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Hochschild cohomology of filtered dg algebras

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Abstract

In this article we extend the main result in [15] to the case where the algebra is not necessarily nonnegatively graded connected. More precisely, we show that, for a nonnegatively filtered connected dg algebra A, it is possible to compute the cup product of the Hochschild cohomology of A at the level of the complex $\mathrm{Hom}_{A^e}(P_\bullet, A)$, where $P_\bullet$ is a semifree resolution of the dg A-bimodule A by making use of the coaugmented curved A_∞-coalgebra structure of a suitable Koszul codual of A, *i.e.* a coaugmented curved A_∞-coalgebra C that is filtered quasi-equivalent to the curved bar construction of A. We do not need to construct any comparison map between $P_\bullet$ and the Hochschild resolution of A, or any lift $\Delta : P \to P \otimes_A P$ of the identity of A.

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1. Introduction

The computation of the algebraic structure of the Hochschild cohomology $HH^\bullet(A)$ of an algebra A is usually a hard task. In particular, the general tools used to compute the cup product usually rely on constructing comparison maps between a "small" projective resolution $P_\bullet$ of the A-bimodule A used to effectively compute A and the Hochschild resolution of A, or a lift $\Delta : P \to P \otimes_A P$ of the identity of A. This involves in general rather tedious (and noncanonical) computations dealing with the previous resolutions. We have however proved in [15] that some of them can be completely circumvented in several situations. To wit, if A is a nonnegatively graded connected algebra, the cup product of $HH^\bullet(A)$ at the level of $\mathrm{Hom}_{A^e}(P_\bullet, A)$ can be directly read from the A_∞-coalgebra structure on $\mathrm{Tor}^A_\bullet(k, k)$. The latter is well-known for instance if A is a (generalized) Koszul algebra (or even a multi-Koszul algebra), which shows that some of the computations in [8] and [25] can be avoided. Let us remark that we do not intend that the calculations in the previous articles are not useful, but that some of them can be replaced by

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homotopical information computed at the level of the Yoneda algebra, if the latter is available (which is the case in several situations).

The goal of this article is to extend the results in [15] to (a large class of) nonaugmented dg algebras, for which $\mathrm{Tor}^A_\bullet(k,k)$ does not even make sense. The key ingredient is the use of curved A_∞-coalgebras that are Koszul codual to the given dg algebra (see [22]). More precisely, the main result of this article is the following (see Theorem 3.2).

Theorem. *Let A be a nonnegatively filtered connected (unitary) dg algebra over a field k. Let C be a filtered coaugmented curved A_∞-coalgebra which is filtered quasi-equivalent to the curved bar construction $B^+(A)$ of A and let τ in $\mathcal{H}om(C,A)$ be the associated twisting cochain. Then, there is a quasi-isomorphism of strictly (unitary) A_∞-algebras (preserving the Adams filtrations) between the (unitary) dg algebra $C^\bullet(A,A)$ computing the Hochschild cohomology of A and $\mathcal{H}om^\tau(C,A)$, which in particular induces an isomorphism of (unitary) graded algebras between $HH^\bullet(A)$ and $H^\bullet(\mathcal{H}om^\tau(C,A))$.*

This also extends the main result of [21]. Moreover, we remark that there is an analogous result for the module structure (over the Hochschild cohomology $HH^\bullet(A)$) of the Hochschild (co)homology of A with coefficients in a dg bimodule M, but since it seems to be less interesting, we have refrained from explicitly stating it.

The proof provided here of the previous theorem –different from the one in [15], Thm. 4.3– is mainly based on homological algebra, following the steps of the main theorem in [21], but taking into account that the objects we deal with are generalizations of those constructed in [15]. More precisely, once the required machinery of (curved) A_∞-(co)algebras is established, the proof of the theorem essentially lies in two main facts in (homological) algebra: any (A_∞-)module N over an (A_∞-)algebra $\mathcal{H}$ is equivalently described by the structure morphism of (A_∞-)algebras $\mathcal{H} \to \mathcal{E}nd(M)$, and for any semifree dg Λ-module K, $\mathcal{H}om_\Lambda(K,-)$ preserves quasi-isomorphisms. Indeed, since $N = A \otimes_\tau C$ is an A_∞-bimodule over $\mathcal{H} = \mathcal{H}om^\tau(C,A)$, and $K = A^e \otimes_\tau C \to A$ is a semifree resolution over $\Lambda = A^e$, we get that $C^\bullet(A,A) \simeq \mathcal{H}om_\Lambda(K,K) \to \mathcal{H}om_\Lambda(K,A) \simeq \mathcal{H}$ is a quasi-isomorphism, which is clearly a left inverse of the structure map of N. The reason for avoiding the steps in [15] is simple. Even though the results in the latter article are somehow optimal in our opinion, for we mainly produced a comparison map between the bar and the minimal A^e-resolutions of A from a quasi-isomorphism of A_∞-algebras between $B^+(A)$ and its minimal model, they essentially rely in the functoriality of the several steps of the constructions. As noted in Remark 2.3, this is already not the case if curvature is involved, without even mentioning the quite problematic category of curved A_∞-algebras. The objects considered in [15] do nonetheless make perfect sense in presence of curvature.

The contents of the article are as follows. In Section 2 we present the basic definitions required to prove the main theorem, namely curved A_∞-algebras and curved A_∞-bimodules (see Subsection 2.1), two basic constructions of those objects (see Subsection 2.2), the basics on twisting cochains (see Subsection 2.3), and the twisted version of the previous constructions (see Subsection 2.4). These sections can be regarded as a natural extension of [15], Sections 2 and 3, on which they heavily rely. We then proceed to prove the main result in Section 3. Finally, we present two applications in Section 4, computing the cup product of the Hochschild cohomology of any PBW deformation of a 3-Koszul algebra of global dimension 3, and of the algebra $k[X]/\langle f\rangle$, where $f \in k[X]$ is any polynomial in one indeterminate. The last one was already computed in [17], using comparison morphisms between the bar resolution of A and a suitable periodic resolution. In our case, the computations follow easily from our main result and the fact that the corresponding filtered coaugmented curved A_∞-coalgebra structure of a (small) Koszul codual of A can be easily calculated in both cases by means of [16], Thm. 6.2.

2. Some definitions

In what follows, k will denote a field. For the fundamentals on curved A_∞-(co)algebras we refer the reader to the nice exposition [22] (see also [23]). However, for A_∞-(co)algebras we shall use the conventions and terminology given in [15], Subsection 2.1, whereas for curved A_∞-coalgebras, we will use those in [16], Subsection 2.1. Moreover, the filtered versions of the previous objects are also clearly defined (see for instance [16], Subsection 5.1). We furthermore remark that all the definitions and results in [15], Sections 2 and 3, for (resp., topological) augmented A_∞-algebras and their (resp., topological) A_∞-bimodules, extend to the case with curvature with precisely the same proofs (except for those involving the notion of quasi-isomorphism, which does not make sense, or those indicated in Remark 2.3). We will provide in this section a very brief presentation of them for the reader's convenience, but we advise checking the mentioned reference.

We will denote by $\mathbb{N}$ the set of (strictly) positive integers, whereas $\mathbb{N}_0$ will be the set of nonnegative integers. We also recall that, if $V = \oplus_{n\in\mathbb{Z}} V^n$ is a (cohomological) graded vector space, we denote by $|v|$ (or $\deg v$) the degree of a nonzero homogeneous element $v \in V^{|v|}$. Moreover, $V[m]$ is the graded vector space over k whose n-th homogeneous component $V[m]^n$ is given by V^{m+n}, for all $n, m \in \mathbb{Z}$, and it is called the *shift* of V. We will denote by $s_V : V \to V[1]$ the *suspension morphism*, whose underlying map is the identity of V. All morphisms between modules will be k-linear (satisfying further requirements if the modules are decorated). All unadorned tensor products $\otimes$ will be over k.

2.1 Basics on curved A_∞-algebras and their curved A_∞-bimodules

The following definition is due to E. Getzler and J. Jones in [11], and it generalizes the notion of A_∞-algebra introduced by J. Stasheff.

A *nonunitary curved A_∞-algebra* A is a coderivation B_A of cohomological degree 1 on the counitary graded tensor algebra $T(A[1]) = \oplus_{n\in\mathbb{N}_0} A[1]^{\otimes n}$ provided with the deconcatenation coproduct, such that $B_A \circ B_A = 0$.[1] The previous counitary dg coalgebra is called the *(nonunitary curved) bar construction* of A and is typically denoted by $B_{nc}(A)$. If $n \in \mathbb{N}$ we will typically denote an element $s(a_1) \otimes \cdots \otimes s(a_n) \in A[1]^{\otimes n}$ by $[a_1|\ldots|a_n]$, where $a_1, \ldots, a_n \in A$, and $s = s_A : A \to A[1]$ is the suspension on A.

As proved in [11], Prop. 1.2, there is a linear bijection between the vector space of coderivations of $T(A[1])$ and the space of linear maps from $T(A[1])$ to $A[1]$. The map is given by sending a coderivation B to $\pi_{A[1]} \circ B$, where $\pi_{A[1]} : T(A[1]) \to A[1]$ is the canonical projection. Hence, B_A is uniquely determined by $\pi_{A[1]} \circ B_A = \sum_{i\in\mathbb{N}_0} b_i$ for $b_i : A[1]^{\otimes i} \to A[1]$. Set $m_i : A^{\otimes i} \to A$ by means of $m_i = -(s_A^{\otimes i})^{-1} \circ b_i \circ s_A$. Then, the collection of maps $m_i : A^{\otimes i} \to A$ for $i \in \mathbb{N}_0$, where m_i is homogeneous of cohomological degree $2 - i$, satisfies the following identities

$$\sum_{(r,s,t)\in\mathcal{I}_n} (-1)^{r+st} m_{r+1+t} \circ (\mathrm{id}_A^{\otimes r} \otimes m_s \otimes \mathrm{id}_A^{\otimes t}) = 0, \qquad (\mathrm{SI}(n))$$

for $n \in \mathbb{N}_0$, where $\mathcal{I}_n = \{(r, s, t) \in \mathbb{N}_0^3 : r + s + t = n\}$. Reciprocally, starting from a collection of maps $m_i : A^{\otimes i} \to A$ fulfilling the previous properties we obtain a nonunitary curved A_∞-algebra

[1] Notice the difference with the usual *nonunitary A_∞-algebra*, which further satisfies that $B_A|_k = 0$.

structure. A *nonunitary curved dg algebra* is a nonunitary curved A_∞-algebra such that $m_i = 0$ for all $i \geq 3$.

Even though we shall not use the following notion, specially for the consequences it entails (see Remark 2.1), let us precise what the typical definition of morphism between curved A_∞-algebras is. Given two nonunitary curved A_∞-algebras $(A, m_\bullet^A)$ and $(A', m_\bullet^{A'})$, a *morphism of nonunitary curved A_∞-algebras* from A to A' is a morphism of counitary dg coalgebras from $B_{nc}(A)$ to $B_{nc}(A')$. It is not hard to show that any tensor coalgebra $T(V) = \oplus_{n\in\mathbb{N}_0} V^{\otimes n}$ has a unique group-like element, given by $1_k \in k = V^{\otimes 0}$. Since any morphism of coalgebras preserves group-likes, this implies that any morphism G of counitary coalgebras from a tensor coalgebra $T(V)$ to a tensor coalgebra $T(W)$ sends 1_k to 1_k. Moreover, it is also easy to show that any morphism G of counitary coalgebras from $T(V)$ to $T(W)$ sends $\bar{T}(V) = \oplus_{n\in\mathbb{N}} V^{\otimes n}$ to $\bar{T}(W) = \oplus_{n\in\mathbb{N}} W^{\otimes n}$, so in particular it is uniquely determined by the morphism $\bar{G}$ it induces between the noncounitary graded coalgebras $\bar{T}(V)$ to $\bar{T}(W)$. By [19], Lemme 1.1.2.2, such a morphism is uniquely determined by its composition with the canonical projection $\pi_W : \bar{T}(W) \to W$. As a consequence, any morphism $\hat{F}$ of dg coalgebras from $B_{nc}(A)$ to $B_{nc}(A')$ is uniquely determined by the map $\pi_{A'[1]} \circ \hat{F}|_{\bar{T}(A[1])} = \sum_{i\in\mathbb{N}} F_i$, where $F_i : A[1]^{\otimes i} \to A'[1]$.[2] Set $f_i : A^{\otimes i} \to A'$ the homogeneous map of cohomological degree $1 - i$ by means of $F_i = s_B \circ f_i \circ (s_A^{\otimes i})^{-1}$, for $i \in \mathbb{N}$. Then, the fact that $\hat{F}$ is a morphism of counitary dg coalgebras means precisely that $\{f_i\}_{i\in\mathbb{N}}$ satisfies that $f_1(m_0^A) = m_0^{A'}$, and

$$\sum_{(r,s,t)\in\mathcal{I}'_n} (-1)^{r+st} f_{r+1+t} \circ (\mathrm{id}_A^{\otimes r} \otimes m_s^A \otimes \mathrm{id}_A^{\otimes t}) = \sum_{q\in\mathbb{N}} \sum_{\bar{i}\in\mathbb{N}^{q,n}} (-1)^w m_q^{A'} \circ (f_{i_1} \otimes \cdots \otimes f_{i_q}), \quad (\mathrm{MI}(n))$$

for $n \in \mathbb{N}$, where $\mathcal{I}'_n = \{(r, s, t) \in \mathbb{N}_0 \times \mathbb{N} \times \mathbb{N}_0 : r + s + t = n\}$, $w = \sum_{j=1}^{q} (q - j)(i_j - 1)$ and $\mathbb{N}^{q,n}$ is the subset of $\mathbb{N}^q$ of elements $\bar{i} = (i_1, \dots, i_q)$ such that $|\bar{i}| = i_1 + \cdots + i_q = n$. Note that (MI($n$)) is exactly the same condition as the one satisfied by morphisms of (noncurved) nonunitary A_∞-algebras. Moreover, the notions of *strict morphism*, *identity* and *composition* are also precisely the same as those in the case of (noncurved) nonunitary A_∞-algebras, as well as the notion of *(strict) unit* (see [20], Section 4).

Remark 2.1. In case that A and A' are nonunitary curved dg algebras, the previous notion of morphism does not include the classical definition of morphism of nonunitary curved dg algebras (see [22], Section 3.1), for the *change-of-connection elements* are lacking when one deals with the generality of curved A_∞-algebras. Unfortunately, there is no straightforward way to overcome this issue, and it seems to lie in the heart of the theory. This has rather striking consequences. Indeed, it is not hard to show that, given any two curved A_∞-algebras A and A' with nonzero curvature terms m_0^A and $m_0^{A'}$, and given any homogeneous linear map $f : A \to B$ of degree zero satisfying $f(m_0^A) = m_0^{A'}$, there exists an isomorphism $f_\bullet$ of nonunitary curved A_∞-algebras from A to B such that $f_1 = f$ (*cf.* the Remark in [22], Subsection 7.3).

For the following definitions we refer to [1], Section 2.6, even though we present a different sign convention. Given a curved A_∞-algebra A, a *curved A_∞-bimodule over* A is a graded vector space M and a bicoderivation B_M on the graded counitary bicomodule $B_{nc}(A) \otimes M[1] \otimes B_{nc}(A)$ over $B_{nc}(A)$ such that $B_M \circ B_M = 0$. We shall denote the previous bicomodule by $B_{nc}(A, M, A)$. Since $B_{nc}(A) \otimes M[1] \otimes B_{nc}(A)$ is a cofree graded bicomodule, a bicoderivation is uniquely determined

[2] Note that we have avoided to use [11], Lemma 1.6, which is unfortunately false. Indeed, the proposed inverse is not well-defined, since the family $\{\Delta_n\}$ in that result is not locally finite.

by its composition with $\epsilon_{B_{nc}(A)} \otimes \mathrm{id}_{M[1]} \otimes \epsilon_{B_{nc}(A)}$, which is a sum of mappings of the form $b_{p,q} : A[1]^{\otimes p} \otimes M[1] \otimes A[1]^{\otimes q} \to M[1]$, for $p, q \in \mathbb{N}_0$. Define $m^M_{p,q} : A^{\otimes p} \otimes M \otimes A^{\otimes q} \to M$ as $-s_M^{-1} \circ b_{p,q} \circ (s_A^{\otimes p} \otimes s_M \otimes s_A^{\otimes q})$. Then, the collection of maps $m^M_{p,q} : A^{\otimes p} \otimes M \otimes A^{\otimes q} \to M$ for $p, q \in \mathbb{N}_0$, where $m^M_{p,q}$ is of homogeneous of cohomological degree $1-p-q$, satisfies the following identities

$$\sum_{(r,s,t)\in\mathcal{I}_{n'+n''+1}} (-1)^{r+st} \tilde{m}^M_{r,t} \circ (\mathrm{id}^{\otimes r} \otimes \tilde{m}_s \otimes \mathrm{id}^{\otimes t}) = 0 \qquad (\mathrm{BI}(n', n''))$$

in $\mathcal{H}om(A^{\otimes n'} \otimes M \otimes A^{\otimes n''}, M)$ for all $n', n'' \in \mathbb{N}_0$, where we recall that $\mathcal{I}_n = \{(r,s,t) \in \mathbb{N}_0^3 : r+s+t = n\}$, and where $\tilde{m}_s$ is interpreted as the corresponding multiplication map m_s of A if either $r+s \leq n'$ or $s+t \leq n''$, and it is understood as $m^M_{n'-r,n''-t}$ else. In the first case, $\tilde{m}^M_{r,t}$ is $m^M_{n'-s+1,n''}$ if $r+s \leq n'$ or $m^M_{n',n''-s+1}$ if $s+t \leq n''$, and it is $m^M_{r,t}$ else. Moreover, $\mathrm{id}^{\otimes r}$ is $\mathrm{id}_A^{\otimes r}$ and $\mathrm{id}^{\otimes t}$ is $\mathrm{id}_A^{\otimes(n'-r-s)} \otimes \mathrm{id}_M \otimes \mathrm{id}_A^{\otimes n''}$ if $r+s \leq n'$; $\mathrm{id}^{\otimes r}$ is $\mathrm{id}_A^{\otimes n'} \otimes \mathrm{id}_M \otimes \mathrm{id}_A^{\otimes(n''-s-t)}$ and $\mathrm{id}^{\otimes t}$ is $\mathrm{id}_A^{\otimes t}$ if $s+t \leq n''$; and $\mathrm{id}^{\otimes r}$ is $\mathrm{id}_A^{\otimes r}$ and $\mathrm{id}^{\otimes t}$ is $\mathrm{id}_A^{\otimes t}$ else. Reciprocally, given any collection of maps $m^M_{p,q} : A^{\otimes p} \otimes M \otimes A^{\otimes q} \to M$ fulfilling the previous properties, it defines a curved A_∞-bimodule structure on M over A. Note that a curved A_∞-algebra is also a curved A_∞-bimodule for the structure maps $m_{p,q} = m_{p+q+1}$, where $p, q \in \mathbb{N}_0$. If A is (strictly) unitary, one further imposes the same conditions as in the case of zero curvature (see [15], Section 2.2). Moreover, if A is a (unitary) curved dg algebra, then a *curved dg A-bimodule* is a curved A_∞-bimodule M satisfying that $m^M_{p,q} = 0$ for all $p, q \in \mathbb{N}_0$ such that $p+q > 1$.

Remark 2.2. As recalled in Remark 2.1 for the case of curved A_∞-algebras, the notion of morphism of curved A_∞-bimodules (which is the same as in the case of zero curvature, see [19], Ch. 2, Section 5) is problematic, for it turns the theory trivial from the point of view of homological algebra. Indeed, any object of the dg category of curved A_∞-bimodules over a curved A_∞-algebra with nonzero curvature is contractible (see the Remark in [22], Subsection 7.3).

Shift and duals of curved A_∞-bimodules are defined exactly in the same way as in the noncurved case (recalled in [14], Section 1.5). For instance, if M is a curved A_∞-bimodule M over a curved A_∞-algebra A, the *(graded) dual A_∞-bimodule* $M^\#$ has as underlying graded space the usual graded dual $M^\#$ of M, together with the multiplication maps ${}^\# m_{p,q} : A^{\otimes p} \otimes M^\# \otimes A^{\otimes q} \to M^\#$ defined by

$${}^\# m_{p,q}(a_1, \dots, a_p, \lambda, a'_1, \dots, a'_q)(m) = -(-1)^{\sigma'} \lambda\big(m_{q,p}(a'_1, \dots, a'_q, m, a_1, \dots, a_p)\big), \qquad (1)$$

where

$$\sigma' = (1+p+q)\deg\lambda + p + q + pq + \Big(\sum_{j=1}^{p} \deg a_j\Big)\Big(\deg m + \deg\lambda + \sum_{i=1}^{q} \deg a'_i\Big),$$

for all homogeneous $m \in M$, $\lambda \in M^\#$ and $a_1, \dots, a_p, a'_1, \dots, a'_q \in A$. In the same way as in the proof in the noncurved case given in [14], Section 1.5, it is long but straightforward to check that this indeed defines a curved A_∞-bimodule structure on $M^\#$.

2.2 Two constructions If $(C, \Delta^C_\bullet, \epsilon_C)$ is a (strictly) counitary curved A_∞-coalgebra and $(A, \mu_A, \eta_A, d_A, h_A)$ is a unitary curved dg algebra, the dg vector space $\mathcal{H} = \mathcal{H}om(C, A)$ has the structure of a (strictly) unitary curved A_∞-algebra, which we call the *convolution curved A_∞-algebra*, where $m_0^{\mathcal{H}} = h_A\epsilon_C + \eta_A \circ \Delta_0^C$, $m_1^{\mathcal{H}}$ is given by the usual differential $d_A \circ \phi - (-1)^{|\phi|}\phi \circ \Delta_1^C$,

$$m_n^{\mathcal{H}}(\phi_1 \otimes \cdots \otimes \phi_n) = (-1)^{n(|\phi_1|+\cdots+|\phi_n|+1)}\mu_A^{(n)} \circ (\phi_1 \otimes \cdots \otimes \phi_n) \circ \Delta_n^C \tag{2}$$

for $n \geq 2$, and $1_{\mathcal{H}om(C,A)} = \eta_A \circ \epsilon_C$. We leave to the reader the tedious but elementary verification that this gives a (strictly) unitary curved A_∞-algebra. In particular, the previous construction defines a (strictly) unitary curved A_∞-algebra structure on the graded dual $C^{\#}$ of C.

Remark 2.3. Notice however that $\mathcal{H}om(C, A)$ is not functorial in C, since a morphism $C \to D$ of (strictly) counitary curved A_∞-coalgebras does not induce in general a morphism of (strictly) unitary A_∞-algebras from $\mathcal{H}om(D, A)$ to $\mathcal{H}om(C, A)$. This is in stark contrast to the case of (noncurved) A_∞-coalgebras (see [15], Subsection 2.3). The reason for this apparent imbalance between curved A_∞-algebras and curved A_∞-coalgebras is due to the fact that morphisms of both theories are quite different (*cf.* Subsection 2.1 and [16], Subsection 2.1).

If M is a curved dg A-bimodule over a unitary curved dg algebra A and C is a counitary curved A_∞-coalgebra, then it is long but straightforward to verify that $\mathcal{H}om(C, M)$ is a curved A_∞-bimodule over $\mathcal{H}om(C, A)$ via $m_{0,0}^{\mathcal{H}om(C,M)}(\omega) = d_M \circ \omega - (-1)^{|\omega|}\omega \circ \Delta_1^C$, and

$$\begin{aligned} &m_{p,q}^{\mathcal{H}om(C,M)}(\phi_1, \ldots, \phi_p, \omega, \phi_{p+1}, \ldots, \phi_{p+q}) \\ &\quad = (-1)^w m_M^{p,q} \circ (\phi_1 \otimes \cdots \otimes \phi_p \otimes \omega \otimes \phi_{p+1} \otimes \cdots \otimes \phi_{p+q}) \circ \Delta_{p+q+1}^C, \end{aligned} \tag{3}$$

for all $p, q \in \mathbb{N}_0$ such that $p+q \geq 1$, where $w = (p+q+1)(1+|\omega|+\sum_{i=1}^{p+q}|\phi_i|)$, $m_M^{p,q} : A^{\otimes p} \otimes M \otimes A^{\otimes q} \to M$ denotes the successive application of the action of A on M, $\phi_1, \ldots, \phi_{p+q} \in \mathcal{H}om(C, A)$ and $\omega \in \mathcal{H}om(C, M)$.

Moreover, $M \otimes C$ is a curved A_∞-bimodule over $\mathcal{H}om(C, A)$ with the structure morphisms given by $m_{0,0}^{M\otimes C} = d_M \otimes \mathrm{id}_C + \mathrm{id}_M \otimes \Delta_1^C$, and, for $p + q \geq 1$,

$$\begin{aligned} &m_{p,q}^{M\otimes C}(\phi_1 \otimes \cdots \otimes \phi_p \otimes (m \otimes c) \otimes \psi_1 \otimes \cdots \otimes \psi_q) \\ &\quad = (-1)^{\epsilon'}\big(\phi_1(c_{(q+2)}) \ldots \phi_p(c_{(q+p+1)})\big).m.\big(\psi_1(c_{(1)}) \ldots \psi_q(c_{(q)})\big) \otimes c_{(q+1)}, \end{aligned} \tag{4}$$

where $\Delta_{p+q+1}^C(c) = c_{(1)} \otimes \cdots \otimes c_{(p+q+1)}$, and

$$\begin{aligned} \epsilon' = pq + |c||m| + (p+q+1)\Big(\sum_{i=1}^{p}|\phi_i| + \sum_{j=1}^{q}|\psi_j|\Big) + \sum_{\substack{1 \leq i \leq p \\ q+2 \leq i' \leq q+i}} |c_{(i')}||\phi_i| \\ + \sum_{\substack{1 \leq j \leq q \\ 1 \leq j' < j}} |c_{(j')}||\psi_j| + \Big(|m| + \sum_{i=1}^{p}|c_{(q+1+i)}| + \sum_{j=1}^{q}|\psi_j|\Big)\Big(\sum_{j=1}^{q+1}|c_{(j)}|\Big). \end{aligned} \tag{5}$$

It is rather long but direct to prove that it is indeed a curved A_∞-bimodule over $\mathcal{H}om(C, A)$.[3] Even more, the previous curved A_∞-bimodule structure on $M \otimes C$ over $\mathcal{H}om(C, A)$ is obtained

[3] There is an unintended typo in [15], eq. (6), where the third term in the right member is incomplete. The sign given in this article corrects it. This typo does not affect however the validity of the signs in other sections of [15], with exception of equation (18), where the first term of σ' should be $l'l''$ instead of l'', and the last displayed equation in Thm. 4.5, where it should be $n_1 + n_1(n_3 + n_5)$ instead of $n_1 + n_3 + n_5$. We also correct another sign typo in [15], Thm. 4.5, unrelated to the previous one, where the right member of the second displayed equation should be -1 instead of $-(-1)^s$.

as follows. Since M is a curved A_∞-bimodule over A, the same hods for $M^\#$, so $\mathcal{H}om(C, M^\#)$ is a curved A_∞-bimodule over $\mathcal{H}om(C, A)$, which in turn implies that $\mathcal{H}om(C, M^\#)^\#$ is too. Consider the canonical injection $i : M \otimes C \to \mathcal{H}om(C, M^\#)^\#$ sending $m \otimes c$ to the functional that sends $\mu \in \mathcal{H}om(C, M^\#)$ to $(-1)^w \mu(c)(m)$, where $w = |c||m| + |c||\mu| + |m||\mu|$. A long but straightforward computation shows that the image of i is a curved A_∞-subbimodule of $\mathcal{H}om(C, M^\#)^\#$ and one then shows that transporting the curved A_∞-bimodule structure on the image of i to $M \otimes C$ by means of i gives (4). This is precisely how we obtained (4) as well as the sign (5).

2.3 Twisting cochains We first remark that, as stated in [16], Subsection 5.1, all the definitions of curved A_∞-(co)algebras can be done for the category of graded vector spaces V further provided with increasing (nonnegative) filtrations $\{F^\bullet V\}_{\bullet \in \mathbb{N}_0}$ of graded vector subspaces and all the maps preserve the filtrations. We will assume that the filtrations are *exhaustive*, *i.e.* $\cup_{n \in \mathbb{N}_0} F^n V = V$. We will talk in that case of *filtered curved A_∞-(co)algebras*, *morphisms of filtered curved A_∞-(co)algebras*, etc. We recall that k is provided with the trivial filtration $F^n k = k$, for all $n \in \mathbb{N}_0$. In the case of a coaugmented curved A_∞-coalgebra C one further imposes $F^0 C$ to be the image of the coaugmentation η_C of C. As a consequence, $\mathrm{Gr}_{F^\bullet C}(C)$ has zero curvature, so it is in fact a coaugmented A_∞-coalgebra. Moreover, we recall that a morphism $f_\bullet : C \to C'$ of filtered coaugmented curved A_∞-coalgebras is called a *filtered quasi-equivalence* if the associated morphism $\mathrm{Gr}(f_\bullet)$ is a quasi-equivalence of coaugmented A_∞-coalgebras.

Let C be a (strictly) counitary curved A_∞-coalgebra and A be a unitary curved dg algebra. We recall that a *twisting cochain* from C to A is a linear map $\tau : C \to A$ of cohomological degree 1 such that $\tau \circ \eta_C$ vanishes and that it satisfies the *Maurer-Cartan equation*

$$d_A \circ \tau + \sum_{i \in \mathbb{N}_0} (-1)^{i(i+1)/2+1} \mu_A^{(i)} \circ \tau^{\otimes i} \circ \Delta_i = 0, \tag{6}$$

where $\mu_A^{(i)} : A^{\otimes i} \to A$ is the iterative application of the product of A if $i \geq 2$, the identity map of A if $i = 1$, and the unit η_A of A if $i = 0$. Note that the sum in (6) is well-defined by the local finiteness assumption on the higher comultiplications of C.

As explained in [22], Section 7.5 (and recalled in [16], Subsection 2.1), given a coaugmented curved A_∞-coalgebra C, there exists a unitary dg algebra associated to it, called the *cobar construction*, and denoted by $\Omega^+(C)$. Its underlying graded vector space is $T(J_C[-1])$, where J_C is the cokernel of the coaugmentation η_C of C. If $\mathrm{Tw}(C, A)$ denotes the set of twisting cochains from C to A, we have a canonical map

$$\mathrm{Hom}_{\text{u-dg-alg}}(\Omega^+(C), A) \to \mathrm{Tw}(C, A) \tag{7}$$

given by $g \mapsto g \circ \tau^C$, where $\tau^C : C \to \Omega^+(C)$ is the composition of the canonical projection $C \to J_C$, $s^{-1}_{J_C[-1]}$ and the canonical inclusion of $J_C[-1]$ inside $\Omega^+(C)$. It is clear that the map (7) is a bijection, and we will denote the image of a twisting cochain τ under its inverse map by F_τ. Indeed, since $\Omega^+(C)$ is a free tensor algebra, a unitary morphism of graded algebras $F : \Omega^+(C) \to A$ is uniquely determined by its restriction to $J_C[-1]$, and a verification shows that (6) is tantamount to the fact that F commutes with the differentials. Furthermore, by means of the previous morphism we can define the *composition twisting cochain* of a morphism of coaugmented curved A_∞-coalgebras $f_\bullet : C' \to C$ with a twisting cochain τ from C to A. Indeed, if $F_\tau \in \mathrm{Hom}_{\text{u-dg-alg}}(\Omega^+(C), A)$ is the morphism such that $F_\tau \circ \tau^C = \tau$, and $\Omega^+(f_\bullet)$ is the morphism of unitary dg algebras from $\Omega^+(C')$ to $\Omega^+(C)$, the composition twisting cochain $\tau \circ f_\bullet$ is defined as $F_\tau \circ \Omega^+(f_\bullet) \circ \tau^{C'}$.

2.4 Twisted convolution algebras and twisted tensor products Instead of dealing with the general theory of twists of topological curved A_∞-algebras and their topological curved A_∞-bimodules (*cf.* [15], Subsections 3.2–3.3, for the case of zero curvature), we will deal with the following restricted version. We refer the reader to [7], Chap. III, §2, n° 5, for the basic definitions on linear topologies (see also [15], Subsection 3.1).

The following result follows the same idea as the one given in [15], Prop. 3.6.

Lemma 2.4. *Let C be a filtered coaugmented curved A_∞-coalgebra with increasing filtration $\{F_iC\}_{i\in\mathbb{N}_0}$, A be a unitary dg algebra A and $\tau \in \mathrm{Tw}(C,A)$ be a twisting cochain. Let $\mathcal{H} = \mathcal{H}om(C,A)$ be the graded vector space with the topology $\{F^i\mathcal{H}\}_{i\in\mathbb{N}_0}$ such that $F^i\mathcal{H}$ is the subset of $\mathcal{H}$ formed by maps which vanish on the subspace $F_{i-1}C$ of C. Then $\mathcal{H}$ is a complete topological graded vector space. Define $m_n^\tau : \mathcal{H}^{\otimes n} \to \mathcal{H}$ for $n \in \mathbb{N}$ by*

$$\begin{aligned} &m_n^\tau(\phi_1 \otimes \cdots \otimes \phi_n) = \delta_{n,1} d_A \circ \phi_1 \\ &+ \sum_{\ell\in\mathbb{N}_0} \sum_{\bar{\ell}\in\mathbb{N}_0^{n+1,\ell}} (-1)^{w'_{\bar{\ell}}} \mu_A^{(n+\ell)} \circ (\tau^{\otimes \ell_1} \otimes \phi_1 \otimes \tau^{\otimes \ell_2} \otimes \cdots \otimes \tau^{\otimes \ell_n} \otimes \phi_n \otimes \tau^{\otimes \ell_{n+1}}) \circ \Delta_{n+\ell}^C, \end{aligned} \tag{8}$$

where $\mathbb{N}_0^{n+1,\ell}$ is the subset of $\mathbb{N}_0^{n+1}$ formed by all $\bar{\ell} = (\ell_1,\dots,\ell_{n+1})$ such that $|\bar{\ell}| = |\ell_1| + \cdots + |\ell_{n+1}| = \ell$, and $w'_{\bar{\ell}}$ is

$$(n+\ell)(|\phi_1| + \cdots + |\phi_n| + \ell + 1) + \frac{\ell(\ell+1)}{2} + \ell n + \sum_{j=2}^{n+1} \ell_j(|\phi_1| + \cdots + |\phi_{j-1}| + j - 1).$$

Then, $(\mathcal{H}, m_\bullet^\tau)$ is a (strictly) unitary A_∞-algebra with unit $1_{\mathcal{H}om(C,A)} = \eta_A \circ \epsilon_C$.

Proof. For the completeness of $\mathcal{H}$, see [15], Lemma 3.12. Note that (8) are well defined by the local finiteness assumption on the higher comultiplications of C. By a tedious but straightforward verification, the reader could verify that the maps (8) indeed satisfy the Stasheff identities, but we will recall a more conceptual proof based on [15], Section 3 (which is turn based on [19], Section 6.2).

Note that the maps (2) are contracting, *i.e.* $m_n(\phi_1,\dots,\phi_n) \in F^p\mathcal{H}$, if $\phi_i \in F^{p_i}\mathcal{H}$, for $i = 1,\dots,n$, where $p = p_1 + \cdots + p_n$. Hence, the differential $B_\mathcal{H}$ of $B_{nc}(\mathcal{H})$ induced by (2) is continuous, where $B_{nc}(\mathcal{H})$ has the induced topology. Consider the complete topological coaugmented graded coalgebra $C(\mathcal{H})$ generated by $\mathcal{H}$ (see [24], Thm. 3.1, or [15], Prop. 3.1), which contains the bar construction $B_{nc}(\mathcal{H})$ of $\mathcal{H}$ as a dense subspace. Since $B_\mathcal{H}$ is continuous, it extends to a unique continuous endomorphism $\hat{B}_\mathcal{H}$ of $C(\mathcal{H})$, which also satisfies the Leibniz property, since it holds on a dense subspace. Denote by $\hat{B}_{nc}(\mathcal{H})$ the complete topological coaugmented graded coalgebra $C(\mathcal{H})$ endowed with this coderivation.

Let $\tau : C \to A$ be any homogeneous map of degree 1 vanishing on the image of the coaugmentation of C, *i.e.* $\tau \in F^1\mathcal{H}$. This fact together with the previous contracting property of (2) implies that the sum in (8) is convergent for the topology of $\mathcal{H}$. Let $\xi_\tau : k \to \hat{B}_{nc}(\mathcal{H})$ be the unique morphism of topological graded counitary coalgebras satisfying that $\hat{\pi}_1 \circ \xi_\tau$ sends 1 to $-[\tau]$, where $\hat{\pi}_1 : \hat{B}_{nc}(\mathcal{H}) \to \mathcal{H}[1]$ is the canonical projection. We first note that τ satisfies the Maurer-Cartan equation (6) if and only if $\hat{B}_\mathcal{H} \circ \xi_\tau = 0$. Let $t_\tau : \hat{B}_{nc}(\mathcal{H}) \to \mathcal{H}[1]$ be the sum of $\hat{\pi}_1$ and the composition of the canonical projection $\epsilon_{\hat{B}_{nc}(\mathcal{H})} : \hat{B}_{nc}(\mathcal{H}) \to k$ together with t_τ. Then, there exists a unique morphism of topological counitary graded coalgebras $T_\tau : \hat{B}_{nc}(\mathcal{H}) \to \hat{B}_{nc}(\mathcal{H})$ such that $\hat{\pi}_1 \circ T_\tau = t_\tau$. Note that $T_\tau \circ \eta_{\hat{B}_{nc}(\mathcal{H})} = \xi_\tau$ and that $\hat{\pi}_1 \circ T_\tau \circ T_{-\tau} = \pi_1 \circ T_{-\tau} \circ T_\tau = \pi_1$. The last chain of identities and the universal property of $C(\mathcal{H})$ imply that T_τ is an isomorphism with inverse $T_{-\tau}$. The reader can verify that the higher multiplications on $\mathcal{H}$ obtained from the differential $T_\tau^{-1} \circ B_{\mathcal{H}} \circ T_\tau$ restricted to $T^c(\mathcal{H})$ are precisely those given by (8). The Stasheff identities for (8) are then a direct consequence of

$$(T_\tau^{-1} \circ B_{\mathcal{H}} \circ T_\tau) \circ (T_\tau^{-1} \circ B_{\mathcal{H}} \circ T_\tau) = T_\tau^{-1} \circ B_{\mathcal{H}} \circ B_{\mathcal{H}} \circ T_\tau = 0,$$

since $B_{\mathcal{H}} \circ B_{\mathcal{H}} = 0$.

Finally, it is clear that m_n^τ for $n \in \mathbb{N} \setminus \{2\}$ vanishes if one of the arguments is the unit $1_{\mathcal{H}om(C,A)} = \eta_A \circ \epsilon_C$, using the counitary condition satisfied by the maps Δ_n^C. The fact that $1_{\mathcal{H}om(C,A)}$ is a unit for m_2^τ is a clear verification. □

We call the previous (strictly) unitary A_∞-algebra the *twisted convolution A_∞-algebra*, and we denote it by $\mathcal{H}om^\tau(C, A)$.

Analogously as in the untwisted case given in (3), if M is a dg A-bimodule over a unitary dg algebra A and C is a (strictly) counitary curved A_∞-coalgebra, then $\mathcal{H}om(C, M)$ has a structure of A_∞-bimodule over $\mathcal{H}om^\tau(C, A)$, with the analogous expressions to (8). We will denote this A_∞-bimodule by $\mathcal{H}om^\tau(C, M)$.

Moreover, and also following the untwisted case given in (4), $M \otimes C$ has a canonical structure of A_∞-bimodule over $\mathcal{H}om^\tau(C, A)$, given by pulling back the structure on $(\mathcal{H}om^\tau(C, M^\#))^\#$ via the canonical injection

$$i : M \otimes C \to \big(\mathcal{H}om^\tau(C, M^\#)\big)^\#$$

recalled in Subsection 2.2. We will denote this A_∞-bimodule by $M \otimes_\tau C$. It clearly coincides with the A_∞-bimodule denoted in the same way in [15], for the case C has zero curvature.

3. The main result

Following [19], Déf. 2.2.1.3 and Prop. 2.2.4.1 (for the case of (co)augmented dg (coalgebras), [22], Section 6.5 (for the case of curved coaugmented dg coalgebras and dg algebras) and [21], Def. 6.1, we say that a twisting cochain $\tau : C \to A$ from a cocomplete coaugmented curved A_∞-coalgebra C to a unitary dg algebra A is *acyclic* if the morphism $\mu_A \otimes \epsilon_C : A^e \otimes_\tau C \to A$ to the standard dg A-bimodule A is a quasi-isomorphism, where we recall that a coaugmented curved A_∞-coalgebra C is *cocomplete* if the cobar construction $\Omega^+(C)$ is a cofibrant dg algebra with respect to the model structure constructed by V. Hinich. (see [22], Rk. 9.4).

We first recall one of the main theorems proved in [16] (see Thm. 5.2), which is in turn an extension of a result announced by B. Keller at the X ICRA, in Toronto, 2002, and it gives an equivalent condition to the existence of an acyclic filtered twisting cochain.

Theorem 3.1. *Let C be a filtered coaugmented curved A_∞-coalgebra and A be a nonnegatively (Adams) filtered unitary dg algebra over a field k, whose associated graded algebra $\mathrm{Gr}_{F^\bullet A}(A)$ is an Adams nonnegatively graded connected dg algebra. Then, the following are equivalent:*

(i) there is a filtered quasi-equivalence of coaugmented curved A_∞-coalgebras

$$F : C \to B^+(A); \tag{9}$$

(ii) there is a filtered twisting cochain $\tau : C \to A$ such that either one of the following equivalent conditions holds

(a) the associated graded of the filtered morphism of unitary dg algebras $F_\tau : \Omega^+(C) \to A$ is a quasi-isomorphism;

(b) the associated graded map of the filtered morphism $\mu_A \otimes \epsilon_C : A^e \otimes_\tau C \to A$ to the standard dg A-bimodule A is a quasi-isomorphism, where $\mu_A : A \otimes A \to A$ is the product of A.

If the hypothesis concerning the associated graded of a nonnegatively filtered unitary dg algebra A is satisfied, we will say it is a *nonnegatively (Adams) filtered connected dg algebra.*

We now present the main theorem of this article, which extends the main result of both [15] (for the Hochschild cohomology) and [21].

Theorem 3.2. *Let A be a nonnegatively filtered connected (unitary) dg algebra over a field k and let C be a filtered coaugmented curved A_∞-coalgebra which is filtered quasi-equivalent to the curved bar construction $B^+(A)$ of A. Take $\tau \in \mathcal{H}om(C, A)$ the corresponding twisting cochain, given by the previous theorem. Then, there is a quasi-isomorphism of (strictly) unitary A_∞-algebras (preserving the Adams filtrations) from the (unitary) dg algebra $C^\bullet(A, A)$ computing the Hochschild cohomology of A to $\mathcal{H}om^\tau(C, A)$, which in particular induces an isomorphism of (unitary) graded algebras from $HH^\bullet(A)$ to $H^\bullet(\mathcal{H}om^\tau(C, A))$.*

Proof. By Theorem 3.1, we can assume that $\mu_A \otimes \epsilon_C : A^e \otimes_\tau C \to A$ is a quasi-isomorphism, where $\mu_A : A \otimes A \to A$ is the product of A. Since A^e is a dg A^e-bimodule, and the definition of $A^e \otimes_\tau C$ only uses one of the two commuting dg A-bimodules structures of A^e, which we will call *inner*, the remaining dg A-bimodule structure, which we call *outer*, induces a structure of dg A-bimodule on $A^e \otimes_\tau C$, such that $\mu_A \otimes \epsilon_C$ is a morphism of dg A-bimodules. Moreover, it is clear that $K = A^e \otimes_\tau C$ is a semifree dg A-bimodule (see [4], Section 8.2, for the definition). Indeed, consider the filtration of K given by $F^n K = A^e \otimes_\tau F^n C$ (this is different from the filtration on K induced by the Adams filtrations on A and C). It is clearly a filtration of K by dg A-bimodules, and its associated graded is a dg A-bimodule of the form $A^e \otimes V$, where V is a dg vector space, and $A^e \otimes V$ is provided with the differential induced by that of A and V. We recall that, given any dg vector space (V, d), there is a closed isomorphism from V to a dg vector space of the form $Z \oplus T$, where Z has zero differential, and $T = T' \oplus T'[1]$, for some graded vector space T', such that the differential d_T of T vanishes on $T'[1]$ and $d_T|_{T'}$ is the suspension $s_{T'} : T' \to T'[1]$. As a consequence, $A^e \otimes V$ is a direct sum of shifts of A^e and of categorically free dg A^e-modules, so it is semifree (see [4], Sections 8.2 and 8.4, for the definitions, and Cor. 8.4.5). This implies that K is semifree (see [4], Cor. 8.2.4), as was to be shown, so $\mu_A \otimes \epsilon_C : A^e \otimes_\tau C \to A$ is a semifree resolution of A. A standard result in homological algebra yields in turn that the (unitary) dg algebra $C^\bullet(A, A)$ is quasi-isomorphic to the (unitary) dg algebra $\mathcal{E}nd_{A^e}(K)$, provided with the product given by composition and the usual differential, where the index indicates we are considering the dg A-bimodule structure on K induced by the outer action of A^e on A^e (see [13], Lemma 3.2). Since K is an A_∞-bimodule over $\mathcal{H}om^\tau(C, A)$, it is in particular a left A_∞-module over $\mathcal{H}om^\tau(C, A)$. The last statement is equivalent to the fact that there is a morphism of (strictly) unitary A_∞-algebras

$$f_\bullet : \mathcal{H}om^\tau(C, A) \to \mathcal{E}nd(K),$$

where the codomain is the (unitary) dg algebra with the product given by composition and the usual differential (see for instance [18], p. 15). Since the A_∞-bimodule structure of K over

$\mathcal{H}om^{\tau}(C,A)$ commutes with the dg A-bimodule structure induced by the outer action of A^e on A^e, we see that each of the components of the morphism $\{f_\bullet\}_{\bullet\in\mathbb{N}}$ factors though the inclusion

$$\mathcal{E}nd_{A^e}(K) \to \mathcal{E}nd(K),$$

of (unitary) dg algebras. As a consequence, we obtain a morphism of (strictly) unitary A_∞-algebras

$$\tilde{f}_\bullet : \mathcal{H}om^{\tau}(C,A) \to \mathcal{E}nd_{A^e}(K).$$

We claim that $\{\tilde{f}_\bullet\}_{\bullet\in\mathbb{N}}$ is a quasi-isomorphism of (strictly) unitary A_∞-algebras, *i.e.* $\tilde{f}_1$ is a quasi-isomorphism of complexes. In order to prove so, we note first that the underlying complex of $\mathcal{H}om^{\tau}(C,A)$ is isomorphic to $\mathcal{H}om_{A^e}(K,A)$. Indeed, any $\bar{\phi} \in \mathcal{H}om^{\tau}(C,A)$ extends to a unique A^e-linear map ϕ from K to A, and given any A^e-linear map ϕ from K to A, the induced map $\bar{\phi}(c) = \phi(1_A \otimes 1_A \otimes c)$ belongs to $\mathcal{H}om^{\tau}(C,A)$. These maps clearly commute with the differentials. On the other hand, since K is semifree, it is homotopically projective, *i.e.* the functor $\mathcal{H}om_{A^e}(K,-)$ preserves quasi-isomorphisms (see [4], Lemma 9.3.5), so $\mathcal{H}om_{A^e}(K,\mu\otimes\epsilon_C)$ is a quasi-isomorphism. It is clear that the latter gives a left inverse of $\tilde{f}_1$, which in turn implies that $\tilde{f}_1$ is a quasi-isomorphism, as was to be shown. □

4. Some applications

4.1 The cup product of the Hochschild cohomology of a PBW deformation of a 3-Koszul algebra of global dimension 3 In this subsection we will apply Theorem 3.2 to compute the cup product of the Hochschild cohomology of any PBW-deformation A of a 3-Koszul algebra A' of global dimension 3. The reason for restricting to this case is because the coaugmented curved A_∞-coalgebra structure on $\mathrm{Tor}^{A'}_\bullet(k,k)$ that is filtered quasi-equivalent to the curved cobar construction of A can be explicitly determined from [16], Thm. 6.2, without any extra calculation. Even though this result can be used in many other situations, this restriction already covers several interesting examples of algebras, such as the PBW deformations A of any of the cubic algebras appearing in the classification by M. Artin and W. Schelter in [3] –which are clearly 3-Koszul by their Thm. 1.5– and that were considered in [10], Section 8, or the *inhomogeneous Yang-Mills algebras* (see [5]), which include the so-called *discrete minimal surface algebras* (see [2]). In particular, this also covers any enveloping algebra A of a Lie algebra $\mathfrak{g}$ of dimension 3, being a PBW deformation of a polynomial algebra $A' = k[x,y,z]$ on three variables. Another particular nice example, that was considered recently, is that of *general down-up algebras*, which can be regarded as PBW deformations of the corresponding *graded down-up algebras* (see [6], Example 3.8).

For the basics on generalized Koszul algebras as well as their PBW deformations and the notation we shall follow, we refer to [16], Sections 3 and 4, and the references therein. Let $A' = TV/\langle R\rangle$ be a 3-homogeneous algebra of global dimension 3 over a field k satisfying the generalized Koszul property, and let $C = \mathrm{Tor}^{A'}_\bullet(k,k)$, where $C_0 = k$, $C_1 = V$, $C_2 = R$ and $C_3 = (V\otimes R)\cap(R\otimes V)$. As recalled in [16], Section 3, C is provided with a structure of coaugmented A_∞-coalgebra, which is quasi-equivalent to the bar construction of A', having only two nonvanishing comultiplications, $\bar{\Delta}_2$ and $\bar{\Delta}_3$. To describe them, we will denote by 1 the unit element in $C_0 = k$. Moreover, given $\omega \in C_3$, we will write $\bar{\omega}_1 \otimes \omega_2$ its image under the canonical injection $C_3 \to V\otimes R$, where $\bar{\omega}_1 \in V$, $\omega_2 \in R$ and a sum is implicit. The element $\omega_1\otimes\bar{\omega}_2 \in R\otimes V$ is analogously defined. Then, the coaugmented A_∞-coalgebra on C is given by

(i) $\bar{\Delta}_2(1) = 1 \otimes 1$, $\bar{\Delta}_2(x) = 1 \otimes x + x \otimes 1$, for all $x \in C_1 \oplus C_2$, and $\bar{\Delta}_2(\omega) = 1 \otimes \omega + \omega \otimes 1 + \bar{\omega}_1 \otimes \omega_2 + \omega_1 \otimes \bar{\omega}_2$, for $\omega \in C_3$;

(ii) $\bar{\Delta}_3(x) = 0$, for all $x \in C_0 \oplus C_1 \oplus C_3$, and $\bar{\Delta}_3(r) \in C_1^{\otimes 3}$ is the image of $r \in R$ under the canonical inclusion $R \subseteq V^{\otimes 3}$.

The counit and coaugmentation of C are the canonical projection $C \to C_0 = k$ and the canonical inclusion $k = C_0 \to C$, respectively.

Example 4.1. We recall that, given $\alpha, \beta \in k$, the *graded down-up algebra* $A(\alpha, \beta, 0)$ is defined as $TV/\langle R\rangle$, with $V = k.u \oplus k.d$, and $R \subseteq V^{\otimes 3}$ is the vector space spanned by

$$r_u = du^2 - \alpha udu - \beta u^2 d \quad \text{and} \quad r_d = d^2u - \alpha dud - \beta ud^2,$$

where we have omitted the tensor products for simplicity. In this case, $C_3 = (V \otimes R) \cap (R \otimes V)$ is the vector space spanned by $\omega = r_d u - \beta r_u d = dr_u - \beta ur_d$. The previous result tells us for example that $\bar{\Delta}_2(\omega) = 1 \otimes \omega + \omega \otimes 1 + r_d \otimes u - \beta r_u \otimes d + d \otimes r_u - \beta u \otimes r_d$, and $\bar{\Delta}_3(r_u) = d \otimes u \otimes u - \alpha u \otimes d \otimes u - \beta u \otimes u \otimes d$. The reader can apply [15], Thm. 4.5, to the Hochschild cocycles obtained in [9] (where a basis of the Hochschild cohomology was computed for some particular values of the parameters $\alpha, \beta \in k$) and obtain in this way their cup product.

Let $A = TV/\langle P\rangle$ be a PBW deformation of $A' = TV/\langle R\rangle$, where $P = \{r - \varphi(r) : r \in R\}$, and $\varphi = \sum_{j=0}^{2} \varphi_j$, where $\varphi_j : R \to V^{\otimes j}$. A direct application of [16], Thm. 6.2, tells us that the curved bar construction of A is filtered quasi-equivalent to the following coaugmented curved A_∞-coalgebra structure on the graded vector space $C = \operatorname{Tor}_\bullet^{A'}(k, k)$. It has only the nontrivial comultiplications Δ_0, Δ_1, Δ_2 and Δ_3, that are given as follows:

$$\begin{aligned} \Delta_0 &= -\varphi_0 \circ \pi_2, & \Delta_1 &= (\mathrm{id}_V \otimes \varphi_2 - \varphi_2 \otimes \mathrm{id}_V) \circ \pi_3 - \varphi_1 \circ \pi_2, \\ \Delta_3 &= \bar{\Delta}_3, & \Delta_2 &= \bar{\Delta}_2 - \varphi_2 \circ \pi_2, \end{aligned} \tag{10}$$

where $\pi_i : C \to C_i$ is the canonical projection for $i \in \{2, 3\}$. The counit and coaugmentation are the same as before. We remark that we are not using the Koszul sign rule in the definition of Δ_1, since φ_2 is only regarded as a morphism of vector spaces.

Example 4.1 (continued)**.** We now recall that, given $\alpha, \beta, \gamma \in k$, the *general down-up algebra* $A(\alpha, \beta, \gamma)$ is defined as $TV/\langle P\rangle$, with $P = \{r - \varphi(r) : r \in R\}$ and $\varphi = \sum_{j=0}^{2} \varphi_j$, where $\varphi_j : R \to V^{\otimes j}$ is given by $\varphi_0 = \varphi_2 = 0$, and $\varphi_1(r_u) = \gamma u$ and $\varphi_1(r_d) = \gamma d$. This implies that the coaugmented curved A_∞-coalgebra structure on the graded vector space $C = \operatorname{Tor}_\bullet^{A'}(k, k)$ that is filtered quasi-equivalent to the curved bar construction of A differs from the coaugmented A_∞-coalgebra structure considered in Example 4.1 only by Δ_1, which vanishes on $C_0 \oplus C_1 \oplus C_3$, and sends r_c to $-\gamma c$, for $c \in \{u, d\}$.

Given any $r \in R$, let us write $\varphi_2(r) = r' \otimes r''$, with $r', r'' \in V$, and the sum is implicit, and $r = r_{(1)} \otimes r_{(2)} \otimes r_{(3)}$, for $r_{(i)} \in V$, $i = 1, 2, 3$, and the sum is also implicit. Then, we obtain the following result, which is a direct consequence of the previous discussion and Theorem 3.2.

Theorem 4.2. *Let $A = TV/\langle P\rangle$ be a PBW deformation of a 3-Koszul algebra $A' = TV/\langle R\rangle$ of global dimension 3, where $P = \{r - \varphi(r) : r \in R\}$, $\varphi = \sum_{j=0}^{2} \varphi_j$, and $\varphi_j : R \to V^{\otimes j}$. Let $C = \operatorname{Tor}_\bullet^{A'}(k, k)$ be the coaugmented curved A_∞-coalgebra that is filtered quasi-equivalent to the curved bar construction of A. By Theorem 3.2, the graded algebras $HH^\bullet(A)$ and $H^\bullet(\mathcal{H}om^\tau(C, A))$ are isomorphic, and the product $\phi_1 \cdot \phi_2$ of two cocycles $\phi_1 \in \mathcal{H}^{p_1} = \mathcal{H}om(C_{p_1}, A)$ and $\phi_2 \in \mathcal{H}^{p_2} = \mathcal{H}om(C_{p_2}, A)$ in the latter algebra with $p_1, p_2 \in \{0, \dots, 3\}$ is given by*

1. *if* $p_1 + p_2 > 3$, $\phi_1 \cdot \phi_2 = 0$;
2. *if* $(p_1, p_2) \neq (1, 1)$ *and* $p_1 + p_2 \leq 3$, *then either* p_1 *or* p_2 *is even, and* $\phi_1 \cdot \phi_2 \in \mathcal{H}^{p_1+p_2}$ *is*

$$(\phi_1 \cdot \phi_2)(c) = \phi_1(c_{(1)}).\phi_2(c_{(2)}),$$

for all $c \in C_{p_1+p_2}$, *where* $c = c_{(1)} \otimes c_{(2)}$, $c_{(1)} \in C_{p_1}$ *and* $c_{(2)} \in C_{p_2}$, *and we have omitted the sum in the expression of* c;

3. *if* $(p_1, p_2) = (1, 1)$, *then* $\phi_1 \cdot \phi_2 \in \mathcal{H}^2$ *is*

$$\begin{aligned}(\phi_1 \cdot \phi_2)(r) = &- \phi_1(r')\phi_2(r'') - r_{(1)}\phi_1(r_{(2)})\phi_2(r_{(3)})\\ &- \phi_1(r_{(1)})r_{(2)}\phi_2(r_{(3)}) - \phi_1(r_{(1)})\phi_2(r_{(2)})r_{(3)},\end{aligned}$$

for all $r \in R$, *where we have not written the products of* A *and we regard* $r_{(i)} \in V$ *inside of* A, *for all* $i = 1, 2, 3$.

Example 4.1 (continued)**.** If A is the general down-up algebra $A(\alpha, \beta, \gamma)$, then the first term in the right member of the product described in item 3 vanishes, since $\varphi_2 = 0$ in this case. Hence, the generic expression of the cup product of cocycles is the same as the one for the corresponding graded down-up algebra $A' = A(\alpha, \beta, 0)$, even though one must take into account that the product of A and that of A' are different.

4.2 The cup product of the Hochschild cohomology of $k[X]/\langle f\rangle$ In this subsection we shall compute the cup product of the Hochschild cohomology of the algebra given as the quotient of the polynomial ring in one indeterminate by a monic polynomial $f \in k[X]$ of the form $f = \sum_{j=0}^{N} a_j X^j$ for an integer $N \geq 2$, *i.e.* $a_N = 1$. Set $A = k[X]/\langle f\rangle$. The cup product of the Hochschild cohomology of A was computed for the first time in [17], using comparison morphisms between the bar resolution and a smaller resolution of A. We will show that our main result gives this structure as a rather easy consequence, without the need of any comparison map.

Note that A is a PBW deformation of the N-Koszul algebra $A' = k[X]/\langle X^N\rangle$, since eq. (3.8), (3.9) and (3.10) in [6], Prop. 3.6, are clearly verified, where $\varphi_j(X^N) = -a_j X^j$, for all $j \in \{0, \dots, N-1\}$. Moreover, it is clear that $C = \mathrm{Tor}^{A'}_{\bullet}(k, k)$ is given by $C_p = k.X^{\xi_N(p)}$, for all $p \in \mathbb{N}_0$, where $\xi_N(2m) = Nm$ and $\xi_N(2m+1) = Nm + 1$, for all $m \in \mathbb{N}_0$. Since A' is N-Koszul, the coaugmented A_∞-coalgebra on C that is quasi-equivalent to the bar construction of A' is given by

$$\begin{aligned}\bar{\Delta}_2(X^{Nk}) &= \sum_{j=0}^{k} X^{Nj} \otimes X^{N(k-j)},\\ \bar{\Delta}_2(X^{Nk+1}) &= \sum_{j=0}^{k} (X^{Nj+1} \otimes X^{N(k-j)} + X^{Nj} \otimes X^{N(k-j)+1}),\\ \bar{\Delta}_N(X^{Nk}) &= \sum_{\substack{(j_1,\dots,j_N) \in \mathbb{N}_0^N\\ j_1+\dots+j_N = k-1}} X^{Nj_1+1} \otimes \dots \otimes X^{Nj_N+1},\\ \bar{\Delta}_N(X^{Nk+1}) &= 0,\end{aligned} \tag{11}$$

for all $k \in \mathbb{N}_0$, and the other comultiplications vanish. The counit and coaugmentation are the same as in the previous subsection.

We leave to the reader the verification that the following coaugmented curved A_∞-coalgebra on $C = \mathrm{Tor}^{A'}_{\bullet}(k, k)$ is filtered quasi-equivalent to the curved bar construction of A. This is a consequence of the fact that a direct application of [16], Prop. 2.1, tells us that the alluded object is indeed a coaugmented curved A_∞-coalgebra, which further fulfills the hypothesis of [16], Thm. 6.2. This curved A_∞-coalgebra on C satisfies that $\Delta_j = 0$, for all $j > N$, Δ_N coincides with $\bar{\Delta}_N$ given in (11),

$$\begin{aligned} \Delta_p(X^{Nk}) &= \delta_{p,2} \sum_{j=0}^{k} X^{Nj} \otimes X^{N(k-j)} \\ &\quad + a_p \sum_{\substack{(j_1,\dots,j_p)\in\mathbb{N}_0^p \\ j_1+\dots+j_p=k-1}} X^{Nj_1+1} \otimes \dots \otimes X^{Nj_p+1}, \\ \Delta_p(X^{Nk+1}) &= \delta_{p,2} \sum_{j=0}^{k} (X^{Nj+1} \otimes X^{N(k-j)} + X^{Nj} \otimes X^{N(k-j)+1}), \end{aligned} \tag{12}$$

for all $k \in \mathbb{N}_0$ and $p \in \{1,\dots,N-1\}$, and $\Delta_0(X^{\xi_N(q)}) = \delta_{q,2} a_0$ for all $q \in \mathbb{N}_0$. The counit and coaugmentation are the same as before.

Note that $A^e \otimes_\tau C$ is isomorphic to the projective resolution of A obtained in [12], Prop. 1.3 (see also [17], Prop. 2.1), where τ is the twisting cochain given as the composition of the canonical projection $C \to V$ and minus the canonical inclusion $V \to A$. By Theorem 3.2, the graded algebras $HH^\bullet(A)$ and $H^\bullet(\mathcal{H})$ are isomorphic. A simple computation shows that the cohomology of $\mathcal{H} = \mathcal{H}om^\tau(C, A)$ is given by

(i) $\mathcal{H}^0 = \mathcal{H}om(k, A) \simeq H^0(\mathcal{H})$,

(ii) $H^{2k}(\mathcal{H}) \simeq \mathcal{H}om(C_{2k}, A)/\{\phi(X^{Nk}) = f'g, \text{ for some } g \in A\}$,

(iii) $\{\phi \in \mathcal{H}om(C_{2k-1}, A) : f'\phi(X^{Nk-N+1}) = 0\} \simeq H^{2k-1}(\mathcal{H})$,

for all $k \in \mathbb{N}$, where $f' \in A$ is the image under the canonical projection $k[X] \to A$ of the formal derivative of f in $k[X]$, the isomorphisms in (i) and (iii) are given by the inclusion, and the one in (ii) is induced by the canonical projection. This also follows from [17], Prop. 2.2, since $\mathcal{H}om^\tau(C, A)$ and $\mathcal{H}om_{A^e}(A^e \otimes_\tau C, A)$ are isomorphic complexes.

The explicit expression (8) for $n = 2$ tells us that the cup product of $\phi \in \mathcal{H}^p$ and $\phi \in \mathcal{H}^q$, with either p or q even, is of the form $\mu_A \circ (\phi \otimes \psi) \circ \Delta_2$, *i.e.*

$$(\phi \cdot \psi)(X^{\xi_N(m)}) = \delta_{m,p+q} \phi(X^{\xi_N(p)}) \psi(X^{\xi_N(q)}), \tag{13}$$

for all $m \in \mathbb{N}_0$. If both p and q are odd, then (8) tells us that $(\phi \cdot \psi)(X^{\xi_N(m)})$ is given by

$$-\delta_{m,p+q} \sum_{\ell=0}^{N-2} \Big(\sum_{k=1}^{\ell+1} k \Big) a_{\ell+2} X^\ell \phi(X^{\xi_N(p)}) \psi(X^{\xi_N(q)}), \tag{14}$$

for all $m \in \mathbb{N}_0$. Note that (13) and (14) coincide with the expressions in [17], Lemmas 3.1 and 4.1, up to sign.[4] Notice moreover that we have written $\sum_{k=1}^{\ell+1} k$ instead of $(\ell+1)(\ell+2)/2$, as in [17], for the equality between them only holds if the field k has characteristic different from 2.

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[4] The sign difference between our last displayed equation and [17], Lemma 4.1, is due to the fact that we have followed the Koszul sign rule in the definition of the cup product, which is not the case of [17].

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