = \mathbf{sSet}_*$ *then the homotopy category is equivalent to the stable homotopy category of Bousfield and Friedlander.*

To prove the last assertion of the theorem we need to outline a result. Let $\mathbf{I}_{\mathscr{M}}$ be the generating set of cofibrations of $\mathscr{M}$. For every $i \in \mathbf{I}_{\mathscr{M}}$ we have introduced in Notation 2.12 and Notation 2.15 two maps α_i and ζ_i in $\text{Arr}(\mathscr{M})$. The map ζ_i is a projective cofibration whereas α_i is an injective cofibration.

Lemma 7.24. *For every* n*, consider the functor* $\Upsilon_n\Gamma \in \text{Hom}(Arr(\mathscr{M})_{\text{proj}}, \mathcal{LS}_{\mathcal{U}}(\mathbb{N}, \mathscr{M})_{proj})$*. Then for every* $i \in \mathbf{I}_{\mathscr{M}}$ *the following hold.*

1. *The map* $\Upsilon_n\Gamma(\alpha_i) = (\sigma_k)$ *is such that for every* $k \geq n+1$*,* σ_k *is an isomorphism in* $\mathscr{M}_{\mathcal{U}}[\mathscr{M}]$
2. *The map* $\Upsilon_n\Gamma(\zeta_i) = (\sigma_k)$ *is such that for every* $k \geq n+1$*,* σ_k *is a level-wise trivial cofibration* $\mathscr{M}_{\mathcal{U}}[\mathscr{M}]$*.*
3. *Taking* $\mathscr{M} = \mathbf{sSet}_*$ *we find that for every* n *and for every* $i \in \mathbf{I}_{\mathscr{M}}$ *the maps* $\Upsilon_n\Gamma(\alpha_i)$*,* $\Upsilon_n\Gamma(\zeta_i)$ *are stable equivalence. In particular since* $\Upsilon_n\Gamma(\zeta_i)$ *is already a cofibration, then it is a trivial stable cofibration in* $\mathcal{S}p_{\Omega}(\mathbb{N}, \mathbf{sSet}_*)$*.*

Proof. The first two assertions follow directly from the construction of Γ (Proposition 4.8) and Υ_n (Lemma 7.13). The third assertion follows from the definition of the stable homotopy groups (see [2], [84]). These groups are defined as direct colimit; so what matters is the stabilization for greater k. The last assertion is clear. □

We are now able to prove Theorem 7.23.

Proof of Theorem 7.23. The existence of the model structure and the fact that every fibrant object is a $\mathcal{U}$-spectrum is simply given by Theorem 2.9.

To prove that the homotopy category is the usual stable homotopy category we will show that we have the same fibrant objects in the model structure in Hovey [45, Theorem 3.4] which is also a left Bousfield localization of the same strict model structure.

If $[\mathcal{X}_\bullet, 1] \in \mathcal{S}\mathrm{p}_\Omega(\mathbb{N}, \mathbf{sSet}_*)$ is fibrant in our new model category, then $[\mathcal{X}_\bullet, 1]$ is level-wise fibrant and satisfies the Segal conditions so it is a $\mathcal{U}$-spectrum, whence it is a fibrant object in the model structure of Hovey. Conversely assume that $[\mathcal{X}_\bullet, 1] \in \mathcal{S}\mathrm{p}_\Omega(\mathbb{N}, \mathbf{sSet}_*)$ is fibrant in Hovey's model structure. Then in a tautological way $[\mathcal{X}_\bullet, 1]$ is σ-local for any stable equivalence σ. Thanks to Lemma 7.24, we know that for every i and for every n, $\Upsilon_n\Gamma(\zeta_i)$ is a stable weak equivalence and indeed a trivial cofibration. This means $[\mathcal{X}_\bullet, 1]$ is $\Upsilon_n\Gamma(\zeta_i)$-local for every i and for every n, therefore $[\mathcal{X}_\bullet, 1]$ is fibrant in the model structure obtained from Theorem 2.9. □

Corollary 7.25. *If $\mathscr{M}$ is the category of pointed simplicial sets, our model structure on the category $\mathcal{S}\mathrm{p}_\Omega(\mathbb{N}, \mathbf{sSet}_*)$ coincides with the stable model structure of Bousfield-Friedlander [24], Hovey [45], Schwede [84].*

Proof. Indeed we are localizing the same original model structure and we have the same fibrant objects as Hovey. It turns out that the new weak equivalences are also the same since we have the same original model structure, thus the same function complexes. □

Remark 7.26. It remains to compare the previous theorems when we consider the various model structure on $\mathrm{Hom}(\mathbb{N}_{\mathrm{disc}}, \mathscr{M}_{\mathcal{U}}[\mathscr{M}])$ and more generally $\mathrm{Hom}(\mathbb{O}_{\mathrm{disc}}, \mathscr{M}_{\mathcal{U}}[\mathscr{M}])$. We also need to extend these results when $\mathscr{M}$ is cellular and left proper. This will be done later. There is much that is left to be done on the subject as well as for the symmetric case.

$\mathcal{U}$-chain complexes We close this paper with the definition of $\mathcal{U}$-chain complexes. They will be studied in the subsequent papers. We assume that $\mathscr{M}$ has a zero object and let $\mathcal{U} : \mathscr{M} \longrightarrow \mathscr{M}$ be a right adjoint which is not necessarily a Quillen functor.

Definition 7.27. Let $\mathcal{LS}_{\mathcal{U}}(\mathbb{Z}, \mathscr{M})$ be the category of linked $\mathbb{Z}$-sequences as before. Say that $[\mathcal{X}_\bullet, 1] \in \mathcal{LS}_{\mathcal{U}}(\mathbb{Z}, \mathscr{M})$ is a *$\mathcal{U}$-chain complex* if for every n the following composite is the zero map:

$$\mathcal{X}_n^0 \xrightarrow{\varepsilon} \mathcal{U}(\mathcal{X}_{n+1}^0) \xrightarrow{\mathcal{U}(\varepsilon)} \mathcal{U}^2(\mathcal{X}_{n+2}^0).$$

Denote by $\mathcal{C}\mathrm{h}_{\mathcal{U}}(\mathbb{Z}, \mathscr{M})$ the full subcategory of $\mathcal{LS}_{\mathcal{U}}(\mathbb{Z}, \mathscr{M})$ spanned by $\mathcal{U}$-chain complexes. We have a forgetful functor $P : \mathcal{C}\mathrm{h}_{\mathcal{U}}(\mathbb{Z}, \mathscr{M}) \longrightarrow \mathrm{Hom}(\mathbb{Z}_{\mathrm{disc}}, \mathscr{M}_{\mathcal{U}}[\mathscr{M}])$

Obviously if $\mathscr{M}$ is an abelian category and $\mathcal{U} = \mathrm{Id}$ we find here the classical definition of chain complexes. We will develop the homotopy theory of them following the classical case. Moreover, note that these constructions can be made for any $\mathbb{O} \subseteq \mathbb{Z}$.

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