IGHER
STRUCTURES

Oplax Hopf algebras

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Abstract

We introduce the notion of an oplax Hopf monoid in any braided monoidal bicategory, generalizing that of a Hopf monoid in a braided monoidal category in an appropriate way. We show that Hopf $\mathcal{V}$-categories introduced in [4] are a particular type of oplax Hopf monoids in the monoidal bicategory $\mathsf{Span}|\mathcal{V}$ described in [5]. Finally, we introduce Frobenius $\mathcal{V}$-categories as the Frobenius objects in the same monoidal bicategory.

1. Introduction

Over the last decades, a growing number of variations on the notion of Hopf algebra have surfaced, and this has led to increasing investigations on the intriguing question of how these various notions are related and can be unified in a single (categorical) framework.

Weak Hopf algebras [9] were introduced to describe extensions of Von Neumann algebras as crossed products (by an action of a weak Hopf C^*-algebra). A weak Hopf algebra is an algebra that also has a coalgebra structure, but their compatibility as well as the antipode properties are weakened compared to those of usual Hopf algebras. Weak Hopf algebras were one of the

inspiring examples to define *Hopf algebroids* [10], which in turn led to Hopf monads [13]. Just as Hopf monads live in the monoidal 2-category of categories, functors and natural transformations, Hopf-type objects can be defined in any monoidal bicategory, and they have been studied in this way, see eg. [20, 18, 39, 7]. In the latter, all previous mentioned Hopf-type objects (as well as a few more) are interpreted as particular opmonoidal monads in suitable monoidal bicategories, and different classical characterizations of Hopf algebras are recovered at an extremely high level of generality.

In a different direction, *multiplier Hopf algebras* [42] provide a generalization of Hopf algebras to the non-unital setting. Recently, the theory of weak and multiplier Hopf algebras have been merged [43, 6] and brought to the categorical setting [8].

An intermediate notion, called *Hopf category*, was introduced recently in [4], as a linearized version of groupoids. Just as a category can be viewed as a 'monoid with many objects', Hopf $\mathcal{V}$-categories are a many-object generalization (or a *categorification*) of Hopf algebras. They are defined as categories enriched over comonoids (in a braided monoidal category $\mathcal{V}$), that admit a suitable notion of antipode. Moreover, a Hopf category can be 'packed' by taking the coproduct of its hom-objects, leading to examples of weak (multiplier) Hopf algebras. In [5], Böhm showed that Hopf $\mathcal{V}$-categories fit in the framework of [7], namely they can be viewed as a particular type of opmonoidal monad in a suitably constructed monoidal bicategory denoted $\mathsf{Span}|\mathcal{V}$, where a braided monoidal category $\mathcal{V}$ is viewed as a 1-object monoidal bicategory.

For H either a classical Hopf algebra, a weak Hopf algebra, a Hopf algebroid or a Hopf category, there are two fundamental points for its bicategorical interpretation as described above: H is always a 1-cell in the considered bicategory, and moreover its underlying monoid and comonoid structures make use of different monoidal products (that together fit into a *duoidal* category structure).

In this work, we aim to provide a next step in this process of settling the notion of Hopf-type objects in higher categories, moving from monoidal bicategories i.e. a tricategory with a single 0-cell, to braided monoidal bicategories [23] i.e. a tetracategory with a single 0-cell and a single 1-cell, see [41, 27]. Within such a setting, we introduce the notion of an *oplax bimonoid* which consists of a 0-cell endowed with the structure of a pseudomonoid and pseudocomonoid, along with oplax compatibility constraints between them. In the spirit of Tannaka-Krein duality, we study the relation between monoidal structures on the category of lax pseudomodules over a pseudomonoid and oplax bimonoid structures on this pseudomonoid.

Particular attention should be paid to the correct notion of an antipode for an oplax Hopf algebra, which is more elaborate than what one might expect at first sight. It is no longer a convolution inverse of the identity, but an *oplax inverse*, a notion inspired by firm Morita contexts. However, we still retain the uniqueness of the antipode if it exists, and we discuss the relation between the existence of an antipode and the bijectivity of fusion morphism for an oplax bimonoid.

A first motivating example for our new notion are groupoids which we show are oplax Hopf monoids in the bicategory of spans. More generally, Hopf categories are a well-described class of oplax Hopf monoids in a variation of Böhm monoidal bicategory $\mathsf{Span}|\mathcal{V}$, where now the monoidal category $\mathcal{V}$ is considered as a monoidal 2-category with trivial 2-cells. We show that this monoidal bicategory has some interesting features, in particular under some mild conditions the forgetful functor to Span is a 2-opfibration.

The conceptual advantage of our approach is that the underlying monoid and comonoid structure of an oplax bimonoid or Hopf monoid live over the same monoidal product, which

places our concept closer to the self-dual notions of classical Hopf algebras and weak Hopf algebras, restoring this feature for Hopf categories which was lost in their interpretation as Hopf monads in [5]. In fact, the latter description and the current one are related via a *dimension shift* in the following sense: a Hopf category in [5] is a 1-cell in a certain monoidal bicategory, whereas in our setting it is a 0-cell in a (different) symmetric monoidal bicategory.

We finish the paper by describing the notion of a Frobenius pseudomonoid in $\mathsf{Span}|\mathcal{V}$, leading to the notion of *Frobenius $\mathcal{V}$-category*. Although both Hopf and Frobenius categories are viewed as pseudomonoids and pseudocomonoids in the same monoidal bicategory (albeit with different compatibility conditions), by taking a closer look it becomes evident that the algebra and co-algebra structures for a Hopf category are of different 'type', whereas for a Frobenius category both algebra and coalgebra structure are of the same 'type'. In the subsequent [15] we show that a Hopf $\mathcal{V}$-category that is locally rigid admits the structure of a Frobenius $\mathcal{V}$-category, providing a generalization of the well-known Larson Sweedler theorem.

Our paper is structured as follows. In Section 2, we recall some known results and fix notation, especially for monoidal bicategories and pseudo(co)monoids. In Section 3 we introduce oplax bimonoids and oplax Hopf monoids, and prove some first results for them. Section 4 consists of some elementary observations about the set X^2 and more generally groupoids viewed as objects in Span, where they give examples of oplax Hopf monoids and Frobenius monoids. In Section 5, we extensively describe the symmetric monoidal bicategory $\mathsf{Span}|\mathcal{V}$ and show that the forgetful functor to Span is a 2-opfibration. In Section 6 we show that Hopf categories are oplax Hopf monoids in $\mathsf{Span}|\mathcal{V}$, as well as intermediate structures like enriched (op)categories and morphisms between them can all be realized in that setting. Finally, in Section 7 we study the Frobenius monoids in $\mathsf{Span}|\mathcal{V}$, which we term Frobenius $\mathcal{V}$-categories. In the appendix we list the various coherence conditions for the objects and morphisms we introduced.

2. Preliminaries

In this section, we provide some background material and fix notation and terminology for what follows.

2.1 Bimonoids, Hopf monoids and Frobenius monoids. We assume familiarity with the theory of (braided) monoidal categories $(\mathcal{V}, \otimes, I, \alpha, \lambda, \sigma)$; see e.g. [33] or [29]. Since any monoidal category is monoidally equivalent to a strict monoidal category, we will assume from now on without loss of generality that the associativity constraint α and the unitality constraint λ are indentities. Recall that a monoid (algebra) in $\mathcal{V}$ is a triple $(A, m\colon A \otimes A \to A, j\colon I \to A)$ satisfying the usual associativity and unitality constraints. Dually, a comonoid (coalgebra) in $\mathcal{V}$ will be denoted as (C, d, ϵ). We denote the categories of monoids and comonoids respectively by $\mathsf{Mon}(\mathcal{V})$ and $\mathsf{Comon}(\mathcal{V})$ $(\cong \mathsf{Mon}(\mathcal{V}^{\mathrm{op}})^{\mathrm{op}})$. When $\mathcal{V}$ is braided, both these categories inherit a monoidal structure from $\mathcal{V}$.

A bimonoid (bialgebra) in a braided monoidal category is an object M with a monoid structure (M, m, j) and a comonoid structure (M, δ, ε) that are compatible, in the sense that a bimonoid is equivalently both a comonoid in $\mathsf{Mon}(\mathcal{V})$ and a monoid in $\mathsf{Comon}(\mathcal{V})$. Diagrammatically

this is expressed by

$$
\begin{array}{ccc}
M \otimes M & \xrightarrow{\ m\ } & M \\
{\scriptstyle \delta \otimes \delta}\downarrow & & \downarrow {\scriptstyle \delta} \\
M \otimes M \otimes M \otimes M & & \\
{\scriptstyle 1 \otimes \sigma \otimes 1}\downarrow & & \downarrow \\
M \otimes M \otimes M \otimes M & \xrightarrow[m \otimes m]{} & M \otimes M
\end{array}
\qquad
\begin{array}{ccc}
I & \xrightarrow{\ j\ } & M \\
{\scriptstyle j \otimes j}\searrow & & \downarrow {\scriptstyle \delta} \\
& M \otimes M &
\end{array}
\qquad
\begin{array}{ccc}
M \otimes M & \xrightarrow{\ m\ } & M \\
{\scriptstyle \varepsilon \otimes \varepsilon}\downarrow & \swarrow {\scriptstyle \varepsilon} & \\
I \otimes I & &
\end{array}
\qquad
\begin{array}{ccc}
I & \xrightarrow{\ j\ } & M \\
{\scriptstyle 1}\searrow & & \downarrow {\scriptstyle \varepsilon} \\
& & I
\end{array}
\qquad (1)
$$

Bimonoids form a category $\mathsf{Bimon}(\mathcal{V})$ whose morphisms are simultaneously monoid and comonoid morphisms.

Moreover, a Hopf monoid (Hopf algebra) is a bimonoid (H, m, j, d, ϵ) equipped with an antipode $s \colon H \to H$ satisfying $m \circ (s \otimes 1_H) \circ d = j \circ \epsilon = m \circ (1_H \otimes s) \circ d$. Since H has both a monoid and comonoid structure, we can define the convolution product $f \odot g$ of any two endomorphisms $f, g \in \mathcal{V}(H, H)$ as the composite $m \circ (f \otimes g) \circ d$; this is an associative binary operation on $\mathcal{V}(H, H)$ with $I^\odot := j \circ \epsilon$ as its unit. Using this terminology, the defining property of an antipode is that it is inverse to 1_H under the convolution product. Hence, the antipode is unique when it exists, and bimonoid morphisms between Hopf monoids can be seen to automatically commute with the antipode. Therefore Hopf monoids form a full subcategory $\mathsf{Hopf}(\mathcal{V})$ of $\mathsf{Bimon}(\mathcal{V})$. Equivalently to the existence of an antipode on a bimonoid H, a Hopf monoid can be characterised via the invertibility of the canonical maps $(1_H \otimes m) \circ (d \otimes 1_H)$ or $(m \otimes 1_H) \circ (1_H \otimes d)$; these are sometimes referred to as *fusion morphisms*, see e.g. [36].

Finally, a Frobenius monoid A also has both a monoid and comonoid structure, but in this case they are compatible via the Frobenius laws $(1_A \otimes m) \circ (d \otimes 1_A) = d \circ m = (m \otimes 1_A) \circ (1_A \otimes d)$. Morphisms between Frobenius monoids are simultaneously monoid and comonoid morphisms. We denote this category by $\mathsf{Frob}(\mathcal{V})$. For more details about Frobenius algebras, see e.g. [38].

2.2   Enriched categories and opcategories. Recall that for a monoidal category $\mathcal{V}$, a $\mathcal{V}$-graph A consists of a set of objects X together with a family of hom-objects $\{A_{x,y}\}_{x,y \in X}$ in $\mathcal{V}$; we will use the notation $A_{x,y}$ rather than the more common $A(x, y)$. Morphisms of $\mathcal{V}$-graphs are functions f between the sets of objects, together with a family of arrows $F_{xy} \colon A_{x,y} \to B_{fx,fy}$ in $\mathcal{V}$. Together these form a category $\mathcal{V}\text{-}\mathsf{Grph}$.

A $\mathcal{V}$-*category* is a $\mathcal{V}$-*graph* $\{A_{x,y}\}_{x,y \in X}$ together with composition (or multiplication) laws $m_{xyz} \colon A_{x,y} \otimes A_{y,z} \to A_{x,z}$ and identities $j_x \colon I \to A_{x,x}$ that satisfy the usual associativity and unity conditions. A $\mathcal{V}$-functor is a $\mathcal{V}$-graph morphism that respects composition and units, and together these form a category $\mathcal{V}\text{-}\mathsf{Cat}$. If $\mathcal{V}$ is braided monoidal, every $\mathcal{V}$-category A has an *opposite* $\mathcal{V}$-category A^{op} with the same objects and hom-objects $A^{\mathrm{op}}_{x,y} := A_{y,x}$, where composition arises from the one in A composed with the braiding. More on the subject of enriched categories can be found in [30].

A $\mathcal{V}$-*opcategory* C is a category enriched in $\mathcal{V}^{\mathrm{op}}$ [20, §9]. Explicitly, there exist cocomposition and coidentity families of arrows in $\mathcal{V}$

$$
d_{xyz} \colon C_{x,z} \to C_{x,y} \otimes C_{y,z}, \quad \epsilon_x \colon C_{x,x} \to I \qquad (2)
$$

satisfying coassociativity and counity axioms. A $\mathcal{V}$-*opfunctor* is a $\mathcal{V}^{\mathrm{op}}$-functor, so there is a category $\mathcal{V}\text{-}\mathsf{opCat} = \mathcal{V}^{\mathrm{op}}\text{-}\mathsf{Cat}$.

2.3   Monoidal bicategories and pseudo(co)monoids. Recall that a *monoidal bicategory* $\mathcal{K}$ is a one-object tricategory [22, Def. 2.6], namely a bicategory equipped with a pseudofunctor

$\otimes \colon \mathcal{K} \times \mathcal{K} \to \mathcal{K}$ and a unit object $I \colon \mathbf{1} \to \mathcal{K}$ which are associative and unital up to coherent equivalence; for the explicit description see [17, 35]. A *monoidal 2-category* is one whose underlying bicategory is really a 2-category. Notice that this is different from a *2-monoidal 2-category* whose tensor product is a strict 2-functor, namely where $(f \otimes g) \circ (h \otimes k) = (f \circ h) \otimes (g \circ k)$ rather than being coherently isomorphic as in the pseudo case. A *weak monoidal pseudofunctor* $\mathcal{F} \colon \mathcal{K} \to \mathcal{L}$ between monoidal bicategories preserves the monoidal structure via pseudonatural maps $\mathcal{F}(A) \otimes \mathcal{F}(B) \to \mathcal{F}(A \otimes B)$ and $\mathcal{F}(1_A) \to 1_{\mathcal{F}A}$ along with associativity and unity invertible modifications subject to coherence axioms, see [20].

It is well-known that any monoidal bicategory is monoidally biequivalent to a monoidal 2-category, and more precisely a *Gray monoid*: this is the one-object case of the coherence theorem for tricategories [24, 22]. A Gray monoid [3, 20], also called a *semistrict monoidal 2-category*, is a monoidal 2-category with a tensor product which is strictly associative and unital on objects, but still a pseudofunctor and not a strict 2-functor. In particular, for given 1-cells $f \colon A \to B$ and $g \colon C \to D$ there exists a suitable invertible 2-cell $c_{f,g} \colon (f \otimes 1) \circ (1 \otimes g) \to (1 \otimes g) \circ (f \otimes 1)$, natural in f and g and satisfying certain coherence conditions. By convention, we denote the 1-cell $(f \otimes 1) \circ (1 \otimes g)$ by $f \otimes g$. In what follows, whenever we consider a monoidal bicategory or monoidal 2-category, we will always suppose that it is a Gray monoid, where we suppress the morphisms $c_{f,g}$ and their inverses or (vertical and horizontal) compositions.

A monoidal 2-category is *braided* when it comes equipped with a pseudonatural equivalence with components $\sigma_{A,B} \colon A \otimes B \to B \otimes A$, and two invertible modifications relating the tensors of three objects, subject to axioms that are detailed in references such as [3, Def. 6], [20, Def. 12] and finalised in [19, Def. 2.2]; it is *sylleptic* when there exists an invertible modification $v \colon \sigma \circ \sigma \Rightarrow 1$ and *symmetric* when $\sigma_{A,B} \circ v_{A,B} = v_{B,A} \circ \sigma_{B,A}$, subject to appropriate axioms. On the other hand, its weakened version of a braided monoidal bicategory as described in [23, § 2.4] is computationally more challenging; due to a coherence theorem though, every braided monoidal bicategory is braided monoidally biequivalent to a braided monoidal 2-category, as shown in [23, Thm 27]. In more detail, there is an induced braided structure on the biequivalent Gray monoid, which is then adjusted in order to constitute an actual braided monoidal 2-categorical structure as defined in [19, Def. 2.2]. Furthermore, the respective coherence result for symmetric monoidal bicategories can be found in [25]. Therefore in what follows, we often work without loss of generality in the simpler context of a braided or symmetric monoidal 2-category (Gray monoid) $\mathcal{K}$.

A *pseudomonoid* [20, §3] (or *monoidale*) $(A, m, j, \alpha, \ell, r)$ in $\mathcal{K}$ is an object A with multiplication $m \colon A \otimes A \to A$, unit $j \colon A \to I$, and invertible 2-cells

$$
\begin{array}{ccc}
A \otimes A \otimes A \xrightarrow{1 \otimes m} A \otimes A & \qquad A \xrightarrow{1 \otimes j} A \otimes A \xleftarrow{j \otimes 1} A \\
{\scriptstyle m \otimes 1}\Big\downarrow \quad \overset{\cong}{\scriptstyle \alpha} \quad \Big\downarrow {\scriptstyle m} & \qquad \searrow \overset{\ell}{\underset{\cong}{}} \Big\downarrow {\scriptstyle m} \overset{r}{\underset{\cong}{}} \swarrow \\
A \otimes A \xrightarrow[\quad m \quad]{} A & \qquad A
\end{array}
\tag{3}
$$

satisfying appropriate coherence conditions found in Section A.1. For example, a pseudomonoid in the cartesian monoidal 2-category $(\mathsf{Cat}, \times, \mathbf{1})$ is a monoidal category. Dually, we have the notions of a *pseudocomonoid* in a monoidal 2-category, denoted by $(C, \delta, \varepsilon, \beta, s, t)$.

An *oplax* (or *opmonoidal*) *morphism* between pseudomonoids is a 1-cell $f \colon A \to B$ equipped

with 2-cells

$$
\begin{array}{ccc}
A \otimes A \xrightarrow{\ m\ } A & \qquad & I \xrightarrow{\ j\ } A \\
{\scriptstyle f \otimes f}\big\downarrow \quad \Downarrow \phi \quad \big\downarrow {\scriptstyle f} & & \quad \Downarrow \phi_0 \quad \big\downarrow {\scriptstyle f} \\
B \otimes B \xrightarrow{\ m\ } B & & \qquad\ \ \searrow^{j}\ B
\end{array}
\tag{4}
$$

such that compatibility conditions for units and multiplications hold, see Section A.1. A *lax morphism* of pseudomonoids is defined similarly, with the 2-cells ϕ, ϕ_0 pointing in the opposite direction and the conditions adjusted accordingly. For example, (op)lax morphisms between pseudomonoids in Cat are precisely (op)lax monoidal functors. If $\mathcal{K}$ is a monoidal 2-category, then pseudomonoids together with (op)lax morphisms form categories $\mathsf{PsMon_{opl}}(\mathcal{K})$ and $\mathsf{PsMon_{lax}}(\mathcal{K})$. Dually, we can talk of oplax morphisms between pseudocomonoids; if $\mathcal{K}^{\mathrm{op}}$ is the monoidal 2-category with reversed 1-cells, it is the case that $\mathsf{PsMon_{opl}}(\mathcal{K}^{\mathrm{op}}) \cong \mathsf{PsComon_{opl}}(\mathcal{K})^{\mathrm{op}}$.

In fact, for any monoidal bicategory $\mathcal{K}$, $\mathsf{PsMon_{opl}}(\mathcal{K})$ is also a bicategory: if (f, ϕ, ϕ_0) and (g, ψ, ψ_0) are two oplax morphisms between pseudomonoids A and B, a 2-cell between them is some $\alpha\colon f \Rightarrow g$ which is compatible with the oplax structure 2-cells, see Section A.1. The similarly defined bicategory $\mathsf{PsMon_{lax}}(\mathcal{K})$ was denoted by $\mathsf{Mon}(\mathcal{K})$ in [18].

Furthermore, when $\mathcal{K}$ is a braided monoidal bicategory with braiding σ, the bicategory $\mathsf{PsMon_{opl}}(\mathcal{K})$ obtains a monoidal structure itself: the multiplication and unit on the tensor product of two pseudomonoids A and B is

$$
A \otimes B \otimes A \otimes B \xrightarrow{\ 1 \otimes \sigma \otimes 1\ } A \otimes A \otimes B \otimes B \xrightarrow{\ m \otimes m\ } A \otimes B
\tag{5}
$$

$$
I \cong I \otimes I \xrightarrow{\ j \otimes j\ } A \otimes B
$$

and the coherence data can be readily constructed also. This can be seen as a result of the fact that the braided structure of a monoidal bicategory endows the pseudofunctor $\otimes$ with a monoidal structure, as discussed in [20, 117], hence it preserves pseudomonoids by [20, Prop. 5].

Finally, a *Frobenius pseudomonoid* inside a monoidal bicategory is an object with a pseudomonoid and pseudocomonoid structure (A, m, j, d, ϵ) together with isomorphisms

$$
\begin{array}{ccc}
A \otimes A \xrightarrow{\ \ \ d \otimes 1\ \ \ } A \otimes A \otimes A \\
{\scriptstyle 1 \otimes d}\big\downarrow \quad {}^{m}\searrow \quad \overset{\phi}{\cong} \quad \big\downarrow {\scriptstyle 1 \otimes m} \\
\qquad\quad A \quad \overset{\psi}{\cong} \quad \searrow^{d} \\
A \otimes A \otimes A \xrightarrow{\ \ \ m \otimes 1\ \ \ } A \otimes A
\end{array}
\tag{6}
$$

satisfying certain coherence conditions, see Section A.5. For a detailed study of such objects and equivalent formulations see [38, 31, 32]. If $f\colon A \to B$ is a 1-cell between Frobenius pseudomonoids, we may choose any of the four combinations of lax or oplax structures between the pseudo(co)monoids for forming a bicategory. For example, we can denote by $\mathsf{Frob_{opl,opl}}(\mathcal{K})$ the category of Frobenius pseudomonoids with oplax pseudomonoid, oplax pseudocomonoid morphisms between them.

3. Oplax bimonoids and Hopf monoids

In this section, we define oplax variations of bimonoids and Hopf monoids in a braided monoidal bicategory and examine some of their basic properties. These serve as basic concepts and tools for the following sections.

3.1   Oplax bimonoids. Previously we described the monoidal bicategory $\mathsf{PsMon}_{\mathsf{opl}}(\mathcal{K})$ of pseudomonoids with oplax morphisms and 2-cells for a braided monoidal bicategory $\mathcal{K}$; one may consider pseudocomonoids therein and establish a notion of 'oplax bimonoid', namely an object with a pseudomonoid and pseudocomonoid structure and an oplax interaction between them.

Definition 3.1.1. In a braided monoidal bicategory $(\mathcal{K}, \otimes, I, \sigma)$, an *oplax bimonoid* is an object M in $\mathcal{K}$ endowed with a pseudomonoid structure (M, m, j) and a pseudocomonoid structure (M, δ, ε) along with 2-cells

$$
\begin{array}{ccc}
M \otimes M & \xrightarrow{\quad m \quad} & M \\
{\scriptstyle \delta \otimes \delta} \downarrow & & \downarrow {\scriptstyle \delta} \\
M \otimes M \otimes M \otimes M & \Downarrow \theta & \\
{\scriptstyle 1 \otimes \sigma \otimes 1} \downarrow & & \downarrow \\
M \otimes M \otimes M \otimes M & \xrightarrow{\quad m \otimes m \quad} & M \otimes M
\end{array}
\tag{7}
$$

$$
\begin{array}{cccccc}
I \xrightarrow{\ j\ } M & \qquad & M \otimes M \xrightarrow{\ m\ } M & \qquad & I \xrightarrow{\ j\ } M \\
{\scriptstyle \sim}\downarrow \quad \Downarrow\theta_0 \quad \downarrow{\scriptstyle \delta} & & {\scriptstyle \varepsilon\otimes\varepsilon}\downarrow \quad \Downarrow\chi \quad \downarrow{\scriptstyle \varepsilon} & & {\scriptstyle 1}\downarrow \quad \Downarrow\chi_0 \quad \downarrow{\scriptstyle \varepsilon} \\
I \otimes I \xrightarrow{\ j\otimes j\ } M \otimes M & & I \otimes I \xrightarrow{\ \sim\ } I & & I \xrightarrow{\ 1\ } I
\end{array}
\tag{8}
$$

expressing that comultiplication and counit are oplax pseudomonoid maps as in (4). These data satisfy certain axioms which are explicitly written in Section A.2.

The above notion indeed generalizes ordinary bimonoids by essentially inserting 2-cells inside the commutative diagrams (1), subject to coherent conditions. For example, any bimonoid in a braided monoidal category $\mathcal{V}$ viewed as a (2-)monoidal 2-category with trivial 2-cells is an oplax bimonoid, strict in that the (co)associativity and (co)unit constraints are identities. Moreover, an oplax bimonoid in that strict sense in the cartesian 2-category $\mathcal{K} = (\mathsf{Cat}, \times, \mathbf{1})$ is a strict monoidal category, since any object has a unique strict comonoid structure. A general oplax bimonoid in Cat is up to isomorphism a monoidal category: due to $\mathbf{1}$ being terminal, a pseudocomonoid is up to isomorphism still the trivial one namely the comultiplication is isomorphic to the diagonal, and $\chi_0, \chi = \mathsf{id}$ whereas θ, θ_0 are uniquely determined isomorphisms. Non-trivial examples will be provided in the subsequent sections.

Morphisms between oplax bimonoids are oplax maps between the corresponding pseudo-comonoids in $\mathsf{PsMon}_{\mathsf{opl}}(\mathcal{K})$. More explicitly, we have the following definition.

Definition 3.1.2. An *oplax bimonoid morphism* between oplax bimonoids M and N with structure 2-cells $(\theta, \theta_0, \chi, \chi_0)$ and $(\xi, \xi_0, \omega, \omega_0)$ in a braided monoidal bicategory $\mathcal{K}$ is a 1-cell $f \colon M \to N$ that is both an oplax pseudomonoid morphism and an oplax pseudocomonoid morphism with structure 2-cells

$$
\begin{array}{cccccccc}
M \otimes M \xrightarrow{\ m\ } M & \quad & I \xrightarrow{\ j\ } M & \quad & M \xrightarrow{\ \delta\ } M \otimes M & \quad & M \\
{\scriptstyle f\otimes f}\downarrow \ \Downarrow\phi\ \downarrow{\scriptstyle f} & & \quad \Downarrow\phi_0 \ \downarrow{\scriptstyle f} & & {\scriptstyle f}\downarrow \ \Downarrow\psi\ \downarrow{\scriptstyle f\otimes f} & & {\scriptstyle f}\downarrow \ \Downarrow\psi_0 \\
N \otimes N \xrightarrow{\ m\ } N & & N & & N \xrightarrow{\ \delta\ } N \otimes N & & N \xrightarrow{\ \varepsilon\ } I
\end{array}
\tag{9}
$$

satisfying four axioms expressing that ψ and ψ_0 are 2-cells between oplax pseudomonoid maps; axioms are explicitly recorded in Section A.3.

Oplax bimonoids form a bicategory $\mathsf{OplBimon}(\mathcal{K}) = \mathsf{PsComon}_{\mathsf{opl}}(\mathsf{PsMon}_{\mathsf{opl}}(\mathcal{K}))$. Notice that the axioms oplax bimonoids satisfy are very similar to those of duoidal categories [2], which can be explained by observing that duoidal categories are pseudomonoids in $\mathsf{PsMon}_{\mathsf{opl}}(\mathcal{K})$ or equivalently in $\mathsf{PsMon}_{\mathsf{lax}}(\mathcal{K})$, for $\mathcal{K} = \mathsf{Cat}$. Hence these notions are 'half-dual' to one other.

As mentioned above, a weak monoidal pseudofunctor between monoidal bicategories preserves pseudomonoids and analogously a (strong) monoidal one preserves pseudocomonoids as well. Below we establish that a braided monoidal pseudofunctor as in [20, Def. 14] preserves oplax bimonoids between braided monoidal bicategories.

Proposition 3.1.3. *Let* $\mathcal{F}\colon \mathcal{K} \to \mathcal{L}$ *be a braided monoidal pseudofunctor of bicategories. If M is an oplax bimonoid in $\mathcal{K}$, then $\mathcal{F}M$ is an oplax bimonoid in $\mathcal{L}$.*

Proof. Suppose $(M, m, j, \delta, \varepsilon, \theta, \theta_0, \chi, \chi_0)$ is an oplax bimonoid as in Definition 3.1.1. Then the pseudomonoid and pseudocomonoid structure on $\mathcal{F}M$ are given by

$$\mathcal{F}M \otimes \mathcal{F}M \to \mathcal{F}(M \otimes M) \xrightarrow{\mathcal{F}m} \mathcal{F}M, \qquad I \to \mathcal{F}I \xrightarrow{\mathcal{F}j} \mathcal{F}M$$

$$\mathcal{F}M \xrightarrow{\mathcal{F}\delta} \mathcal{F}(M \otimes M) \to \mathcal{F}M \otimes \mathcal{F}M, \qquad \mathcal{F}M \xrightarrow{\mathcal{F}\varepsilon} \mathcal{F}I \to I$$

where the unnamed arrows are the structure maps of the strong monoidal pseudofunctor $\mathcal{F}$, namely two equivalences and its pseudoinverses. We can then equip $\mathcal{F}M$ with an oplax bimonoid structure (7) and (8) as follows: the 2-cell $\theta_{\mathcal{F}M}$ is formed as the composite

where the two unnamed invertible 2-cells are due to pseudonaturality of the monoidal structure maps of $\mathcal{F}$, whereas $(*)$ comes from $\mathcal{F}$ being (strong) monoidal and braided. Similarly the rest three 2-cells are constructed as composites of $\mathcal{F}\theta_0$, $\mathcal{F}\chi$, $\mathcal{F}\chi_0$ with coherent isomorphisms, and the conditions of Section A.2 are satisfied.   □

In the spirit of [2], one could define a notion of 'bilax' monoidal pseudofunctor between braided monoidal bicategories which would be precisely such that it preserves oplax bimonoids: essentially a weak and an opweak monoidal pseudofunctor which comes with coherent isomorphisms of the form $(*)$. For our purposes though, the above setting suffices for our example at hand – the strict monoidal forgetful functor of bicategories of Section 5.2.

3.2   Oplax modules for oplax bimonoids.   One of the fundamental properties of a bimonoid in an ordinary monoidal category is that there exists a monoidal structure on its category of modules, with a strict monoidal forgetful functor. In order to generalize this for oplax bimonoids in monoidal bicategories, we introduce the notion of an oplax module.

Definition 3.2.1. Let (M, m, j) be a pseudomonoid in a monoidal bicategory $\mathcal{K}$. A *(right) oplax M-module* is an object $X \in \mathcal{K}$ equipped with a 1-cell $\rho \colon X \otimes M \to X$ and 2-cells

$$
\begin{array}{ccc}
X \otimes M \otimes M & \xrightarrow{1 \otimes m} & X \otimes M \\
{\scriptstyle \rho \otimes 1} \downarrow & \Downarrow \xi & \downarrow {\scriptstyle \rho} \\
X \otimes M & \xrightarrow{\rho} & X
\end{array}
\qquad
\begin{array}{ccc}
X & \xrightarrow{1 \otimes j} & X \otimes M \\
 & \Downarrow \xi_0 & \downarrow {\scriptstyle \rho} \\
 & \xrightarrow{\mathrm{id}} & X
\end{array}
\tag{10}
$$

satisfying appropriate compatibility axioms listed in Section A.4. When ξ and ξ_0 are invertible, we say that X is a *pseudo M-module*.

A *(right) oplax M-module morphism* $(f, \phi) \colon (X, \rho_X, \xi, \xi_0) \to (Y, \rho_Y, \zeta, \zeta_0)$ consists of a 1-cell $f \colon X \to Y$ in $\mathcal{K}$ and a 2-cell

$$
\begin{array}{ccc}
X \otimes M & \xrightarrow{\rho_X} & X \\
{\scriptstyle f \otimes 1} \downarrow & \Downarrow \phi & \downarrow {\scriptstyle f} \\
Y \otimes M & \xrightarrow{\rho_Y} & Y
\end{array}
\tag{11}
$$

satisfying coherence conditions (70) and (71). If ϕ is invertible, we say that (f, ϕ) is an oplax *M-module pseudomorphism*. Finally, a *transformation* between oplax *M*-module morphisms $(f, \phi), (g, \psi)$ is a 2-cell $\alpha \colon f \Rightarrow g$ which is compatible with the structure morphisms, see (72).

Oplax M-modules together with their oplax morphisms and transformations form a bicategory $\mathsf{OplMod}_M^{\mathsf{opl}}$, which comes with an evident forgetful strict functor to $\mathcal{K}$. Similarly, we have the bicategory $\mathsf{OplMod}_M^{\mathsf{ps}}$ of oplax M-modules with pseudomorphisms and transformations.

Remark 3.2.2. Clearly, one could also consider (op)lax (co)modules over a pseudo(co)monoid. In fact, the notion of pseudomodule as defined above is closely related to the notion of algebra for the pseudomonad $(- \otimes M)$ on $\mathcal{K}$ as considered in [34]. However, it is important to remark that some (but not all) structure 2-cells in our context have the opposite direction of those e.g. in [34, p.96].

Since the appearance of the first pre-print version of this work, oplax modules over a *skew monoidale* also have been studied under the name of *oplax actions* in [1]. Therein, oplax modules are required to satisfy an additional axiom, which however follows from the other axioms in our setting (see [34]).

Proposition 3.2.3. *For any oplax bimonoid M, the bicategories $\mathsf{OplMod}_M^{\mathsf{ps}}$ and $\mathsf{OplMod}_M^{\mathsf{opl}}$ have a monoidal structure, such that the forgetful functors $\mathsf{OplMod}_M^{\mathsf{ps}} \to \mathsf{OplMod}_M^{\mathsf{opl}} \to \mathcal{K}$ are strict monoidal.*

Proof. Suppose $(\mathcal{K}, \otimes, I, \sigma)$ is a braided monoidal bicategory. Take an oplax bimonoid $(M, m, j, \delta, \varepsilon)$ with structure 2-cells $(\theta, \theta_0, \chi, \chi_0)$ as in Definition 3.1.1, and two oplax M-modules (X, ρ_X, ξ, ξ_0) and $(Y, \rho_Y, \zeta, \zeta_0)$ as in Definition 3.2.1. We can endow $X \otimes Y$ with an oplax M-module structure as follows: the M-action is the composite

$$
\rho_{X \otimes Y} \colon X \otimes Y \otimes M \xrightarrow{1 \otimes 1 \otimes \delta} X \otimes Y \otimes M \otimes M \xrightarrow{1 \otimes \sigma \otimes 1} X \otimes M \otimes Y \otimes M \xrightarrow{\rho_X \otimes \rho_Y} X \otimes Y \tag{12}
$$

and the structure 2-cells are given by

$$
\begin{array}{c}
\text{(diagram)}
\end{array}
\tag{13}
$$

The first diagram has vertices:

- Top row: $X \otimes Y \otimes M \otimes M \xrightarrow{11m} X \otimes Y \otimes M$
- $X \otimes Y \otimes M \otimes M \otimes M \xrightarrow{1111\delta} X \otimes Y \otimes M \otimes M \otimes M \otimes M \xrightarrow{111\sigma 1} X \otimes Y \otimes M \otimes M \otimes M \otimes M \xrightarrow{11mm} X \otimes Y \otimes M \otimes M$
- $X \otimes M \otimes Y \otimes M \otimes M \qquad X \otimes M \otimes Y \otimes M \otimes M \otimes M \xrightarrow{11\sigma 11} X \otimes M \otimes M \otimes Y \otimes M \otimes M \xrightarrow{1m1m} X \otimes M \otimes Y \otimes M$
- $X \otimes Y \otimes M \xrightarrow{1\delta} X \otimes Y \otimes M \otimes M \xrightarrow{1\sigma 1} X \otimes M \otimes Y \otimes M \xrightarrow{\rho\rho} X \otimes Y$

with vertical arrows $11\delta 1$, $11\delta\delta$, 11δ; $1\sigma 11$, $1\sigma 111$, $1\sigma 1$, $1\sigma 1$; $\rho\rho 1$, $\rho\rho 11$, $\rho 1\rho 1$, $\rho\rho$; and 2-cells $\Downarrow 11\theta$, $\Downarrow \xi\zeta$.

The second diagram:

- $X \otimes Y \xrightarrow{11j} X \otimes Y \otimes M$
- $X \otimes Y \xrightarrow{11jj}$, $\Downarrow 11\theta_0$, 11δ
- $1j1j$, $X \otimes Y \otimes M \otimes M$
- $\Downarrow \xi_0\zeta_0$, $1\sigma 1$
- $X \otimes M \otimes Y \otimes M$
- 1, $\rho_X\rho_Y$
- $X \otimes Y$

where the empty squares should be filled by the coherence isomorphisms of the braided monoidal bicategory $\mathcal{K}$ (or its equivalent Gray monoid), and we have suppressed the $\otimes$-symbol for morphisms. Combining the axioms (68) and (69) for X and Y with the oplax bimonoid axioms for M as found in Section A.2, we can verify that this constitutes an oplax module structure on $X \otimes Y$.

Moreover, the monoidal unit I of $\mathcal{K}$ can also be endowed with the structure of an oplax M-module, with action and structure 2-cells

$$
\rho_I : I \otimes M \cong M \xrightarrow{\varepsilon} I
\tag{14}
$$

$$
\begin{array}{ccc}
I \otimes M \otimes M \cong M \otimes M \xrightarrow{m} M & \qquad & I \xrightarrow{j} M \\
\varepsilon\otimes 1 \downarrow \quad \varepsilon\otimes\varepsilon \quad \Downarrow \chi \quad \downarrow \varepsilon & & \Downarrow \chi_0 \quad \downarrow \varepsilon \\
M \xrightarrow{\varepsilon} I & & 1 \searrow \quad I
\end{array}
$$

Given morphisms of oplax modules $(f, \phi) : X \to Y$ and $(g, \psi) : Z \to U$, define $(f, \phi) \otimes (g, \psi)$

as $(f \otimes g, \tau)$, where $f \otimes g$ is the tensor product in $\mathcal{K}$ and τ is given by

$$
\begin{array}{ccc}
X \otimes Z \otimes M & \xrightarrow{\;f \otimes g \otimes 1\;} & Y \otimes U \otimes M \\
{\scriptstyle 1 \otimes 1 \otimes \delta}\downarrow & & \downarrow{\scriptstyle 1 \otimes 1 \otimes \delta} \\
X \otimes Z \otimes M \otimes M & \xrightarrow{\;f \otimes g \otimes 1 \otimes 1\;} & Y \otimes U \otimes M \otimes M \\
{\scriptstyle 1 \otimes \sigma \otimes 1}\downarrow & & \downarrow{\scriptstyle 1 \otimes \sigma \otimes 1} \\
X \otimes M \otimes Z \otimes M & \xrightarrow{\;f \otimes 1 \otimes g \otimes 1\;} & Y \otimes M \otimes U \otimes M \\
{\scriptstyle \rho_X \otimes \rho_Z}\downarrow & \Downarrow \phi \otimes \psi & \downarrow{\scriptstyle \rho_Y \otimes \rho_U} \\
X \otimes Z & \xrightarrow{\qquad f \otimes g \qquad} & Y \otimes U
\end{array}
\tag{15}
$$

Clearly, if the morphisms (f, ϕ) and (g, ψ) are pseudo, namely if ϕ and ψ above are invertible, then $f \otimes g$ is a pseudo morphism as well, since (15) is a composite of invertible 2-cells. The tensor product of oplax module transformations is given by the tensor product of 2-cells in $\mathcal{K}$; there is no extra structure.

With the above descriptions, it can be verified that the category of oplax modules over an oplax bimonoid is a monoidal bicategory, and it directly follows that the forgetful functor to $\mathcal{K}$ strictly preserves the monoidal structure. $\qquad\square$

Remark 3.2.4. Notice that if X and Y are pseudo M-modules, $X \otimes Y$ is naturally still only an oplax module even in the stricter 2-monoidal 2-category case, due to the non-invertible 2-cells θ and θ_0 of the oplax bimonoid used for the coherence 2-cells (13) of $X \otimes Y$ in the above proof. Hence the bicategory of pseudomodules over an oplax bimonoid is in general not a monoidal bicategory.

For a monoid M in an ordinary monoidal category $\mathcal{C}$ (under some conditions, see e.g. [46]), it is known that there is a bijective correspondence between bimonoid structures on M and monoidal structures on the category of M-modules such that the forgetful functor to $\mathcal{C}$ is strict monoidal. Since the previous proposition lifts one part of this correspondence to oplax bimonoids, we now consider part of the converse direction by showing that the monoidal structure on the bicategory of oplax modules over an oplax bimonoid completely determines its pseudocomonoid structure.

Let us assume that for a pseudomonoid (M, m, j), the bicategory $\mathsf{OplMod}_M^{\mathsf{ps}}$ is monoidal in such a way that the forgetful functor to $\mathcal{K}$ is strict monoidal. Since M is a pseudomonoid, it is a pseudo M-module (M, m) and so $M \otimes M$ is an oplax M-module: denote its action by $\rho_{M \otimes M}$ and structure 2-cells $(\xi^{M \otimes M}, \xi_0^{M \otimes M})$ as in (10). Then we can define a comultiplication 1-cell as the composite

$$
\delta \colon M \xrightarrow{\;j \otimes j \otimes 1\;} M \otimes M \otimes M \xrightarrow{\;\rho_{M \otimes M}\;} M \otimes M. \tag{16}
$$

Moreover, the monoidal unit I is also an oplax M-module with action denoted ρ_I and 2-cells (ξ^I, ξ_0^I); therefore a counit can be defined as

$$
\varepsilon \colon M \cong I \otimes M \xrightarrow{\;\rho_I\;} I. \tag{17}
$$

Using notation as in Definition 3.1.1, the required 2-cells $(\theta, \theta_0, \chi, \chi_0)$ for an oplax bimonoid

structure on M can be constructed as follows: θ is built from $\xi^{M\otimes M}$ as the composite

$$
\begin{array}{c}
(18)
\end{array}
$$

Moreover, θ_0 can be built using $\xi_0^{M\otimes M}$ as the composite

$$
\begin{array}{c}
I \xrightarrow{j\otimes j} M\otimes M \xrightarrow{1\otimes j} M\otimes M\otimes M \\
\Downarrow \xi_0^{M\otimes M} \qquad \rho_{M\otimes M} \\
1 \\
j\otimes j \qquad M\otimes M
\end{array}
\qquad (19)
$$

In a similar way, χ and χ_0 can be obtained from ξ^I and ξ_0^I as follows

$$
\begin{array}{c}
M\otimes M \xrightarrow{m} M \qquad\qquad I \xrightarrow{j} M \\
\varepsilon\otimes\varepsilon \quad \varepsilon\otimes 1 \quad \Downarrow \xi^I \quad \varepsilon \qquad\qquad \Downarrow \xi_0^I \quad \varepsilon \\
M \\
\varepsilon \\
I\otimes I \xrightarrow{\quad\quad} I \qquad\qquad 1 \qquad I
\end{array}
\qquad (20)
$$

The following result shows that when we consider the monoidal structure on $\mathsf{OplMod}_M^{\mathsf{ps}}$ as described in Proposition 3.2.3, by the above construction we recover the initial pseudocomonoid structure of the oplax bimonoid M, which is a so-called *Tannakian recontruction theorem* for oplax bimonoids.

Proposition 3.2.5. *Suppose that M is an oplax bimonoid. Then the 1-cells (16) and (17) and the 2-cells (18) to (20) constructed using the monoidal structure of $\mathsf{OplMod}_M^{\mathsf{opl}}$ as described in Proposition 3.2.3 endow the underlying pseudomonoid M with an oplax bimonoid structure which is isomorphic to the initial one.*

Proof. The following diagram can be filled with invertible 2-cells, showing that the the reconstructed comultiplication (16) using (12) is isomorphic to the initial comultiplication d.
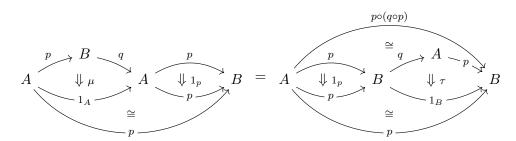
$$
\begin{array}{ccc}
M \xrightarrow{m\otimes m\otimes 1} M\otimes M\otimes M \xrightarrow{1\otimes 1\otimes d} M\otimes M\otimes M\otimes M \\
d \downarrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad 1\otimes\sigma\otimes 1 \downarrow \\
M\otimes M \xRightarrow{j\otimes 1 j\otimes 1} M\otimes M\otimes M\otimes M \\
\qquad\qquad\qquad\qquad\qquad\qquad m\otimes m \downarrow \\
M\otimes M
\end{array}
$$

Furthermore, by definition it is obvious that the reconstructed counit coincides with the initial one since ρ_I itself is defined as isomorphic to the counit (14). In the same way, one observes that the reconstructed 2-cells $\theta, \theta_0, \chi, \chi_0$ coincide with the initial ones and hence we obtain by reconstruction an isomorphic oplax bimonoid structure on M. $\qquad\square$

Remark 3.2.6. A next step would be a *recognition theorem* for oplax bimonoids, in the following sense. Given a pseudomonoid M in $\mathcal{K}$ such that the category $\mathsf{OplMod}^{\mathsf{ps}}_M$ is monoidal and the forgetful functor to $\mathcal{K}$ is strict monoidal, when do the constructions described above Proposition 3.2.5 provide an oplax bimonoid structure on M, such that the monoidal structure of $\mathsf{OplMod}^{\mathsf{ps}}_M$ coincides with the one given in Proposition 3.2.3? In view of the 1-categorical case (see e.g. [46]), in order to obtain such a theorem one would need a suitable generator condition on the monoidal unit of the bicategory $\mathcal{K}$. Such a result, which would lead to a bijective correspondence between oplax bimonoid structures on a given pseudo monoid M and monoidal structures on its category of oplax modules with a strict monoidal forgetful functor to $\mathcal{K}$, is beyond the scope of the present paper and left open for future work.

3.3 Oplax inverses. Having established the notion of an oplax bimonoid in Definition 3.1.1, we now investigate the appropriate notion of an antipode in this generalized setting. For that purpose, we introduce the notion of an oplax inverse in a monoidal category as a special case Morita context. We begin with some motivating definitions; in what follows, $\circ$ denotes vertical composition of 2-cells, $*$ denotes horizontal composition of 2-cells and we suppress horizontal composition of 1-cells.

Definition 3.3.1. [21] A *(wide) Morita context* (A, B, p, q, μ, τ) in a bicategory $\mathcal{K}$ consists of two objects A and B, two 1-cells $p\colon A \to B$ and $q\colon B \to A$ and two 2-cells $\mu\colon qp \Rightarrow 1_A$ and $\tau\colon pq \Rightarrow 1_B$ such that $1_p * \mu = \tau * 1_p\colon pqp \Rightarrow p$ explicitly as in

and similarly $1_q * \tau = \mu * 1_q\colon qpq \Rightarrow q$.

A wide Morita context is called *strict* if the 2-cells μ and τ are invertible.

We will henceforth call a wide Morita context *firm* if the above composite 2-cells $1_p * \mu, \tau * 1_p, 1_q * \tau, \mu * 1_q$ are in fact invertible. The motivation for this terminology comes from the following example. Notice that a strict wide Morita context is always firm, but the inverse is not true; firmness serves as an intermediate notion.

There exists a straightforward notion of (iso)morphism of wide Morita contexts. From now on, we will suppress the word 'wide' for Morita contexts.

Examples 3.3.2. In the bicategory BMod of algebras, bimodules and bilinear maps, wide Morita contexts as in Definition 3.3.1 coincide with the classical notion of a Morita context. In that case, it is well-known that a Morita context (A, B, p, q, μ, τ) is strict if and only if p is finitely

generated and projective both as a left A-module and a right B-module and there are isomorphisms $B \cong {}_A\mathsf{End}(p)$, $A \cong \mathsf{End}_B(p)^{\mathrm{op}}$, $q \cong {}_A\mathsf{Hom}(p, A) \cong \mathsf{Hom}_B(p, B)$, if and only if the bimodules p and q induce an equivalence between the categories of A-modules and B-modules. Similarly, such a classical Morita context is firm if and only if the rings $R = p \otimes_B q$ and $S = q \otimes_A p$ are firm and p is R-firmly projective as left A-module and S-firmly projective as right B-module; in this case, there is an equivalence between the categories of firm R-modules and firm S-modules, see [11, 45].

In BMod, it is therefore clear that any strict Morita context is firm, but not vice versa: not every firmly projective module is necessarily finitely generated and projective.

Similarly to the essential uniqueness of adjoints in a bicategory, for a given 1-cell $p \colon A \to B$ it is known that there can exist only one (up to isomorphism) strict Morita context (A, B, p, q, μ, τ). The subsequent theorem generalizes this result for firm Morita contexts in bicategories, and is fundamental for what follows; before that, a lemma establishes some required identities.

Lemma 3.3.3. *Let (A, B, p, q, μ, τ) and $(A, B, p, q', \mu', \tau')$ be two firm Morita contexts. If α, β, α' and β' are the respective inverse 2-cells of $1_p * \mu$, $1_q * \tau$, $1_p * \mu'$ and $1_{q'} * \tau'$, the following hold:*

(i) $1_q * \alpha = \beta * 1_p$ *(iii)* $(\alpha' * 1_{qp}) \circ \alpha = (1_{pq'} * \alpha) \circ \alpha'$
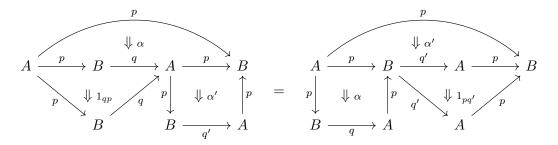*(ii)* $\alpha * 1_p = 1_q * \beta$

as well as appropriate identities including α' and β'.

Proof. (i). By the defining axioms of a Morita context, all arrows (2-cells in $\mathcal{K}$) from left to right in the next diagram are equal, and therefore have equal inverses. By definition, the inverse of the upper arrows is $1_q * \alpha$ and the inverse of the lower arrows is $\beta * 1_p$.
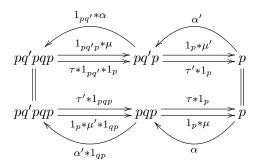

Clause (ii) is proven in a similar way.
(iii). This 2-dimensional equality can be depicted as

Similarly to the proof of (i), we will show that both 2-cells have the same inverse. Consider the following diagram, where the parallel arrows are the same due to Morita context axioms, and

the inner rectangle commutes due to the interchange law.

$$\square$$

Theorem 3.3.4. *If (A, B, p, q, μ, τ) and $(A, B, p, q', \mu', \tau')$ are two firm Morita contexts, there is an isomorphism $q \cong q'$ that makes the above Morita contexts isomorphic.*

Proof. Denote by α and β the inverses of respectively $1_p * \mu$ and $1_q * \tau$, and similarly α' and β' for the corresponding 2-cells of the alternate firm Morita context, as in Lemma 3.3.3. We will show that the following 2-cells between q and q'

$$\phi : q \xrightarrow{\quad \beta \quad} qpq \xrightarrow{1_q * \alpha' * 1_q} qpq'pq \xrightarrow{\mu * 1_{q'} * \tau} q'$$

$$\psi : q' \xrightarrow{\quad \beta' \quad} q'pq' \xrightarrow{1_{q'} * \alpha * 1_{q'}} q'pqpq' \xrightarrow{\mu' * 1_q * \tau'} q$$

are mutual inverses, therefore providing the required isomorphism. Indeed, consider the following identities, where the first one is due to the interchange law, and the unnamed ones are due to firmness of the Morita context:

$$
\begin{aligned}
\psi \circ \phi \quad &= \quad (\mu * \mu' * 1_q * \tau' * \tau) \circ (1_{qpq'} * \alpha * 1_{q'pq}) \circ (1_{qp} * \beta' * 1_{pq}) \circ (1_q * \alpha' * 1_q) \circ \beta \\[4pt]
&\overset{((i))}{=} \quad (\mu * \mu' * 1_q * \tau' * \tau) \circ (1_{qpq'} * \alpha * 1_{q'pq}) \circ (1_{qpq'} * \alpha' * 1_q) \circ (1_q * \alpha' * 1_q) \circ \beta \\[4pt]
&\overset{((iii))}{=} \quad (\mu * \mu' * 1_q * \tau' * \tau) \circ (1_q * \alpha' * 1_{qpq'pq}) \circ (1_{qpq'} * \alpha * 1_q) \circ (1_q * \alpha' * 1_q) \circ \beta \\[4pt]
&\overset{((iii))}{=} \quad (\mu * \mu' * 1_q * \tau' * \tau) \circ (1_q * \alpha' * 1_{qpq'pq}) \circ (1_q * \alpha' * 1_{qpq}) \circ (1_q * \alpha * 1_q) \circ \beta \\[4pt]
&= \quad (\mu * \mu' * 1_q * \tau) \circ (1_q * \alpha' * 1_{qpq}) \circ (1_q * \alpha * 1_q) \circ \beta \\[4pt]
&= \quad (\mu * 1_q * \tau) \circ (1_q * \alpha * 1_q) \circ \beta \\[4pt]
&\overset{((i))}{=} \quad (\mu * 1_q * \tau) \circ (\beta * 1_{qp}) \circ \beta \\[4pt]
&= \quad (1_q * \tau) \circ \beta \\[4pt]
&= \quad 1_q
\end{aligned}
$$

In a very analogous way, $\phi \circ \psi = 1_{q'}$ and hence q and q' are isomorphic. $\square$

We now consider the particular case of an one-object bicategory, i.e. a monoidal category. A firm Morita context $(\star, \star, X, Y, t_1, t_2)$ in a bicategory $\mathcal{K}$ with one object $\star$ and $\mathcal{V} = \mathcal{K}(\star, \star)$ gives rise to the following definition.

Definition 3.3.5. An *oplax inverse* for an object X in a monoidal category $\mathcal{V}$ is an object Y, together with morphisms $\tau_1 \colon X \otimes Y \to I$ and $\tau_2 \colon Y \otimes X \to I$ such that $1_Y \otimes \tau_1 = \tau_2 \otimes 1_Y$ and $\tau_1 \otimes 1_X = 1_X \otimes \tau_2$, and these are invertible morphisms in $\mathcal{V}$.

As a result of Theorem 3.3.4, such a notion is unique up to isomorphism. Moreover, as a special case of the fact that pseudofunctors between bicategories preserve Morita contexts [21, Prop. 1.10], we obtain that a strong monoidal functor between monoidal categories preserve oplax inverses.

In many immediate examples, oplax inverses are trivial. Indeed, in the monoidal category $\mathcal{V} = \mathsf{Vect}_k$ of vector spaces over a field, a simple dimension argument shows that an object has an oplax inverse if and only if it is isomorphic to the monoidal unit k. Nevertheless, as will be evident in the next sections, oplax inverses provide the required structure to capture the proper notion of antipodes for oplax bialgebras.

3.4   Oplax Hopf monoids.   Let us first recall the *convolution* monoidal structure [20, Prop.4] on a hom-category between a pseudomonoid and pseudocomonoid in a monoidal bicategory, that naturally generalizes the classical convolution for (co)monoids in monoidal categories.

Lemma 3.4.1. *Let (M, m, j) be a pseudomonoid and (C, δ, ε) a pseudocomonoid in $\mathcal{K}$. The hom-category $\mathcal{K}(C, M)$ is a monoidal category, with tensor product $\odot$ defined on 1-cells $f, g \colon C \to M$ by*

$$f \odot g := C \xrightarrow{\ \delta\ } C \otimes C \xrightarrow{\ f \otimes g\ } M \otimes M \xrightarrow{\ m\ } M$$

and on two-cells $\alpha \colon f \Rightarrow f'$, $\beta \colon g \Rightarrow g'$ by

$$\alpha \odot \beta := C \xrightarrow{\ \delta\ } C \otimes C \overset{f \otimes g}{\underset{f' \otimes g'}{\Longrightarrow \alpha \otimes \beta}} M \otimes M \xrightarrow{\ m\ } M \ .$$

The monoidal unit is given by $I^{\odot} = j \circ \varepsilon \colon C \to I \to M$.

If M is an oplax bimonoid in $\mathcal{K}$ as in Definition 3.1.1, the above lemma assures that $\mathcal{K}(M, M)$ has a monoidal structure given by convolution. This is clearly different to the standard monoidal structure of every endo-hom category $(\mathcal{K}(M, M), \circ, 1_M)$ via horizontal composition, for an arbitrary (not necessarily monoidal) bicategory $\mathcal{K}$; for a meaningful relation between these two monoidal structures, see Lemma 3.4.5. Another relation, albeit not relevant to our current development, is that if M is a map monoidale these form a duoidal structure on $\mathcal{K}(M, M)$, see e.g. [7, Section 3.3].

Since in the classical case, the antipode of a Hopf algebra is a convolution inverse to the identity, the following definition sets the respective oplax version in an analogous setting.

Definition 3.4.2. An *oplax antipode* for an oplax bimonoid M in a braided monoidal bicategory $\mathcal{K}$ is an oplax inverse of 1_M in the convolution monoidal category $\mathcal{K}(M, M)$.

To unpack the above definition, consider an oplax bimonoid $(M, m, j, \delta, \varepsilon)$ in $\mathcal{K}$. Definition 3.3.5 implies that an oplax antipode is a 1-cell $s \colon M \to M$ along with 2-cells $\tau_1 \colon 1_M \odot s \Rightarrow I^{\odot}$ and $\tau_2 \colon s \odot 1_M \Rightarrow I^{\odot}$ as in

$$\tag{21}$$

such that $1_s \odot \tau_1 = \tau_2 \odot 1_s$ and $\tau_1 \odot 1_{1_M} = 1_{1_M} \odot \tau_2$, namely

$$(22)$$

and all 2-cells $1_s \odot \tau_1$, $\tau_2 \odot 1_s$, $\tau_1 \odot 1_{1_M}$ and $1_{1_M} \odot \tau_2$ are invertible.
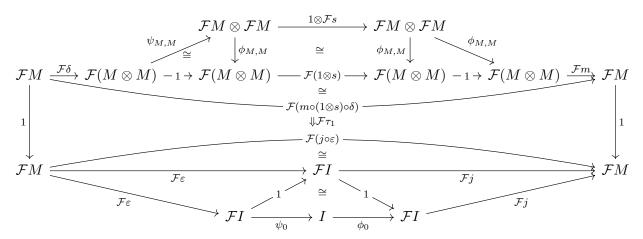
The above says that $s \odot 1_M \odot s \cong s$ and $1_M \odot s \odot 1_M \cong 1_M$ via specific 2-isomorphisms.

Definition 3.4.3. An *oplax Hopf monoid* in a monoidal bicategory $\mathcal{K}$ is an oplax bimonoid with an oplax antipode. A *morphism* of oplax Hopf monoids is an oplax bimonoid morphism which preserves the antipode.

Notice that from Theorem 3.3.4 it follows that the antipode is essentially unique when it exists; hence 'being Hopf' is a property on oplax bimonoids and not structure. We obtain a (bi)category $\mathsf{OplHopf}(\mathcal{K})$ of oplax Hopf monoids and morphisms between them, with a faithful forgetful functor to $\mathsf{OplBimon}(\mathcal{K})$ of oplax bimonoids. Moreover, Proposition 3.1.3 extends to the case of oplax Hopf monoids as follows.

Proposition 3.4.4. *If $\mathcal{F}\colon \mathcal{K} \to \mathcal{L}$ is a braided monoidal pseudofunctor and M is an oplax Hopf monoid in $\mathcal{K}$ with antipode s, then $\mathcal{F}M$ is an oplax Hopf monoid in $\mathcal{L}$ with antipode $\mathcal{F}s$.*

Proof. Suppose the monoidal pseudofunctor $\mathcal{F}$ has structure maps the pseudonatural equivalences $\phi_{X,Y}\colon \mathcal{F}X \otimes \mathcal{F}Y \to \mathcal{F}(X \otimes Y)$ and $\psi_{X,Y}\colon \mathcal{F}(X \otimes Y) \to \mathcal{F}X \otimes \mathcal{F}Y$ with $\phi\psi \cong \psi\phi \cong \mathsf{id}$, and similarly for ϕ_0, ψ_0. The 1-cell $\mathcal{F}s\colon \mathcal{F}M \to \mathcal{F}M$ along with the composite 2-cells formed from $\mathcal{F}\tau_1$ as in

and similarly the one formed from $\mathcal{F}\tau_2$ constitute an oplax antipode for the oplax bimonoid $\mathcal{F}M$. This can be also deduced from the fact that $\mathcal{F}_{M,M}\colon \mathcal{K}(M,M) \to \mathcal{K}(\mathcal{F}M,\mathcal{F}M)$ is a strong $\odot$-monoidal functor with structure maps essentially those on the top and the bottom of the diagram, therefore preserves oplax inverses. $\square$

Finally, we can also express the definition of an oplax Hopf monoid in terms of fusion morphisms, similarly to the classical case recalled in Section 2.1. For an oplax bimonoid $(M, m, j, \delta, \varepsilon)$, consider $M \otimes M$ as a (strict) left M-module and (strict) right M-comodule with actions given by

$$\rho = M \otimes M \otimes M \xrightarrow{m \otimes 1_M} M \otimes M$$

$$\chi = M \otimes M \xrightarrow{1_M \otimes \delta} M \otimes M \otimes M$$

If we denote by ${}_M\mathcal{K}^M(M \otimes M, M \otimes M)$ the monoidal subcategory of the endo-hom category $(\mathcal{K}(M \otimes M, M \otimes M), \circ, 1_{M \otimes M})$ that consists of strict M-module and M-comodule morphisms, we have the following result.

Lemma 3.4.5. *For M an oplax bimonoid in a monoidal bicategory $\mathcal{K}$, there exists an isomorphism of monoidal categories*

$$({}_M\mathcal{K}^M(M \otimes M, M \otimes M), \circ, 1_{M \otimes M}) \cong (\mathcal{K}(M, M), \odot, I^\odot).$$

Proof. Define $F \colon \mathcal{K}(M, M) \to {}_M\mathcal{K}^M(M \otimes M, M \otimes M)$ mapping any $f \colon M \to M$ to

$$F(f) = M \otimes M \xrightarrow{1_M \otimes \delta} M \otimes M \otimes M \xrightarrow{1_M \otimes f \otimes 1_M} M \otimes M \otimes M \xrightarrow{m \otimes 1_M} M \otimes M.$$

One easily verifies that this is left M-linear and right M-colinear. Conversely, given any left M-module and right M-comodule map $g \colon M \otimes M \to M \otimes M$, define

$$G(g) = M \xrightarrow{j \otimes 1_M} M \otimes M \xrightarrow{g} M \otimes M \xrightarrow{1_M \otimes \varepsilon} M$$

The bijection is established using the strict (co)unity conditions for M, and furthermore it can be checked that $F(f \odot f') = F(f) \circ F(f')$, hence this is a monoidal isomorphism. $\square$

Under the above isomorphism, the identity $1_M \in \mathcal{K}(M, M)$ corresponds to the so-called *fusion* 1-cell

$$M \otimes M \xrightarrow{1_M \otimes \delta} M \otimes M \otimes M \xrightarrow{m \otimes 1_M} M \otimes M \tag{23}$$

and the following can be easily deduced from Definition 3.4.2.

Proposition 3.4.6. *An oplax bimonoid is an oplax Hopf monoid if and only if the fusion 1-cell $(m \otimes 1_M) \circ (1_M \otimes \delta)$ has an oplax inverse in ${}_M\mathcal{K}^M(M \otimes M, M \otimes M)$.*

4. The object X^2 in Span

In this section, we describe various algebraic properties of X^2 as an object in the symmetric monoidal bicategory Span; these results will be necessary to capture the structure of Hopf and Frobenius $\mathcal{V}$-categories in Sections 6 and 7, where X serves as their set of objects. Although the results of this section are valid for arbitrary groupoids as discussed in Remark 4.5.2, we provide explicit proofs only for X^2.

4.1 Spans. Recall that in any category $\mathcal{C}$, a span $X \twoheadrightarrow Y$ is a diagram of the form $X \xleftarrow{f} S \xrightarrow{g} Y$. A morphism between spans is a map $u : S \to T$ making both left and right triangles commute:

$$
\begin{array}{ccc}
 & S & \\
f \swarrow & \downarrow u & \searrow g \\
 & T & \\
 & h \swarrow \quad \searrow k & \\
X & & Y
\end{array}
\tag{24}
$$

If $\mathcal{C}$ has pullbacks, the composite $X \twoheadrightarrow Y \twoheadrightarrow Z$ is $X \longleftarrow S \times_Y R \longrightarrow Z$ in the usual way. Since pullbacks are unique only up to isomorphism, the above data forms a bicategory. As explained in Section 2.3, using coherence we can work with Span as if it was a strict 2-category; on the other hand, taking isomorphism classes of spans we can work with a 1-category also denoted Span, and it should be clear from the context which one we mean. From this point on, we restrict ourselves to Span for $\mathcal{C} = \mathsf{Set}$, where spans can also be expressed as $(f, g) \colon S \to X \times Y$ as is the case in any category with products.

Remark 4.1.1. An evident observation, which will nevertheless be very useful in checking various axioms in the following sections, is that there is at most one span morphism with target a span with a monic leg. Explicitly, if there were two maps of spans $u, u' \colon S \to T$ as in (24) and for example the leg $h \colon T \to X$ is monic, then $u = u'$ since $h \circ u = f = h \circ u'$. More generally, a similar argument shows that there is at most one span morphism with target a span with jointly monic legs. As a result, in later axioms that involve equalities of pasted composites of 2-cells in Span, whenever the target span has that property, the axiom verification is straightforward since there can only be one such 2-cell.

The bicategory $(\mathsf{Span}, \times, 1 = \{\star\})$ is symmetric monoidal, with $\sigma \colon X \times Y \twoheadrightarrow Y \times X$ given by the span $X \times Y \xleftarrow{\mathsf{id}} X \times Y \xrightarrow{\mathsf{sw}} Y \times X$ where sw is the switch map $(x, y) \mapsto (y, x)$ in Set. If we view Set as a 2-category with trivial 2-cells, there is a faithful (strict) monoidal (strict) functor
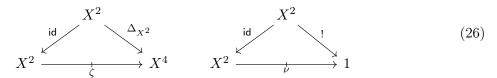
$$
\mathsf{Set} \to \mathsf{Span}
\tag{25}
$$

which acts as the identity on objects, and maps a function $f \colon X \to Y$ to the span $X \xleftarrow{\mathsf{id}} X \xrightarrow{f} Y$. Indeed $g \circ f \colon X \to Y \to Z$ is mapped to the span $X \xleftarrow{\mathsf{id}} X \xrightarrow{g \circ f} Z$ by choosing pullbacks along the identity to be identity in Set. In a similar way there is an embedding from $\mathsf{Set}^{\mathrm{op}}$ to Span.

4.2 Trivial monoid and comonoid structures. Since every set X has a canonical comonoid structure in Set given by the diagonal map $\Delta \colon X \to X \times X$ and unique map $! \colon X \to 1$, the monoidal embedding $\mathsf{Set} \to \mathsf{Span}$ as in (25) yields a comonoid structure on X in Span with $(\mathsf{id}, \Delta) \colon X \twoheadrightarrow X \times X$ as the comultiplication and $(\mathsf{id}, !) \colon X \twoheadrightarrow 1$ as the counit; we refer to this as the *trivial comonoid structure* on X in Span. Similarly, the contravariant embedding yields a monoid structure on X which is the reverse of the comonoid structure and is called the *trivial monoid structure* on X. Notice that these structures are *strict* rather than pseudo in the monoidal bicategory of spans, namely the (co)associator and (co)unitors are identities and the axioms (51) are trivially satisfied (since for example $c_{m,m}$ in this case is the identity).
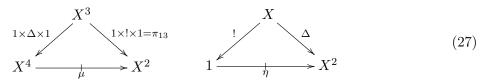
In particular, on an object of the form X^2 in Span, we will denote the trivial comonoid

structure as (X^2, ζ, ν) where

$$
\begin{array}{c}
X^2 \\
\text{id} \swarrow \quad \searrow \Delta_{X^2} \\
X^2 \xrightarrow[\zeta]{} X^4
\end{array}
\qquad
\begin{array}{c}
X^2 \\
\text{id} \swarrow \quad \searrow ! \\
X^2 \xrightarrow[\nu]{} 1
\end{array}
\tag{26}
$$

Remark 4.2.1. Notice that while $\mathsf{Comon(Set)} \cong \mathsf{Set}$ as is the case in any cartesian monoidal category, here $\mathsf{Comon(Span)} \not\cong \mathsf{Span}$ since the latter is no longer cartesian: the categorical product in this category is given by the disjoint union. Hence our so-called 'trivial' comonoid structure above is not so in that usual way.

4.3   Groupoid pseudomonoid and pseudocomonoid structures.

For any set X, one can form the codiscrete category whose objects are elements of X and where there is a single arrow $(x, y) \in X^2$ between any pair of objects $x, y \in X$. The identity arrows are the pairs (x, x) and composition is $((x, y), (y, z)) \mapsto (x, z)$. This category is in fact a groupoid: inverses are computed by $(x, y) \mapsto (y, x)$. This groupoid gives rise to a pseudomonoid structure on X^2 in Span with the following multiplication and unit

$$
\begin{array}{c}
X^3 \\
{\scriptstyle 1\times\Delta\times 1} \swarrow \quad \searrow {\scriptstyle 1\times!\times 1 = \pi_{13}} \\
X^4 \xrightarrow[\mu]{} X^2
\end{array}
\qquad
\begin{array}{c}
X \\
! \swarrow \quad \searrow \Delta \\
1 \xrightarrow[\eta]{} X^2
\end{array}
\tag{27}
$$

We call this the *groupoid pseudomonoid structure* on X^2. The reverse structure

$$
u = (\pi_{13}, 1\times\Delta\times 1)\colon X^3 \to X^2 \times X^4 \qquad v = (\Delta, !)\colon X \to X^2 \times 1
$$

is called the *groupoid pseudocomonoid structure* on X^2.

Explicitly, the coassociator $\beta\colon (1 \times u) \circ u \cong (u \times 1) \circ u$ and counitors $t\colon (v \times 1) \circ u \cong 1$, $s\colon (1 \times v) \circ u \cong 1$ are formed between

$$\tag{28}$$

where P and Q are isomorphic to X^4, and A, B are isomorphic to X^2. The precise span isomorphisms can be computed in a clear but tedious way: for example, taking the usual choice of pullbacks in Set, we can compute that

$$
\begin{aligned}
X^3{\times}X^2{\times}X^3 \supseteq P &= \{((x_1, x_2, x_3), (x_4, x_5), (x_6, x_7, x_8)) \mid (x_1, x_2, x_2, x_3){=}(x_4, x_5, x_6, x_8)\} \\
&\cong \{(x_1, x_2, x_7, x_3)\} \\
X^3{\times}X^3{\times}X^2 \supseteq Q &= \{((y_1, y_2, y_3), (y_4, y_5, y_6), (y_7, y_8)) \mid (y_1, y_2, y_2, y_3){=}(y_4, y_6, y_7, y_8)\} \\
&\cong \{(y_1, y_5, y_2, y_3)\}
\end{aligned}
$$

and the precise isomorphism that commutes with the legs can be written according to the above representation, namely $x_1 \mapsto y_1$, $x_2 \mapsto y_5$, $x_7 \mapsto y_2$ and $x_3 \mapsto y_3$.

Notice that by Remark 4.1.1, the above structure span isomorphisms are unique: the counitors involve the identity, and the coassociator has monic legs on the right side. For the same reasons, the pseudocomonoid axioms – dual to (51) – hold: both span 2-isomorphisms involve a span with a monic leg, therefore they must be equal.

4.4 Oplax bimonoid and oplax Hopf monoid structures. The following result establishes the oplax bimonoid structure of X^2 in the symmetric monoidal bicategory Span.

Proposition 4.4.1.

    (i) For any set X, the groupoid pseudomonoid (μ, η) (27) and trivial comonoid (ζ, ν) (26) structures on X^2 make it an oplax bimonoid in Span.

    (ii) For any set X, the trivial monoid and groupoid pseudocomonoid structures on X^2 make it an oplax bimonoid in Span.

Proof. Since Span is a symmetric monoidal bicategory and X^2 is a strict comonoid with the trivial comonoid structure, $X^2 \times X^2$ is a pseudocomonoid with comultiplication and counit similarly to (5)

$$
X^2 \times X^2 \xrightarrow{\;\zeta \times \zeta\;} X^2 \times X^2 \times X^2 \times X^2 \xrightarrow{\;1 \times \sigma_{X^2, X^2} \times 1\;} X^8, \qquad\qquad X^2 \times X^2 \xrightarrow{\;\nu \times \nu\;} 1
$$

The data for the oplax bimonoid structure (Definition 3.1.1) are span morphisms $\theta, \theta_0, \chi, \chi_0$ defined below.

$$(29)$$

$$
\tag{30}
$$

where $X^4 \times X^6 \supseteq P \cong X^3$ and θ is the (unique) span isomorphism that fits therein, like above. The conditions listed in Section A.2 can all be verified again by Remark 4.1.1 since all have a span with monic leg as target, therefore $(X^2, \mu, \eta, \zeta, \nu, \mathsf{id}, \Delta, 1\times\Delta\times1, !)$ is an oplax bimonoid in the symmetric monoidal bicategory Span.
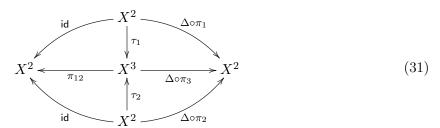
Since Span is isomorphic to $\mathsf{Span}^{\mathrm{op}}$, the two statements are equivalent. $\qquad\square$

Remark 4.4.2. In general, asking for a pseudomonoid and pseudocomonoid to be an oplax bimonoid amounts to specifying structure, not a property; it requires the existence of four 2-cells subject to axioms. However, there are cases like above where that structure is unique when it exists, according to Remark 4.1.1: in all cases (29) and (30), the target span has at least one monic leg.

Finally, X^2 is an oplax Hopf monoid in Span as in Definition 3.4.3.

Proposition 4.4.3. *For any set X, the oplax bimonoid structure $(X^2, \mu, \eta, \zeta, \nu)$ uniquely extends to an oplax Hopf monoid structure on X^2.*

Proof. The oplax antipode is given by the span $s\colon X^2 \xleftarrow{\mathsf{id}} X^2 \xrightarrow{\mathsf{sw}} X^2$, along with two 2-cells τ_1, τ_2 (21) in Span, computed to be the upper and lower span morphisms

$$
\tag{31}
$$

with $\tau_1\colon (a,b) \mapsto (a,b,a)$ and $\tau_2\colon (a,b) \mapsto (a,b,b)$. These are monomorphisms, and moreover according to Remark 4.4.2 they are unique. One can now verify that s and 1_{X^2} by means of the 2-cells τ_1 and τ_2 are oplax inverses in the convolution category $\mathsf{Span}(X^2, X^2)$. $\qquad\square$

Remark 4.4.4. We could alternatively consider the fusion 1-cell (23) for $(X^2, \mu, \eta, \zeta, \nu)$

$$
c\colon X^4 \xrightarrow{1_{X^2}\times\zeta} X^6 \xrightarrow{\mu\times1_{X^2}} X^4
$$

which is computed to be a span $X^4 \leftarrow X^3 \to X^4$ defined by $(a,b,b,c) \leftarrow\!\shortmid (a,b,c) \mapsto (a,c,b,c)$. As expected from Proposition 3.4.6, this is not invertible in Span, but it allows an oplax inverse in $_{X^2}\mathsf{Span}^{X^2}(X^4, X^4)$ namely the following composite

$$
\bar{c}\colon X^4 \xrightarrow{1_{X^2}\times\zeta} X^6 \xrightarrow{1_{X^2}\times\sigma\times1_{X^2}} X^6 \xrightarrow{\mu\times1_{X^2}} X^4
$$

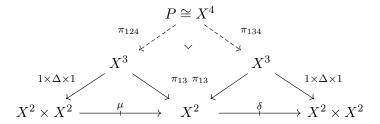Once we recognize that both legs of the span c are monomorphisms, it is immediate that $\bar{c}$ and c are oplax inverses. Notice that $\bar{c}$ given by $(a,b,c,b) \leftarrow\!\shortmid (a,b,c) \mapsto (a,c,c,b)$ is the reverse span of c up to a switch between b and c.

4.5 Frobenius monoid structure. The following proposition exhibits how different combinations of the earlier monoid and comonoid structures give rise to Frobenius (pseudo)monoids in the symmetric monoidal bicategory Span.
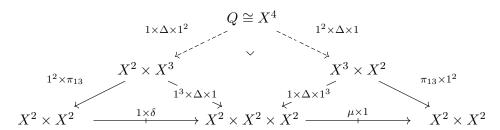
Proposition 4.5.1.

(i) *For any set X, the trivial monoid $((\Delta, \mathsf{id}), (!, \mathsf{id}))$ and its reverse trivial comonoid structures make X into a Frobenius (strict) monoid in Span; clearly also X^2 with (26) and its reverse.*

(ii) *For any set X, the groupoid monoid (μ, η) (27) and its reverse groupoid comonoid structure (u, v) on X^2 make it a Frobenius pseudomonoid in Span.*

Proof. The first statement can be easily verified. For the second one, we need to provide the structure isomorphisms of (6) and verify the conditions of Section A.5. The middle composite is

which explicitly acts as $(a, b, b, d) \leftarrow\!\shortmid (a, b, c, d) \mapsto (a, c, c, d)$. The downside composite on the other hand is

So the span isomorphism is between $P \subseteq X^3 \times X^3$ and $Q \subseteq X^2 \times X^3 \times X^3 \times X^3$, and the other isomorphism is formed similarly. Even though the spans involved do not have monic legs in this case as earlier, they are jointly monic; again by Remark 4.1.1, this fact renders the structure span isomorphisms unique. Finally, all conditions (73) to (76) hold for the same reasons. $\qquad\square$

Remark 4.5.2. The above structures arise not just on the set X^2, but for an arbitrary groupoid $\mathcal{G} = (\, G_1 \underset{t}{\overset{s}{\rightrightarrows}} G_0 \,)$ with set of objects G_0, set of morphisms G_1, and s, t the source and target maps. Any $\mathcal{G}$ gives rise to a pseudomonoid (and a pseudocomonoid) in Span with multiplication and counit as follows

$$(32)$$

where $G_2 = G_{1\,s}\times_t G_1$ are the composable morphisms of $\mathcal{G}$ and $e : G_0 \to G_1$ gives the identity; this determines the 'groupoid' structures of Section 4.3. Moreover, the trivial monoid and groupoid pseudocomonoid structures on $\mathcal{G}_1$ make it an oplax bimonoid in Span that uniquely extends to an oplax Hopf monoid, analogously to Propositions 4.4.1 and 4.4.3. Finally, the groupoid pseudomonoid and pseudocomonoid structures on $\mathcal{G}_1$ make it a Frobenius pseudomonoid in Span as in Proposition 4.5.1.

The results of this section can be summarised in the following table. The combinations of structures in the columns give oplax Hopf monoid structures on X^2 in Span, and in the rows give Frobenius monoid structures.

Table 1: Summary

	Hopf	Hopf
Frob	Trivial monoid	Trivial comonoid
Frob	Groupoid comonoid	Groupoid monoid

A similar observation was made by Street in [37], where the possible Frobenius and Hopf algebra structures for a group algebra kG were described.

5. The symmetric monoidal bicategory Span$|\mathcal{V}$

In [5], starting from a bicategory $\mathcal{K}$ a new bicategory Span$|\mathcal{K}$ is constructed, with the property that when $\mathcal{K}$ is monoidal so is Span$|\mathcal{K}$. The two special cases that are addressed (due to the Hopf objects of interest) are Span$|$Cat and Span$|\mathcal{V}$, for $\mathcal{V}$ a braided monoidal category considered as a one-object monoidal bicategory.

In our context, we take $\mathcal{V}$ to be any monoidal category, this time viewed as a monoidal 2-category with trivial 2-cells; this induces some sort of 'level-shift' compared to Böhm's primary examples. We give an explicit description of the monoidal bicategory Span$|\mathcal{V}$ [5, §2.1] in our particular case of interest, and in the following Sections 6 and 7 we show how it serves as a common framework for Hopf and Frobenius $\mathcal{V}$-categories: they are expressed as oplax Hopf monoids and Frobenius pseudomonoids in Span$|\mathcal{V}$ respectively.

5.1   Monoidal bicategory structure. The 0-cells in Span$|\mathcal{V}$ are pairs (X, M) where X is a set and $M\colon X \to \mathcal{V}$ is a functor, given by a family of objects $\{M_x\}_{x\in X}$ in $\mathcal{V}$; we use the shorthand notation M_X for the pair (X, M). A 1-cell from M_X to N_Y in Span$|\mathcal{V}$ consists of a span $X \xleftarrow{f} S \xrightarrow{g} Y$ of sets, along with a natural transformation

$$S \underset{g}{\overset{f}{\rightrightarrows}} \begin{matrix} X \\ \Downarrow\alpha \\ Y \end{matrix} \overset{M}{\underset{N}{\rightrightarrows}} \mathcal{V} \tag{33}$$

whose components are arrows $\alpha_s\colon M_{fs} \to N_{gs}$ in $\mathcal{V}$. We use the shorthand ${}^f\alpha^g\colon M_X \to N_Y$ and specify S separately when needed. A 2-cell in Span$|\mathcal{V}$ denoted by $\phi_u\colon {}^f\alpha^g \Rightarrow {}^h\beta^k\colon M_X \to N_Y$ is a map of spans $u\colon S \to T$, i.e. $hu = f$ and $ku = g$, such that α factorizes through β in the sense that $\alpha = \beta * 1_u$:

$$S \xrightarrow{u} T \underset{k}{\overset{h}{\rightrightarrows}} \begin{matrix} X \\ \Downarrow\beta \\ Y \end{matrix} \overset{M}{\underset{N}{\rightrightarrows}} \mathcal{V} \quad = \quad S \underset{g}{\overset{f}{\rightrightarrows}} \begin{matrix} X \\ \Downarrow\alpha \\ Y \end{matrix} \overset{M}{\underset{N}{\rightrightarrows}} \mathcal{V} \tag{34}$$

Componentwise, this means that $\alpha_s = \beta_{us}\colon M_{fs} \to N_{gs}$ in $\mathcal{V}$. Notice that a 2-cell is invertible if and only if the underlying map of spans is invertible. Vertical composition is performed by the

usual composition of maps of spans, and horizontal composition $M_X \xrightarrow{\alpha} N_Y \xrightarrow{\gamma} O_Z$ of 1-cells is obtained via the pasted composite of natural transformations
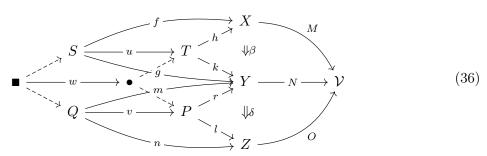
$$\begin{array}{c}\text{(diagram 35)}\end{array} \qquad (35)$$

given by $\mathcal{V}$-arrows $(\gamma \circ \alpha)_{(s,q)} \colon M_{fs} \xrightarrow{\alpha_s} N_{gs} \xrightarrow{\equiv} N_{mq} \xrightarrow{\beta_q} O_{nq}$ for any $(s,q) \in S \times_Y Q$. The identity $1_M \colon M_X \to M_X$ is given by the identity span and natural transformation. Finally, horizontal composition of 2-cells in $\mathsf{Span}|\mathcal{V}$ follows from horizontal composition of 2-cells in Span: in detail,

$$\phi * \psi = \quad M_X \underset{\beta}{\overset{\alpha}{\Downarrow \phi}} N_Y \underset{\delta}{\overset{\gamma}{\Downarrow \psi}} O_Z$$

is given by the map of spans w below,

$$\begin{array}{c}\text{(diagram 36)}\end{array} \qquad (36)$$

where $\blacksquare$ is the pullback of g and m and $\bullet$ is the pullback of r and k, and the factorization of $\gamma \circ \alpha$ through $\delta \circ \beta$ is

$$\gamma_q \circ \alpha_s = (\gamma \circ \alpha)_{(s,q)} = (\delta \circ \beta)_{w(s,q)} = (\delta \circ \beta)_{(us,vq)} = \delta_{vq} \circ \beta_{us}$$

for any (s,q) in the square pullback, since $\alpha_s = \beta_{us}$ and $\gamma_q = \delta_{vq}$ by ϕ and ψ.

With the above data, $\mathsf{Span}|\mathcal{V}$ becomes a bicategory. Notice that $\mathsf{Span}|(\mathcal{V}^{\mathrm{op}}) \cong (\mathsf{Span}|\mathcal{V})^{\mathrm{op}}$.

Remark 5.1.1. By definition of a 2-cell in $\mathsf{Span}|\mathcal{V}$, namely a map of spans which satisfies a factorization property (34), it follows that two 2-cells are equal if and only if their underlying maps of spans in Span are equal; the accompanying factorizations are conditions that already hold. This fact is very useful when verifying axioms including 2-cells in subsequent sections.

Since $\mathcal{V}$ is a monoidal 2-category (with trivial 2-cells), the bicategory $\mathsf{Span}|\mathcal{V}$ has an induced monoidal structure as follows. On objects, the tensor product $M_X \otimes N_Y = (M \otimes N)_{X \times Y}$ is given by $(M \otimes N)_{x,y} = M_x \otimes N_y$ in $\mathcal{V}$, namely the composite

$$X \times Y \xrightarrow{M \times N} \mathcal{V} \times \mathcal{V} \xrightarrow{\otimes \times \otimes} \mathcal{V}.$$

For 1-cells, ${}^f \alpha^g \otimes {}^k \beta^l$ is of the form

$$\begin{array}{c}\text{(diagram 37)}\end{array} \qquad (37)$$

given by morphisms $M_{fs} \otimes Q_{kt} \xrightarrow{\alpha_s \otimes \beta_t} N_{gs} \otimes R_{lt}$ in $\mathcal{V}$. For 2-cells, it is given by the product of the factorizing functions (maps of spans) in (34), and the monoidal unit is $I_\mathcal{V} \colon \mathbf{1} \to \mathcal{V}$ that picks out the unit in $\mathcal{V}$. Finally, if $(\mathcal{V}, \otimes, I, \sigma)$ is braided, then so is $\mathsf{Span}|\mathcal{V}$ via $\bar{\sigma}_{MN} \colon M_X \otimes N_Y \to N_Y \otimes M_X$ given by

$$
\begin{array}{ccc}
& \xrightarrow{\;\mathsf{id}\;} X \times Y \xrightarrow{\;M \otimes N\;} & \\
X \times Y & \Downarrow \bar{\sigma}_{MN} & \mathcal{V} \\
& \xrightarrow{\;\mathsf{sw}\;} Y \times X \xrightarrow{\;N \otimes M\;} &
\end{array}
$$

with components $M_x \otimes N_y \xrightarrow{\sigma_{M_x, N_y}} N_y \otimes M_x$ in $\mathcal{V}$.

5.2   The underlying span functor. There is an evident strict functor of bicategories $U \colon \mathsf{Span}|\mathcal{V} \to \mathsf{Span}$ that forgets the data associated with $\mathcal{V}$. In more detail, U maps a 0-cell A_X to the set X, a 1-cell ${}^f\alpha^g$ to the span $X \leftarrow\!f\!- S -\!g\!\to Y$ and a 2-cell ϕ_u to the map of spans u; it is clear that the composition is preserved on the nose. This functor is strict monoidal, since

$$
U(A_X) \otimes_{\mathsf{Span}} U(B_Y) = X \times Y = U(A_X \otimes_{\mathsf{Span}|\mathcal{V}} B_Y), \qquad U(I_1) = 1 \tag{38}
$$

and similarly for morphisms. Since U is a (symmetric) strict monoidal strict functor of bicategories, Proposition 3.1.3 ensures that any pseudomonoid or pseudocomonoid object in $\mathsf{Span}|\mathcal{V}$ has an underlying pseudo(co)monoid in Span which by the definitions is part of the structure; e.g. if A_X is a pseudomonoid in $\mathsf{Span}|\mathcal{V}$, X must be a pseudomonoid in Span.

In fact, this strict functor of bicategories is a kind of *2-opfibration*, see [14]. Briefly, therein the usual Grothendieck construction (see [12, §8]) extends to give a correspondence between 2-functors $F \colon \mathbb{X}^{\mathrm{op}} \to$ 2-Cat and 2-fibrations $\int F \to \mathbb{X}$, and a similar correspondence between trihomomorphisms from a bicategory into Bicat and fibrations of bicategories. The covariant case is not explicitly described therein, but is easily obtained by modifying the contravariant construction. In the case of a covariant F, the resulting functor is a 2-opfibration and the 2-category $\int F$ has objects $(X, A \in FX)$, morphisms pairs $(f \colon X \to Y \in \mathbb{X}, \alpha \colon Ff(A) \to B \in FY)$ and 2-cells $(f, \alpha) \Rightarrow (g, \beta)$

$$
\begin{array}{ccccc}
& \xrightarrow{\;f\;} & & Ff(A) & \\
X & \Downarrow \phi & Y & \text{in } \mathbb{X} \text{ such that} \quad (F\alpha)_A \Big\downarrow \quad \searrow \alpha & \text{in } FY. \\
& \xrightarrow{\;g\;} & & Fg(A) \xrightarrow{\;\beta\;} B &
\end{array} \tag{39}
$$

The work of Buckley also omits any version of this construction where F is not as strict as it can be (a 2-functor from a 2-category into 2-Cat) or as weak as it can be (a trihomomorphism from a bicategory into Bicat), but those intermediate versions can be generated by adapting the construction accordingly. In our case, we need the covariant version where $\mathbb{X}$ is a bicategory and F is a pseudofunctor into 2-Cat, when $\int F$ becomes a bicategory and $\int F \to \mathbb{X}$ a strict functor.

Theorem 5.2.1. *If $\mathcal{V}$ has colimits, $U \colon \mathsf{Span}|\mathcal{V} \to \mathsf{Span}$ is a 2-opfibration.*

Proof. We build a pseudofunctor $\mathsf{Span} \to \mathsf{Cat} \hookrightarrow$ 2-Cat and apply a modified Grothendieck construction from [14]; the resulting 2-opfibration is isomorphic to U.

Briefly recall that the left Kan extension of a functor $K \colon \mathcal{A} \to \mathcal{C}$ along some $F \colon \mathcal{A} \to \mathcal{B}$ is a functor $\mathrm{Lan}_F K \colon \mathcal{B} \to \mathcal{C}$ equipped with a natural transformation $\eta_K \colon K \Rightarrow (\mathrm{Lan}_F K) \circ F$ such

that $\mathrm{Lan}_F \dashv F_* = - \circ F$ with unit η. When $\mathcal{C}$ has colimits, the left Kan extension always exists and can be computed as the coend [30, (4.25)]

$$\mathrm{Lan}_F K(b) := \int^{a \in \mathcal{A}} \mathcal{B}(Fa, b) \cdot Ka$$

where $\cdot$ denotes the set-theoretic copower.

Define a pseudofunctor $\mathsf{Span} \to \mathsf{Cat}$ which maps X to $[X, \mathcal{V}]$, a span $X \xleftarrow{f} S \xrightarrow{g} Y$ to $\mathrm{Lan}_g(- \circ f) \colon [X, \mathcal{V}] \to [Y, \mathcal{V}]$ and a map of spans $u \colon (f, g) \to (h, k)$ (24) to a natural transformation $\bar{u}$ whose components are

$$\bar{u}_A \colon \mathrm{Lan}_g(Af) = \mathrm{Lan}_g(Ahu) \cong \mathrm{Lan}_k(\mathrm{Lan}_u(Ahu)) \xrightarrow{\mathrm{Lan}_k(\varepsilon_{Ah})} \mathrm{Lan}_k(Ah) \tag{40}$$

for all $A \in [X, \mathcal{V}]$, where the isomorphism is a standard property for Kan extensions of composites [30, (4.48)]. This action on 1- and 2-cells defines a functor for all X, Y in Span. For pseudofunctoriality, we need natural isomorphisms $\mathrm{Lan}_{\mathsf{id}}(- \circ \mathsf{id}) = \mathsf{id}_{[X, \mathcal{V}]}$ and $\mathrm{Lan}_k(\mathrm{Lan}_g(- \circ f) \circ h) \cong \mathrm{Lan}_{k\pi_2}(- \circ f\pi_1)$ for every pullback of spans

The identity is straightforward, while the second isomorphism arises as the composite

where the middle isomorphism is realised via the following coends computation, for any $B \in [S, \mathcal{V}]$:

$$\mathrm{Lan}_g B(ht) = \int^{s \in S} Y(gs, ht) \cdot Bs \cong \int^{s \in S}_{gs = ht} Bs = \sum_{gs = ht} Bs \qquad (Y, S \text{ are sets})$$

$$\mathrm{Lan}_{\pi_2}(B\pi_1)t = \int^{(s, t') \in Q} T(\pi_2(s, t'), t) \cdot (B\pi_1)(s, t') \cong \int^{(s, t') \in Q}_{t = t'} Bs = \sum_{gs = ht} Bs.$$

The axioms for pseudofunctoriality can be shown to hold using the universal properties of these coends. If we compose this functor with the usual 2-functor $\mathsf{Cat} \to \mathsf{2\text{-}Cat}$ and apply the Grothendieck construction, we obtain the bicategory whose objects are $(X, A \in [X, \mathcal{V}])$, 1-cells are pairs

$$\begin{cases} \mathrm{Lan}_g(Af) \xrightarrow{\alpha} B & \text{in } [Y, \mathcal{V}] \\ X \xleftarrow{f} S \xrightarrow{g} Y & \text{in } \mathsf{Span} \end{cases} \tag{41}$$

and 2-cells are, for u as in (24) and $\bar{u}$ as in (40),

$$
(f,g) \xrightarrow{u} (h,k) \text{ in Span such that} \quad
\begin{array}{c}
\mathrm{Lan}_g(Af) \\
{\scriptstyle \bar{u}_A}\downarrow \qquad \searrow {\scriptstyle \alpha} \\
\mathrm{Lan}_k(Ah) \xrightarrow[\beta]{} B
\end{array}
\quad \text{in } [Y,\mathcal{V}]. \tag{42}
$$

This bicategory has exactly the same objects as $\mathsf{Span}|\mathcal{V}$, and there is an isomorphism between their hom-categories: the necessary bijections between (41) and (33), (42) and (34) are exactly the universal property of left Kan extension. This isomorphism commutes with the projections to Span and thus makes $U\colon \mathsf{Span}|\mathcal{V} \to \mathsf{Span}$ a 2-opfibration. $\qquad \square$

The above theorem shows that the definition of the bicategory $\mathsf{Span}|\mathcal{V}$ in fact arises from a series of quite natural constructions.

6. Hopf $\mathcal{V}$-categories as oplax Hopf monoids

In this section, we recall the notion of Hopf $\mathcal{V}$-category [4] for a braided monoidal category $\mathcal{V}$, and we realize them as oplax Hopf monoids in the monoidal bicategory $\mathsf{Span}|\mathcal{V}$, described respectively in Sections 3 and 5. This allows us to understand exactly how such a concept can be described as a 'Hopf object' internal to some monoidal structure, generalizing the classical setting via this relaxed notion of a Hopf monoid in higher dimensions.

Our notation in what follows uses Latin letters to denote 'global' operations that relate hom-objects of different indices (like category composition), and Greek letters to denote 'local' operations that relate hom-objects of fixed indices (like monoid multiplication).

6.1   Hopf $\mathcal{V}$-categories. Intuitively, a category can be thought of as a many-object generalization of a monoid. Since a Hopf monoid — internal to any braided monoidal category — has both a monoid and a comonoid structure, it is reasonable to ask what its many-object generalization should be. A natural answer to this question is as follows.

Definition 6.1.1. [4, §2] If $(\mathcal{V}, \otimes, I, \sigma)$ is braided, a *semi-Hopf $\mathcal{V}$-category H* is a $\mathsf{Comon}(\mathcal{V})$-enriched category. Explicitly, it consists of a collection of objects H_0 and for every $x, y \in H_0$ an object $H_{x,y}$ of $\mathcal{V}$, together with families of morphisms in $\mathcal{V}$

$$
\begin{aligned}
m_{xyz} &\colon H_{x,y} \otimes H_{y,z} \to H_{x,z} & j_x &\colon I \to H_{x,x} \\
\delta_{ab} &\colon H_{a,b} \to H_{a,b} \otimes H_{a,b} & \varepsilon_{ab} &\colon H_{a,b} \to I
\end{aligned}
$$

which make H a $\mathcal{V}$-category, each $H_{x,y}$ a comonoid in $\mathcal{V}$, and satisfy

$$
\begin{array}{ccc}
H_{x,y} \otimes H_{y,z} & \xrightarrow{\delta_{xy} \otimes \delta_{yz}} & H_{x,y} \otimes H_{x,y} \otimes H_{y,z} \otimes H_{y,z} \\
\downarrow m_{xyz} & & \downarrow 1 \otimes \sigma \otimes 1 \\
& & H_{x,y} \otimes H_{y,z} \otimes H_{x,y} \otimes H_{y,z} \\
& & \downarrow m_{xyz} \otimes m_{xyz} \\
H_{x,z} & \xrightarrow{\quad \delta_{xz} \quad} & H_{x,z} \otimes H_{x,z}
\end{array}
\tag{43}
$$

$$
\begin{array}{ccc}
I \xrightarrow{\;\sim\;} I \otimes I & \qquad H_{x,y} \otimes H_{y,z} \xrightarrow{\varepsilon_{xy} \otimes \varepsilon_{yz}} I \otimes I & \qquad I \xrightarrow{\;\mathrm{id}\;} I \\
{\scriptstyle j_x}\downarrow \quad \downarrow {\scriptstyle j_x \otimes j_x} & {\scriptstyle m_{xyz}}\downarrow \qquad \qquad \downarrow {\scriptstyle \sim} & {\scriptstyle j_x}\downarrow \qquad \downarrow {\scriptstyle \mathrm{id}} \\
H_{x,x} \xrightarrow{\delta_{xx}} H_{x,x} \otimes H_{x,x} & \qquad H_{x,z} \xrightarrow{\quad \varepsilon_{xz} \quad} I & \qquad H_{x,x} \xrightarrow{\varepsilon_{xx}} I.
\end{array}
$$

Semi-Hopf $\mathcal{V}$-categories together with $\mathsf{Comon}(\mathcal{V})$-functors and $\mathsf{Comon}(\mathcal{V})$-natural transformations form the 2-category $\mathsf{Comon}(\mathcal{V})$-$\mathsf{Cat}$, denoted by sHopf-$\mathcal{V}$-Cat.

Examples 6.1.2.
1. Every bimonoid in a braided monoidal category $\mathcal{V}$ can be viewed as a 1-object semi-Hopf $\mathcal{V}$-category, therefore semi-Hopf categories are indeed many-object generalizations of bial-gebras.
2. For any cartesian monoidal category $\mathcal{V}$, where by default $\mathsf{Comon}(\mathcal{V}) \cong \mathcal{V}$, any $\mathcal{V}$-enriched category is automatically a semi-Hopf $\mathcal{V}$-category.
3. In [28], it is established that when $\mathcal{V}$ is a locally presentable braided monoidal closed category, the category of monoids $\mathsf{Mon}(\mathcal{V})$ is enriched in $\mathsf{Comon}(\mathcal{V})$ via the existence of *universal measuring comonoids*, first introduced by Sweedler in [40]. Therefore $\mathsf{Mon}(\mathcal{V})$ is a semi-Hopf $\mathcal{V}$-category. The assumptions on $\mathcal{V}$ are satisfied by a wide class of examples, like vector spaces, modules over a commutative ring, chain complexes over a commutative ring, Grothendieck toposes etc.

Definition 6.1.3. [4, Def. 2.3] A *Hopf $\mathcal{V}$-category* is a semi-Hopf $\mathcal{V}$-category equipped with a family of maps $s_{xy} \colon H_{x,y} \to H_{y,x}$ in $\mathcal{V}$ satisfying

$$
\begin{array}{ccccc}
& & H_{x,y} \otimes H_{x,y} \xrightarrow{H_{x,y} \otimes s_{xy}} H_{x,y} \otimes H_{y,x} & & \\
& {\scriptstyle \delta_{xy}}\nearrow & & {\scriptstyle m_{xyx}}\searrow & \\
H_{x,y} & \xrightarrow{\quad \varepsilon_{xy} \quad} & I & \xrightarrow{\quad j_x \quad} & H_{x,x}
\end{array}
\tag{44}
$$

$$
\begin{array}{ccccc}
& & H_{x,y} \otimes H_{x,y} \xrightarrow{s_{xy} \otimes H_{x,y}} H_{y,x} \otimes H_{x,y} & & \\
& {\scriptstyle \delta_{xy}}\nearrow & & {\scriptstyle m_{yxy}}\searrow & \\
H_{x,y} & \xrightarrow{\quad \varepsilon_{xy} \quad} & I & \xrightarrow{\quad j_y \quad} & H_{y,y} \ .
\end{array}
$$

Such an identity-on-objects $\mathcal{V}$-graph map $s \colon H \to H^{\mathrm{op}}$ is called the *antipode* of H.

If H and K are Hopf $\mathcal{V}$-categories, a $\mathsf{Comon}(\mathcal{V})$-functor $F \colon H \to K$ is called a *Hopf $\mathcal{V}$-functor* if $s_{f(x)f(y)} \circ F_{xy} = F_{yx} \circ s_{xy}$ for all $x, y \in X$. In fact, any $\mathsf{Comon}(\mathcal{V})$-functor automatically satisfies that condition [4, 2.10]; hence we have a full 2-subcategory Hopf-$\mathcal{V}$-Cat of sHopf-$\mathcal{V}$-Cat.

Examples 6.1.4.

1. Every Hopf algebra H in a braided monoidal $\mathcal{V}$ is a 1-object Hopf $\mathcal{V}$-category; in particular, each 'diagonal' hom-object $H_{x,x}$ of a Hopf $\mathcal{V}$-category H is a Hopf monoid in $\mathcal{V}$.

2. A Hopf Set-category is precisely a groupoid; in general, for a cartesian monoidal $\mathcal{V}$, a Hopf $\mathcal{V}$-category is a $\mathcal{V}$-*groupoid*, namely a $\mathcal{V}$-category whose hom-objects are equipped with inversion maps.

3. If we replace $\mathcal{V}$ with $\mathcal{V}^{\mathrm{op}}$, we obtain the notion of a *(semi-)Hopf $\mathcal{V}$-opcategory*, called *dual* Hopf category in [4]. Since $\mathsf{Comon}(\mathcal{V}^{\mathrm{op}}) \cong \mathsf{Mon}(\mathcal{V})^{\mathrm{op}}$, a semi-Hopf $\mathcal{V}$-opcategory $(C, d, \epsilon, \mu, \eta)$ is precisely a $\mathsf{Mon}(\mathcal{V})$-opcategory, i.e. is equipped with 'global' cocomposition and coidentity morphisms d_{xyz}, ϵ_x as in (2), together with 'local' multiplication and unit morphisms $\mu_{xy} \colon C_{x,y} \otimes C_{x,y} \to C_{x,y}$, $\eta_{xy} \colon I \to C_{x,y}$ making each hom-object a monoid in $\mathcal{V}$, subject to compatibility conditions. Moreover, a Hopf $\mathcal{V}$-opcategory comes with arrows $s_{xy} \colon C_{y,x} \to C_{x,y}$ satisfying dual axioms to (44).

Remark 6.1.5. Hopf $\mathcal{V}$-categories can be realized as *Hopf monads* inside two different monoidal bicategories as follows. In [5, §4.8], Hopf $\mathcal{V}$-categories are captured as certain Hopf monads in $\mathsf{Span}|\mathcal{V}$ (those with indiscrete underlying category), for a one-object monoidal bicategory $\mathcal{V}$. On the other hand, Hopf $\mathcal{V}$-categories are in bijection to Hopf monads in $\mathcal{V}$-Mat, the bicategory of $\mathcal{V}$-matrices; the relevant structure is sketched in [44, Remark 4.8].

6.2   Oplax Hopf monoid structure. In the following theorem, we summarize the main results of this section regarding oplax bimonoid and Hopf monoid structures on an object in $\mathsf{Span}|\mathcal{V}$. Recall that by Propositions 3.1.3 and 3.4.4, the strict monoidal structure (38) of the forgetful functor $U \colon \mathsf{Span}|\mathcal{V} \to \mathsf{Span}$ ensures that the underlying set of an oplax Hopf monoid in $\mathsf{Span}|\mathcal{V}$ is an oplax Hopf monoid in Span; the relevant structures on X^2 were described in Section 4.

Theorem 6.2.1. *Let $\mathcal{V}$ be a braided monoidal category, and X any set.*

1. *A pseudomonoid (resp. pseudocomonoid) in $\mathsf{Span}|\mathcal{V}$ over the groupoid pseudomonoid (resp. pseudocomonoid) X^2 in Span is exactly a $\mathcal{V}$-category (resp. $\mathcal{V}$-opcategory) with objects X.*

2. *A monoid (resp. comonoid) in $\mathsf{Span}|\mathcal{V}$ over the trivial monoid (resp. comonoid) X^2 in Span is exactly a $\mathsf{Mon}(\mathcal{V})$-graph (resp. $\mathsf{Comon}(\mathcal{V})$-graph) with objects X.*

3. *An oplax bimonoid in $\mathsf{Span}|\mathcal{V}$ over the (groupoid pseudomonoid, trivial comonoid) oplax bimonoid X^2 in Span is exactly a semi-Hopf $\mathcal{V}$-category.*

4. *An oplax bimonoid in $\mathsf{Span}|\mathcal{V}$ over the (trivial monoid, groupoid pseudocomonoid) oplax bimonoid X^2 in Span is exactly a semi-Hopf $\mathcal{V}$-opcategory.*

5. *An oplax Hopf bimonoid in $\mathsf{Span}|\mathcal{V}$ over the (groupoid pseudomonoid, trivial comonoid) oplax Hopf bimonoid X^2 in Span is exactly a Hopf $\mathcal{V}$-category.*

6. *An oplax Hopf bimonoid in $\mathsf{Span}|\mathcal{V}$ over the (trivial monoid, groupoid pseudocomonoid) oplax Hopf bimonoid X^2 in Span is exactly a Hopf $\mathcal{V}$-opcategory.*

Similar results are obtained for the morphisms between such objects, leading to isomorphisms of the corresponding categories. We provide proofs of some parts of the theorem below, and the rest follow analogously.

Proposition 6.2.2. *A pseudocomonoid in $\mathsf{Span}|\mathcal{V}$ over the set X^2 with the groupoid pseudocomonoid structure in Span is precisely a $\mathcal{V}$-opcategory with set of objects X. Dually, a pseu-*

domonoid in Span$|\mathcal{V}$ *over the set* X^2 *with the groupoid pseudomonoid structure in* Span *is a* $\mathcal{V}$*-category.*

Proof. Consider a pseudocomonoid C_{X^2} in Span$|\mathcal{V}$, with underlying pseudocomonoid X^2 in Span equipped with the reversed (27). It consists of an object $C\colon X^2 \to \mathcal{V}$, i.e. a family of objects $\{C_{x,y}\}_{x,y\in X}$ in $\mathcal{V}$, along with $d\colon C\otimes C \to C$ and $\epsilon\colon I \to C$ in Span$|\mathcal{V}$, i.e. natural transformations

with components

$$d_{xyz}\colon C_{x,z} \to C_{x,y} \otimes C_{y,z} \qquad \text{and} \qquad \epsilon_x\colon C_{x,x} \to I.$$

The coassociativity isomorphism β dual to (3), using the composition formula (35), amounts to

which is the invertible map of spans (28), satisfying the factorization (34) expressed by

$$((1_C \otimes d)\circ d)_{xyzw} \colon C_{x,w} \xrightarrow{d_{xyw}} C_{x,y}\otimes C_{y,w} \xrightarrow{1\otimes d_{yzw}} C_{x,y}\otimes C_{y,z}\otimes C_{z,w}$$

$$((d \otimes 1_C)\circ d)_{xyzw} \colon C_{x,w} \xrightarrow{d_{xzw}} C_{x,z}\otimes C_{z,w} \xrightarrow{d_{xyz}\otimes 1} C_{x,y}\otimes C_{y,z}\otimes C_{z,w}$$

which is precisely the first $\mathcal{V}$-opcategory axiom, see Section 2.2. Analogously, from the counity isomorphisms we obtain the coidentity axioms for enriched opcategories. □

On the level of morphisms between such structures, the following indicates the relation of $\mathcal{V}$-functors and oplax morphisms between pseudomonoids in Span$|\mathcal{V}$.

Proposition 6.2.3. *Suppose we have two pseudomonoids A_{X^2} and B_{Y^2} in* Span$|\mathcal{V}$ *with the groupoid pseudomonoid structure on their underlying sets, i.e. two $\mathcal{V}$-categories. An oplax pseudomonoid morphism in* Span$|\mathcal{V}$ *of the form*

(45)

is precisely a $\mathcal{V}$-functor. Dually, an oplax pseudocomonoid morphism of the same form is a $\mathcal{V}$*-opfunctor between two $\mathcal{V}$-opcategories.*

Proof. The map $^{\mathrm{id}}\alpha^{f\times f}$ has components $\alpha_{xy}\colon A_{x,y}\to B_{fx,fy}$ and comes equipped with two 2-cells ϕ and ϕ_0 as in (4). The first 2-cell has to be of the form

$$(46)$$

where we compute $S = \{(x,y,z,w)\in X^4 \mid fy = fz\}\subseteq X^4$. On the left and on the right, respectively, we have components

$$(\alpha\circ m)_{xyz}\colon A_{x,y}\otimes A_{y,z}\xrightarrow{m} A_{x,z}\xrightarrow{\alpha_{xz}} B_{fx,fz} \tag{47}$$

$$(m\circ(\alpha\otimes\alpha))_{xyzw}\colon A_{x,y}\otimes A_{z,w}\xrightarrow{\alpha_{xy}\otimes\alpha_{zw}} B_{fx,fy}\otimes B_{fz,fw}\xrightarrow{m} B_{fx,fz}\qquad (fy=fz\ \text{in } S)$$

To define such a 2-cell ϕ in $\mathsf{Span}|\mathcal{V}$, we need a map of spans between the induced outer $Y^2\longleftarrow X^3\longrightarrow X^4$ and $Y^2\longleftarrow S\longrightarrow X^4$ above, that gives a factorization as in (34). The function $1\times\Delta\times 1\colon X^3\to S$ is indeed a map of spans, and the factorization of $(\alpha\circ m)$

requires the equality of the two composites (47), but the second only applied to quadruples (x,y,y,z); this is precisely the first axiom for a $\mathcal{V}$-functor. The second 2-cell is

$$(48)$$

between natural transformations with components $I\xrightarrow{j_x} A_{x,x}\xrightarrow{\alpha_{xx}} B_{fx,fx}$ and $I\xrightarrow{j_y} B_{y,y}$ respectively. Hence define ϕ_0 by the map of spans $f\colon X\to Y$ with a factorization

which gives $\alpha_{xx}\circ j_x = j_{fx}$, precisely the second axiom for a $\mathcal{V}$-functor.

In order to verify that ϕ,ϕ_0 endow α with the structure of an oplax pseudomonoid morphism, we compute the appropriate pasting diagrams (52) using composition formulas like (35) and (36).

Their verifications are greatly simplified by Remark 5.1.1 since they reduce to ones for 2-cells in Span.

Notice that both ϕ and ϕ_0 are uniquely defined as above, since at each case there is only one span morphism possible, see Remark 4.1.1. Therefore $\mathcal{V}$-functors are in bijection with oplax monoid morphisms $^{\mathrm{id}}\alpha^{f \times f}$.

Dually, taking into account that $\mathcal{V}$-opfunctors are $\mathcal{V}^{\mathrm{op}}$-functors hence live inside

$$\mathsf{PsMon}_{\mathsf{opl}}(\mathsf{Span}|(\mathcal{V}^{\mathrm{op}})) \simeq \mathsf{PsMon}_{\mathsf{opl}}((\mathsf{Span}|\mathcal{V})^{\mathrm{op}}) \simeq \mathsf{PsComon}_{\mathsf{opl}}(\mathsf{Span}|\mathcal{V})^{\mathrm{op}}$$

we deduce that an oplax comonoid map of the form $^{\mathrm{id}}\alpha^{f \times f}$ is a $\mathcal{V}$-opfunctor between the corresponding $\mathcal{V}$-opcategories. $\qquad\square$

As a result, we can realize $\mathcal{V}$-Cat as a subcategory of $\mathsf{PsMon}_{\mathsf{opl}}(\mathsf{Span}|\mathcal{V})$, the bicategory of pseudomonoids and oplax morphisms between them, and $\mathcal{V}$-opCat as a subcategory of $\mathsf{PsComon}_{\mathsf{opl}}(\mathsf{Span}|\mathcal{V})^{\mathrm{op}}$. The strict associativity and unitality of composition of 1-cells (45) is due to the fact that they actually come from functions, i.e. are in the image of (25).

We now turn to the expression of semi-Hopf categories as oplax bimonoids (Definition 3.1.1) in $\mathsf{Span}|\mathcal{V}$, using the oplax bimonoid structure on $X^2 \in \mathsf{Span}$ (Proposition 4.4.1); we first consider (pseudo)comonoids over the trivial comonoid structure of X^2 as discussed in Section 4.2.

Proposition 6.2.4. *A comonoid in* $\mathsf{Span}|\mathcal{V}$ *over* X^2 *with the trivial comonoid structure is precisely a* $\mathsf{Comon}(\mathcal{V})$-*graph, i.e. an* X^2-*indexed family of comonoids in* $\mathcal{V}$.

Proof. Such an object consists of $M \colon X^2 \to \mathcal{V}$ together with natural transformations

which are given by components

$$\delta_{xy} \colon M_{x,y} \to M_{x,y} \otimes M_{x,y} \quad \text{and} \quad \varepsilon_{xy} \colon M_{x,y} \to I.$$

Using the same machinery as for earlier proofs, we deduce that these maps satisfy coassociativity and counity conditions which render each $M_{x,y} \in \mathcal{V}$ a strict comonoid; as discussed in Section 4.2, this comultiplication and counit only form a pseudocomonoid structure with identity structure 2-cells. $\qquad\square$

Proposition 6.2.5. *An oplax bimonoid in* $\mathsf{Span}|\mathcal{V}$ *over the oplax bimonoid* X^2 *in* Span *is precisely a semi-Hopf* $\mathcal{V}$-*category.*

Proof. By Propositions 6.2.2 and 6.2.4, we already know that an object A in $\mathsf{Span}|\mathcal{V}$ with the structure of a pseudomonoid and (pseudo)comonoid over the {groupoid pseudomonoid, trivial comonoid} X^2 is a $\mathcal{V}$-category which is $\mathsf{Comon}(\mathcal{V})$-enriched as a graph. We will now show that requiring $(A, m, j, \delta, \varepsilon)$ to be an oplax bimonoid in $\mathsf{Span}|\mathcal{V}$ in fact endows its enriched composition law and identities with a comonoid morphism structure, as in Definition 6.1.1.

Following Definition 3.1.1, A comes equipped with four 2-cells $\theta, \theta_0, \chi, \chi_0$ in $\mathsf{Span}|\mathcal{V}$, satisfying certain coherence conditions. We address each one of them in detail below.
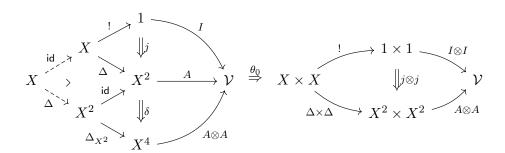
$\boxed{\theta}$

The left hand side pasted composite, using (37) for the tensor of 1-cells, is given by

$$A_{x,y} \otimes A_{y,z} \xrightarrow{\delta_{x,y} \otimes \delta_{y,z}} A_{x,y}^{\otimes 2} \otimes A_{y,z}^{\otimes 2} \xrightarrow{1 \otimes \sigma \otimes 1} A_{x,y} \otimes A_{y,z} \otimes A_{x,y} \otimes A_{y,z} \xrightarrow{m_{xyz}^{\otimes 2}} A_{x,z} \otimes A_{x,z}$$

whereas the right hand side is given by $A_{x,y} \otimes A_{y,z} \xrightarrow{m_{xyz}} A_{x,z} \xrightarrow{\delta_{xz}} A_{x,z} \otimes A_{x,z}$. The required 2-cell θ is given by the (unique) span isomorphism $P \cong X^3$ from (29) with the condition $(\delta \circ m)_{xyz} = ((\delta \otimes \delta) \circ (1 \otimes \sigma \otimes 1) \circ (m \otimes m))_{xyz}$, which is precisely the first commutative diagram in (43).

$\boxed{\theta_0}$

The left hand side composite is given by arrows $I \xrightarrow{j_x} A_{x,x} \xrightarrow{\delta_{x,x}} A_{x,x} \otimes A_{x,x}$ and the right hand side by $I \xrightarrow{j_x \otimes j_y} A_{x,x} \otimes A_{y,y}$ in $\mathcal{V}$. The 2-cell θ_0 is given by the morphism of spans $X \xrightarrow{\Delta} X^2$ from (30) and the factorization

i.e. $(\delta \circ j)_x = (j \otimes j)_{\Delta x} = j_x \otimes j_x$ which gives the second commutative diagram of (43).

$\boxed{\chi}$

Similarly to above, we have components $A_{x,y} \otimes A_{y,z} \xrightarrow{m_{xyz}} A_{x,z} \xrightarrow{\varepsilon_{xz}} I$ and $A_{x,y} \otimes A_{z,w} \xrightarrow{\varepsilon_{xy} \otimes \varepsilon_{zw}}$ $I \otimes I \cong I$, and the 2-cell χ is the a map of spans $X^3 \xrightarrow{1 \times \Delta \times 1} X^4$ as in (30) with the factorization

i.e. $(\varepsilon \circ m)_{xyz} = (\varepsilon \otimes \varepsilon)_{xyyz} = \varepsilon_{xy} \otimes \varepsilon_{yz}$ which is the third commutative diagram of (43).

$\boxed{\chi_0}$

with components $I \xrightarrow{j_x} A_{x,x} \xrightarrow{\varepsilon_x,x} I$ and $I \xrightarrow{id} I$ in $\mathcal{V}$ respectively. The 2-cell χ_0 is given by the map of spans $X \xrightarrow{!} 1$ of (29) and the factorization

i.e. $(\varepsilon \otimes j)_x = (1_I)_x$ which coincides with the fourth diagram of (43).

Due to Remark 5.1.1, the proof of Proposition 4.4.1 guarantees that the above 2-cells satisfy axioms (54) to (63) of an oplax bimonoid, hence the proof is complete. $\qquad\square$

Notice that the 2-cells rendering X^2 an oplax bimonoid in Span in Proposition 4.4.1 are all unique, based on Remark 4.4.2. Hence the above proposition establishes a bijection between semi-Hopf $\mathcal{V}$-categories and oplax bimonoids over X^2 in Span$|\mathcal{V}$.

Moving on to the morphisms between semi-Hopf categories, we initially address the comonoid part of our structures.

Proposition 6.2.6. *Suppose we have two comonoids C_{X^2}, D_{Y^2} in $\mathsf{Span}|\mathcal{V}$ with the trivial comonoid structure on their underlying sets, i.e. two $\mathsf{Comon}(\mathcal{V})$-graphs. A (strict) comonoid map of the form $^{\mathrm{id}}\alpha^{f \times f}$ is precisely a $\mathsf{Comon}(\mathcal{V})$-graph morphism.*

Proof. The 1-cell α is of the form (45), but now the comonoid structures are the trivial and not the groupoid ones; if we denote by δ and ε the comultiplication and counit, the coassociativity isomorphism can again only be the identity, namely $(\alpha \otimes \alpha) \circ \delta = \delta \circ \alpha$ which is

$$(49)$$

For the $\mathcal{V}$-components, this requires the commutativity of

$$
\begin{array}{ccc}
C_{x,y} & \xrightarrow{\;\delta_{xy}\;} & C_{x,y} \otimes C_{x,y} \\
\alpha_{xy} \downarrow & & \downarrow \alpha_{xy} \otimes \alpha_{xy} \\
D_{fx,fy} & \xrightarrow[\;\delta_{fxfy}\;]{} & D_{fx,fy} \otimes D_{fx,fy}
\end{array}
$$

and similarly for counity; these precisely translate in all α_{xy} being comonoid morphisms. $\qquad\square$

Due to Propositions 6.2.4 and 6.2.6, we conclude that the category $\mathsf{Comon}(\mathcal{V})\text{-}\mathsf{Grph}$ is a subcategory of $\mathsf{Comon}(\mathsf{Span}|\mathcal{V})$ of those objects over X^2 with the trivial comonoid structure and of strict morphisms $^{\mathrm{id}}\alpha^{f \times f}$.

Using Definition 3.1.2, we can express $\mathsf{Comon}(\mathcal{V})$-functors in $\mathsf{Span}|\mathcal{V}$ as well.

Proposition 6.2.7. *An oplax bimonoid morphism of the form $^{\mathrm{id}}\alpha^{f \times f}$ between two oplax bimonoids A_{X^2} and B_{Y^2} in $\mathsf{Span}|\mathcal{V}$ as in Proposition 6.2.5 is a semi-Hopf $\mathcal{V}$-functor.*

Proof. The 2-cells $\phi, \phi_0, \psi, \psi_0$ like (9) are respectively given by (46), (48) and identities (49). Propositions 6.2.3 and 6.2.6 ensure that these make $^{\mathrm{id}}\alpha^{f \times f}$ into an oplax morphism of pseudomonoids and a morphism of comonoids, producing the four diagrams of a $\mathsf{Comon}(\mathcal{V})$-functor through factorizations as usual. The conditions of Definition 3.1.2 can be verified in a straightforward way, by comparing the appropriate maps of spans due to Remark 5.1.1. $\qquad\square$

Consequently, $\mathsf{sHopf}\text{-}\mathcal{V}\text{-}\mathsf{Cat}$ is a subcategory of $\mathsf{OplBimon}(\mathsf{Span}|\mathcal{V})$ of oplax bimonoids and morphisms between them, spanned by objects over X^2 as in Proposition 4.4.1 and maps of the form $^{\mathrm{id}}\alpha^{f \times f}$. As mentioned earlier, even though $\mathsf{OplBimon}(\mathsf{Span}|\mathcal{V})$ is in general a bicategory, these structures form a category due to the form of morphisms coming from sheer functions.

Finally, we turn on to the subcategory of Hopf $\mathcal{V}$-categories, Definition 6.1.3. The oplax Hopf monoid $(X^2, \mu, \eta, \zeta, \nu)$ in Span, Proposition 4.4.3, underlies our oplax Hopf monoids in $\mathsf{Span}|\mathcal{V}$, as was the case in all the above propositions.
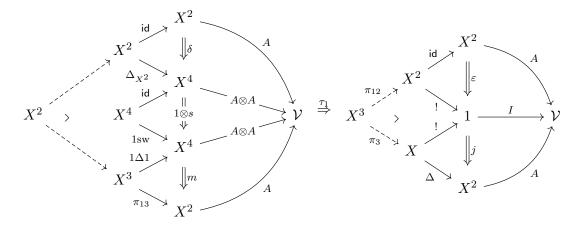
Proposition 6.2.8. *An oplax Hopf monoid in $\mathsf{Span}|\mathcal{V}$ over the oplax Hopf monoid X^2 in Span is precisely a Hopf $\mathcal{V}$-category.*

Proof. Following Definition 3.4.3, we wish to endow the oplax bimonoid $(A, m, j, \delta, \varepsilon, \theta, \theta_0, \chi, \chi_0)$ from Proposition 6.2.5 with an antipode $s\colon A \to A$ in $\mathsf{Span}|\mathcal{V}$ over the span $X^2 \xleftarrow{\mathsf{id}} X^2 \xrightarrow{\mathsf{sw}} X^2$

given by families $s_{xy}\colon A_{x,y} \to A_{y,x}$, along with two 2-cells τ_1, τ_2 in $\mathsf{Span}|\mathcal{V}$ (34) where for example, the first one is

The left hand side and right hand side components are, respectively,

$$(m \circ (1 \otimes s) \circ \delta)_{xy}\colon A_{x,y} \xrightarrow{\delta_{xy}} A_{x,y} \otimes A_{x,y} \xrightarrow{1 \otimes s_{xy}} A_{x,y} \otimes A_{y,x} \xrightarrow{m_{xyx}} A_{x,x}$$

$$(j \circ \varepsilon)_{xyz}\colon A_{x,y} \xrightarrow{\varepsilon_{xy}} I \xrightarrow{j_z} A_{z,z}$$

and the 2-cell τ_1 is given by the map of spans $\rho_1 = (1 \times \mathsf{sw})(\Delta \times 1)\colon X^2 \to X^3$ just like the upper (31), such that

This factorization can be written as the upper commutative diagram of the Hopf $\mathcal{V}$-category axiom (44), and similarly for τ_2. $\qquad\square$

This establishes a bijection between Hopf $\mathcal{V}$-categories and oplax Hopf monoids in $\mathsf{Span}|\mathcal{V}$ over X^2, and a Hopf $\mathcal{V}$-functor is merely an oplax bimonoid morphism in $\mathsf{Span}|\mathcal{V}$ like before.

7. Frobenius $\mathcal{V}$-categories

Hopf algebras are closely related to Frobenius algebras in the classical context, and more generally to Frobenius monoids in an arbitrary monoidal category $\mathcal{V}$. As a result, we may also ask what a many-object generalization of a Frobenius object could be. In this section, we initially provide an ad-hoc definition of a Frobenius $\mathcal{V}$-category, which is then realized as a Frobenius pseudomonoid

in the same monoidal bicategory $\mathsf{Span}|\mathcal{V}$ where Hopf $\mathcal{V}$-categories live as oplax Hopf monoids, see Sections 5 and 6. In subsequent work [15], we study in detail the relation between Frobenius and Hopf categories, providing a generalized version of the standard Larson-Sweedler theorem. For the time being, we restrict to the solid introduction of this notion and its expression as a Frobenius pseudomonoid.

7.1   Frobenius categories. Naturally, the Frobenius counterpart of Hopf $\mathcal{V}$-categories will still have a notion of multiplication and comultiplication. However, the 'local comonoid' part previously captured via the enrichment in comonoids is now replaced by a global operation, expressed as follows.

Definition 7.1.1. A *Frobenius $\mathcal{V}$-category* A is a $\mathcal{V}$-category that is also a $\mathcal{V}$-opcategory and satisfies 'indexed' Frobenius conditions. Explicitly, it consists of a set of objects A_0 and for every $x, y \in A_0$ an object $A_{x,y}$ of $\mathcal{V}$ together with maps

$$m_{xyz} \colon A_{x,y} \otimes A_{y,z} \to A_{x,z} \qquad j_x \colon I \to A_{x,x}$$
$$d_{abc} \colon A_{a,c} \to A_{a,b} \otimes A_{b,c} \qquad \epsilon_y \colon A_{y,y} \to I$$

which satisfy the $\mathcal{V}$-category and $\mathcal{V}$-opcategory axioms, as well as the commutativity of

$$(50)$$

Definition 7.1.2. A *Frobenius $\mathcal{V}$-functor* between two Frobenius categories A and B is a $\mathcal{V}$-graph morphism simultaneously in $\mathcal{V}$-Cat and $\mathcal{V}$-$\mathsf{opCat}^{\mathrm{op}}$. This amounts to a function $f \colon A_0 \to B_0$ between the sets of objects, along with families of arrows $F_{xy} \colon A_{x,y} \to B_{fx,fy}$ in $\mathcal{V}$ subject to the following axioms:

Remark 7.1.3. The above introduced notions should not be confused with (related but different) ones existing in literature. The name *Frobenius category* is also used for an exact category which has enough injectives and enough projectives and where the class of projectives coincides with the class of injectives, see [26]. The name *Frobenius functor* is also used for a functor that has an identical left and right adjoint, see [16].

Frobenius $\mathcal{V}$-categories and Frobenius $\mathcal{V}$-functors form a category Frob-$\mathcal{V}$-Cat.

Examples 7.1.4.

1. Every Frobenius monoid in a monoidal category $\mathcal{V}$ can be viewed a one-object Frobenius $\mathcal{V}$-category; as a result, this definition indeed serves as many-object generalization of Frobenius algebras. In particular, each 'diagonal' hom-object $A_{x,x}$ of a Frobenius $\mathcal{V}$-category is a Frobenius monoid in $\mathcal{V}$.

2. For a commutative ring k, let Mat be the category whose objects are the natural numbers and whose hom-sets $\mathsf{Mat}_{m,n}$ are the sets of $m \times n$ matrices with entries in k. This is a Mod_k-category if we take the usual composition of matrices and identity matrices. The comultiplication and counit are defined by

$$d_{n,p,m}\colon \mathsf{Mat}_{n,m} \to \mathsf{Mat}_{n,p} \otimes \mathsf{Mat}_{p,m}\colon e_{i,j}^{n,m} \mapsto \sum_{t=1}^{p} e_{i,t}^{n,p} \otimes e_{t,j}^{p,m}$$

$$\epsilon_{n,m}\colon \mathsf{Mat}_{n,m} \to k\colon e_{i,j}^{n,m} \mapsto \delta_{i,j}$$

for all $1 \leq i \leq n, 1 \leq j \leq m$, where $e_{i,j}^{n,m}$ denote the elementary matrices of $\mathsf{Mat}_{n,m}$ with a single 1 in the i-th row and j-th column and zeroes elsewhere. This makes Mat into a Frobenius k-linear category, generalizing the classical example of each $\mathsf{Mat}_{n,n}$ being a Frobenius algebra.

7.2 Frobenius pseudomonoid structure. The following results extend Theorem 6.2.1 to the case of Frobenius monoids; they are also relevant to Table 1. Again, the fact that $U\colon \mathsf{Span}|\mathcal{V} \to \mathsf{Span}$ is strict monoidal via (38) implies that the underlying set of a Frobenius pseudomonoid in $\mathsf{Span}|\mathcal{V}$ is also a Frobenius pseudomonoid Span similarly to Proposition 3.1.3, and the set X^2 will once more play this role, with structures described in Section 4.5.

Proposition 7.2.1. *A Frobenius pseudomonoid in $\mathsf{Span}|\mathcal{V}$ over the Frobenius X^2 with groupoid pseudomonoid and groupoid pseudocomonoid structure in Span is a Frobenius $\mathcal{V}$-category.*

Proof. Having established by Proposition 6.2.2 that a pseudomonoid and pseudocomonoid (A, m, j, d, ϵ) in $\mathsf{Span}|\mathcal{V}$ over X^2 of Proposition 4.5.1 is a $\mathcal{V}$-category and $\mathcal{V}$-opcategory, we only need to check they form a Frobenius pseudomonoid structure, namely there exist isomorphisms as in (6) satisfying the axioms of Section A.5. The lower isomorphism is

which uses the isomorphism from Proposition 4.5.1. The existence of such a 2-isomorphism in $\mathsf{Span}|\mathcal{V}$ comes with an equality (34) of composite arrows in $\mathcal{V}$

$$A_{xy} \otimes A_{yz} \xrightarrow{1 \otimes d_{ywz}} A_{xy} \otimes A_{yw} \otimes A_{wz} \xrightarrow{m_{xyw} \otimes 1} A_{xw} \otimes A_{wz}$$

$$A_{xy} \otimes A_{yz} \xrightarrow{m_{xyz}} A_{xz} \xrightarrow{d_{xwz}} A_{xw} \otimes A_{wz}$$

which is precisely the one of the two conditions for a Frobenius $\mathcal{V}$-category (50). The second condition can be checked similarly, and the axioms hold since those in Span hold by Remark 5.1.1 as in earlier proofs. $\square$

Hence we have a bijective correspondence between Frobenius monoids in $\mathsf{Span}|\mathcal{V}$ over X^2 and Frobenius $\mathcal{V}$-categories, for any monoidal category $\mathcal{V}$. Regarding Frobenius $\mathcal{V}$-functors of Definition 7.1.2, by Proposition 6.2.3 we can express them as morphisms between Frobenius pseudomonoids.

Corollary 7.2.2. *An oplax pseudomonoid and oplax pseudocomonoid morphism of the form* ${}^{\mathsf{id}}\alpha^{f\times f}$ *in* $\mathsf{Span}|\mathcal{V}$ *is a Frobenius $\mathcal{V}$-functor.*

We can thus realize $\mathsf{Frob}\text{-}\mathcal{V}\text{-}\mathsf{Cat}$ as the subcategory of $\mathsf{Frob}_{\mathsf{opl},\mathsf{opl}}(\mathsf{Span}|\mathcal{V})$ of Frobenius pseudomonoids in $\mathsf{Span}|\mathcal{V}$ over X^2 with the groupoid pseudo(co)monoid structures, and morphisms of the form ${}^{\mathsf{id}}\alpha^{f\times f}$.

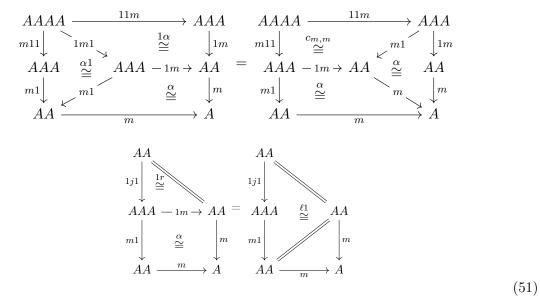For purposes of completeness, we conclude with the following result.

Proposition 7.2.3. *A Frobenius (strict) monoid in* $\mathsf{Span}|\mathcal{V}$ *over X^2 with the trivial monoid and comonoid structure is a* $\mathsf{Frob}(\mathcal{V})$*-graph, whereas a oplax monoid and comonoid morphism* ${}^{\mathsf{id}}\alpha^{f\times f}$ *is a* $\mathsf{Frob}(\mathcal{V})$*-graph morphism.*

Proof. This can be deduced in a straightforward way from Proposition 6.2.4 and its dual by checking the Frobenius conditions. For morphisms, Proposition 6.2.6 and its dual suffice. $\square$

Based on the above, the braided monoidal bicategory $\mathsf{Span}|\mathcal{V}$ serves indeed as the common framework for Hopf $\mathcal{V}$-categories and Frobenius $\mathcal{V}$-categories, which are respectively expressed as oplax Hopf monoids and Frobenius pseudomonoids therein. The fact that subsequent work [15] extends the Larson-Sweedler theorem between such many-object generalized structures may in particular indicate a correspondence between these relaxed notions of oplax bimonoids and their Frobenius counterparts in higher structures, yet to be investigated.

Appendix A:

A.1   Pseudomonoid, oplax maps and 2-cells axioms. A pseudomonoid A with constraints α, ℓ, r as in (3) satisfy the following two axioms
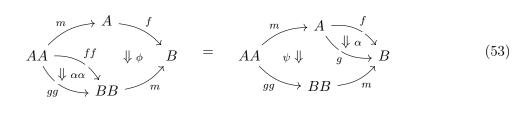

$$\tag{51}$$

A 1-cell $f\colon A \to B$ between pseudomonoids comes with 2-cells (ϕ, ϕ_0) as in (4) that satisfy

$$\text{(52)}$$

A 2-cell $\beta\colon f \Rightarrow g$ between two oplax 1-cells of pseudomonoids A and B satisfies
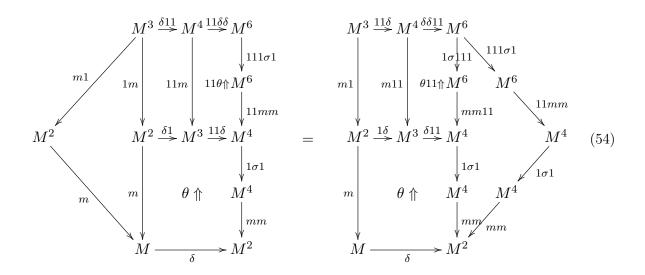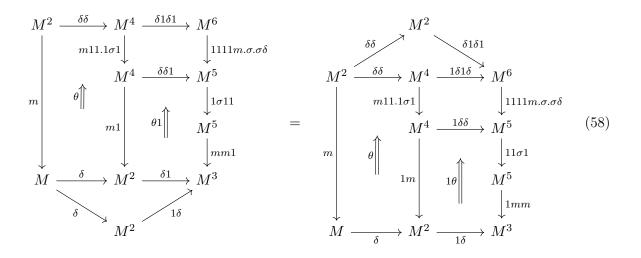
$$\text{(53)}$$

A.2 Oplax bimonoid axioms. An oplax bimonoid $(M, m, j, \delta, \varepsilon, \theta, \theta_0, \chi, \chi_0)$ in a braided monoidal bicategory $\mathcal{K}$ as in Definition 3.1.1, namely an object in $\mathsf{PsComon}(\mathsf{PsMon}_{\mathsf{opl}}(\mathcal{K}))$, satisfies a number of axioms listed below.

Explicitly, (M, m, j) comes with invertible associativity and unit 2-cells (α, ℓ, r) satisfying (51) and (M, δ, ε) come with invertible coassociativity and unit 2-cells (β, s, t) satisfying dual axioms. Moreover, (54) and (55) express that $\delta\colon M \to M \otimes M$ equipped with θ, θ_0 is an oplax morphism between pseudomonoids as in (52); conditions (56) and (57) express that $\varepsilon\colon M \to I$ equipped with χ, χ_0 is an oplax morphism between pseudomonoids; conditions (58) and (59) express that the coassociativity isomorphism β is a monoidal 2-cell namely satisfies (53); and conditions (60) to (63) do the same for the left and right counit isomorphisms s, t.

All empty faces are filled appropriately (co)associativity and (co)unit isomorphisms, or by coherence isomorphisms coming from the braided monoidal structure on the bicategory (Gray monoid) $\mathcal{K}$.

$$(54)$$

$$(55)$$

$$(56)$$

$$(57)$$

$$
\begin{array}{c}
\begin{array}{ccc}
M^2 & \xrightarrow{\delta\delta} & M^4 \xrightarrow{\delta1\delta1} M^6 \\
\end{array}
\end{array}
\qquad (58)
$$

Diagram (58): left side with vertices $M^2, M^4, M^6, M^4, M^5, M^5, M, M^2, M^3, M^2$; arrows $\delta\delta$, $\delta1\delta1$, $m11.1\sigma1$, $1111m.\sigma.\sigma\delta$, $\delta\delta1$, $1\sigma11$, m, θ, $m1$, $\theta1$, $mm1$, δ, $\delta1$, δ, 1δ; equals right side with vertices $M^2, M^2, M^4, M^6, M^4, M^5, M, M^2, M^5, M^3$; arrows $\delta\delta$, $\delta1\delta1$, $\delta\delta$, $1\delta1\delta$, $m11.1\sigma1$, $1111m.\sigma.\sigma\delta$, $1\delta\delta$, m, θ, $1m$, 1θ, $11\sigma1$, δ, 1δ, $1mm$.

$$
\qquad (59)
$$

Diagram (59): left side with vertices $I, II, III, IM, IIM, M, M^2, M^3, M^2$; arrows $\sim$, $\sim$, $1j$, $11j$, $\sim$, $j1$, $1\theta_0$, $jj1$, j, θ_0, δ, $\delta1$, δ, 1δ; equals right side with vertices $II, I, II, III, MI, MII, M, M^2, M^3$; arrows $\sim$, $\sim$, $\sim$, $\sim$, $j1$, $j11$, $\sim$, j, θ_0, $1j$, θ_01, $1jj$, δ, 1δ.

$$
\qquad (60)
$$

Diagram (60): left side with vertices $M^2, M^4, MIMI, M^3, MII, M, M^2, MI$; arrows $\sim$, $\delta\delta$, $1\varepsilon1\varepsilon$, $m11.1\sigma1$, $m11.1\sigma1$, θ, $1\varepsilon\varepsilon$, 1χ, m, $1m$, $\sim$, δ, 1ε, $\sim$; equals right side with vertices $M^2, MIMI, MII, M, MI$; arrows $\sim$, $m11.1\sigma1$, m, $\sim$, $\sim$.

$$
\qquad (61)
$$

Diagram (61): left side with vertices $I, II, II, MI, MI, M, M^2, MI$; arrows $\sim$, $\sim$, 1, $j1$, $j1$, θ_0, 1, $1j$, $1\chi_0$, 1, j, δ, 1ε, $\sim$; equals right side with vertices I, II, MI, M, MI; arrows $\sim$, $j1$, j, id, $\sim$.

$$\text{(62)}$$

$$\text{(63)}$$

A.3   Oplax bimonoid morphism axioms. Below we give the axioms for Definition 3.1.2, of an oplax bimonoid morphism $(f, \phi, \phi_0, \psi, \psi_0) \colon M \to N$, namely a morphism in $\mathsf{PsComon}_{\mathsf{opl}}(\mathsf{PsMon}_{\mathsf{opl}}(\mathcal{K}))$, which includes the oplax pseudomonoid and pseudocomonoid morphism axioms (52), and the condition that ψ, ψ_0 are monoidal 2-cells (53), i.e.

$$\text{(64)}$$

$$\text{(65)}$$

$$\text{(66)}$$

$$(67)$$

More precisely, for example recall that the 2-cell $\psi\colon (f \otimes f) \circ \delta \Rightarrow \delta \circ f$ from (9) has as codomain the pseudomonoid morphism whose structure maps arise as composites from the ones for δ and f, for example

$$
\begin{array}{ccc}
MM & \xrightarrow{\ m\ } & M \\
{\scriptstyle ff}\downarrow & \Downarrow\phi & \downarrow{\scriptstyle f} \\
NN & \xrightarrow{\ m\ } & N \\
{\scriptstyle \delta\delta}\downarrow & \Downarrow\xi & \downarrow{\scriptstyle \delta} \\
NNNN & \xrightarrow{1\sigma1} NNNN \xrightarrow{mm} & NN
\end{array}
$$

and similarly for its domain, where the monoidal product of pseudomonoid maps using the braiding is needed, giving

$$
\begin{array}{ccc}
MM & \xrightarrow{\ m\ } & M \\
{\scriptstyle \delta\delta}\downarrow & \Downarrow\theta & \downarrow{\scriptstyle \delta} \\
MMMM & \xrightarrow{1\sigma1} MMMM \xrightarrow{mm} & MM \\
{\scriptstyle ffff}\downarrow & \downarrow{\scriptstyle f^4} \quad \Downarrow\phi\phi & \downarrow{\scriptstyle ff} \\
NNNN & \xrightarrow{1\sigma1} NNNN \xrightarrow{mm} & NN
\end{array}
$$

which are both used in axiom (64).

A.4 Oplax module, morphism and transformation axioms. We detail the conditions satisfied by the structure maps of a right oplax module (X, ρ, ξ, ξ_0) for a pseudomonoid $(M, m, j, \alpha, r, \ell)$ as defined in Definition 3.2.1.

$$(68)$$

$$(69)$$

An oplax module morphism $f\colon X \to Y$ comes with a 2-cell $\phi\colon f \circ \rho \Rightarrow \rho \circ f \otimes 1$ as in (11) satisfying

$$(70)$$

$$(71)$$

An oplax module transformation $\alpha\colon (f,\phi) \Rightarrow (g,\psi)$ satisfies

$$(72)$$

A.5   Frobenius pseudomonoid axioms.

In [31, Prop. 25], a definition for a pseudomonoid A in a Gray monoid to be Frobenius is given, in terms of the existence of a counit and a map $I \to A \otimes A$ along with two 2-isomorphisms satisfying axioms. We will here use an alternative definition that naturally generalizes the monoid-comonoid definition in the ordinary monoidal setting, using a subset of axioms of [31, Lem. 32].

An object which is a pseudomonoid $(A, m, j, \alpha, \ell, r)$ and a pseudocomonoid $(A, d, \epsilon, \beta, s, t)$ is Frobenius when it comes equipped with 2-isomorphisms ψ, ϕ as in (6) subject to axioms that express that $d\colon A \to A{\otimes}A$ is a left A, right A pseudomodule morphism and also $m\colon A{\otimes}A \to A$ is a left A, right A pseudocomodule morphism. Explicitly, we have the following eight axioms where axioms (73) express that d is a right A-pseudomodule pseudomap between $(A, m\colon A \otimes A \to A)$ and $(A \otimes A, 1 \otimes m\colon A \otimes A \otimes A \to A \otimes A)$ similarly to (70) and (71), axioms (74) express that d is a left A-pseudomodule pseudomap, axioms (75) express that m is a left A-pseudocomodule map between $(A, d\colon A \to A \otimes A)$ and $(A \otimes A, d \otimes 1\colon A \otimes A \to A \otimes A \otimes A)$, and axioms (76) express that m is a right A-pseudocomodule map.

$$(73)$$

$$(74)$$

$$(75)$$

$$(76)$$

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