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On the étale homotopy type of higher stacks

David Carchedi[a]

[a]George Mason University, Fairfax, Virginia 22030, USA

Abstract

A new approach to étale homotopy theory is presented which applies to a much broader class of objects than previously existing approaches, namely it applies not only to all schemes (without any local Noetherian hypothesis), but also to arbitrary higher stacks on the big étale site, and in particular to all algebraic stacks. This approach also produces a more refined invariant than the original construction of Artin-Mazur [2], namely we produce a pro-object in the infinity category of spaces, rather than in the homotopy category. We prove a profinite comparison theorem at this level of generality, which states that if $\mathcal{X}$ is an arbitrary higher stack on the étale site of affine schemes of finite type over $\mathbb{C}$, then the étale homotopy type of $\mathcal{X}$ agrees with the homotopy type of the underlying stack $\mathcal{X}_{top}$ on the topological site, after profinite completion. In particular, if $\mathcal{X}$ is an Artin stack locally of finite type over $\mathbb{C}$, our definition of the étale homotopy type of $\mathcal{X}$ agrees up to profinite completion with the homotopy type of the underlying topological stack $\mathcal{X}_{top}$ of $\mathcal{X}$ in the sense of Noohi [35]. We also show this comparison is compatible in a suitable sense with the comparison theorem of Friedlander for simplicial schemes [17]. In order to prove our comparison theorem, we provide a modern reformulation of the theory of local systems and their cohomology using the language of ∞-categories which we believe to be of independent interest.

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1. Introduction

Given a complex variety V, one can associate topological invariants to this variety by computing invariants of its underlying topological space V_{an}, equipped with the complex analytic topology. However, for a variety over an arbitrary base ring, there is no good notion of underlying topological space which plays the same role. (It is well known that the Zariski topology is too coarse.) Étale cohomology gives a way of partly circumventing this problem, since it associates cohomology groups to a scheme, and it is well known that if V is a complex variety, then its étale

Email address: davidcarchedi@gmail.com (David Carchedi)

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cohomology with coefficients in any finite abelian group A agrees with the singular cohomology of its underlying space V_{an} with the same coefficients. The étale homotopy type of a scheme takes things one giant step further. Although it does not associate a genuine topological space to a scheme, it associates a (pro-)homotopy type, and this allows one to associate to a scheme much more refined topological invariants, e.g. higher homotopy groups.

The original notion of étale homotopy type goes back to seminal work of Artin and Mazur [2] in 1969. They give a way of associating to any locally Noetherian scheme a pro-object in the homotopy category of simplicial sets. From the étale homotopy type of a scheme, one can recover its étale cohomology, and also its étale fundamental group, and higher homotopy groups. Étale homotopy types have made many important impacts in mathematics, perhaps most famously in the proof of the Adams conjecture [42, 47, 16]. More recently, étale homotopy theory has been an important tool in studying the rational points of algebraic varieties [19, 39, 46], and also has an interesting connection with motivic homotopy theory [22, 44].

Artin and Mazur also introduced the notion of *profinite completion*, which was motivated by the notion of profinite completion of groups, and they proved the following celebrated comparison theorem:

Theorem 1.0.1. *[2, Theorem 12.9] Let X be a pointed connected scheme of finite type over $\mathbb{C}$. Then there is a canonical map*

$$[X_{an}] \to [X_{ét}]$$

from the homotopy type of the analytification X_{an} to the étale homotopy type of X which induces an isomorphism on profinite completions.

The above theorem is a vast generalization of the comparison theorem for étale cohomology.

1.1 Comparison with other work Although étale homotopy theory, as developed by Artin and Mazur, has been quite a successful endeavor, there are limitations to their framework. The most serious limitation is that, although their notion of étale homotopy type naturally extends to Deligne-Mumford stacks, it does not easily extend to more general objects, such as Artin stacks. A more subtle limitation is related to the notion of homotopy coherence: the étale homotopy type of a scheme in the sense of Artin and Mazur produces a pro-object— a diagram of a certain shape — in the homotopy category of spaces (or simplicial sets); it is well known that a diagram in the homotopy category need not lift to a diagram of actual spaces— this is the issue of homotopy coherence. A third limitation is that the schemes in question are required to be locally Noetherian. This excludes many natural examples. One such example, is that Vakil and Wickelgren show in [52] that for a quasicompact and quasiseparated scheme X, there exists a universal cover $\widetilde{X}$, which itself surprisingly is always a scheme, however it may fail to be locally Noetherian even when X is, so one cannot apply the machinery of Artin and Mazur to it.

The first two limitations were partly remedied by subsequent work of Friedlander [17], as he refined the construction of Artin and Mazur to define the étale homotopy type of a locally Noetherian scheme as a pro-object in the actual category of simplicial sets, rather than its homotopy category. He also extended the construction to simplicial schemes, and proved a comparison theorem similar to the above, but for pointed connected simplicial schemes of finite type over $\mathbb{C}$. Unfortunately, the profinite comparison result that Friedlander proves uses the same notion of profinite completion as Artin-Mazur, which happens at the level of pro-objects in the homotopy category of spaces, and also Friedlander's approach still has a locally Noetherian hypothesis.

Using shape theory for ∞-topoi, Lurie gives a definition of the étale homotopy type of Deligne-Mumford stacks in [29]. This has the advantage of no longer needing a local Noetherian hypothesis (which is used, e.g. in the Artin-Mazur approach to have a locally connected site, which is needed to define the Verdier functor), and also of producing a pro-object in the ∞-*category* of spaces.

1.2 Overview of our approach to étale homotopy theory In this paper, we present a new approach to étale homotopy theory which offers a refinement of the original construction which produces a pro-object in the ∞-category of spaces rather than its homotopy category and applies to a much broader class of objects, namely to arbitrary higher stacks on the étale site of affine schemes over an arbitrary base, with no Noetherian hypothesis. It is obtained, *a priori* by adapting Lurie's approach for Deligne-Mumford stacks to arbitrary stacks, by extending the assignment to Deligne-Mumford stacks their underlying ∞-topos to an assignment to arbitrary stacks an ∞-topos, by left Kan extension (however the existence of this Kan extension is not automatic and relies on the theory of étale geometric morphisms). Then, in Section 2.5, we unwind this abstract construction and arrive at a simple concrete formula for the etale homotopy type of an arbitrary stack, which to our knowledge, is new even for schemes, and may be taken, *a post priori* as the definition.

Furthermore, our approach to profinite completion follows Lurie and produces a pro-object in the ∞-category of π-finite spaces (Definition 2.1.4). We also prove a generalization of Artin and Mazur's comparison theorem which holds for any higher stack on the étale site of affine schemes of finite type over $\mathbb{C}$. In particular, the comparison result holds for any algebraic stack locally of finite type over $\mathbb{C}$, or more generally, any n-geometric stack locally of finite type over $\mathbb{C}$, in the sense of [51]. We furthermore show that this is a refinement of Friedlander's comparison theorem for simplicial schemes. We believe this is the most general form of this comparison result.

As our machinery applies to much more general objects than previous frameworks, it is ripe for future applications. One application which has already been explored is related to the profinite homotopy type of log schemes, and is explained in Section 7 of [8]. The results of [8] imply that if X is a (fine saturated) log scheme locally of finite type over $\mathbb{C}$, then the homotopy type of its Kato-Nakayama space [25] agrees after profinite completion with the homotopy type of the underlying topological stack of its infinite root stack— which is a pro-algebraic stack [48]. Our comparison theorem, Theorem 5.13, implies that both of these profinite homotopy types also agree with the profinite étale homotopy type of the infinite root stack. As the later makes sense for log schemes over a more general base, this gives a suitable replacement for the Kato-Nakayama space in positive characteristics. The previously existing comparison theorems were not robust enough to apply in this situation.

The machinery and formulation of our approach is quite different than the work of Artin-Mazur and Friedlander, however for locally Noetherian schemes, our definition of étale homotopy type turns out to be essentially the same after unwinding the definitions (see Section 3). The principal difference between the definition of the étale homotopy type of such a scheme as computed according to our machinery and its definition computed according to the machinery of Artin-Mazur is two-fold, namely our approach uses Čech covers and theirs uses hypercovers, and our definition yields a pro-object in the ∞-category of spaces, and theirs yields a pro-object in the *homotopy category* of spaces (see Section 3). By a recent result of Hoyois [20], for locally Noetherian schemes, the former is the only real difference between the definition using our approach and the definition using the approach of Friedlander (see Proposition 3.2.2) in the case of schemes, and we expand this result to the setting of simplicial schemes in Theorem 3.2.4. We choose Čech covers over hypercovers because small étale sites are almost never hypercomplete. (Note: the intrinsic cohomology is still étale sheaf cohomology, not Čech cohomology, even when using Čech descent to define sheaves). The difference between using Čech covers as opposed to using hypercovers is moreover erased by passing to profinite homotopy types.

Our approach necessitates the use of the powerful framework of ∞-categories. The use of this language makes our definition of étale homotopy type much more simple and transparent than previous definitions. This is partly due to the fact that the language of ∞-categories allows for a simple definition of pro-spaces and profinite spaces. At the same time, this approach to pro-spaces is equivalent to the approach of Edwards-Hastings and Isaksen [13, 21] using model categories [3], and moreover, the ∞-categorical approach to profinite spaces is equivalent to Quick's model-categorical approach as in [40, 41]. However, the ∞-categorical approach to pro-spaces and profinite spaces is much easier to work with, e.g. (c.f. [29]): If $\mathcal{S}$ is the ∞-category of spaces, succinctly, the ∞-category $\operatorname{Pro}(\mathcal{S})$ of *pro-spaces* is the full subcategory of $\operatorname{Fun}(\mathcal{S},\mathcal{S})^{op}$ - the opposite of the ∞-category of functors from spaces to spaces - on those functors which preserve finite limits (and are accessible).

Let $\mathcal{X}$ be an arbitrary higher stack on the étale site of affine schemes over k. Let $\mathcal{G}$ be an arbitrary space in $\mathcal{S}$ (i.e. an ∞-groupoid). Denote by $\Delta^{\acute{e}t}(\mathcal{G})$ the étale stackification of the constant presheaf with value $\mathcal{G}$. Then, as a pro-space, the étale homotopy type $\Pi_\infty^{\acute{e}t}(\mathcal{X})$ of $\mathcal{X}$ as a functor from spaces to spaces sends the space $\mathcal{G}$ to the space of maps

$$\operatorname{Hom}\left(\mathcal{X}, \Delta^{\acute{e}t}(\mathcal{G})\right).$$

This assignment produces a functor

$$\operatorname{Sh}_\infty(\mathbf{Aff}_k) \xrightarrow{\Pi_\infty^{\acute{e}t}} \operatorname{Pro}(\mathcal{S})$$

from higher stacks on the étale site to pro-spaces, sending a stack $\mathcal{X}$ to its étale homotopy type.

Strictly speaking, the above description of the étale homotopy type of an arbitrary stack is not by definition; this description is the content of Theorem 4.2. Our definition is of more geometric origin, as our approach has its roots in the philosophy of Grothendieck which led to the inception of the concept of a topos; topoi were invented in order to associate to a scheme an underlying "space" whose cohomology is (by definition) the *étale cohomology* of the scheme in question, and this was an important first step towards producing a Weil cohomology theory and proving the Weil conjectures. We take seriously the idea that the correct geometric "space" underlying a scheme X is its small étale topos $\operatorname{Sh}(X_{\acute{e}t})$, and therefore all the topological invariants of a scheme X should actually be invariants of the topos $\operatorname{Sh}(X_{\acute{e}t})$. To make this precise, one needs a way of associating to a topos a pro-homotopy type. This can be accomplished by using the theory of ∞-topoi. Indeed, given an ∞-topos, there is a simple construction, originally due to Toën and Vezzosi, which associates to an ∞-topos $\mathcal{E}$ a pro-space *Shape* $(\mathcal{E})$ called its *shape*, which is to be thought of as the pro-homotopy type of the ∞-topos in question, and as any topos can be in a natural way regarded as an ∞-topos, this gives a way of associating to any scheme a pro-homotopy type.

The above discussion works well for schemes. It also works well for Deligne-Mumford stacks, as they can be modeled geometrically as ringed topoi. However, to extend the definition of étale homotopy type to an arbitrary higher stack, one needs a new idea. We accomplish this by formally extending the functor associating to a scheme its small étale ∞-topos $\mathrm{Sh}_\infty(X_{\acute{e}t})$ (the ∞-topos associated to its small étale topos) to a colimit preserving functor

$$\mathrm{Sh}_\infty(\mathbf{Aff}_k, \acute{e}t) \to \mathfrak{Top}_\infty$$

from the ∞-category of higher stacks on the étale site of affine schemes over k to the ∞-category of ∞-topoi. The étale homotopy type of a higher stack is then defined to be the shape of its associated ∞-topos via the above functor. Although this definition is not very tractable for stacks which are not Deligne-Mumford, this is rectified by Theorem 4.2, and moreover, in light of our comparison theorem, Theorem 5.13, it is still a reasonable definition for Artin stacks, which may a priori be non-obvious due to the use of the étale topology rather than say the smooth topology.

1.3 The comparison theorem We will now explain in detail the content of our comparison theorem:

In [8], we extend two important classical constructions for schemes and topological spaces to higher stacks, namely the analytification functor and the functor sending a topological space to its homotopy type.

Analytification: Consider the classical analytification functor

$$(\,\cdot\,)_{an} : \mathbf{Sch}^{LFT}_{\mathbb{C}} \to \mathbf{Top},$$

from schemes locally of finite type over $\mathbb{C}$ to topological spaces. It sends a scheme X to its space of $\mathbb{C}$-points equipped with the complex analytic topology. Motivated by the desire to associate to an algebraic stack over $\mathbb{C}$ a natural topological object from which one can extract topological invariants, Noohi extends this construction in [35] to a functor

$$(\,\cdot\,)_{top} : \mathfrak{AlgSt}^{LFT}_{\mathbb{C}} \to \mathfrak{TopSt}$$

from Artin stacks locally of finite type over $\mathbb{C}$ to topological stacks. In [8], we extend this further to a colimit preserving functor

$$(\,\cdot\,)_{top} : \mathrm{Sh}_\infty\left(\mathbf{Aff}^{LFT}_{\mathbb{C}}, \text{ét}\right) \to \mathbb{H}\mathrm{ypSh}_\infty(\mathbf{Top}_{\mathbb{C}})$$

from ∞-sheaves on the étale site of affine schemes of finite type over $\mathbb{C}$, to hypersheaves on a suitable category $\mathbf{Top}_{\mathbb{C}}$ of topological spaces.

The homotopy type of a stack: In [36], Noohi defines a functor

$$ho : \mathfrak{TopSt} \to Ho(\mathbf{Top})$$

from the 2-category of topological stacks to the homotopy category of topological spaces, sending a topological stack $\mathcal{X}$ to its weak homotopy type. Explicitly, if $\mathcal{G}$ is a topological groupoid presentation for $\mathcal{X}$, $ho(\mathcal{X})$ has the weak homotopy type of the classifying space of $\mathcal{G}$. In [37] Noohi and Coyne refine this to a functor to the ∞-category of spaces $\mathcal{S}$.

In [8], we extend this further to a colimit preserving functor

$$\Pi_\infty : \mathbb{H}\mathrm{ypSh}_\infty(\mathbf{Top}_{\mathbb{C}}) \to \mathcal{S}$$

from the ∞-category of hypersheaves on $\mathbf{Top}_{\mathbb{C}}$ to the ∞-category of spaces.

The final important construction we need in order to explain our comparison result is

Profinite completion: In [29], Lurie constructs the profinite completion functor

$$\widehat{(\,\cdot\,)} : \mathcal{S} \to \mathrm{Prof}\,(\mathcal{S})$$

from the ∞-category of spaces to the ∞-category of profinite spaces. In fact, this is the restriction of a profinite completion functor

$$\mathrm{Pro}\,(\mathcal{S}) \to \mathrm{Prof}\,(\mathcal{S})$$

from pro-spaces to profinite spaces, and composing this functor with our étale homotopy type functor

$$\mathrm{Sh}_{\infty}\,(\mathbf{Aff}_k) \xrightarrow{\Pi_{\infty}^{\acute{e}t}} \mathrm{Pro}\,(\mathcal{S})$$

produces a functor

$$\mathrm{Sh}_{\infty}\,(\mathbf{Aff}_k) \xrightarrow{\widehat{\Pi}_{\infty}^{\acute{e}t}} \mathrm{Prof}\,(\mathcal{S})$$

which sends a stack $\mathcal{X}$ to its *profinite étale homotopy type.*

We now state our main result:

Theorem 1.3.1. *Let $\mathbf{Aff}_{\mathbb{C}}^{LFT}$ denote the category of affine schemes of finite type over $\mathbb{C}$. The following diagram commutes up to equivalence:*

$$\begin{tikzcd}
Sh_{\infty}\left(\mathbf{Aff}_{\mathbb{C}}^{LFT}, \acute{e}t\right) \arrow[r, "\widehat{\Pi}_{\infty}^{\acute{e}t}"] \arrow[d, "(\,\cdot\,)_{top}"'] & Prof\,(\mathcal{S}) \\
\mathbb{H}\mathrm{ypSh}_{\infty}\,(\mathbf{Top}_{\mathbb{C}}) \arrow[r, "\Pi_{\infty}"] & \mathcal{S}. \arrow[u, "\widehat{(\,\cdot\,)}"']
\end{tikzcd}$$

In particular, for any ∞-sheaf F on $\left(\mathbf{Aff}_{\mathbb{C}}^{LFT}, \acute{e}t\right)$, there is an equivalence of profinite spaces

$$\widehat{\Pi}_{\infty}^{\acute{e}t}\,(F) \simeq \widehat{\Pi}_{\infty}\,(F_{top}),$$

between the profinite étale homotopy type of F and the profinite completion of the homotopy type of the underlying stack F_{top} on $\mathbf{Top}_{\mathbb{C}}$.

This theorem has the following immediate cor:

Corollary 1.3.2. *Let $\mathcal{X}$ be an Artin stack locally of finite type over $\mathbb{C}$, then there is an equivalence of profinite spaces*

$$\widehat{\Pi}_{\infty}^{\acute{e}t}\,(\mathcal{X}) \simeq \widehat{\Pi}_{\infty}\,(\mathcal{X}_{top}),$$

between the profinite étale homotopy type of $\mathcal{X}$ and the profinite completion of the homotopy type of the underlying topological stack $\mathcal{X}_{top}$.

1.4 Overview of our strategy for proving the comparison theorem It turns out that all of the functors in the statement of Theorem 1.3.1 are colimit preserving, so, by the Yoneda lemma, the comparison result for higher stacks in fact follows formally from the comparison result for affine schemes. Therefore, in order to prove Theorem 1.3.1, one must prove an analogue of Artin and Mazur's classical comparison theorem for (affine) schemes of finite type over $\mathbb{C}$ (without the additional assumptions of being pointed or connected) and for our new ∞-categorical definition of étale homotopy type.

Our strategy is close in spirit to the original strategy of Artin and Mazur, but uses more modern machinery. The key ideas are the following:

1) For X a separated scheme of finite type over $\mathbb{C}$, the shape of the ∞-topos $\mathrm{Sh}_\infty(X_{an})$ of ∞-sheaves on its underlying space X_{an}, *Shape* $(\mathrm{Sh}_\infty(X_{an}))$ is canonically equivalent to the underlying homotopy type $\Pi_\infty(X_{an})$ of X_{an}.
2) Analytification induces a geometric morphism of topoi
$$\varepsilon : \mathrm{Sh}(X_{an}) \to \mathrm{Sh}(X_{\acute{e}t})$$
from the topos of sheaves on X_{an} and the small étale topos of X, which canonically extends to a geometric morphism of ∞-topoi
$$\varepsilon : \mathrm{Sh}_\infty(X_{an}) \to \mathrm{Sh}_\infty(X_{\acute{e}t}).$$
3) A π-*finite space* is a space V with only finitely many connected components and only finitely many homotopy groups all of which are finite. The geometric morphism ε induces a profinite homotopy equivalence if and only if for every π-finite space V, the induced map between global sections of the constant stack with value V
$$\Gamma_{\acute{e}t}\Delta^{\acute{e}t}(V) \to \Gamma_{an}\Delta^{an}(V)$$
is a homotopy equivalence.
4) By GAGA, ε induces an isomorphism on profinite fundamental groups, and by results in [1], it induces an isomorphism in cohomology with coefficients in any local system of finite abelian groups.

In order to deduce that ε is a profinite homotopy equivalence from 3) and 4), one needs to understand the interpretation of cohomology classes of a space with coefficients in a local system in terms of classifying spaces, and one needs to know this interpretation is valid in any ∞-topos. We therefore dedicate Section A of this paper to carefully working this out. This allows us to prove the maps in 3) are homotopy equivalences by induction using Postnikov towers.

Remark 1.4.1. There is substantial overlap of our results with those of Chough, which were developed at essentially the same time as ours, but independently. (We only became aware of Chough's results, which were part of their PhD thesis still in preparation at the time, after we finished writing the original version of this article in 2015.) Specifically, Chough developed an independent model-theoretic approach to defining the étale homotopy type of algebraic stacks, and proved a profinite comparison theorem for algebraic stacks locally of finite type over $\mathbb{C}$, completely analogous to ours, but using his definition. In subsequent work [9], he showed that our cor 5.14 can be obtained from his comparison result, by using a translation of model-theoretic language to that of ∞-categories. In the final version of this paper, we have included a proof that Chough's definition agrees with ours for hypercomplete objects (cor 4.3.) Although our overall approaches are quite different, we believe both approaches will prove valuable to the mathematical community, and are nicely compatible with each other.

1.5 Conventions and notation By an ∞-category, we mean an $(\infty, 1)$-category. We will model these using quasicategories. We follow very closely the notational conventions and terminology from [31], and refer the reader to the index and notational index op. cit. However, we do make a few small deviations from the notational conventions just mentioned:

1. We shall interchangeably use the notation Gpd_∞ and $\mathcal{S}$ for the ∞-category of ∞-groupoids, or the ∞-category of spaces, since these are in fact the same ∞-category. (We find it useful to use one terminology over another in certain instances to emphasize how we are viewing the objects in question.)
2. For $\mathscr{C}$ an ∞-category, we denote by $\mathrm{Hom}_{\mathscr{C}}(C, D)$ the space of morphisms from C to D rather than using the notation $\mathrm{Map}_{\mathscr{C}}(C, D)$.
3. For $\mathscr{C}$ an ∞-category, we denote by $\mathrm{Psh}_\infty(\mathscr{C})$ the ∞-category of ∞-presheaves, i.e. the functor category
$$\mathrm{Fun}(\mathscr{C}^{op}, \mathcal{S}) = \mathrm{Fun}(\mathscr{C}^{op}, \mathrm{Gpd}_\infty).$$

2. Étale Homotopy Theory

In this section we will present a refinement of the construction of Artin and Mazur for the étale homotopy type of a scheme. Our construction is defined for an arbitrary higher stack on the étale site, and agrees with the definition of Lurie for Deligne-Mumford stacks.

2.1 Pro-spaces and profinite spaces In this subsection, we give a brief recollection of the concepts of pro-objects, pro-spaces, and profinite spaces. For more detail, we refer the reader to [8, Section 2].

Definition 2.1.1. Let $\mathscr{C}$ be any ∞-category. Then there is an ∞-category $\mathrm{Pro}(\mathscr{C})$ together with a fully faithful functor

$$j : \mathscr{C} \hookrightarrow \mathrm{Pro}(\mathscr{C}).$$

The ∞-category $\mathrm{Pro}(\mathscr{C})$ is called the ∞-category of **pro-objects** of $\mathscr{C}$, and it satisfies the following universal property:

$\mathrm{Pro}(\mathscr{C})$ admits small cofiltered limits, and if $\mathcal{D}$ is any ∞-category admitting small cofiltered limits, then composition with j induces an equivalence of ∞-categories

$$\mathrm{Fun}_{co-filt.}(\mathrm{Pro}(\mathscr{E}), \mathscr{D}) \to \mathrm{Fun}(\mathscr{E}, \mathscr{D}), \tag{1}$$

where $\mathrm{Fun}_{co-filt.}(\mathrm{Pro}(\mathscr{E}), \mathscr{D})$ is the full subcategory of $\mathrm{Fun}(\mathrm{Pro}(\mathscr{E}), \mathscr{D})$ spanned by those functors which preserve small cofiltered limits.

In many cases, the ∞-category $\mathrm{Pro}(\mathscr{C})$ can be described explicitly. When $\mathscr{C}$ is small, then we can identify $\mathrm{Pro}(\mathscr{C})$ with the full subcategory of $\mathrm{Fun}(\mathscr{C}, \mathcal{S})^{op}$ spanned by those functors which are cofiltered limits of co-representable functors (those of the form $\mathrm{Hom}_{\mathscr{C}}(C, \cdot)$, for C an object of $\mathscr{C}$). In this case, the functor j is simply the Yoneda embedding (of $\mathscr{C}^{op}$). In fact, this description persists for $\mathscr{C}$ a large (but locally small) ∞-category, provided we replace $\mathcal{S}$ with the ∞-category of large spaces, $\widehat{\mathcal{S}}$, and we demand that the cofiltered limits we are considering are small. However, if $\mathscr{C}$ is accessible and admits finite limits, then there is a more concrete description of $\mathrm{Pro}(\mathscr{C})$, namely it is the full subcategory of $\mathrm{Fun}(\mathscr{C}, \mathcal{S})^{op}$ on those functors which are left exact and accessible [29, Proposition 3.1.6].

Remark 2.1.2. In all the cases above, the functor

$$j : \mathscr{C} \to \mathrm{Pro}(\mathscr{C})$$

can be identified with a restriction of the opposite functor of the Yoneda embedding

$$y : \mathscr{C}^{op} \hookrightarrow \mathrm{Psh}_{\infty}(\mathscr{C}^{op}) = \mathrm{Fun}(\mathscr{C}, \mathcal{S}),$$

and since y is fully faithful and preserves limits, j is fully faithful and preserve colimits.

Definition 2.1.3. The ∞-category $\mathrm{Pro}(\mathcal{S})$ is the ∞-category of **pro-spaces**.

2.1.1 Profinite spaces

Definition 2.1.4. A space X in $\mathcal{S}$ is π**-finite** if all its homotopy groups are finite, it has only finitely many non-trivial homotopy groups, and finitely many connected components.

Definition 2.1.5. Let $\mathcal{S}^\pi$ denote the full subcategory of the ∞-category $\mathcal{S}$ on the π-finite spaces. $\mathcal{S}^\pi$ is essentially small and idempotent complete (and hence accessible). The ∞-category of **profinite spaces** is defined to be the ∞-category

$$\mathrm{Prof}(\mathcal{S}) := \mathrm{Pro}(\mathcal{S}^\pi).$$

Denote by i the canonical inclusion $i : \mathcal{S}^\pi \hookrightarrow \mathcal{S}$. It induces a fully faithful embedding

$$\mathrm{Pro}(i) : \mathrm{Prof}(\mathcal{S}) \hookrightarrow \mathrm{Pro}(\mathcal{S})$$

of profinite spaces into pro-spaces [29, Remark 3.1.7]. It is the functor corresponding under (1) with the composite

$$\mathcal{S}^\pi \overset{i}{\hookrightarrow} \mathcal{S} \overset{j}{\hookrightarrow} \mathrm{Pro}(\mathcal{S}).$$

Moreover, i is accessible and preserves finite limits, hence the above functor has a left adjoint

$$i^* : \mathrm{Pro}(\mathcal{S}) \to \mathrm{Prof}(\mathcal{S})$$

induced by composition with i, by loc. cit.

Definition 2.1.6. We denote by $\widehat{(\,\cdot\,)}$ the composite

$$\mathcal{S} \overset{j}{\hookrightarrow} \mathrm{Pro}(\mathcal{S}) \xrightarrow{i^*} \mathrm{Prof}(\mathcal{S})$$

and call it the **profinite completion functor**. Concretely, if X is a space in $\mathcal{S}$, then $\widehat{X}$ corresponds to the composite

$$\mathcal{S}^\pi \overset{i}{\hookrightarrow} \mathcal{S} \xrightarrow{\mathrm{Hom}(X,\,\cdot\,)} \mathcal{S}.$$

This functor has a right adjoint given by the composite

$$\mathrm{Prof}(\mathcal{S}) \overset{\mathrm{Pro}(i)}{\hookrightarrow} \mathrm{Pro}(\mathcal{S}) \xrightarrow{T} \mathcal{S},$$

where T sends a functor

$$F : \mathcal{S} \to \mathcal{S}$$

corresponding to a pro-space to $F(*)$, that is, the evaluation of F on the one-point space [8, Proposition 2.8]. Concretely, T sends a pro-space of the form $\varprojlim\limits_i j(X_i)$ to the actual limit in $\mathcal{S}$

$$\varprojlim_i X_i,$$

see [8, Proposition 2.10].

2.1.2 Comparison with Artin and Mazur's profinite completion Denote by $h : \mathcal{S} \to \mathcal{H}$ the canonical functor from spaces to the homotopy category of spaces. We hence have an induced functor

$$\mathrm{Pro}\,(h) : \mathrm{Pro}\,(\mathcal{S}) \to \mathrm{Pro}\,(\mathcal{H})\,.$$

Denote by $\mathcal{H}^\pi$ the full subcategory of the homotopy category on π-finite spaces. In [2, Theorem 3.4], Artin and Mazur show that the canonical inclusion

$$\mathrm{Pro}\,(\mathcal{H}^\pi) \to \mathrm{Pro}\,(\mathcal{H})$$

has a left adjoint- which we will denote by $(\,\cdot\,)^{\wedge_{AM}}$.

Lemma 2.1.7. *Let Z be a space and $X = \varprojlim_{\alpha} j\,(X_\alpha)$ a pro-space. Then*

$$\pi_0\left(\mathrm{Hom}_{Pro(\mathcal{S})}\,(X, j\,(Z))\right) \cong \mathrm{Hom}_{Pro(\mathcal{H})}\,(Pro\,(h)\,(X)\,, j\,(h\,(Z)))\,.$$

Proof.

$$\begin{aligned}
\pi_0\left(\mathrm{Hom}_{\mathrm{Pro}(\mathcal{S})}\,(X, j\,(Z))\right) &\simeq \pi_0\left(\varinjlim_{\alpha} \mathrm{Hom}_{\mathcal{S}}\,(X_\alpha, Z)\right)\\
&\simeq \varinjlim_{\alpha} \pi_0\,(\mathrm{Hom}_{\mathcal{S}}\,(X_\alpha, Z))\\
&\simeq \varinjlim_{\alpha} \mathrm{Hom}_{\mathcal{H}}\,(h\,(X_\alpha)\,, h\,(Z))\\
&\simeq \mathrm{Hom}_{Pro(\mathcal{H})}\left(\varprojlim_{\alpha} j\,(h\,(X_\alpha))\,, j\,(h\,(Z))\right)\\
&\simeq \mathrm{Hom}_{Pro(\mathcal{H})}\,(\mathrm{Pro}\,(h)\,(X)\,, j\,(h\,(Z)))\,.
\end{aligned}$$

□

Proposition 2.1.8. *Let X be a pro-space. Then $Pro\,(h)\left(\widehat{X}\right) \cong (Pro\,(h)\,(X))^{\wedge_{AM}}$ in $Pro\,(\mathcal{H})$.*

Proof. Analyzing the proof of [2, Theorem 3.4], they construct $(Y)^{\wedge_{AM}}$ as the unique object in $\mathrm{Pro}\,(\mathcal{H}^\pi)$, such that for all $V \in \mathcal{H}^\pi$,

$$\mathrm{Hom}_{\mathrm{Pro}(\mathcal{H}^\pi)}\left((Y)^{\wedge_{AM}}, j\,(V)\right) \cong \mathrm{Hom}_{\mathrm{Pro}(\mathcal{H})}\,(Y, j\,(V))\,.$$

But, $\mathrm{Pro}\,(h)\left(\widehat{X}\right)$ satisfies this by Lemma 2.1.7. □

Remark 2.1.9. What Artin and Mazur call profinite completion is actually the above, but without the condition that the objects in $\mathcal{H}^\pi$ being truncated- i.e. it is the collection of all spaces all of whose truncations are π-finite. Let us call such spaces *almost π-finite*. Thus, being an isomorphism on profinite completions in their sense is a stronger condition than being a profinite equivalence in our sense- however, these two notions agree in any truncated setting. In particular, in the proof of [2, Theorem 12.9], establishing a profinite homotopy equivalence between the étale homotopy type $X_{\acute{e}t}$ of a pointed connected scheme X of finite type over $\mathbb{C}$, and the analytification of said scheme X_{an}, they first argue it is a profinite homotopy equivalence for π-finite spaces, and then appeal to [1, X, cor 4.3] to show that both objects are already homotopy truncated. Moreover, the notion of profinite homotopy equivalence that Friedlander uses in his version of the comparison theorem [17, Theorem 8.4] for simplicial schemes uses π-finite spaces, not almost π-finite spaces. (More precisely, he asks for the map to induce an equivalence between any finite truncation of the respective profinite completions in the sense of Artin-Mazur.)

2.2 Profinite shape theory We first begin by recalling how to associate to a space X in $\mathcal{S}$, an ∞-topos. To do this, it is conceptually simpler to view such an object X as an ∞-groupoid, as then we have a natural candidate for such an ∞-topos, namely the ∞-topos of ∞-presheaves on X, $\mathrm{Psh}_\infty(X)$. Thinking more topos-theoretically, viewing X as an object of the ∞-topos $\mathcal{S}$ of spaces, another natural candidate for such an ∞-topos is the slice ∞-topos $\mathcal{S}/X$, and these two natural choices agree by [31, cor 5.3.5.4]. By [31, Remark 6.3.5.10, Theorem 6.3.5.13, and Proposition 6.3.4.1], it follows that there is a fully faithful colimit preserving functor

$$\begin{array}{rcl} \mathcal{S}/(\,\cdot\,) : \mathcal{S} & \to & \mathfrak{Top}_\infty \\ X & \mapsto & \mathcal{S}/X \end{array}$$

from the ∞-category of spaces to the ∞-category of ∞-topoi.

Remark 2.2.1. The above functor is not to be confused with the functor

$$\begin{array}{rcl} \mathrm{Sh}_\infty(\,\cdot\,) : \mathbf{Top} & \to & \mathfrak{Top}_\infty \\ T & \mapsto & \mathrm{Sh}_\infty(T) \end{array}$$

sending a *topological* space T to its ∞-topos of ∞-sheaves. The above functor however is also fully faithful, once one restricts it to the full subcategory of sober topological spaces. If T is a (sober) topological space and $\Pi_\infty T$ is its associated ∞-groupoid, then $\mathrm{Sh}_\infty(T)$ remembers the space T up to homeomorphism, whereas $\mathcal{S}/(\Pi_\infty T)$ only captures the weak homotopy type of T. For nice spaces, one can recover $\Pi_\infty T$ however as the *shape* of $\mathrm{Sh}_\infty(T)$ (or its hypercompletion), see Proposition 2.2.3 and Proposition 2.2.7.

The functor

$$\mathcal{S}/(\,\cdot\,) : \mathcal{S} \to \mathfrak{Top}_\infty,$$

by the equivalence (1), induces a well-defined functor

$$\mathrm{Pro}(\mathcal{S}) \to \mathfrak{Top}_\infty$$

which sends a representable pro-space $j(X)$ to $\mathcal{S}/X$, and sends a pro-space of the form $\varprojlim_{i\in\mathcal{I}} X_i$ to the cofiltered limit of ∞-topoi $\varprojlim_{i\in\mathcal{I}} \mathcal{S}/X_i$. Denote this functor by $\mathcal{S}^{pro}/(\,\cdot\,)$. By [31, Remark 7.1.6.15], this functor has a left adjoint *Shape*. We now will describe this construction, which originates from [50]:

Recall that a morphism in $\mathfrak{Top}_\infty$

$$f : \mathcal{E} \to \mathcal{F},$$

called a *geometric morphism*, consists of an adjunction $f^* \dashv f_*$, such that the left adjoint f^* preserves finite limits. Let $\mathcal{E}$ be an ∞-topos. Consider the essentially unique geometric morphism $e : \mathcal{E} \to \mathcal{S}$ to the terminal ∞-topos of spaces. Then the composite

$$\mathcal{S} \xrightarrow{e^*} \mathcal{S} \xrightarrow{e_*} \mathcal{S}$$

is a left-exact functor, i.e. a pro-space. We typically denote the inverse image functor e^* as Δ and the direct image functor e_* as Γ. The functor Γ is the global sections functor, i.e. it sends an object E to the space $\mathrm{Hom}_{\mathcal{E}}(1, E)$. We denote by *Shape* $(\mathcal{E})$ the pro-space $\Gamma \circ \Delta$.

Definition 2.2.2. Let $\mathcal{E}$ be an ∞-topos. Then the pro-space $\mathit{Shape}\,(\mathcal{E})$ is called the **shape** of the ∞-topos $\mathcal{E}$.

We have the following useful proposition:

Proposition 2.2.3. *([20, Example 2.4]) Let T be topological space homotopy equivalent to a CW-complex. Then*

$$\mathit{Shape}\,(Sh_\infty\,(T)) \simeq j\,(\Pi_\infty T)\,.$$

However, it is not always true that the $\mathit{Shape}\,(\mathrm{Sh}_\infty\,(T))$ agrees with the weak homotopy type of T if T does not have the homotopy type of a CW complex:

Example 2.2.4. There exists a topological space X which is compact Hausdorff and locally contractible, such that $\mathit{Shape}\,(\mathrm{Sh}_\infty\,(X))$ does not agree with the weak homotopy type of X.

Proof. We learned this argument from discussion with Jacob Lurie. Let X be a locally contractible compact Hausdorff space which has non-zero Betti numbers in every dimension. The existence of such a space is [4, Theorem 1]. Since X is compact Hausdorff, by [31, cor 7.3.4.12],

$$\Gamma_X : \mathrm{Sh}_\infty\,(X) \to \mathcal{S}$$

preserves filtered colimits, and since $\Delta_X \dashv \Gamma_X$,

$$\mathit{Shape}\,(\mathrm{Sh}_\infty\,(X)) : \mathcal{S} \to \mathcal{S}$$

preserves filtered colimits. If $\mathit{Shape}\,(\mathrm{Sh}_\infty\,(X))$ agreed with the weak homotopy type of X, in particular it would be in the image of j, so there would be $K \in \mathcal{S}$ such that

$$\mathit{Shape}\,(\mathrm{Sh}_\infty\,(X)) = \mathrm{Hom}_{\mathcal{S}}\,(K,\,\cdot\,)\,.$$

But since $\mathit{Shape}\,(\mathrm{Sh}_\infty\,(X))$ preserves filtered colimits, this means K would be a compact object of $\mathcal{S}$, and hence represented by a retract of a finite CW-complex [30, Warning 1.4.2.7]. This would imply that there exists an n, such that for all $k > n$, the k^{th} Betti number is zero. Hence K cannot be weakly equivalent to X. □

However, if we use hypersheaves in place of sheaves, then we *can* recover the weak homotopy type of X for any locally contractible space (See Proposition 2.2.7 below). To show this, we will need the following concept, which is expanded upon in much greater detail in Section 3.1:

Definition 2.2.5. An ∞-topos $\mathcal{E}$ is **locally ∞-connected** if the inverse image functor

$$\Delta : \mathcal{S} \to \mathcal{E}$$

has a left adjoint $\Pi^{\mathcal{E}}_\infty$.

Remark 2.2.6. If $\mathcal{E}$ is a locally ∞-connected ∞-topos, then the pro-space $\mathit{Shape}\,(\mathcal{E})$ is corepresented by the space

$$\Pi^{\mathcal{E}}_\infty\,(1)\,.$$

This follows from the fact that if $\mathcal{G}$ is any space in $\mathcal{S}$, by adjunction we have the following natural equivalences

$$\begin{aligned} \mathrm{Hom}_{\mathcal{S}}\left(\Pi^{\mathcal{E}}_\infty\,(1)\,,\mathcal{G}\right) &\simeq \mathrm{Hom}_{\mathcal{E}}\,(1,\Delta\,(\mathcal{G})) \\ &= \Gamma\Delta\,(\mathcal{G}) \\ &= \mathit{Shape}\,(\mathcal{E})\,(\mathcal{G})\,. \end{aligned}$$

We have the following proposition:

Proposition 2.2.7. *Let T be a locally contractible topological space. Then the shape*

$$\mathit{Shape}\left(\mathbb{H}\mathrm{ypSh}_{\infty}\left(T\right)\right)$$

of its ∞-topos of hypersheaves is equivalent to $j\left(\Pi_{\infty}T\right)$. Moreover, $\mathbb{H}\mathrm{ypSh}_{\infty}\left(T\right)$ is locally ∞-connected.

Proof. Denote by $\Pi_{\infty}:\mathbf{Top}\to\mathcal{S}$ the canonical functor sending a space to its weak homotopy type (as an ∞-groupoid). Denote by $Op\left(T\right)$ the poset of open subsets of T. Denote by l the composite

$$Op\left(T\right)\to\mathbf{Top}\xrightarrow{\Pi_{\infty}}\mathcal{S},$$

where the functor $Op\left(T\right)\to\mathbf{Top}$ sends each open subset U of T to itself. Denote by

$$L=\mathrm{Lan}_{y}l:\mathrm{Psh}_{\infty}\left(Op\left(T\right)\right)\to\mathcal{S}$$

the left Kan extension of l along the Yoneda embedding, i.e. the unique colimit preserving functor which agrees with l on representables. It follows from the Yoneda lemma that this functor has a right adjoint R which sends an ∞-groupoid $\mathcal{G}$ to the ∞-presheaf

$$R\left(\mathcal{G}\right):U\mapsto\mathrm{Hom}\left(l\left(U\right),\mathcal{G}\right).$$

We claim that $R\left(\mathcal{G}\right)$ is a hypersheaf. To see this, it suffices to observe that if $V^{\bullet}$ is a hypercover of U, then, regarding it in the natural way as a simplicial topological space, the colimit of the composite

$$\Delta^{op}\xrightarrow{V^{\bullet}}\mathbf{Top}\xrightarrow{\Pi_{\infty}}\mathcal{S}$$

is $l\left(U\right)$, which follows from [11, Theorem 1.3]. It follows that R and L restrict to adjoint functors

$$\mathbb{H}\mathrm{ypSh}_{\infty}\left(T\right)\underset{L}{\overset{R}{\leftrightarrows}}\mathcal{S}.$$

Denote by $Op^{c}\left(T\right)$ the subposet of $Op\left(T\right)$ on those open subsets which are contractible. Then, since T is locally contractible, by the Comparison Lemma [1, III], it follows that

$$\mathrm{Sh}\left(Op\left(T\right)\right)\simeq\mathrm{Sh}\left(Op^{c}\left(T\right)\right),$$

where the latter topos is the topos of sheaves with respect to covers by contractible open subsets. It now follows from [23, Theorem 5] and [31, Proposition 6.5.2.14] that there is a canonical equivalence

$$\mathbb{H}\mathrm{ypSh}_{\infty}\left(Op\left(T\right)\right)\simeq\mathbb{H}\mathrm{ypSh}_{\infty}\left(Op^{c}\left(T\right)\right).$$

The left adjoint Δ to global sections in $\mathbb{H}\mathrm{ypSh}_{\infty}\left(Op^{c}\left(T\right)\right)$ is defined so that $\Delta\left(\mathcal{G}\right)$ can be computed as the hypersheafification of constant presheaf with value $\mathcal{G}$. Note however that for U in $Op^{c}\left(T\right)$, $R\left(\mathcal{G}\right)$ is a hypersheaf, and

$$\begin{aligned}R\left(\mathcal{G}\right)\left(U\right)&\simeq\mathrm{Hom}\left(l\left(U\right),\mathcal{G}\right)\\&\simeq\mathrm{Hom}\left(*,\mathcal{G}\right)\\&\simeq\mathcal{G},\end{aligned}$$

since U is contractible, and hence the constant presheaf is already a hypersheaf on $Op^c(T)$. Hence we can identify R with Δ. It follows that $\mathbb{H}\mathrm{ypSh}_\infty(T)$ is locally ∞-connected with

$$\Pi^T_\infty = L \dashv \Delta.$$

By Remark 2.2.6, it follows that the shape of $\mathbb{H}\mathrm{ypSh}_\infty(T)$ is corepresented by $j\left(\Pi^T_\infty(1)\right)$. But 1 is the representable presheaf corresponding to the open subset T, and hence $\Pi^T_\infty(1)$ is canonically equivalent to

$$l(T) = \Pi_\infty(T). \qquad \square$$

Consider the profinite completion functor from pro-spaces to profinite spaces

$$i^* : \mathrm{Pro}(\mathcal{S}) \to \mathrm{Prof}(\mathcal{S}).$$

By composition we get a functor

$$\mathfrak{Top}_\infty \xrightarrow{\mathit{Shape}} \mathrm{Pro}(\mathcal{S}) \xrightarrow{i^*} \mathrm{Prof}(\mathcal{S}),$$

which we shall denote by $\mathit{Shape}^{\mathrm{Prof}}$, whose right adjoint is given by the composition

$$\mathrm{Prof}(\mathcal{S}) \xrightarrow{\mathrm{Pro}(i)} \mathrm{Pro}(\mathcal{S}) \xrightarrow{\mathcal{S}^{\mathrm{Pro}}/(\,\cdot\,)} \mathfrak{Top}_\infty,$$

which we shall denote by $\mathcal{S}^{\mathrm{Prof}}/(\,\cdot\,)$.

Remark 2.2.8. Combining Example 2.2.4 with Proposition 2.2.7, we see in particular that the shape of sheaves may not always agree with the shape of hypersheaves.

Definition 2.2.9. Let $\mathcal{E}$ be an ∞-topos. Then the profinite space $\mathit{Shape}^{\mathrm{Prof}}(\mathcal{E})$ is called the **profinite shape** of the ∞-topos $\mathcal{E}$.

Definition 2.2.10. Let $\mathcal{E} \to \mathcal{F}$ be a geometric morphism of ∞-topoi. Such a morphism is a **profinite homotopy equivalence** if the induced map

$$\mathit{Shape}^{\mathrm{Prof}}(\mathcal{E}) \to \mathit{Shape}^{\mathrm{Prof}}(\mathcal{F})$$

is an equivalence of profinite spaces.

Remark 2.2.11. Unraveling the definitions, we see that a geometric morphism $\varphi : \mathcal{E} \to \mathcal{F}$ is a profinite homotopy equivalence, if and only if for every π-finite space V, the canonical morphism

$$f_* f^*(V) \to e_* e^*(V)$$

is an equivalence of spaces, where $e : \mathcal{E} \to \mathcal{S}$ and $f : \mathcal{F} \to \mathcal{S}$ are the (essentially unique) maps to the terminal ∞-topos, and the above map is induced by the unit of the adjunction $\varphi^* \dashv \varphi_*$, using the equivalence $e \simeq f \circ \varphi$.

Proposition 2.2.12. *Let $\mathcal{E}$ be an ∞-topos and let $a : \widehat{\mathcal{E}} \to \mathcal{E}$ be the canonical map from its hypercompletion. Then a is a profinite homotopy equivalence.*

Proof. Using Remark 2.2.11, it suffices to show for any π-finite space V that the canonical map

$$e_*e^*(V) \to e_*\left(a_*a^*\left(e^*(V)\right)\right)$$

is an equivalence, where the map above is induced from the map

$$e^*(V) \to a_*a^*e^*(V).$$

However, the latter map is the canonical map from $e^*(V)$ to its hypersheafification. Since V is π-finite, it is n-truncated for some n, and hence so is $e^*(V)$ by [31, Proposition 5.5.6.16], since e^* is left exact. However, every n-truncated sheaf automatically satisfies hyperdescent, so this map is an equivalence. □

2.3 The small étale ∞-topos. For this subsection and the next, we will work over an arbitrary commutative ring k.

Definition 2.3.1. A subcategory $\mathbf{Aff}_k^{\mathfrak{U}}$ of the category of affine schemes over k, $\mathbf{Aff}_k$, is **étale closed** if for any object X in $\mathbf{Aff}_k^{\mathfrak{U}}$, if $Y \to X$ is an étale morphism from an affine scheme, then Y is in $\mathbf{Aff}_k^{\mathfrak{U}}$. We denote the corresponding subcategory of commutative k-algebras by $\mathbf{Alg}_k^{\mathfrak{U}}$.

Example 2.3.2. Since étale maps are of finite presentation, the subcategory $\mathbf{Aff}_k^{LFT}$ of affine k-schemes of finite type is étale closed.

Remark 2.3.3. Since étale maps between affine schemes are of finite presentation, any essentially small subcategory of $\mathbf{Aff}_k$ is contained in an essentially small étale closed subcategory.

In the rest of this section, we will work over a fixed essentially small étale closed subcategory $\mathbf{Aff}_k^{\mathfrak{U}}$ of $\mathbf{Aff}_k$, unless otherwise specified. The notion of being étale closed was specifically chosen so that the étale pretopology on $\mathbf{Aff}_k$ naturally restricts.

Recall that for a scheme X, its **small étale site** is the following Grothendieck site: As a category, $X_{\acute{e}t}$ consists of étale morphisms

$$U \to X,$$

with U another scheme, and the morphisms are commutative triangles over X. The Grothendieck pretopology on this category is given by étale covering families. The small étale topos $\mathrm{Sh}(X_{\acute{e}t})$ is the topos of sheaves over this site.

Definition 2.3.4. Let X be a scheme. Its **small étale ∞-topos** is the ∞-topos $\mathrm{Sh}_\infty(X_{\acute{e}t})$.

Remark 2.3.5. Since étale maps are stable under pullback, the category $X_{\acute{e}t}$ has finite limits. It follows then from [31, Lemma 6.4.5.6] that $\mathrm{Sh}_\infty(X_{\acute{e}t})$ is the 1-localic ∞-topos corresponding to $\mathrm{Sh}(X_{\acute{e}t})$ under the equivalence of ∞-categories between the $(2,1)$-category of topoi and the ∞-category of 1-localic ∞-topoi.

This definition naturally carries over for Deligne-Mumford stacks and their higher analogues. It will be technically convenient to work straightaway with higher Deligne-Mumford stacks. We start by briefly recalling some material from [28] and [6].

Let A be a commutative k-algebra, where k is our base ring. Denote by $A_{\acute{e}t}$ the category whose objects consists of étale morphisms $U \to \mathrm{Spec}\,(A)$, with U another *affine* scheme. There is a canonical inclusion of sites

$$A_{\acute{e}t} \hookrightarrow \mathrm{Spec}\,(A)_{\acute{e}t},$$

which satisfies the conditions of the Comparison Lemma [1, III], hence one has

$$\mathrm{Sh}\left(A_{\acute{e}t}\right) \simeq \mathrm{Sh}\left(\mathrm{Spec}\ (A)_{\acute{e}t}\right).$$

As both sites have finite limits, it follows from [31, Proposition 6.4.5.4] that

$$\mathrm{Sh}_\infty\left(A_{\acute{e}t}\right) \simeq \mathrm{Sh}_\infty\left(\mathrm{Spec}\ (A)_{\acute{e}t}\right).$$

Notice that there is a canonical sheaf of rings $\mathcal{O}_A$ on the site $A_{\acute{e}t}$, which assigns an étale map

$$\mathrm{Spec}\ (B) \to \mathrm{Spec}\ (A)$$

the ring B. The stalks of this sheaf $\mathcal{O}_A$ along geometric points are not only local k-algebras, but in fact *strictly Henselian*. This is important in order to get the correct notion of morphism between ringed topoi. Just as a map of ringed spaces need not be a map of *locally* ringed spaces, since one must demand that the induced map along stalks is a map of local rings, i.e. preserves the maximal ideals, a map of strictly Henselian ringed topoi needs to respect the Henselian ring structure along stalks, i.e. be a Henselian map. Using this idea one can define an ∞-category of strictly Henselian ringed ∞-topoi. Let us denote this ∞-category by $\mathfrak{Top}_\infty^{Hens.}$. (To be precise, $\mathfrak{Top}_\infty^{Hens.}$ is the ∞-category $\mathcal{L}\mathcal{T}op\left(\mathcal{G}\right)^{op}$ defined in [28, Definition 1.4.8], with $\mathcal{G}$ the étale geometry in the sense of Section 2.6 of op. cit.) By [28, Theorem 2.2.12] with $\mathcal{G}$ the étale geometry as in Section 2.6 of op. cit., the construction

$$A \mapsto \mathrm{Sh}_\infty\left(A_{\acute{e}t}\right)$$

can be turned into a fully faithful functor

$$\mathrm{Spec}_{\acute{e}t} : \mathbf{Aff}_k^{\mathcal{U}} \hookrightarrow \mathfrak{Top}_\infty^{Hens.}$$

from affine k-schemes of finite type over k to ∞-topoi locally ringed in strict Henselian rings.

The ∞-category $\mathfrak{Top}_\infty^{Hens.}$ carries a natural Grothendieck topology [6, Definition 4.3.2], also called the *étale topology*, which is a natural extension of the classical étale topology on $\mathbf{Aff}_k^{\mathcal{U}}$ with respect to the functor $\mathrm{Spec}_{\acute{e}t}$. We say that a strictly Henselian ringed ∞-topos $(\mathcal{E}, \mathcal{O}_{\mathcal{E}})$ is ($\mathcal{U}$-)**Deligne-Mumford** if there exists an étale covering family

$$\left(\left(\mathcal{E}_\alpha, \mathcal{O}_{E_\alpha}\right) \to \left(\mathcal{E}, \mathcal{O}_E\right)\right)_\alpha$$

such that for each α,

$$\left(\mathcal{E}_\alpha, \mathcal{O}_{E_\alpha}\right) \simeq \mathrm{Spec}_{\acute{e}t}\left(A_\alpha\right),$$

for $A_\alpha \in \mathbf{Alg}_k^{\mathcal{U}}$. We will call a Deligne-Mumford strictly Henselian ringed ∞-topos a **Deligne-Mumford scheme**, as these are precisely $\mathcal{G}$-schemes in the sense of [28, Definition 2.3.9], where $\mathcal{G}$ is the étale geometry in the sense of Section 2.6 of op. cit. (and the étale cover is by affines in $\mathbf{Aff}_k^{\mathcal{U}}$). We denote the ∞-category of Deligne-Mumford schemes by $\mathfrak{DM}\mathbf{Sch}_k^{\mathcal{U}}$.

Restriction along $\mathrm{Spec}_{\acute{e}t}$ defines for each strictly Henselian ringed ∞-topos $(\mathcal{E}, \mathcal{O}_E)$ a functor of points

$$\begin{array}{rcl} \tilde{y}\left(\left(\mathcal{E}, \mathcal{O}_E\right)\right) : \left(\mathbf{Aff}_k^{\mathcal{U}}\right)^{op} & \to & \mathrm{Gpd}_\infty \\ \mathrm{Spec}\ (A) & \mapsto & \mathrm{Hom}_{\mathfrak{Top}_\infty^{Hens.}}\left(\mathrm{Spec}_{\acute{e}t}\left(A\right), \left(\mathcal{E}, \mathcal{O}_E\right)\right). \end{array}$$

Each functor $\tilde{y}((\mathcal{E}, \mathcal{O}_E))$ satisfies étale descent and the restriction of $\tilde{y}$ to Deligne-Mumford schemes defines a fully faithful functor

$$\tilde{y} : \mathfrak{DM}\mathbf{Sch}_k^{\mathfrak{U}} \hookrightarrow \mathrm{Sh}_\infty\left(\mathbf{Aff}_k^{\mathfrak{U}}, \acute{e}t\right)$$

[28, Theorem 2.4.1, Lemma 2.4.13], [6, Theorem 5.2.2, Remark 5.2.3].

Definition 2.3.6. A **Deligne-Mumford ∞-stack** is an ∞-stack $\mathcal{X}$ on the étale site of $\mathbf{Aff}_k^{\mathfrak{U}}$, equivalent to the functor of points of a Deligne-Mumford scheme. We denote the ∞-category of such stacks by $\mathfrak{DM}(k)_\infty^{\mathfrak{U}}$.

Remark 2.3.7. By [28, Theorem 2.6.18], $\mathfrak{DM}(k)_\infty^{\mathfrak{U}}$ contains the classical $(2,1)$-category of Deligne-Mumford stacks that can be modeled on affines in $\mathbf{Aff}_k^{\mathfrak{U}}$ as a full subcategory, but also contains more general objects as there are no separation conditions imposed. E.g., $B(\mathbb{Z})$ we will be considered a Deligne-Mumford stack in this setting. This will cause no problems and will in fact simplify the proofs considerably.

Definition 2.3.8. Let $\mathcal{X}$ be a Deligne-Mumford (∞-)stack. Its **small étale ∞-topos** is the ∞-topos of ∞-sheaves over $\mathcal{X}_{\acute{e}t}$, $\mathrm{Sh}_\infty(\mathcal{X}_{\acute{e}t})$, where $\mathcal{X}_{\acute{e}t}$ is the (∞-)category of (not necessarily representable) étale maps $U \to \mathcal{X}$ with U a scheme, equipped with the Grothendieck topology generated by étale covering families. If $\mathcal{X} \simeq \tilde{y}((\mathcal{E}, \mathcal{O}_{\mathcal{E}}))$, by an étale morphism, we mean a morphism $U \to \mathcal{X}$ which under the Yoneda lemma corresponds to a morphism

$$\left(\mathrm{Sh}_\infty(U_{\acute{e}t}), \mathcal{O}_U\right) \to (\mathcal{E}, \mathcal{O}_{\mathcal{E}})$$

of Deligne-Mumford schemes which is étale in the sense of [28, Definition 2.3.1].

Lemma 2.3.9. *Let $\mathcal{X}$ be a Deligne-Mumford ∞-stack. By definition, $\mathcal{X}$ is the functor of points of a Deligne-Mumford scheme $(\mathcal{E}, \mathcal{O}_{\mathcal{E}})$. In this case, one has a canonical equivalence*

$$Sh_\infty(\mathcal{X}_{\acute{e}t}) \simeq \mathcal{E}.$$

Proof. Let $\mathfrak{DM}(k)_\infty^{\mathfrak{U}}$ denote the ∞-category of Deligne-Mumford ∞-stacks built out of affine schemes in $\mathbf{Aff}_k^{\mathfrak{U}}$, and define similarly $\mathbf{Sch}_k^{\mathfrak{U}}$ to be the analogously defined category of schemes. By [6, Remarks 5.19 and 5.23], we see that $\mathbf{Sch}_k^{\mathfrak{U}}$ is a locally small strong étale blossom in the sense of [6, Definition 5.1.7], or more precisely, $\mathbf{Sch}_k^{\mathfrak{U}}$ is canonically equivalent to the strong étale blossom whose objects are the Deligne-Mumford schemes which are classical schemes built out of affines in $\mathbf{Aff}_k^{\mathfrak{U}}$. By [6, Theorems 5.3.6 and 5.3.7] combined with Proposition 5.3.2 of op. cit., it follows in fact that

$$\mathcal{E} \simeq \mathrm{Sh}_\infty\left(\mathbf{Sch}_k^{\mathfrak{U},\acute{e}t}\right)/\tilde{y}^{\acute{e}t}(\mathcal{X}),$$

where $\tilde{y}^{\acute{e}t}(\mathcal{X})$ is the stack assigning a scheme X the ∞-groupoid of étale maps

$$X \to \mathcal{X}.$$

By [6, Remark 2.2.4 and Proposition 2.2.1], we conclude that $\mathcal{E} \simeq \mathrm{Sh}_\infty(\mathcal{X}_{\acute{e}t})$. ∎

Lemma 2.3.10. *There is a colimit preserving functor*

$$Sh_\infty\left((\,\cdot\,)_{\acute{e}t}\right) : Sh_\infty\left(\mathbf{Aff}_k^{\mathfrak{U}}, \acute{e}t\right) \to \mathfrak{Top}_\infty$$

which sends an affine scheme $\mathrm{Spec}\,(A)$ *to* $Sh_\infty(A_{\acute{e}t})$.

Proof. We start by constructing such a functor out of ∞-presheaves, which can be accomplished simply by taking the left Kan extension of the functor

$$\mathbf{Aff}_k^{\mathfrak{U}} \xrightarrow{\quad \mathrm{Spec}_{\acute{e}t} \quad} \mathfrak{Top}_{\infty}^{Hens.} \to \mathfrak{Top}_{\infty}$$

along the Yoneda embedding

$$y : \mathbf{Aff}_k^{\mathfrak{U}} \hookrightarrow \mathrm{Psh}_{\infty}\left(\mathbf{Aff}_k^{\mathfrak{U}}\right),$$

where

$$\mathfrak{Top}_{\infty}^{Hens.} \to \mathfrak{Top}_{\infty}$$

is the canonical functor which forgets the structure sheaf. Denote this left Kan extension by L. By [31, Proposition 5.5.4.20 and Theorem 5.1.5.6], it suffices to show that L sends each covering sieve

$$S_{\mathfrak{U}} \hookrightarrow y\left(\mathrm{Spec}\ (A)\right)$$

for $\mathfrak{U} = \left(U_i \to \mathrm{Spec}\ (A)\right)_i$ an étale covering family, to an equivalence. Note however that this covering sieve is the colimit of the Čech nerve

$$N_{\mathfrak{U}} : \Delta^{op} \to \mathrm{Psh}_{\infty}\left(\mathbf{Aff}_k^{\mathfrak{U}}\right)$$

of $\mathfrak{U}$. Since L preserves colimits, it thus suffices to show that the canonical map

$$\varinjlim L \circ N_{\mathfrak{U}} \to L\left(y\left(\mathrm{Spec}\ (A)\right)\right) \simeq \mathrm{Sh}_{\infty}\left(A_{\acute{e}t}\right)$$

is an equivalence.

The functor $N_{\mathfrak{U}}$ has a canonical lift to an augmented simplicial diagram

$$\widehat{N_{\mathfrak{U}}} : \left(\Delta^{op}\right)^{\triangleright} \cong \Delta_{+}^{op} \longrightarrow \mathrm{Psh}_{\infty}\left(\mathbf{Aff}_k^{\mathfrak{U}}\right)$$

defining the canonical cocone for $N_{\mathfrak{U}}$ with vertex $y\left(\mathrm{Spec}\ (A)\right)$ (which corresponds to the inclusion of the subobject $S_{\mathfrak{U}} \hookrightarrow y\left(\mathrm{Spec}\ (A)\right)$). At the level of simplicial sets, the formation of right cones is strictly left adjoint to the formation of slice quasicategories, so the map of simplicial sets $\widehat{N_{\mathfrak{U}}}$ is adjoint to a map

$$\widetilde{N}_{\mathfrak{U}} : \Delta^{op} \to \mathrm{Psh}_{\infty}\left(\mathbf{Aff}_k^{\mathfrak{U}}\right)/y\left(\mathrm{Spec}\ (A)\right).$$

Now L induces a colimit preserving functor

$$\widetilde{L} : \mathrm{Psh}_{\infty}\left(\mathbf{Aff}_k^{\mathfrak{U}}\right)/y\left(\mathrm{Spec}\ (A)\right) \to \mathfrak{Top}_{\infty}/\mathrm{Sh}_{\infty}\left(A_{\acute{e}t}\right)$$

By [28, Example 2.3.8] together with the fact that along representables L agrees with ∞-sheaves on the small étale site, the diagram $\widetilde{L} \circ \widetilde{N}_{\mathfrak{U}}$ consists of étale geometric morphisms of ∞-topoi over $\mathrm{Sh}_{\infty}\left(A_{\acute{e}t}\right)$ and therefore there is a factorization of $\widetilde{L} \circ \widetilde{N}_{\mathfrak{U}}$ of the form

$$\Delta^{op} \xrightarrow{F} \mathfrak{Top}^{\acute{e}t}/\mathrm{Sh}_{\infty}\left(A_{\acute{e}t}\right) \to \mathfrak{Top}/\mathrm{Sh}_{\infty}\left(A_{\acute{e}t}\right),$$

where $\mathfrak{Top}^{\acute{e}t}$ denotes the ∞-category of ∞-topoi and étale geometric morphisms, and moreover the composite

$$\Delta^{op} \xrightarrow{F} \mathfrak{Top}^{\text{ét}}/\mathrm{Sh}_{\infty}\left(A_{\acute{e}t}\right) \to \mathfrak{Top}_{\infty}^{\acute{e}t} \to \mathfrak{Top}_{\infty}$$

agrees up to equivalence with $L \circ N_{\mathfrak{U}}$. Note that by [31, Remark 6.3.5.10], there is a canonical equivalence of ∞-categories $\mathfrak{Top}^{\acute{e}t}/\mathrm{Sh}_\infty(A_{\acute{e}t}) \simeq \mathrm{Sh}_\infty(A_{\acute{e}t})$ under which F corresponds to the Čech nerve of the same étale cover, except regarded as a simplicial diagram

$$\overline{N}_{\mathfrak{U}} : \Delta^{op} \to \mathrm{Sh}_\infty(A_{\acute{e}t}).$$

The colimit of this diagram is the terminal object. Notice that $\mathfrak{Top}^{\acute{e}t}/\mathrm{Sh}_\infty(A_{\acute{e}t}) \to \mathfrak{Top}^{\acute{e}t}_\infty$ preserves colimits and so does $\mathfrak{Top}^{\acute{e}t}_\infty \to \mathfrak{Top}_\infty$ by [31, Theorem 6.3.5.13]. The result now follows since the terminal object gets sent to $\mathrm{Sh}_\infty(A_{\acute{e}t})$ under the composite

$$\mathfrak{Top}^{\acute{e}t}/\mathrm{Sh}_\infty(A_{\acute{e}t}) \to \mathfrak{Top}^{\acute{e}t}_\infty \to \mathfrak{Top}_\infty. \qquad \square$$

Definition 2.3.11. Let F be an ∞-sheaf on $\left(\mathbf{Aff}^{\mathfrak{U}}_k, \acute{e}t\right)$. Then the **small étale ∞-topos of** F is

$$\mathrm{Sh}_\infty(F_{\acute{e}t}) := \mathrm{Sh}_\infty\left((\,\cdot\,)_{\acute{e}t}\right)(F).$$

We will proceed to justify this definition by showing it agrees with Definition 2.3.8 when F is a Deligne-Mumford (∞-)stack. First, we will show that the definition does not depend on the ambient étale closed subcategory:

Remark 2.3.12. Suppose that $\mathbf{Aff}^{\mathfrak{V}}_k$ is an essentially small étale closed category of affine schemes which contains $\mathbf{Aff}^{\mathfrak{U}}_k$. Denote by i the inclusion

$$i : \mathbf{Aff}^{\mathfrak{U}}_k \hookrightarrow \mathbf{Aff}^{\mathfrak{V}}_k.$$

Then i induces a restriction functor

$$i^* : \mathrm{Sh}_\infty\left(\mathbf{Aff}^{\mathfrak{V}}_k, \acute{e}t\right) \to \mathrm{Sh}_\infty\left(\mathbf{Aff}^{\mathfrak{U}}_k, \acute{e}t\right)$$

which has a fully faithful left adjoint

$$i_! : \mathrm{Sh}_\infty\left(\mathbf{Aff}^{\mathfrak{U}}_k, \acute{e}t\right) \hookrightarrow \mathrm{Sh}_\infty\left(\mathbf{Aff}^{\mathfrak{V}}_k, \acute{e}t\right).$$

Concretely, $i_!$ is the unique colimit preserving functor sending each affine scheme $\mathrm{Spec}(A)$ to itself. Denote by

$$S : \mathrm{Sh}_\infty\left(\mathbf{Aff}^{\mathfrak{U}}_k, \acute{e}t\right) \to \mathfrak{Top}_\infty$$

the colimit preserving functor from Lemma 2.3.10, and similarly denote by

$$R : \mathrm{Sh}_\infty\left(\mathbf{Aff}^{\mathfrak{V}}_k, \acute{e}t\right) \to \mathfrak{Top}_\infty$$

the corresponding functor for the étale closed category $\mathbf{Aff}^{\mathfrak{V}}_k$. Then we have a canonical equivalence

$$S(F) \simeq R(i_!(F)).$$

To see this, note the composition

$$\mathrm{Sh}_\infty\left(\mathbf{Aff}^{\mathfrak{U}}_k, \acute{e}t\right) \xrightarrow{i_!} \mathrm{Sh}_\infty\left(\mathbf{Aff}^{\mathfrak{V}}_k, \acute{e}t\right) \xrightarrow{R} \mathfrak{Top}_\infty$$

is colimit preserving and agrees with S along representables. It follows that the above composition

$$R \circ i_! \simeq S.$$

We will now justify the notation for the functor $\mathrm{Sh}_\infty\left(\left(\,\cdot\,\right)_{\acute{e}t}\right)$ by showing it agrees with Definition 2.3.8:

Proposition 2.3.13. *Let $S = Sh_\infty\left(\left(\,\cdot\,\right)_{\acute{e}t}\right)$ be the functor from Lemma 2.3.10, and let $\mathfrak{X}$ be a Deligne-Mumford ∞-stack. Then $S\left(\mathfrak{X}\right)$ is equivalent to the small étale ∞-topos of $\mathfrak{X}$, in the sense of Definition 2.3.8.*

Proof. Denote by $\mathfrak{DM}\left(k\right)_\infty^{\mathfrak{U},\acute{e}t}$ the subcategory of Deligne-Mumford ∞-stacks built out of affines in $\mathfrak{U}$, where the morphisms are (not necessarily representable) étale maps. By [6, Proposition 5.2.11 and Remark 5.3.11], it follows that the composition

$$\mathfrak{DM}\left(k\right)_\infty^{\mathfrak{U},\acute{e}t} \to \mathfrak{DM}\left(k\right)_\infty^{\mathfrak{U}} \hookrightarrow \mathrm{Sh}_\infty\left(\mathbf{Aff}_k^{\mathfrak{U}}, \acute{e}t\right)$$

preserves colimits. Hence

$$\mathfrak{DM}\left(k\right)_\infty^{\mathfrak{U},\acute{e}t} \to \mathfrak{DM}\left(k\right)_\infty^{\mathfrak{U}} \hookrightarrow \mathrm{Sh}_\infty\left(\mathbf{Aff}_k^{\mathfrak{U}}, \acute{e}t\right) \xrightarrow{S} \mathfrak{Top}_\infty$$

preserves colimits as well. By [28, Remark 5.3.11, Lemma 5.1.1, and Proposition 4.3.1], so does the composite

$$\mathfrak{DM}\left(k\right)_\infty^{\mathfrak{U},\acute{e}t} \simeq \mathfrak{DM}\mathbf{Sch}_k^{\mathfrak{U},\acute{e}t} \to \mathfrak{Top}_\infty^{\acute{e}t},$$

where $\mathfrak{DM}\mathbf{Sch}_k^{\mathfrak{U},\acute{e}t}$ is the ∞-category of Deligne-Mumford schemes and their étale morphisms, and hence, by [31, Theorem 6.3.5.13], the composite

$$\mathfrak{DM}\left(k\right)_\infty^{\mathfrak{U},\acute{e}t} \simeq \mathfrak{DM}\mathbf{Sch}_k^{\mathfrak{U},\acute{e}t} \to \mathfrak{Top}_\infty^{\acute{e}t} \to \mathfrak{Top}_\infty$$

also preserves colimits. Note that the latter composite sends a Deligne-Mumford ∞-stack $\mathfrak{X}$ which is the functor of points of a Deligne-Mumford scheme $\left(\mathcal{E}, \mathcal{O}_{\mathcal{E}}\right)$ to the ∞-topos $\mathcal{E}$. So both composites

$$\mathfrak{DM}\left(k\right)_\infty^{\mathfrak{U},\acute{e}t} \simeq \mathfrak{DM}\mathbf{Sch}_k^{\mathfrak{U},\acute{e}t} \to \mathfrak{Top}_\infty^{\acute{e}t} \to \mathfrak{Top}_\infty$$

and

$$\mathfrak{DM}\left(k\right)_\infty^{\mathfrak{U},\acute{e}t} \to \mathfrak{DM}\left(k\right)_\infty^{\mathfrak{U}} \hookrightarrow \mathrm{Sh}_\infty\left(\mathbf{Aff}_k^{\mathfrak{U}}, \acute{e}t\right) \xrightarrow{S} \mathfrak{Top}_\infty$$

are colimit preserving and send an affine scheme $\mathrm{Spec}\left(A\right)$ to $\mathrm{Sh}_\infty\left(A_{\acute{e}t}\right)$. By [6, Theorem 5.37] (combined with Remark 4.31 of op. cit.) we see that there is a canonical equivalence of ∞-categories

$$\mathrm{Sh}_\infty\left(\mathbf{Aff}^{\mathfrak{U},\acute{e}t}\right) \simeq \mathfrak{DM}\left(k\right)_\infty^{\mathfrak{U},\acute{e}t}$$

which sends the sheaf of étale points of an affine scheme $\mathrm{Spec}\left(A\right)$ to $\mathrm{Spec}\left(A\right)$ itself. The result now follows from [31, Proposition 5.5.4.20 and Theorem 5.1.5.6], together with Lemma 2.3.9 of this article. □

Remark 2.3.14. The results of this section readily generalize to the settings of derived and spectral algebraic geometry. The proofs are exactly the same, once one replaces the étale geometry ([28, Definition 2.6.12]) with the derived étale geometry ([28, Definition 4.3.13]) or the spectral étale geometry ([32, Definition 8.11]) respectively. We presented the results in the setting of non-derived schemes merely to avoid overburdening the reader with new concepts. In fact, nothing is lost, since for a derived Deligne-Mumford stack, its underlying classical Deligne-Mumford stack has the same underlying ∞-topos.

2.4 Étale homotopy type We now present the definition of the étale homotopy type of a general ∞-sheaf on the étale site of $\mathbf{Aff}_k^{\mathfrak{U}}$, for some small étale closed subcategory of affine k-schemes, in the sense of Definition 2.3.1.

Definition 2.4.1. The **étale fundamental ∞-groupoid** functor is the composite

$$\mathrm{Sh}_\infty\left(\mathbf{Aff}_k^{\mathfrak{U}}, \acute{e}t\right) \xrightarrow{\mathrm{Sh}_\infty((\cdot)_{\acute{e}t})} \mathfrak{Top}_\infty \xrightarrow{Shape} \mathrm{Pro}\,(\mathcal{S}),$$

and is denoted by $\Pi_\infty^{\acute{e}t}$. For F an ∞-sheaf on $\left(\mathbf{Aff}_k^{\mathfrak{U}}, \acute{e}t\right)$, its **étale homotopy type** is $Shape\,(\mathrm{Sh}_\infty(F_{\acute{e}t}))$, the shape of its small étale ∞-topos.

We also introduce a slight variant:

Definition 2.4.2. The **hyper-étale fundamental ∞-groupoid** functor is the composite

$$\mathrm{Sh}_\infty\left(\mathbf{Aff}_k^{\mathfrak{U}}, \acute{e}t\right) \xrightarrow{\mathrm{Sh}_\infty((\cdot)_{\acute{e}t})} \mathfrak{Top}_\infty \xrightarrow{\mathbb{H}yp} \mathfrak{Top}_\infty \xrightarrow{Shape} \mathrm{Pro}\,(\mathcal{S}),$$

where $\mathbb{H}yp$ is the hypercompletion functor. We denote this composite by $\Pi_\infty^{\mathbb{H}\text{-}\acute{e}t}$. For F an ∞-sheaf on $\left(\mathbf{Aff}_k^{\mathfrak{U}}, \acute{e}t\right)$, its **hyper-étale homotopy type** is $Shape\,(\mathbb{H}\mathrm{yp}\mathrm{Sh}_\infty(F_{\acute{e}t}))$, the shape of the hypercompletion of its small étale ∞-topos.

Remark 2.4.3. The process of hypercompleting an ∞-topos is indeed functorial. Let $\widehat{\mathrm{Psh}}_\infty(\mathfrak{Top}_\infty)$ denote the ∞-category of large presheaves of ∞-groupoids on the ∞-category of ∞-topoi, and consider the inclusion

$$q : \mathfrak{Top}_\infty^{\mathbb{HC}} \hookrightarrow \mathfrak{Top}_\infty$$

of the full subcategory of hypercomplete ∞-topoi. Then, by [31, Proposition 6.5.2.13], for any ∞-topos $\mathcal{E}$, if $y(\mathcal{E})$ is its associated representable (large) presheaf, the presheaf $q^*y(\mathcal{E})$ on $\mathfrak{Top}_\infty^{\mathbb{HC}}$ is representable by the hypercompletion $\widehat{\mathcal{E}}$. It follows that there is a canonical natural equivalence making the following diagram commute

$$\begin{tikzcd}
\mathfrak{Top}_\infty \arrow[r, hook, "y"] \arrow[drr, dashed, "\mathbb{H}yp"'] & \widehat{\mathrm{Psh}}_\infty(\mathfrak{Top}_\infty) \arrow[r, "q^*"] & \widehat{\mathrm{Psh}}_\infty\left(\mathfrak{Top}_\infty^{\mathbb{HC}}\right) \\
& & \mathfrak{Top}_\infty^{\mathbb{HC}}. \arrow[u, hook, "y"']
\end{tikzcd}$$

Moreover, this canonical equivalence component-wise

$$y\left(\widehat{\mathcal{E}}\right) \xrightarrow{\sim} q^*y(\mathcal{E})$$

under the Yoneda lemma corresponds to the canonical geometric morphism

$$\epsilon_{\mathcal{E}} : \widehat{\mathcal{E}} \hookrightarrow \mathcal{E},$$

and hence $\mathbb{H}yp$ is right adjoint to the canonical inclusion

$$\mathfrak{Top}^{\mathbb{HC}} \hookrightarrow \mathfrak{Top}_\infty,$$

with counit

$$\epsilon : q \circ \mathbb{H}yp \Rightarrow id_{\mathfrak{Top}_\infty}.$$

Example 2.4.4. There exists a pseudo-algebraically closed field $\mathbb{F}$ such that

$$\Pi_\infty^{\acute{e}t}(\mathrm{Spec}\,(\mathbb{F})) \not\simeq \Pi_\infty^{\mathbb{H}\text{-}\acute{e}t}(\mathrm{Spec}\,(\mathbb{F})).$$

Proof. Fix a prime p, and consider the p-adic integers $\mathbb{Z}_p$ as a profinite group. Since $\mathbb{Z}_p$ is projective, by [15, 23.1.3], there exists a pseudo-algebraically closed field $\mathbb{F}$ such that the absolute Galois group

$$\mathbf{Gal}\,(\mathbb{F}^{sep}/\mathbb{F}) \cong \mathbb{Z}_p.$$

By [7, Proposition 4.9], we have that $\mathrm{Sh}_\infty(\mathbb{F}_{\acute{e}t}) \simeq \mathcal{S}/B\mathbf{Gal}\,(\mathbb{F}^{sep}/\mathbb{F})$, where $\mathcal{S}/\cdot$ is the right adjoint to *Shape*. But as $B\mathbf{Gal}\,(\mathbb{F}^{sep}/\mathbb{F})$ is a profinite space, by [27, Theorem E.2.4.1], it follows that

$$\mathit{Shape}\,(\mathrm{Sh}_\infty(\mathbb{F}_{\acute{e}t})) \simeq B\mathbf{Gal}\,(\mathbb{F}^{sep}/\mathbb{F}).$$

Consider the sheaves F and F' defined as in [31, Warning 7.2.2.31]. There it is argued that F is not a hypersheaf. Now, the canonical ∞-connective map $\alpha : F \to F'$ induces an equivalence

$$a(\alpha) : a(F) \xrightarrow{\sim} a\left(F'\right),$$

where a denote hypersheafification. Since $a(\alpha)$ is invertible, there is an induced map

$$F' \to a\left(F'\right) \to a(F)$$

inducing a global section of $a(F)$ not in the image of F. It follows that the ∞-topos $\mathrm{Sh}_\infty(\mathbb{F}_{\acute{e}t})$ has a different shape than its hypercompletion, since their corresponding shapes, as functors from $\mathcal{S} \to \mathcal{S}$ do not agree on K. Hence

$$\Pi_\infty^{\acute{e}t}(\mathrm{Spec}\,(\mathbb{F})) \not\simeq \Pi_\infty^{\mathbb{H}\text{-}\acute{e}t}(\mathrm{Spec}\,(\mathbb{F})). \qquad \square$$

Remark 2.4.5. This is another case of the shape of the hypercompletion of an ∞-topos differing from the shape of the ∞-topos itself. Also, it implies that the "well-known" fact that the étale homotopy type of $\mathrm{Spec}\,(k_{\acute{e}t})$ is $B\mathbf{Gal}\,(k^{sep}/k)$ can actually be false if we use hypersheaves. However, most of the literature is in the setting of working with $\sharp$-equivalences of pro-spaces, so this distinction is erased.

Definition 2.4.6. The **profinite étale fundamental ∞-groupoid** functor is the composite

$$\mathrm{Sh}_\infty\left(\mathbf{Aff}_k^{\mathcal{U}}, \acute{e}t\right) \xrightarrow{\mathrm{Sh}_\infty((\,\cdot\,)_{\acute{e}t})} \mathfrak{Top}_\infty \xrightarrow{\mathit{Shape}^{\mathrm{Prof}}} \mathrm{Prof}\,(\mathcal{S}),$$

and is denoted by $\widehat{\Pi}_\infty^{\acute{e}t}$. For F an ∞-sheaf on $\left(\mathbf{Aff}_k^{\mathcal{U}}, \acute{e}t\right)$, its **profinite étale homotopy type** is $\mathit{Shape}^{\mathrm{Prof}}(\mathrm{Sh}_\infty(F_{\acute{e}t}))$, the shape of its small étale ∞-topos.

Remark 2.4.7. There is no need to introduce a hyper-étale variant of the profinite étale ∞-groupoid functor since by Proposition 2.2.12, for each étale ∞-sheaf F we have an induced equivalence of profinite spaces

$$\mathit{Shape}^{\mathrm{Prof}}(\mathbb{H}\mathrm{ypSh}_\infty(F_{\acute{e}t})) \xrightarrow{\sim} \mathit{Shape}^{\mathrm{Prof}}(\mathrm{Sh}_\infty(F_{\acute{e}t})).$$

In particular, this has an effect of "erasing the difference" between the étale homotopy type and the hyper-étale homotopy type of a stack.

3. Comparison to Artin-Mazur and Friedlander's approach

In this section, we will explain in what sense our definition of étale homotopy type agrees with Friedlander's definition, when his makes sense, e.g. in the locally Noetherian setting, and in what sense our definition is a refinement of that of Artin and Mazur.

It should be noted however that one needs a locally connected hypothesis in order for either the Artin-Mazur or Friedlander étale homotopy type to be defined, whereas our definition does not need this assumption. The reason the approaches of Artin-Mazur and Friedlander need additional assumptions than ours is because both of these definitions ([2, Definition 9.6] and [17, Definition 4.4]) use Verdier's connected-component functor, which is only defined for locally connected sites. The small étale site of a locally Noetherian scheme is locally connected, but this is not true for arbitrary schemes. As we will shall see, Verdier's connected-component functor arises naturally when one tries to derive an explicit formula for the (hyper)étale homotopy type in the locally connected setting. Therefore, before explaining this comparison, we will make a short excursion into the theory of locally connected topoi. In the process, we will prove some basic properties that we will need later.

3.1 Locally connected ∞-topoi.

Definition 3.1.1. An object E in a topos $\mathcal{E}$ is **connected** if whenever there is an isomorphism $E \cong U \coprod V$ in $\mathcal{E}$, then exactly one of U and V is not an initial object.

Remark 3.1.2. An object E in a topos $\mathcal{E}$ is connected if and only if the functor

$$\mathrm{Hom}_{\mathcal{E}}(E, \cdot\,) : \mathcal{E} \to Set$$

preserves coproducts. (See Proposition 3.1.16.)

Definition 3.1.3. A topos $\mathcal{E}$ is **locally connected** if and only if every object E in $\mathcal{E}$ can be written as a coproduct of connected objects. (The initial object is an empty coproduct).

Lemma 3.1.4. *[24, Lemma C.3.3.6] A topos $\mathcal{E}$ is **locally connected** if and only if the inverse image functor*

$$\Delta : Set \to \mathcal{E}$$

has a left adjoint Π_0.

Remark 3.1.5. Let U be a connected object of a locally connected topos $\mathcal{E}$, and let S be a set. Then we have:

$$\begin{aligned}
\mathrm{Hom}\left(\Pi_0(U), S\right) &\cong \mathrm{Hom}\left(U, \Delta(S)\right)\\
&\cong \mathrm{Hom}\left(U, \coprod_{s\in S} 1\right)\\
&\cong \coprod_{s\in S} \mathrm{Hom}\left(U, 1\right)\\
&\cong \coprod_{s\in S} *\\
&\cong S,
\end{aligned}$$

where the second to last isomorphism follows from Remark 3.1.2, and thus $\Pi_0(U) \cong *$. It follows that if $E = \coprod_{i\in I} U_i$ is a decomposition of E into connected objects, then

$$\Pi_0(E) \cong I,$$

hence the "set of connected components of E" is well defined up to isomorphism, and isomorphic to $\Pi_0(E)$.

Definition 3.1.6. The functor Π_0 (for historical reasons) is called the **Verdier connected component** functor.

Definition 3.1.7. A locally connected topos $\mathcal{E}$ is **connected** if and only if the terminal object 1 is connected. Equivalently, if and only if

$$\Pi_0(\mathcal{E}) := \Pi_0(1) \cong *.$$

The following proposition is standard:

Proposition 3.1.8. *Let $\mathscr{C}$ be a locally connected Grothendieck site as in [2, Section 9], then the topos of sheaves of sets $Sh(\mathscr{C})$ is locally connected.*

Example 3.1.9. Let X be a locally connected topological space (in the strong sense that the each point x has a neighborhood basis of connected open subsets). Then the open cover Grothendieck topology on the poset of open subsets $Op(X)$ is a locally connected site, and hence $\mathrm{Sh}(X)$ is locally connected. Any sheaf of sets F on X is the sections of a local homeomorphism $L(F) \to X$, and such an F is connected if and only if the space $L(F)$ is.

Example 3.1.10. Let X be a locally Noetherian scheme. Then its small étale site $X_{\acute{e}t}$ is locally connected (see [18, I 6.1.9]). It follows that the small étale topos $\mathrm{Sh}(X_{\acute{e}t})$ is locally connected. Concretely, a representable sheaf $Y \to X$ in $\mathrm{Sh}(X_{\acute{e}t})$, i.e. an étale map from a scheme Y, is connected if and only Y is a connected scheme. More generally, as any étale sheaf over a scheme is representable by an étale map $P \to X$ from an algebraic space (with no separation conditions), a sheaf F in $\mathrm{Sh}(X_{\acute{e}t})$ corresponding to such a map is connected if and only if the algebraic space P is.

Definition 3.1.11. An ∞-topos $\mathcal{E}$ is **locally connected** if its underlying topos $\mathrm{Disc}(\mathcal{E})$ of discrete objects is a locally connected topos, where $\mathrm{Disc}(\mathcal{E})$ is the full subcategory of $\mathcal{E}$ spanned by the 0-truncated objects.

Remark 3.1.12. It might be tempting to think an ∞-topos is locally connected if and only if the inverse image functor

$$\Delta : \mathrm{Gpd}_\infty \to \mathcal{E}$$

has a left adjoint Π_∞. However, this is a strictly stronger condition; an ∞-topos satisfying this property is said to be **locally ∞-connected**. For example, a locally connected space X may not have $\mathbb{H}\mathrm{ypSh}_\infty(X)$ locally ∞-connected, but this will hold if X is locally contractible.

Definition 3.1.13. An object E in an ∞-topos $\mathcal{E}$ is **connected** if whenever there is an equivalence $E \simeq U \coprod V$ in $\mathcal{E}$, then exactly one of U and V is not an initial object.

Lemma 3.1.14. *An object E in an ∞-topos $\mathcal{E}$ is connected if and only if its 0-truncation $\pi_0(E)$ is connected in $Disc(\mathcal{E})$.*

Proof. Suppose that $\pi_0(E)$ is connected, and we have $E \simeq U \coprod V$. Then since π_0 is a left adjoint, we have

$$\pi_0(E) \cong \pi_0(U) \coprod \pi_0(V).$$

So, without loss of generality, $\pi_0(U)$ is an initial object, and hence so is U. Hence E is connected.

Conversely, suppose that E is connected and that

$$\pi_0(E) \cong U \coprod V.$$

Then since colimits are universal,

$$E \simeq E \times_{\pi_0(E)} U \coprod E \times_{\pi_0(E)} V,$$

and hence, without loss of generality, $E \times_{\pi_0(E)} U$ is an initial object. However, since

$$E \to \pi_0(E)$$

is an epimorphism, it follows that so is $\emptyset = E \times_{\pi_0(E)} U \to U$, therefore U is initial. □

Lemma 3.1.15. *Let $\mathcal{E}$ be a locally connected ∞-topos. Then any object E can be written as a coproduct of connected objects.*

Proof. Let E be an object of a locally connected ∞-topos. Then by definition, $\pi_0(E)$ is an object of a locally connected topos, hence we can write

$$\pi_0(E) = \coprod_{i \in I} U_i$$

where each U_i is connected. But then, since colimits are universal, it follows that

$$E \simeq \coprod_{i \in I} E \times_{\pi_0(E)} U_i.$$

Now, since,

$$\pi_0\left(E \times_{\pi_0(E)} U_i\right) \cong U_i,$$

each E $\times_{\pi_0(E)} U_i$ is connected by Lemma 3.1.14. □

Proposition 3.1.16. *Let E be an object of a locally connected ∞-topos $\mathcal{E}$. Then E is connected if and only if the functor*

$$\mathrm{Hom}_{\mathcal{E}}(E, \cdot) : \mathcal{E} \to \mathit{Gpd}_\infty$$

preserves coproducts.

Proof. Let E be connected and let $X = \coprod\limits_{i \in I} X_i$ be an object of $\mathcal{E}$. Let

$$f : E \to X$$

be a map in $\mathcal{E}$. Then since colimits are universal, we have

$$E \simeq \coprod_{i \in I} E \times_X X_i.$$

Fix $j \in I$ and write

$$E \simeq E \times_X X_j + \coprod_{i \neq j} E \times_X X_i.$$

Since E is connected, only one the above factors can be non-initial. Moreover, we cannot have that $E \simeq E \times_X X_j$ is initial for all $j \in I$, for this would imply that E was initial. Now suppose by way of contradiction that there is $j \neq k$ in I such that $E \simeq E \times_X X_j$ and $E \simeq E \times_X X_k$ are both non-initial. Then since E is connected,

$$\coprod_{i \neq j} E \times_X X_i$$

is initial, but

$$\coprod_{i \neq j} E \times_X X_i = E \times_X X_k + \coprod_{i \neq j, i \neq k} E \times_X X_i,$$

and $E \times_X X_k$ is non-initial, which then leads to a contradiction. So $E \times_X X_i$ is non-initial for exactly one i, and hence for this i,

$$E \times_X X_i \simeq E.$$

It follows that f factors through the inclusion $X_i \to X$. Hence $\mathrm{Hom}_{\mathcal{E}}(E, \cdot\,)$ preserves coproducts.

Conversely, suppose that $\mathrm{Hom}_{\mathcal{E}}(E, \cdot\,)$ preserves coproducts and that $E \simeq U \coprod V$. Then

$$\mathrm{Hom}_{\mathcal{E}}\left(E, U \coprod V\right) \simeq \mathrm{Hom}_{\mathcal{E}}(E, U) \coprod \mathrm{Hom}_{\mathcal{E}}(E, V),$$

so the equivalence

$$E \xrightarrow{\sim} U \coprod V,$$

must factor through one of the factors, and hence the other factor must be initial. □

3.2 Comparing definitions of étale homotopy type

Remark 3.2.1. Let us derive, from a modern perspective, the formula of Artin-Mazur for the étale homotopy type of a locally Noetherian scheme X [2, Definition 9.6 on p. 114], by unwinding the definitions to derive an explicit formula for the hyper-étale homotopy type as in Definition 2.4.2. Since X is locally Noetherian, its small étale site is locally connected by [18, I 6.1.9]. Let Z be a space in $\mathcal{S}$. Then, as a left exact functor

$$\Pi_\infty^{\mathbb{H}\text{-}\acute{e}t}(X) : \mathcal{S} \to \mathcal{S},$$

we have

$$\Pi_\infty^{\mathbb{H}\text{-}\acute{e}t}(X)(Z) = \Gamma_{\mathbb{H}\mathrm{ypSh}_\infty(X_{\acute{e}t})} \Delta_{\mathbb{H}\mathrm{ypSh}_\infty(X_{\acute{e}t})}(Z),$$

that is, it assigns Z the space of sections of the hypersheafification on the constant presheaf with value Z.

Hypersheafification of a presheaf F can be constructed in one-step ([31, p. 672]) as follows:

$$F^\dagger(X) = \varinjlim_{U^\bullet \to X} \left[\varprojlim F(U^\bullet)\right],$$

with the colimit ranging over a suitable filtered category of split hypercovers by connected objects in the small étale site for X. (See [2, Lemma 8.8] for a justification as to why we can restrict to *split* hypercovers, and see [20, Section 5] for a more precise meaning of "suitable filtered category.") Such a hypercover is a simplicial object in presheaves which is degree-wise the coproduct of representables. Explicitly, one has for all n

$$U^n = \coprod_{i \in I_n} y(C_i^n), \tag{2}$$

with each C_i^n a connected object in the small étale site of X, and the notation

$$\varprojlim F\left(U^\bullet\right)$$

is shorthand for

$$\varprojlim_{n\in\Delta}\left[\prod_{i\in I_0} F\left(C_i^0\right) \rightrightarrows \prod_{j\in I_1} F\left(C_j^1\right) \substack{\rightarrow\\ \rightarrow\\ \rightarrow} \prod_{k\in I_2} F\left(C_k^2\right)\ldots\right].$$

For such a hypercover

$$U^\bullet \to X$$

let $\Pi_0\left(U^n\right)$ be the set of connected components of U^n in the sense of Definition 3.1.6 (also [2, p. 111]), and denote the corresponding simplicial set by $\pi\left(U^\bullet\right)$. Explicitly, in the notation (2), $\Pi_0\left(U^n\right) \cong I_n$, since each representable in (2) is connected. By abuse of notation, we will denote the associated object in $\mathcal{S}$ by the same symbol $\pi\left(U^\bullet\right)$. Note that we have

$$\pi\left(U^\bullet\right) = \varinjlim_{n\in\Delta^{op}} \Pi_0\left(U^n\right).$$

We thus have that

$$\Pi_\infty^{\mathbb{H}\text{-}\acute{e}t}\left(X\right)\left(Z\right) = \varinjlim_{U^\bullet\to X}\left[\varprojlim_{n\in\Delta}\prod_{a\in\Pi_0(U^n)} Z\right],$$

i.e.

$$\begin{aligned}
\operatorname{Hom}_{\operatorname{Pro}(\mathcal{S})}\left(\Pi_\infty^{\mathbb{H}\text{-}\acute{e}t}\left(X\right), j\left(Z\right)\right) &= \varinjlim_{U^\bullet\to X}\left[\varprojlim_{n\in\Delta}\prod_{a\in\Pi_0(U^n)} Z\right]\\
&\simeq \varinjlim_{U^\bullet\to X}\operatorname{Hom}_{\mathcal{S}}\left(\varinjlim_{n\in\Delta^{op}}\Pi_0\left(U^n\right), Z\right)\\
&\simeq \varinjlim_{U^\bullet\to X}\operatorname{Hom}_{\mathcal{S}}\left(\pi\left(U^\bullet\right), Z\right)\\
&\simeq \operatorname{Hom}_{\operatorname{Pro}(\mathcal{S})}\left(\varprojlim_{U^\bullet\to X}\pi\left(U^\bullet\right), j\left(Z\right)\right),
\end{aligned}$$

so we conclude that $\Pi_\infty^{\mathbb{H}\text{-}\acute{e}t}\left(X\right)$ can be identified with the pro-space $\varprojlim_{U^\bullet\to X}\pi\left(U^\bullet\right)$. Comparing this to the Verdier functor [2, p. 112], we see the only difference between $\Pi_\infty^{\mathbb{H}\text{-}\acute{e}t}\left(X\right)$ and the Artin-Mazur étale homotopy type of X is that $\Pi_\infty^{\mathbb{H}\text{-}\acute{e}t}\left(X\right)$ is a pro-object in the ∞-category of spaces, whereas the Artin-Mazur étale homotopy type is a pro-object in the *homotopy* category of spaces.

We will make the above argument more precise in what follows, by instead, directly comparing with Friedlander's approach. Let us introduce some notation. Let

$$q : Set^{\Delta^{op}} \to \mathcal{S}$$

be the functor sending a simplicial set to its associated ∞-groupoid. We can realize this concretely e.g. as

$$Set^{\Delta^{op}} = \operatorname{Psh}\left(\Delta\right) \hookrightarrow \operatorname{Psh}_\infty\left(\Delta\right) \xrightarrow{\varinjlim(\,\cdot\,)} \mathcal{S}.$$

Notice that q induces a well-defined functor

$$\operatorname{Pro}(q) : \operatorname{Pro}\left(Set^{\Delta^{op}}\right) \to \operatorname{Pro}(\mathcal{S}).$$

The following is a result of Hoyois:

Proposition 3.2.2. *([20, cor 3.4])*
Let X be a pointed connected locally Noetherian scheme. Denote by $\mathfrak{Fr}^{ét}(X)$ the étale homotopy type of X, as defined by Friedlander ([17, Definition 4.4]). Then

$$Pro(q)\left(\mathfrak{Fr}^{ét}(X)\right) \simeq Shape\left(\mathbb{H}\mathrm{ypSh}_{\infty}(X_{ét})\right),$$

i.e. the pro-space associated to $\mathfrak{Fr}^{ét}(X)$ agrees with $\Pi_{\infty}^{\mathbb{H}\text{-}ét}(X)$.

Corollary 3.2.3. *Let X be a pointed connected locally Noetherian scheme. Then the image of $\Pi_{\infty}^{\mathbb{H}\text{-}ét}(X)$ in the category $Pro(\mathcal{H}_*)$ of pro-objects in the pointed homotopy category, agrees with the homotopy type of X as defined by Artin and Mazur.*

Proof. This follows from Proposition 3.2.2 and [17, Proposition 4.5]. □

We finally show that Friedlander's definition for the étale homotopy type of a simplicial scheme, also agrees with our definition, up to profinite completion:

Theorem 3.2.4. *Let $X_\bullet$ be a pointed connected locally Noetherian simplicial scheme. Let*

$$[X_\bullet] = \varinjlim X_n$$

denote its associated stack in $Sh_{\infty}\left(\mathbf{Aff}_k^{\mathfrak{U}}, ét\right)$ (e.g. if $X_\bullet$ is the nerve of a groupoid object in schemes, $[X_\bullet]$ is the associated algebraic stack). Then

$$\widehat{\Pi_{\infty}^{ét}([X_\bullet])} \simeq Pro(q)\,\widehat{\left(\mathfrak{Fr}^{ét}(X_\bullet)\right)}.$$

Proof. Note that since $\Pi_{\infty}^{ét}$ preserves colimits,

$$\Pi_{\infty}^{ét}([X_\bullet]) \simeq \varinjlim_{n\in\Delta^{op}} \Pi_{\infty}^{ét}(X_n).$$

Now, from Proposition 2.2.12 and Proposition 3.2.2, it follows that this in turn is equivalent to

$$\varinjlim_{n\in\Delta^{op}} \operatorname{Pro}(q)\left(\mathfrak{Fr}^{ét}(X_n)\right).$$

For a simplicial scheme $Y_\bullet$, denote by $HR(Y)$ and $HRR(Y_\bullet)$ the homotopy category of hypercovers and rigid hypercovers of $Y_\bullet$ respectively, in the sense of [17, Definitions 3.3 and Proposition 4.3]. Notice that

$$\operatorname{Pro}(q)\left(\mathfrak{Fr}^{ét}(X_\bullet)\right) \simeq \varprojlim_{U\in HRR(X_\bullet)} j\left(\varinjlim_{n\in\Delta^{op}} \pi(U_{n,n})\right).$$

The proof of [17, Proposition 4.5] implies that the canonical map $HRR(Y_\bullet) \to HR(Y_\bullet)$ is left final. Also, for any n, the canonical restriction map

$$HR(Y_\bullet) \to HR(Y_n)$$

is left final for all n. Using these facts, combined with [20, cor 5.5], we deduce that for each n,

$$\operatorname{Pro}(q)\left(\mathfrak{Fr}^{\acute{e}t}(X_n)\right) \simeq \varprojlim_{U\in HRR(X_\bullet)} j\left(\varinjlim_{k\in\Delta^{op}} \pi(U_{k,n})\right).$$

Let V be a π-finite space. Then:

$$\begin{aligned}
\widehat{\Pi_\infty^{\acute{e}t}([X_\bullet])}(V) &\simeq \left(\varinjlim_{n\in\Delta^{op}} \operatorname{Pro}(q)\left(\mathfrak{Fr}^{\acute{e}t}(X_n)\right)\right)(V)\\
&\simeq \operatorname{Hom}\left(\varinjlim_{n\in\Delta^{op}} \varprojlim_{U\in HRR(X_\bullet)} j\left(\varinjlim_{k\in\Delta^{op}} \pi(U_{k,n})\right), j(V)\right)\\
&\simeq \varprojlim_{n\in\Delta} \operatorname{Hom}\left(\varprojlim_{U\in HRR(X_\bullet)} j\left(\varinjlim_{k\in\Delta^{op}} \pi(U_{k,n})\right), j(V)\right)\\
&\simeq \varprojlim_{n\in\Delta} \varinjlim_{U\in HRR(X_\bullet)^{op}} \operatorname{Hom}\left(j\left(\varinjlim_{k\in\Delta^{op}} \pi(U_{k,n})\right), j(V)\right).
\end{aligned}$$

But, since V is π-finite, it is m-truncated for some m, and so by [8, Lemma 2.21], this is furthermore equivalent to

$$\varprojlim_{n\in\Delta_{\leq m}} \varinjlim_{U\in HRR(X_\bullet)^{op}} \operatorname{Hom}\left(j\left(\varinjlim_{k\in\Delta^{op}} \pi(U_{k,n})\right), j(V)\right)$$

and since filtered colimits commute with finite limits we have this is in turn equivalent to

$$\begin{aligned}
&\varinjlim_{U\in HRR(X_\bullet)^{op}} \varprojlim_{n\in\Delta_{\leq m}} \operatorname{Hom}\left(j\left(\varinjlim_{k\in\Delta^{op}} \pi(U_{k,n})\right), j(V)\right)\\
\simeq\ &\varinjlim_{U\in HRR(X_\bullet)^{op}} \varprojlim_{n\in\Delta} \operatorname{Hom}\left(j\left(\varinjlim_{k\in\Delta^{op}} \pi(U_{k,n})\right), j(V)\right)\\
\simeq\ &\varinjlim_{U\in HRR(X_\bullet)^{op}} \operatorname{Hom}\left(\varinjlim_{n\in\Delta^{op}} j\left(\varinjlim_{k\in\Delta^{op}} \pi(U_{k,n})\right), j(V)\right)\\
\simeq\ &\varinjlim_{U\in HRR(X_\bullet)^{op}} \operatorname{Hom}\left(j\left(\varinjlim_{n\in\Delta^{op}} \pi(U_{n,n})\right), j(V)\right)\\
\simeq\ &\operatorname{Hom}\left(\varprojlim_{U\in HRR(X_\bullet)} j\left(\varinjlim_{n\in\Delta^{op}} \pi(U_{n,n})\right), j(V)\right)\\
\simeq\ &\operatorname{Hom}\left(\varprojlim_{U\in HRR(X_\bullet)} j\left(\varinjlim_{n\in\Delta^{op}} \pi(U_{n,n})\right), j(V)\right)\\
\simeq\ &\operatorname{Hom}\left(\operatorname{Pro}(q)\left(\mathfrak{Fr}^{\acute{e}t}(X_\bullet)\right), j(V)\right)\\
\simeq\ &\operatorname{Pro}(q)\left(\mathfrak{Fr}^{\acute{e}t}(X_\bullet)\right)(V).
\end{aligned}$$

□

Remark 3.2.5. All that was needed was that each π-finite space is truncated, so we actually get an equivalence between their pro-truncated spaces.

4. A concrete description of the étale homotopy type

Thus far, we have succeeded in generalizing the previously existing definitions of étale homotopy type to a definition that makes sense for arbitrary higher stacks on the big étale site. However, when the stack in question is not Deligne-Mumford, this construction is a bit opaque, since it involves taking the shape of a colimit of ∞-topoi indexed by the right fibration associated to the stack in question. This may seem unsatisfying since the definition of Friedlander, which agrees with ours in the locally Noetherian setting, has an explicit (albeit complex) formula. In this subsection, we will rectify this by showing the pro-space associated to our construction of the étale homotopy type, even when applied to an arbitrary higher stack, has a natural concrete description given by a very simple formula.

Consider the essentially unique geometric morphism

$$\mathrm{Sh}_\infty\left(\mathbf{Aff}_k^{\mathfrak{U}}, \acute{e}t\right) \to \mathcal{S}$$

to the terminal ∞-topos. This is represented by an adjoint pair $\Delta^{\acute{e}t} \dashv \Gamma_{\acute{e}t}$, and $\Delta^{\acute{e}t}$ is left exact. Moreover, $\Delta^{\acute{e}t}$ assigns a space Z the sheafification of the constant presheaf with value Z, and $\Gamma_{\acute{e}t}$ assigns an ∞-sheaf F the value $F\left(\mathrm{Spec}\ (k)\right).$

Let F be an object of $\mathrm{Sh}_\infty\left(\mathbf{Aff}_k^{\mathfrak{U}}, \acute{e}t\right)$. Consider the composition of functors

$$\mathcal{S} \xrightarrow{\Delta^{\acute{e}t}} \mathrm{Sh}_\infty\left(\mathbf{Aff}_k^{\mathfrak{U}}, \acute{e}t\right) \xhookrightarrow{y} \mathrm{Psh}_\infty\left(\mathrm{Sh}_\infty\left(\mathbf{Aff}_k^{\mathfrak{U}}, \acute{e}t\right)\right) \xrightarrow{ev_F} \mathcal{S},$$

where

$$ev_F : \mathrm{Psh}_\infty\left(\mathrm{Sh}_\infty\left(\mathbf{Aff}_k^{\mathfrak{U}}, \acute{e}t\right)\right) \to \mathcal{S}$$

is the functor evaluating a presheaf G on $\mathrm{Sh}_\infty\left(\mathbf{Aff}_k^{\mathfrak{U}}, \acute{e}t\right)$ at the object F. Since limits in a functor ∞-category are computed object-wise, and since $\Delta^{\acute{e}t}$ is left exact, the above composition is also left-exact, hence a pro-space. Let us denote this pro-space by $l\left(F\right).$ The pro-space $l\left(F\right)$ is given by the simple formula

$$l\left(F\right)\left(Z\right) = \mathrm{Hom}_{\mathrm{Sh}_\infty\left(\mathbf{Aff}_k^{\mathfrak{U}}, \acute{e}t\right)}\left(F, \Delta^{\acute{e}t}\left(Z\right)\right). \tag{3}$$

This can be formulated more abstractly as follows. Let $\mathcal{E}$ be an arbitrary ∞-topos and let

$$\Delta^{\mathcal{E}} \dashv \Gamma_{\mathcal{E}}$$

denote the essentially unique geometric morphism $e : \mathcal{E} \to \mathcal{S}$. Let E be an object in $\mathcal{E}$ and let

$$\pi_E : \mathcal{E}/E \to \mathcal{E}$$

denote the associated étale geometric morphism. Then the composite

$$\mathcal{E}/E \xrightarrow{\pi_E} \mathcal{E} \xrightarrow{e} \mathcal{S}$$

is the essentially unique geometric morphism from $\mathcal{E}/E$ to $\mathcal{S}$, so

$$\Delta^{\mathcal{E}/E} \simeq \pi_E^* \circ \Delta^{\mathcal{E}}.$$

It follows that for Z in $\mathcal{S}$ we have:

$$\begin{aligned}
\mathit{Shape}\left(\mathcal{E}/E\right)\left(Z\right) &\simeq \Gamma_{\mathcal{E}/E}\left(\Delta^{\mathcal{E}/E}\left(Z\right)\right)\\
&\simeq \Gamma\left(E \times \Delta^{\mathcal{E}}\left(Z\right) \to E\right)\\
&\simeq \mathrm{Hom}_{\mathcal{E}}\left(E, \Delta^{\mathcal{E}}\left(Z\right)\right).
\end{aligned}$$

By [31, Proposition 6.3.5.14], the assignment $E \mapsto \mathcal{E}/E$ assembles into a colimit preserving functor

$$\chi : \mathcal{E} \to \mathfrak{Top}_\infty.$$

By composition, we get a colimit preserving functor

$$\mathcal{E} \xrightarrow{\chi} \mathfrak{Top}_\infty \xrightarrow{Shape} \operatorname{Pro}(\mathcal{S})$$

sending an object E of $\mathcal{E}$ to $Shape\,(\mathcal{E}/E)$. By the above discussion, we see that for F an object of the ∞-topos $\operatorname{Sh}_\infty\left(\mathbf{Aff}^{\mathfrak{U}}_k, ét\right)$, the pro-space $l\,(F)$ is nothing but $Shape\left(\operatorname{Sh}_\infty\left(\mathbf{Aff}^{\mathfrak{U}}_k, ét\right)/F\right)$, and hence the assignment $F \mapsto l\,(F)$ assembles into a colimit preserving functor

$$\operatorname{Sh}_\infty\left(\mathbf{Aff}^{\mathfrak{U}}_k, ét\right) \xrightarrow{\chi} \mathfrak{Top}_\infty \xrightarrow{Shape} \operatorname{Pro}(\mathcal{S}).$$

Lemma 4.1. *Consider the ringed ∞-topos* $\operatorname{Spec}_{ét}(k)$ *and denote by*

$$\delta : \mathcal{S} \to Sh_\infty(k_{ét})$$

the inverse image functor of the unique geometric morphism

$$Sh_\infty(k_{ét}) \to \mathcal{S}$$

from the underlying ∞-topos of $\operatorname{Spec}_{ét}(k)$ *to the terminal ∞-topos of spaces. Let Z be an arbitrary space. Then* $\operatorname{Spec}_{ét}(k)/\delta(Z)$ *is a Deligne-Mumford scheme, whose functor of points is the stack $\Delta^{ét}(Z)$.*

Proof. It follows from [28, Proposition 2.3.10] that $\operatorname{Spec}_{ét}(k)/\delta(Z)$ is a Deligne-Mumford scheme. It therefore suffices to show that its functor of points is $\Delta^{ét}(Z)$. Consider the composition of functors

$$\mathcal{S} \xrightarrow{\delta} \operatorname{Sh}_\infty(k_{ét}) \xrightarrow{\sim} \mathfrak{DM}(k)^{\mathfrak{U},ét}_\infty/\operatorname{Spec}_{ét}(k) \to \mathfrak{DM}(k)^{\mathfrak{U},ét}_\infty \to \mathfrak{DM}(k)^{\mathfrak{U}}_\infty \hookrightarrow \operatorname{Sh}_\infty\left(\mathbf{Aff}^{\mathfrak{U}}_k, ét\right).$$

By [28, Proposition 2.3.5] and [6, Proposition 5.2.11], the composition preserves small colimits. Moreover, it sends the one point space $*$ to $\operatorname{Spec}(k)$, which is terminal. The functor $\Delta^{ét}$ also has this property, and since the above composite and $\Delta^{ét}$ are both colimit preserving functors out of $\mathcal{S} = \operatorname{Psh}_\infty(*)$, they must agree, but this is exactly what we wanted to show, since the composite sends a space Z to the functor of points of $\operatorname{Spec}_{ét}(k)/\delta(Z)$. □

Theorem 4.2. *There is a canonical equivalence of functors*

$$\Pi^{ét}_\infty \xrightarrow{\sim} Shape \circ \chi.$$

In particular, for any ∞-sheaf F on $\left(\mathbf{Aff}^{\mathfrak{U}}_k, ét\right)$, there is a canonical equivalence of pro-spaces

$$\Pi^{ét}_\infty(F) \xrightarrow{\sim} l\,(F),$$

where $l\,(F)$ is defined as in equation (3).

Proof. Since both functors $Shape \circ \chi$ and $\Pi_\infty^{\acute{e}t}$ are colimit preserving functors

$$\mathrm{Sh}_\infty\left(\mathbf{Aff}_k^{\mathfrak{U}}, \acute{e}t\right) \to \mathrm{Pro}(\mathcal{S}),$$

by [31, Proposition 5.5.4.20 and Theorem 5.1.5.6], it suffices to show that both functors agree up to equivalence when restricted to affine schemes. Note that both $\mathrm{Spec}\,(A)$ and $\Delta^{\acute{e}t}(Z)$ are functors of points of Deligne-Mumford schemes, and the functor of points construction is a fully faithful embedding of Deligne-Mumford schemes into $\mathrm{Sh}_\infty\left(\mathbf{Aff}_k^{\mathfrak{U}}, \acute{e}t\right)$. It follows that the canonical map

$$\begin{tikzcd} \mathrm{Hom}_{\mathfrak{DMSch}_k^{\mathfrak{U}}}\left(\mathrm{Spec}_{\acute{e}t}(A), \mathrm{Spec}_{\acute{e}t}(k)/\delta(Z)\right) \arrow[r] & \mathrm{Hom}_{\mathrm{Sh}_\infty\left(\mathbf{Aff}_k^{\mathfrak{U}},\acute{e}t\right)}\left(\mathrm{Spec}\,(A), \Delta^{\acute{e}t}(Z)\right) \arrow[d, phantom, sloped, "\simeq"] \\ & l\left(\mathrm{Spec}\,(A)\right)(Z) \end{tikzcd}$$

is an equivalence of ∞-groupoids.

By [28, Remark 2.3.20], the following is a pullback diagram in $\mathfrak{DM}(k)_\infty^{\mathfrak{U}}$:

$$\begin{tikzcd} \mathrm{Spec}_{\acute{e}t}(A)/f^*\delta(Z) \arrow[r] \arrow[d] & \mathrm{Spec}_{\acute{e}t}(k)/\delta(Z) \arrow[d] \\ \mathrm{Spec}_{\acute{e}t}(A) \arrow[r, "f"] & \mathrm{Spec}_{\acute{e}t}(k), \end{tikzcd}$$

where f is the map whose functor of points is the unique map to $\mathrm{Spec}\,(k)$. Since $\mathrm{Spec}_{\acute{e}t}(k)$ is the terminal Deligne-Mumford scheme, it follows that

$$\mathrm{Hom}_{\mathfrak{DMSch}_k^{\mathfrak{U}}}\left(\mathrm{Spec}_{\acute{e}t}(A), \mathrm{Spec}_{\acute{e}t}(k)/\delta(Z)\right)$$

is equivalent to the space of sections of the étale map

$$\mathrm{Spec}_{\acute{e}t}(A)/f^*\delta(Z) \to \mathrm{Spec}_{\acute{e}t}(A),$$

and since any section of an étale map is étale, this is in turn the space of maps in the slice category

$$\mathrm{Hom}_{\mathfrak{DM}(k)_\infty^{\mathfrak{U},\acute{e}t}/\mathrm{Spec}_{\acute{e}t}(A)}\left(id_{\mathrm{Spec}_{\acute{e}t}(A)}, \mathrm{Spec}_{\acute{e}t}(A)/f^*\delta(Z) \to \mathrm{Spec}_{\acute{e}t}(A)\right).$$

Now, by [28, Proposition 2.3.5], this is equivalent to the space of maps

$$\mathrm{Hom}_{\mathrm{Sh}_\infty(A_{\acute{e}t})}\left(1, f^*\delta(Z)\right) = \Gamma^A\left(f^*\delta(Z)\right).$$

Since $\mathcal{S}$ is the terminal ∞-topos, it follows that $f^*\delta \simeq \Delta_A^{\acute{e}t}$, (where $\Delta_A^{\acute{e}t}$ is the inverse image of the unique geometric morphism to $\mathcal{S}$) and hence

$$\begin{aligned} \Gamma^A\left(f^*\delta(Z)\right) &\simeq \Gamma^A\left(\Delta_A^{\acute{e}t}(Z)\right) \\ &= Shape\left(\mathrm{Spec}\,(A)\right)(Z) \\ &= \Pi_\infty^{\acute{e}t}\left(\mathrm{Spec}\,(A)\right)(Z). \end{aligned}$$

□

For $X_\bullet$ a simplicial object in the 1-topos $\mathrm{Sh}\left(\mathbf{Aff}_k^{\mathfrak{U}}, \acute{e}t\right)$. We can associate to it, its colimit $[X_\bullet] := \varinjlim_{n \in \Delta^{op}} X_\bullet$ in the ∞-topos $\mathrm{Sh}\left(\mathbf{Aff}_k^{\mathfrak{U}}, \acute{e}t\right)$.

Corollary 4.3. *In the situation above, the étale homotopy type $\Pi^{\acute{e}t}([X_\bullet])$ agrees up to profinite completion with the topological type of $X_\bullet$ in the sense of Chough [10, Definition 3.2.6] (and agrees without the completion when $X_\bullet$ is n-truncated for some n).*

Proof. Translating into ∞-categorical language, Chough's topological type of $X_\bullet$ is the shape of the ∞-topos $\mathbb{H}\mathrm{ypSh}_\infty\left(\mathbf{Aff}_k^{\mathfrak{U}}, \acute{e}t\right)/[X_\bullet]^\wedge$, where $[X_\bullet]^\wedge$ is the hypersheafification of $[X_\bullet]$. The result now follows from Proposition 2.2.12 and Theorem 4.2. □

5. A profinite comparison theorem

In this section, we extend the results of [2] to show that the profinite étale homotopy type of any higher stack on the site of affine schemes of finite type over $\mathbb{C}$ agrees with the profinite homotopy type of its underlying topological stack.

We start by recalling some notions and results from Section 3 of [8].

Let **Top** be the category of topological spaces and let $\mathbf{Top}^s_{\mathbb{C}}$ denote the full subcategory on all those spaces which are contractible and locally contractible spaces which are homeomorphic to a subspace of $\mathbb{R}^n$ for some n. Denote by $\mathbf{Top}_{\mathbb{C}}$ the following subcategory of topological spaces:

Definition 5.1. A topological space T is in $\mathbf{Top}_{\mathbb{C}}$ if T has an open cover $(U_\alpha \hookrightarrow T)_\alpha$ such that each U_α is an object of $\mathbf{Top}^s_{\mathbb{C}}$.

The reason for decorating the notation with "$\mathbb{C}$" is that $\mathbf{Top}_{\mathbb{C}}$ is a good recipient for the analytification functor from complex schemes. Recall from [49] that there is an analytification functor

$$(\,\cdot\,)_{an} : \mathbf{Sch}^{LFT}_{\mathbb{C}} \to \mathbf{Top},$$

from schemes locally of finite type over $\mathbb{C}$ to topological spaces, and this functor preserves finite limits. When X is a scheme, $X_{an} = X(\mathbb{C})$ is its space of $\mathbb{C}$-points equipped with the complex analytic topology. X_{an} is locally (over any affine) a triangulated space by [26], so in particular X_{an} is locally contractible, and since X_{an} is locally cut-out of $\mathbb{C}^n$ by polynomials, so it follows that X_{an} is in $\mathbf{Top}_{\mathbb{C}}$.

In [35, Section 20], Noohi extends the analytification functor to a left exact functor

$$(\,\cdot\,)_{top} : \mathfrak{AlgSt}^{LFT}_{\mathbb{C}} \to \mathfrak{TopSt}$$

from Artin stacks locally of finite type over $\mathbb{C}$ to topological stacks. For $\mathcal{X}$ an Artin stack, $\mathcal{X}_{top}$ is called its *underlying topological stack.* In [8, Theorem 3.1 and cor 3.11], we extend this further to a left exact colimit preserving functor

$$(\,\cdot\,)_{top} : \mathrm{Sh}_\infty\left(\mathbf{Aff}^{LFT}_{\mathbb{C}}, \text{ét}\right) \to \mathbb{H}\mathrm{ypSh}_\infty\left(\mathbf{Top}_{\mathbb{C}}\right)$$

from ∞-sheaves on the étale site of affine schemes of finite type over $\mathbb{C}$, to hypersheaves on $\mathbf{Top}_{\mathbb{C}}$ (with respect to the open cover topology). For $\mathcal{X}$ any ∞-stack on $\left(\mathbf{Aff}^{LFT}_{\mathbb{C}}, \text{ét}\right)$, we refer to $\mathcal{X}_{top}$ as its *underlying stack on* $\mathbf{Top}_{\mathbb{C}}$.

Remark 5.2. Even though $\mathbf{Top}_{\mathbb{C}}$ is not a small category, $\mathbb{H}\mathrm{ypSh}_\infty\left(\mathbf{Top}_{\mathbb{C}}\right)$ is an ∞-topos, since there is a canonical equivalence of ∞-categories

$$\mathbb{H}\mathrm{ypSh}_\infty\left(\mathbf{Top}_{\mathbb{C}}\right) \simeq \mathbb{H}\mathrm{ypSh}_\infty\left(\mathbf{Top}^s_{\mathbb{C}}\right).$$

See [8, Section 3.1].

In Section 3.2 of [8], we also prove the following theorem:

Theorem 5.3. *[8, Proposition 3.12,cor 3.13, and Remark 3.14] The ∞-topos $\mathbb{H}\mathrm{ypSh}_{\infty}(\mathbf{Top}_{\mathbb{C}})$ is locally ∞-connected, and the left adjoint to the constant functor Δ is a colimit preserving functor*

$$\Pi_{\infty} : \mathbb{H}\mathrm{ypSh}_{\infty}(\mathbf{Top}_{\mathbb{C}}) \to \mathcal{S}$$

which sends a space X in $\mathbf{Top}_{\mathbb{C}}$, to its underlying weak homotopy type.

The functor Π_{∞} is called the **fundamental ∞-groupoid** functor. (This is an extension of the results of [36].)

We now state our main result:

For any ∞-sheaf F on $\left(\mathbf{Aff}_{\mathbb{C}}^{LFT}, \text{ét}\right)$, there is an equivalence of profinite spaces

$$\widehat{\Pi}_{\infty}^{\acute{e}t}(F) \simeq \widehat{\Pi}_{\infty}(F_{top}),$$

between the profinite étale homotopy type of F and the profinite completion of the homotopy type of the underlying stack F_{top} on $\mathbf{Top}_{\mathbb{C}}$ (see Theorem 5.13).

We will need a few preliminaries. Notice that there is a canonical functor $\mathbf{Top}_{\mathbb{C}} \to \mathfrak{Top}_{\infty}$ which factors as

$$\mathbf{Top}_{\mathbb{C}} \hookrightarrow \mathbf{Top} \xrightarrow{\mathrm{Sh}} \mathfrak{Top} \hookrightarrow \mathfrak{Top}_{\infty}$$

the canonical inclusion, followed by the canonical functor $T \mapsto \mathrm{Sh}(T)$ from topological spaces to topoi (which is fully faithful when restricted to sober spaces), followed by the canonical inclusion identifying topoi with 1-localic ∞-topoi.

Since the poset of open subsets of a topological space has finite limits, it follows from [31, Proposition 6.4.5.4] that the total composite sends a topological space T to the ∞-topos $\mathrm{Sh}_{\infty}(T)$ of ∞-sheaves over T. Denote by

$$\mathbb{H}\mathrm{yp} : \mathfrak{Top}_{\infty} \to \mathfrak{Top}_{\infty}$$

the hypercompletion functor (see Remark 2.4.3).

Recall that by Remark 5.2, there is a canonical equivalence of ∞-categories

$$\mathbb{H}\mathrm{ypSh}_{\infty}(\mathbf{Top}_{\mathbb{C}}) \simeq \mathbb{H}\mathrm{ypSh}_{\infty}(\mathbf{Top}_{\mathbb{C}}^{s}).$$

With this in mind, the following lemma's proof is completely analogous to that of Lemma 2.3.10. We leave the details to the reader:

Lemma 5.4. *There exists a colimit preserving functor*

$$\mathbb{H}\mathrm{ypSh}_{\infty}(\,\cdot\,) : \mathbb{H}\mathrm{ypSh}_{\infty}(\mathbf{Top}_{\mathbb{C}}) \to \mathfrak{Top}_{\infty}$$

which sends a representable sheaf $y(T)$, where T is a topological space, to the ∞-topos of hypersheaves on T.

Lemma 5.5. *The following diagram commutes up to equivalence:*

$$\begin{tikzcd}
\mathbb{H}\mathrm{ypSh}_{\infty}(\mathbf{Top}_{\mathbb{C}}) \arrow[r, "\mathbb{H}\mathrm{ypSh}_{\infty}(\,\cdot\,)"] \arrow[d, "\Pi_{\infty}"'] & \mathfrak{Top}_{\infty} \arrow[d, "Shape"] \\
\mathcal{S} \arrow[r, hook, "j"] & Pro(\mathcal{S}).
\end{tikzcd}$$

Proof. Recall that the there is a canonical equivalence

$$\mathbb{H}\mathrm{ypSh}_\infty\left(\mathbf{Top}_\mathbb{C}\right) \simeq \mathbb{H}\mathrm{ypSh}_\infty\left(\mathbf{Top}^s_\mathbb{C}\right).$$

Since all the functors in the above diagram preserve colimits, it suffices by [31, Proposition 5.5.4.20, Theorem 5.1.5.6] to prove that there is a natural equivalence of functors

$$\mathit{Shape} \circ \mathbb{H}\mathrm{ypSh}_\infty\left(\,\cdot\,\right) \circ y \simeq j \circ \Pi_\infty \circ y$$

where

$$y : \mathbf{Top}_\mathbb{C} \hookrightarrow \mathbb{H}\mathrm{ypSh}_\infty\left(\mathbf{Top}_\mathbb{C}\right)$$

is the Yoneda embedding. Let T be an object of $\mathbf{Top}_\mathbb{C}$. In particular, T is locally contractible. By the proof of Proposition 2.2.7, we have a canonical identification

$$\mathit{Shape}\left(\mathbb{H}\mathrm{ypSh}_\infty\left(T\right)\right) \simeq j\left(\Pi_\infty\left(T\right)\right),$$

and by construction, there is a canonical equivalence

$$\mathit{Shape}\left(\mathbb{H}\mathrm{ypSh}_\infty\left(T\right)\right) \simeq \mathit{Shape} \circ \mathbb{H}\mathrm{ypSh}_\infty\left(\,\cdot\,\right)\left(y\left(T\right)\right). \qquad \square$$

Proposition 5.6. *There is a canonical natural transformation*

$$\begin{tikzcd} \mathit{Sh}_\infty\left(\mathbf{Aff}^{LFT}_\mathbb{C}, \acute{e}t\right) \arrow[rr, bend left, "\mathbb{H}\mathrm{ypSh}_\infty(\,\cdot\,)\circ(\,\cdot\,)_{top}"] \arrow[rr, bend right, "\mathit{Sh}_\infty((\,\cdot\,)_{\acute{e}t})"'] \arrow[rr, phantom, "\Downarrow\xi"] & & \mathfrak{Top}_\infty \end{tikzcd}$$

Such that the induced natural transformation

$$\mathit{Shape}^{\mathit{Prof}} \circ \mathbb{H}\mathrm{ypSh}_\infty\left(\,\cdot\,\right) \circ \left(\,\cdot\,\right)_{top} \xRightarrow{\mathit{Shape}^{\mathit{Prof}}(\xi)} \mathit{Shape}^{\mathit{Prof}} \circ \mathit{Sh}_\infty\left(\left(\,\cdot\,\right)_{\acute{e}t}\right) = \widehat{\Pi}^{\acute{e}t}_\infty$$

is an equivalence.

To prove the above proposition, since all the functors involved are colimit preserving, by [31, Proposition 5.5.4.20, Theorem 5.1.5.6] it suffices to prove the result after restricting the functors to affine schemes of finite type over $\mathbb{C}$. The affine assumption will not play a role, so we will establish the result for any scheme X of finite type over $\mathbb{C}$.

Following [1, exposé XI.4]:

Denote by $\mathbf{Top}^{\acute{e}t}/X_{an}$ the category of local homeomorphisms over X_{an}. Let

$$\alpha : X_{\acute{e}t} \to \mathbf{Top}^{\acute{e}t}/X_{an}$$

be the restriction of the analytification functor; it sends an étale map of schemes $f : Y \to X$ to the local homeomorphism $f_{an} : Y_{an} \to X_{an}$. Note also that via the étalé space construction, there is a canonical equivalence of categories $\mathbf{Top}^{\acute{e}t}/X_{an} \simeq \mathrm{Sh}\left(X_{an}\right)$. Since $\mathrm{Sh}\left(X_{an}\right)$ has enough points, and since α is left exact, it follows that α is flat, and hence by [24, B3.2.7], α induces a geometric morphism

$$\varphi : \mathrm{Sh}\left(X_{an}\right) \to \mathrm{Psh}\left(X_{\acute{e}t}\right).$$

Explicitly, φ^* is the left Kan extension $\mathrm{Lan}_y(\alpha)$ of α along the Yoneda embedding, and for F a sheaf on X_{an} corresponding to a local homeomorphism $LF \to X_{an}$, $\varphi_*(F)$ evaluated on an étale morphism $Y \to X$ is $\mathrm{Hom}_{X_{an}}(Y_{an}, LF)$. The functor α sends étale covering families in $X_{\acute{e}t}$ to families of jointly surjective local homeomorphisms. These are exactly the effective epimorphisms in $\mathbf{Top}^{\acute{e}t}/X_{an}$. Hence, identifying α with a functor $X_{\acute{e}t} \to \mathrm{Sh}(X_{an})$, we see that α pulls back sheaves on $\mathrm{Sh}(X_{an})$ equipped with the canonical topology to sheaves, since the canonical topology $\mathrm{Sh}(X_{an})$ is precisely generated by jointly epimorphic families. It follows that $\varphi_*(F)$ is always a sheaf, and hence φ^* restricts to a left exact colimit preserving functor

$$\varepsilon_X^* : \mathrm{Sh}(X_{\acute{e}t}) \to \mathrm{Sh}(X_{an}),$$

hence constitutes a geometric morphism

$$\varepsilon_X : \mathrm{Sh}(X_{an}) \to \mathrm{Sh}(X_{\acute{e}t}).$$

Since both $X_{\acute{e}t}$ and the poset of open subsets of X_{an} have finite limits, this canonically extends to a geometric morphism of ∞-topoi

$$\varepsilon_X : \mathrm{Sh}_\infty(X_{an}) \to \mathrm{Sh}_\infty(X_{\acute{e}t})$$

by [31, Proposition 6.4.5.4].

We define ξ_X as the composition

$$\mathbb{H}\mathrm{ypSh}_\infty(X_{an}) \xrightarrow{\epsilon_X} \mathrm{Sh}_\infty(X_{an}) \xrightarrow{\varepsilon_X} \mathrm{Sh}_\infty(X_{\acute{e}t}).$$

Suppose that $f : X \to Y$ is a morphism of schemes, and consider the following diagram of categories:

$$\begin{tikzcd} \mathrm{Sh}(X_{\acute{e}t}) \arrow[r, "\varepsilon_X^*"] & \mathrm{Sh}(X_{an}) \\ \mathrm{Sh}(Y_{\acute{e}t}) \arrow[u, "f^*"] \arrow[r, "\varepsilon_Y^*"] & \mathrm{Sh}(Y_{an}). \arrow[u, "f_{an}^*"'] \end{tikzcd}$$

Since all the functors involved preserve colimits, and since analytification preserves finite limits, it follows that there is a canonical 2-morphism

$$\varepsilon(f) : \varepsilon_X^* \circ f^* \overset{\sim}{\Rightarrow} f_{an}^* \circ \varepsilon_Y^*.$$

That is to say, $\varepsilon(f)$ represents a 2-morphism in the $(2,1)$-category $\mathfrak{Top}$ of topoi, making the following diagram commute

$$\begin{tikzcd} \mathrm{Sh}(X_{an}) \arrow[r, "\varepsilon_X"] \arrow[d] & \mathrm{Sh}(X_{\acute{e}t}) \arrow[d] \\ \mathrm{Sh}(Y_{an}) \arrow[r, "\varepsilon_Y"] & \mathrm{Sh}(Y_{\acute{e}t}). \end{tikzcd}$$

Moreover, it is easy to check that the various geometric morphisms ε_X together with these 2-morphisms assemble into a lax natural-transformation

$$\begin{tikzcd} \mathbf{Aff}_{\mathbb{C}}^{LFT} \arrow[r, bend left, "\mathrm{Sh}(\cdot)\circ(\cdot)_{an}", ""{name=U, below}] \arrow[r, bend right, "\mathrm{Sh}((\cdot)_{\acute{e}t})."', ""{name=D, above}] & \mathfrak{Top}. \arrow[from=U, to=D, Rightarrow, "\varepsilon"] \end{tikzcd}$$

(The necessary coherency conditions follow by a similar argument by pasting diagrams.) By abuse of notation, composition with the canonical inclusion

$$\mathfrak{Top} \hookrightarrow \mathfrak{Top}_\infty$$

induces a natural transformation

$$\begin{tikzcd} \mathbf{Aff}_{\mathbb{C}}^{LFT} \arrow[r, bend left, "\mathrm{Sh}_\infty(\,\cdot\,)\circ(\,\cdot\,)_{an}"] \arrow[r, bend right, "\mathrm{Sh}_\infty((\,\cdot\,)_{\acute{e}t})."'] \arrow[r, phantom, "\Downarrow\varepsilon"] & \mathfrak{Top}_\infty \end{tikzcd}$$

Finally, by composing with the counit

$$\epsilon : q \circ \mathbb{H}yp \Rightarrow id_{\mathfrak{Top}_\infty}$$

of the coreflective subcategory of hypercomplete ∞-topoi (Remark 2.4.3), we get a natural transformation

$$\begin{tikzcd} \mathbf{Aff}_{\mathbb{C}}^{LFT} \arrow[r, bend left, "\mathbb{H}\mathrm{ypSh}_\infty(\,\cdot\,)\circ(\,\cdot\,)_{an}"] \arrow[r, bend right, "\mathrm{Sh}_\infty((\,\cdot\,)_{\acute{e}t})"'] \arrow[r, phantom, "\Downarrow\xi"] & \mathfrak{Top}_\infty. \end{tikzcd}$$

Again, since all the functors involved in the statement of Proposition 5.6 are colimit preserving, by [31, Proposition 5.5.4.20, Theorem 5.1.5.6] this natural transformation lifts to one of the form

$$\begin{tikzcd} \mathrm{Sh}_\infty\left(\mathbf{Aff}_{\mathbb{C}}^{LFT}, \acute{e}t\right) \arrow[r, bend left, "\mathbb{H}\mathrm{ypSh}_\infty(\,\cdot\,)\circ(\,\cdot\,)_{top}"] \arrow[r, bend right, "\mathrm{Sh}_\infty((\,\cdot\,)_{\acute{e}t})"'] \arrow[r, phantom, "\Downarrow\xi"] & \mathfrak{Top}_\infty, \end{tikzcd}$$

and to prove that it is an equivalence after applying $\mathit{Shape}^{\mathrm{Prof}}$, it suffices to show that each geometric morphism ξ_X is a profinite homotopy equivalence, when X is a scheme of finite type over $\mathbb{C}$. Note that by Proposition 2.2.12,

$$\epsilon_X : \mathbb{H}\mathrm{ypSh}_\infty\left(X_{an}\right) \to \mathrm{Sh}_\infty\left(X_{an}\right)$$

is a profinite homotopy equivalence for all X, so it suffices to prove that each geometric morphism

$$\varepsilon_X : \mathrm{Sh}_\infty\left(X_{an}\right) \to \mathrm{Sh}_\infty\left(X_{\acute{e}t}\right)$$

is a profinite homotopy equivalence as well.

Remark 5.7. The morphism between profinite shapes induced by ε_X, regarded as map in pro-objects in the homotopy category, agrees with the comparison map of Artin-Mazur in [2, Theorem 12.9]; their map is also directly induced from the same map of sites.

Let us fix a scheme X of finite type over $\mathbb{C}$ and denote ε_X from now on by ε.

The main ingredient in showing that ε is a profinite homotopy equivalence is the following classical result from [1]:

Theorem 5.8. *[1, exposé XI.4 Theorem 4.3, Theorem 4.4, and exposé XVI.4, Theorem 4.1] Let X $\mathbb{C}$-scheme. Then*

1) If X is locally of finite type, then analytification functor $\alpha : X_{\acute{e}t} \to \mathbf{Top}^{\acute{e}t}/X_{an}$ induces an equivalence of categories between finite étale maps over X and finite covering spaces of X_{an}.

2) If X is finite type, then ε induces an isomorphism in cohomology with coefficients in any local system of finite abelian groups.

Lemma 5.9. *Let G be a finite group, and X a scheme of finite type over $\mathbb{C}$. Then the analytification functor*

$$X_{\acute{e}t} \to \mathbf{Top}^{\acute{e}t}/X_{an}$$

induces an equivalence of categories between the category of G-torsors on X and the category of principal G-bundles over X_{an}.

Proof. Let H be a group object in a Cartesian monoidal category $\mathscr{C}$. Recall that a H**-torsor** in $\mathscr{C}$ is an inhabited H-object

$$\rho : H \times P \to P$$

such that the canonical map

$$H \times P \to P \times P$$

is an isomorphism. Note that any finite group G is canonically and simultaneously a group object both in the category of finite étale maps over X and the category of finite covering maps of X_{an}. A G-torsor in these categories is a G-torsor over X and principal G-bundle over X_{an} respectively. The result now follows from Theorem 5.8, 1). □

Recall that from Remark 2.2.11 that ε is a profinite homotopy equivalence if and only if for every π-finite space V, the induced map

$$\Gamma_{\acute{e}t}\Delta^{\acute{e}t}(V) \to \Gamma_{an}\Delta^{an}(V)$$

is an equivalence, where

$$\mathrm{Sh}_\infty(X_{\acute{e}t}) \underset{\Gamma_{\acute{e}t}}{\overset{\Delta^{\acute{e}t}}{\rightleftarrows}} \mathcal{S}$$

and

$$\mathrm{Sh}_\infty(X_{an}) \underset{\Gamma_{an}}{\overset{\Delta^{an}}{\rightleftarrows}} \mathcal{S}$$

are the unique geometric morphisms to the terminal ∞-topos $\mathcal{S}$. Our method of proof will be to first establish this for connected π-finite spaces by using induction on Postnikov towers.

Lemma 5.10. *Let $f : \mathcal{E} \to \mathcal{F}$ be a geometric morphism of ∞-topoi and let $\mathcal{A}$ be an abelian sheaf in $\mathcal{F}$. Suppose that for all $n \geq 0$, f induces an isomorphism in sheaf cohomology groups*

$$H^n(\mathcal{F}, \mathcal{A}) \xrightarrow{\sim} H^n(\mathcal{E}, f^*\mathcal{A}).$$

Then for all n, then induced map

$$\Gamma_{\mathcal{F}}(K(\mathcal{A},n)) \to \Gamma_{\mathcal{E}}(K(f^*\mathcal{A},n))$$

is an equivalence of ∞-groupoids.

Proof. First of all, it follows immediately from [31, Remark 6.5.1.4] that

$$f^*K(\mathcal{A},n) = K(f^*\mathcal{A},n),$$

which explains the induced map above; it is induced by the functor f^* (since f^* preserves the terminal object). Notice that for any abelian sheaf $\mathcal{B}$ on an ∞-topos $\mathcal{X}$, for $n>0$, $K(\mathcal{B},n)$ has the structure of a grouplike $\mathbb{E}_\infty$-object in $\mathcal{X}$, and consequently,

$$\Gamma_{\mathcal{X}}(K(\mathcal{B},n)) = \mathrm{Hom}_{\mathcal{X}}(1, K(\mathcal{B},n))$$

is a grouplike $\mathbb{E}_\infty$-space. Let $e: 1 \to K(\mathcal{B},n)$ be the group unit. We can identify e with a map from the one-point space

$$* \xrightarrow{e} \mathrm{Hom}_{\mathcal{X}}(1, K(\mathcal{B},n)).$$

Since the $\mathbb{E}_\infty$-space $\mathrm{Hom}_{\mathcal{X}}(1, K(\mathcal{B},n))$ is grouplike, it suffice to prove that

$$\Gamma_{\mathcal{F}}(K(\mathcal{A},n)) \to \Gamma_{\mathcal{E}}(K(f^*\mathcal{A},n))$$

induces an isomorphism on π_0 and on all higher homotopy groups *at the canonical base point e*. So it suffices to show that for all $k>0$ the induced map

$$\pi_0\Omega^k_e(\Gamma_{\mathcal{F}}(K(\mathcal{A},n))) \to \pi_0\Omega^k_e(\Gamma_{\mathcal{E}}(K(f^*\mathcal{A},n)))$$

is an isomorphism.

Notice that the canonical base point e is in the image of the global sections functor $\Gamma_{\mathcal{F}}$, i.e.

$$e = \Gamma_{\mathcal{F}}(e) : \Gamma_{\mathcal{F}}(1) \to \Gamma_{\mathcal{F}}(K(\mathcal{A},n)),$$

and since $\Gamma_{\mathcal{F}}$ preserves finite limits, it then follows that

$$\Omega^k_e(\Gamma_{\mathcal{F}}(K(\mathcal{A},n))) \simeq \Gamma_{\mathcal{F}}\left(\Omega^k_e((K(\mathcal{A},n)))\right).$$

When $k \le n$, $\Omega^k_e((K(\mathcal{A},n))) \simeq K(\mathcal{A},n-k)$ and for $k>n$, it's the terminal object. Consequently, we have that

$$\pi_k(\Gamma_{\mathcal{F}}(K(\mathcal{A},n)),e) \cong \pi_0\Omega^k_e(\Gamma_{\mathcal{F}}(K(\mathcal{A},n))) \cong H^{n-k}(\mathcal{F},\mathcal{A})$$

for $k \le n$, and otherwise is zero, and similarly for $\pi_k(\Gamma_{\mathcal{E}}(K(f^*\mathcal{A},n)),e)$. □

Proposition 5.11. *Let X be a connected scheme of finite type over $\mathbb{C}$. Then*

$$\varepsilon : Sh_\infty(X_{an}) \to Sh_\infty(X_{\acute{e}t})$$

is a profinite homotopy equivalence.

Proof. It suffices to prove that for every π-finite space, the induced map

$$\Gamma_{\acute{e}t}\Delta^{\acute{e}t}(V) \to \Gamma_{an}\Delta^{an}(V)$$

is an equivalence of ∞-groupoids. Denote by $\mathscr{C}$ the full subcategory of $\mathcal{S}$ spanned by all spaces V for which the above map is an equivalence. Note that since the functors $\Gamma_{\acute{e}t}, \Delta^{\acute{e}t}, \Gamma_{an}, \Delta^{an}$ all preserve finite limits, if follows that $\mathscr{C}$ is closed under finite limits in $\mathcal{S}$. Note by Theorem 5.8, 2), together with Lemma 5.10 it follows in particular that $\mathscr{C}$ contains all Eilenberg-MacLane spaces of the form $K(A,n)$, with A a finite abelian group. Also, it follows from Lemma 5.9 that $\mathscr{C}$ contains all $K(G,1)$ for all finite groups G.

Fix a finite abelian group A and let $n > 0$ be an integer. We claim that $B\mathbf{Aut}(K(A,n))$ is also in $\mathscr{C}$. Let us establish this claim. We have already seen that the canonical map

$$\Gamma_{\acute{e}t}\Delta^{\acute{e}t}(K(\mathbf{Aut}(A),1)) \to \Gamma_{an}\Delta^{an}(K(\mathbf{Aut}(A),1))$$

is an equivalence of ∞-groupoids. Consider the canonical map

$$\psi : B\mathbf{Aut}(K(A,n)) \to K(\mathbf{Aut}(A),1)$$

induced by identifying $K(\mathbf{Aut}(A),1)$ as the 1-truncation of $B\mathbf{Aut}(K(A,n))$. It suffices to prove that for every base point

$$\tau : * \to \Gamma_{\acute{e}t}\Delta^{\acute{e}t}(K(\mathbf{Aut}(A),1)),$$

the induced maps between the (homotopy) fiber of

$$\Gamma_{\acute{e}t}\Delta^{\acute{e}t}(\psi)$$

over τ and the (homotopy) fiber of

$$\Gamma_{an}\Delta^{an}(\psi)$$

over $\varepsilon^*\tau$ is an equivalence of ∞-groupoids. Let F_n, denote the fiber of

$$\Gamma_{\acute{e}t}\Delta^{\acute{e}t}(\psi)$$

over τ, i.e. the pullback

$$\begin{tikzcd} F_n \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathrm{Sh}_\infty(X_{\acute{e}t})}\left(1, \Delta^{\acute{e}t}(B\mathbf{Aut}(K(A,n)))\right) \arrow[d, "\Gamma_{\acute{e}t}\Delta^{\acute{e}t}(\psi)"] \\ * \arrow[r, "\tau"] & \mathrm{Hom}_{\mathrm{Sh}_\infty(X_{\acute{e}t})}\left(1, \Delta^{\acute{e}t}(K(\mathbf{Aut}(A),1))\right). \end{tikzcd}$$

By [31, Proposition 5.5.5.12], we have a canonical identification

$$F_n \simeq \mathrm{Hom}_{\mathrm{Sh}_\infty(X_{\acute{e}t})/\Delta^{\acute{e}t}(K(\mathbf{Aut}(A),1))}\left(\tau, \Delta^{\acute{e}t}(\psi)\right).$$

The latter space of maps is the space of lifts

$$\begin{tikzcd} 1 \times_{\Delta^{\acute{e}t}(K(\mathbf{Aut}(A),1))} \Delta^{\acute{e}t}(B\mathbf{Aut}(K(A,n))) \arrow[r] \arrow[d] & \Delta^{\acute{e}t}(B\mathbf{Aut}(K(A,n))) \arrow[d, "\Delta^{\acute{e}t}(\psi)"] \\ 1 \arrow[r, "\tau"] \arrow[ur, dashed] & \Delta^{\acute{e}t}(K(\mathbf{Aut}(A),1)) \end{tikzcd}$$

which is canonically equivalent to the space

$$\Gamma^{\acute{e}t}\left(1\times_{\Delta^{\acute{e}t}(K(\mathbf{Aut}(A),1))}\Delta^{\acute{e}t}\left(B\mathbf{Aut}\left(K\left(A,n\right)\right)\right)\right).$$

Since $\Delta^{\acute{e}t}$ preserves finite limits, it follows from Lemma A.1.8 that the following is a pullback diagram in $\mathrm{Sh}_\infty\left(X_{\acute{e}t}\right)$:

$$\begin{tikzcd}
\Delta^{\acute{e}t}\left(B\mathbf{Aut}\left(K\left(A,n\right)\right)\right) \arrow[r] \arrow[d, "\Delta^{\acute{e}t}(\psi)"'] & \Delta^{\acute{e}t}\left(K\left(\mathbf{Aut}\left(A\right),1\right)\right) \arrow[d, "\Delta^{\acute{e}t}(\theta_{n+1})"] \\
\Delta^{\acute{e}t}\left(K\left(\mathbf{Aut}\left(A\right),1\right)\right) \arrow[r, "\Delta^{\acute{e}t}(\theta_{n+1})"] & \Delta^{\acute{e}t}\left(B\mathbf{Aut}\left(K\left(A,n+1\right)\right)\right).
\end{tikzcd}$$

In light of this, by the pullback square at the end of the proof of Theorem A.2.13, we have a canonical identification

$$1\times_{\Delta^{\acute{e}t}(K(\mathbf{Aut}(A),1))}\Delta^{\acute{e}t}\left(B\mathbf{Aut}\left(K\left(A,n\right)\right)\right)\simeq K\left(\mathcal{F}_\tau,n+1\right),$$

where $\mathcal{F}_\tau$ is the abelian sheaf classified by the local system τ. In summary, we have that the fiber of $\Gamma_{\acute{e}t}\Delta^{\acute{e}t}\left(\psi\right)$ over τ can canonically be identified with $\Gamma_{\acute{e}t}\left(K\left(\mathcal{F}_\tau,n+1\right)\right)$. Hence the induced map between fibers can be identified with the induced map

$$\Gamma_{\acute{e}t}\left(K\left(\mathcal{F}_\tau,n+1\right)\right)\to\Gamma_{an}\left(K\left(\mathcal{F}_{\varepsilon^*\tau},n+1\right)\right).$$

By Theorem 5.8, 2), for all n, the induced map

$$H^n\left(\mathrm{Sh}_\infty\left(X_{\acute{e}t}\right),\mathcal{F}_\tau\right)\xrightarrow{\sim}H^n\left(\mathrm{Sh}_\infty\left(X_{an}\right),\mathcal{F}_{\varepsilon^*\tau}\right)$$

is an isomorphism. The claim now follows from Lemma 5.10.

Since X is connected, it follows that so is X_{an}, and hence the terminal objects of both $\mathrm{Sh}_\infty\left(X_{an}\right)$ and $\mathrm{Sh}_\infty\left(X_{\acute{e}t}\right)$ are connected, by Lemma 3.1.14. It follows then from Proposition 3.1.16 that both $\Gamma_{\acute{e}t}$ and Γ_{an} preserve coproducts, and hence $\mathscr{C}$ is closed under coproducts in $\mathcal{S}$. This reduces our job to checking that

$$\Gamma_{\acute{e}t}\Delta^{\acute{e}t}\left(V\right)\to\Gamma_{an}\Delta^{an}\left(V\right)$$

is an equivalence for all *connected* π-finite spaces.

Let us prove by induction on n that $\mathscr{C}$ contains all connected n-truncated π-finite spaces. We have already established that this holds for $n=1$. Suppose by hypothesis that $n\geq 2$ and $\mathscr{C}$ contains all $(n-1)$-truncated connected π-finite spaces. We wish to show that $\mathscr{C}$ contains all n-truncated connected π-finite spaces. Let Z be such a space. Denote by Z_{n-1} the $(n-1)^{st}$-truncation of Z. Then $Z\to Z_{n-1}$ has fiber $K\left(\pi_n\left(Z\right),n\right)$. Let A be the abelian group $\pi_n\left(Z\right)$. Then by Proposition A.1.5, we have a pullback square

$$\begin{tikzcd}
Z \arrow[r] \arrow[d] & K\left(\mathbf{Aut}\left(A\right),1\right) \arrow[d, "\theta_n"] \\
Z_{n-1} \arrow[r] & B\mathbf{Aut}\left(K\left(A,n\right)\right).
\end{tikzcd}$$

Since $\mathscr{C}$ is stable under finite limits, it now follows that Z is in $\mathscr{C}$ as well. □

Proposition 5.12. *Let X be a scheme of finite type over $\mathbb{C}$. Then*

$$\varepsilon : Sh_\infty\left(X_{an}\right) \to Sh_\infty\left(X_{\acute{e}t}\right)$$

is a profinite homotopy equivalence.

Proof. Notice that both ∞-topoi involved are locally connected. With this in mind, let $X = \coprod\limits_\alpha X_\alpha$, with each X_α a connected scheme. For any space V,

$$\begin{aligned} \Gamma^{\acute{e}t}\Delta^{\acute{e}t}(V) &= \mathrm{Hom}_{\mathrm{Sh}_\infty(X_{\acute{e}t})}\left(\coprod_\alpha X_\alpha, \Delta^{\acute{e}t}(V)\right) \\ &\simeq \prod_\alpha \mathrm{Hom}_{\mathrm{Sh}_\infty(X_{\acute{e}t})}\left(X_\alpha, \Delta^{\acute{e}t}(V)\right) \\ &\simeq \prod_\alpha \mathrm{Hom}_{\mathrm{Sh}_\infty((X_\alpha)_{\acute{e}t})}\left(1, \Delta^{\acute{e}t}_\alpha(V)\right) \\ &\simeq \prod_\alpha \Gamma^{\acute{e}t}_\alpha \Delta^{\acute{e}t}_\alpha(V). \end{aligned}$$

The analytification of X is

$$X_{an} = \coprod_\alpha (X_\alpha)_{an}$$

and each X_α is connected as a topological space. By analogous reasoning as above we have

$$\Gamma^{an}\Delta^{an}(V) \simeq \prod_\alpha \Gamma^{an}_\alpha \Delta^{an}_\alpha(V).$$

The result now follows from Proposition 5.11. □

This establishes the proof of Proposition 5.6. We now prove our main theorem:

Theorem 5.13. *The following diagram commutes up to equivalence:*

$$\begin{tikzcd} Sh_\infty\left(\mathbf{Aff}^{LFT}_{\mathbb{C}}, \acute{e}t\right) \arrow[r, "\widehat{\Pi}^{\acute{e}t}_\infty"] \arrow[d, "(\,\cdot\,)_{top}"'] & Prof(\mathcal{S}) \\ \mathbb{H}\mathrm{ypSh}_\infty\left(\mathbf{Top}_{\mathbb{C}}\right) \arrow[r, "\Pi_\infty"] & \mathcal{S}. \arrow[u, "\widehat{(\,\cdot\,)}"'] \end{tikzcd}$$

In particular, for any ∞-sheaf F on $\left(\mathbf{Aff}^{LFT}_{\mathbb{C}}, \acute{e}t\right)$, there is an equivalence of profinite spaces

$$\widehat{\Pi}^{\acute{e}t}_\infty(F) \simeq \widehat{\Pi}_\infty\left(F_{top}\right),$$

between the profinite étale homotopy type of F and the profinite completion of the homotopy type of the underlying stack F_{top} on $\mathbf{Top}_{\mathbb{C}}$.

Proof. By Proposition 5.6, we have an equivalence

$$Shape^{\mathrm{Prof}} \circ \mathbb{H}\mathrm{ypSh}_\infty(\,\cdot\,) \circ (\,\cdot\,)_{top} \xRightarrow[\sim]{Shape^{\mathrm{Prof}}(\xi)} \widehat{\Pi}^{\acute{e}t}_\infty.$$

Note that by definition we have

$$Shape^{\mathrm{Prof}} = i^* \circ Shape,$$

so

$$Shape^{\mathrm{Prof}} \circ \mathbb{H}\mathrm{ypSh}_\infty(\,\cdot\,) \circ (\,\cdot\,)_{top} = i^* \circ Shape \circ \mathbb{H}\mathrm{ypSh}_\infty(\,\cdot\,) \circ (\,\cdot\,)_{top}.$$

By Lemma 5.5 we have an equivalence

$$Shape \circ \mathbb{H}\mathrm{ypSh}_\infty(\,\cdot\,) \simeq j \circ \Pi_\infty.$$

Furthermore, by definition the profinite completion functor

$$\widehat{(\,\cdot\,)} : \mathcal{S} \to \mathrm{Prof}$$

is $i^* \circ j$, so finally

$$Shape^{\mathrm{Prof}} \circ \mathbb{H}\mathrm{ypSh}_\infty(\,\cdot\,) \circ (\,\cdot\,)_{top} \simeq \widehat{(\,\cdot\,)} \circ \Pi_\infty \circ (\,\cdot\,)_{top}. \qquad \square$$

Corollary 5.14. *Let $\mathfrak{X}$ be an Artin stack locally of finite type over $\mathbb{C}$, then there is an equivalence of profinite spaces*

$$\widehat{\Pi}^{\acute{e}t}_\infty(\mathfrak{X}) \simeq \widehat{\Pi}_\infty(\mathfrak{X}_{top}),$$

between the profinite étale homotopy type of $\mathfrak{X}$ and the profinite completion of the homotopy type of the underlying topological stack $\mathfrak{X}_{top}$.

Remark 5.15. In light of Remark 5.7, we conclude that for pointed schemes of finite type over $\mathbb{C}$, our comparison map in Theorem 5.13 induces the same one as Artin and Mazur.

Example 5.16. Consider the moduli stack $\mathcal{M}_{g,n}$ of proper smooth curves of genus g with n marked points, and let $\Gamma_{g,n}$ be the mapping class group of a surface of genus g with n marked points. Fix an embedding

$$\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}.$$

Then it was shown in [38] that the homotopy type of the analytification of

$$\mathcal{M}_{g,n} \otimes \overline{\mathbb{Q}}$$

is that of $B\Gamma_{g,n}$. It follows that there is an equivalence of profinite spaces

$$\widehat{\Pi}^{\acute{e}t}_\infty\left(\mathcal{M}_{g,n} \otimes \overline{\mathbb{Q}}\right) \simeq \widehat{B\Gamma_{g,n}}.$$

An analogous result was shown in [38] using the machinery of étale homotopy type of Friedlander, and the notion of profinite completion of Artin-Mazur. Similarly, it follows from [12] that

$$\widehat{\Pi}^{\acute{e}t}_\infty\left(\overline{\mathcal{M}_{g,n}} \otimes \overline{\mathbb{Q}}\right) \simeq \widehat{B\mathcal{CL}_{g,n}},$$

where $\overline{\mathcal{M}_{g,n}}$ is the Deligne-Mumford compactification, and $\mathcal{CL}_{g,n}$ is the Charney-Lee category.

Example 5.17. Consider the moduli stack of elliptic curves $\mathcal{M}_{ell}$, and fix an embedding

$$\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}.$$

Then it is shown in [14] than the homotopy type of the analytification is that of $BSL(2,\mathbb{Z})$, from which it follows that

$$\widehat{\Pi}^{\acute{e}t}_\infty\left(\mathcal{M}_{ell} \otimes \overline{\mathbb{Q}}\right) \simeq B\widehat{SL(2,}\mathbb{Z}).$$

As above, an analogous result was shown in the same paper, but using the machinery of étale homotopy type of Friedlander, and the notion of profinite completion of Artin-Mazur.

Example 5.18. Let G be an algebraic group over $\mathbb{C}$. Then

$$\widehat{\Pi}^{\acute{e}t}_{\infty}(BG) \simeq \widehat{BG_{an}}.$$

Note: If G is discrete and good in the sense of Serre [45], then $\widehat{BG_{an}} \simeq B\widehat{G}$.

Example 5.19. Let X be a fine saturated log scheme locally of finite type over $\mathbb{C}$, and let $\sqrt[\infty]{X}$ be its infinite-root stack in the sense of [48]. (This is a pro-object in algebraic stacks). It was shown in [8] that the homotopy type of the underlying (pro-)topological stack, after profinite completion agrees with the Kato-Nakayama space X_{log} of X in the sense of [25]. It thus follows that

$$\widehat{\Pi}^{\acute{e}t}_{\infty}\left(\sqrt[\infty]{X}\right) \simeq \widehat{X_{log}}.$$

As the Kato-Nakayama space is only defined for log schemes over $\mathbb{C}$, this suggests that the infinite root stack could be a suitable replacement for it in positive characteristics.

Appendix A: Cohomology with coefficients in a local system

In this appendix we give a careful introduction to the concept of cohomology with coefficients in a local system of abelian groups using the modern language of ∞-categories. We work this out first for the case of spaces, and then for an arbitrary ∞-topos, and link these definitions with the classical definition of cohomology with twisted coefficients in a topos. The material in this section plays a pivotal role in proving the main theorem of this paper.

A.1 Topological case In this subsection, we will explain how to define the cohomology of a space X with coefficients in a local system of abelian groups by using classifying spaces. The basic idea is not new and goes back to [43, 5], and we benefited greatly from discussion with Achim Krause. In what follows, we formulate cohomology of local systems on spaces in the natural setting of the ∞-category $\mathcal{S}$ of spaces.

A.1.1 Preliminaries on ∞-groupoids Let X be a space in $\mathcal{S}$. Regarding X as an ∞-groupoid (and hence as an ∞-category), by the proof of [31, cor 5.3.5.4], there is a canonical equivalence of ∞-categories

$$\mathcal{S}/X \simeq \mathrm{Psh}_{\infty}(X).$$

Let us unravel this equivalence a little. First note that since X is an ∞-*groupoid*, X is naturally equivalent to its opposite ∞-category X^{op}, so we have a canonical equivalence $\mathrm{Psh}_{\infty}(X) \simeq \mathrm{Fun}(X, \mathcal{S})$. It will be convenient to phrase things in terms of $\mathrm{Fun}(X, \mathcal{S})$ instead:

Given an object $f: Y \to X$ of $\mathcal{S}/X$, it defines a functor $F: X \to \mathcal{S} = \mathrm{Gpd}_{\infty}$, by assigning to each object x of the ∞-groupoid X, the ∞-groupoid of lifts

$$\begin{tikzcd} & Y \arrow[d, "f"] \\ * \arrow[ur, dashed] \arrow[r, "x"] & X. \end{tikzcd}$$

By abstract nonsense, this is the same as the ∞-groupoid of lifts in the pullback diagram

$$\begin{tikzcd} * \times_X^{x,f} Y \arrow[r] \arrow[d] & Y \arrow[d, "f"] \\ * \arrow[u, dashed, bend left] \arrow[r, "x"] & X, \end{tikzcd}$$

in other words the space

$$\operatorname{Hom}\left(*, * \times_X^{x,f} Y\right) \simeq * \times_X^{x,f} Y.$$

In particular, if $f : X \to Y$ arises from a continuous map $\tilde{f}$ of topological spaces, $F(x)$ is the homotopy fiber of $\tilde{f}$ over x. Conversely, given an arbitrary functor $F : X \to \mathrm{Gpd}_\infty$, it corresponds to a left fibration

$$\pi_F : \int_X F \to X,$$

which since X is a Kan complex (when regarded as a quasicategory) is also a Kan fibration, and hence π_F is a map of Kan complexes, corresponding to an object of $\mathcal{S}/X$. It follows from the proof of [31, cor 5.3.5.4] that as an ∞-category, using the identification $\mathrm{Fun}(X, \mathcal{S}) \simeq \mathrm{Psh}_\infty(X)$, $\int_X F$ is the full subcategory of the slice category $\mathrm{Psh}_\infty(X)/F$ on those morphisms whose domain is a representable presheaf, and this ∞-category is an ∞-groupoid.

Proposition A.1.1. *If $F : X \to \mathcal{S}$ is a functor, its associated left fibration*

$$\pi_F : \int_X F \to X$$

can canonically be identified with the canonical map

$$\varinjlim F \to X.$$

Proof. This statement is essentially [31, cor 3.3.4.6], but we provide here another proof which the reader comfortable with standard 1-categorical arguments may find more conceptual: Consider the composite

$$\mathrm{Psh}_\infty(X) \xrightarrow{\sim} \mathcal{S}/X \to \mathcal{S},$$

where the functor $\mathcal{S}/X \to \mathcal{S}$ is the forgetful functor, which is colimit preserving. Let x be an object of the ∞-groupoid X. Then the representable presheaf $y(x)$ gets mapped as

$$y(x) \mapsto x : * \to X \mapsto *$$

under the above composite. It follows from [31, Theorem 5.1.5.6] that the composite must in fact be equivalent to the left Kan extension of the terminal functor $X \to \mathcal{S}$ (sending everything to the contractible space) along the Yoneda embedding

$$y : X \hookrightarrow \mathrm{Psh}_\infty(X).$$

It follows that the composite is the functor assigning a presheaf F its colimit. The result now follows from [31, Proposition 1.2.13.8]. ∎

A.1.2 Universal fibrations Let Gpd_∞ be the ∞-category of small ∞-groupoids. Recall that for $\mathscr{C}$ an ∞-category, given a functor

$$F : \mathscr{C} \to \mathrm{Gpd}_\infty$$

there is an associated left fibration

$$\int_{\mathscr{C}} F \to \mathscr{C}.$$

This is an ∞-categorical analogue of the classical Grothendieck construction from category theory, which associates a functor

$$F : \mathscr{D} \to \mathrm{Gpd},$$

with $\mathscr{D}$ a small category, to its associated cofibered category

$$\int_{\mathscr{D}} F \to \mathscr{D}.$$

This construction is part of an equivalence of ∞-categories between the functor category $\mathrm{Fun}\left(\mathscr{C}, \mathrm{Gpd}_\infty\right)$ and the ∞-category of left fibrations over $\mathscr{C}$ (more precisely the ∞-category associated to the covariant model structure on marked simplicial sets over $\mathscr{C}$; see [31, Section 2.1]).

Definition A.1.2. Let $id : \mathrm{Gpd}_\infty \to \mathrm{Gpd}_\infty$ be the identity functor. The **universal left fibration** is its associated left fibration

$$\mathcal{Z}_{\mathcal{S}} \to \mathrm{Gpd}_\infty \simeq \mathcal{S}.$$

The universal property of the above left fibration, which justifies its name, is that if $F : \mathscr{C} \to \mathrm{Gpd}_\infty$ is a functor, then the following is a pullback diagram

$$\begin{tikzcd} \int_{\mathscr{C}} F \arrow[r] \arrow[d] & \mathcal{Z}^{\mathcal{S}} \arrow[d] \\ \mathscr{C} \arrow[r, "F"] & \mathcal{S}. \end{tikzcd}$$

(in the ∞-category $\widehat{\mathrm{Cat}}_\infty$ of large ∞-categories). See [31, Section 3.3.2].

The following lemma follows immediately:

Lemma A.1.3. *Let $f : \mathscr{C} \to \mathscr{D}$ be a functor between ∞-categories and let $G : \mathscr{D} \to \mathcal{S}$ be another functor. Then the following diagram is a pullback diagram:*

$$\begin{tikzcd} \int_{\mathscr{C}} (G \circ f) \arrow[r] \arrow[d] & \int_{\mathscr{D}} G \arrow[d] \\ \mathscr{C} \arrow[r, "f"] & \mathscr{D}. \end{tikzcd}$$

A.1.3 Classifying spaces and cohomology with coefficients local systems Fix A an abelian group and let $n > 0$ be an integer. Denote by $\mathcal{S}_{K(A,n)}$ the full subcategory of spaces on the single space $K(A, n)$, and denote by $B\mathbf{Aut}\left(K(A, n)\right)$ its maximal sub-Kan-complex, i.e. the ∞-groupoid obtained by throwing out all the arrows which are not equivalences. Define the space $\mathbf{Aut}\left(K(A, n)\right)$ to be the mapping space $\mathrm{Hom}_{B\mathbf{Aut}(K(A,n))}\left(K(A, n), K(A, n)\right)$. Concretely, $\mathbf{Aut}\left(K(A, n)\right)$ is the space of self homotopy equivalences of $K(A, n)$. Denote by Θ_n the canonical inclusion

$$B\mathbf{Aut}\left(K(A, n)\right) \to \mathcal{S}_{K(A,n)} \hookrightarrow \mathcal{S}.$$

Denote by $\theta_n : U_n^A \to B\mathbf{Aut}\left(K\left(A,n\right)\right)$ the left fibration classified by the functor Θ_n, which is simply a map of spaces.

Denote by $\mathbf{Aut}_*\left(K\left(A,n\right)\right)$ the space of homotopy equivalences of $K\left(A,n\right)$ that preserve the base-point. It is the subcategory of the ∞-groupoid

$$\mathbf{End}_*\left(K\left(A,n\right)\right) = \mathrm{Hom}_{\mathcal{S}_*}\left(K\left(A,n\right), K\left(A,n\right)\right)$$

of pointed endomorphisms of $K\left(A,n\right)$ on those which are equivalences. By [30, Theorem 5.1.3.6] the n-fold loop space functor

$$\Omega^n : \mathcal{S}_*^{\geq n} \to \mathrm{Mon}^{gp}_{\mathbb{E}_n}\left(\mathcal{S}\right)$$

from the ∞-category of pointed n-connective spaces to the ∞-category of grouplike $\mathbb{E}_n$-spaces is an equivalence. Since $K\left(A,n\right)$ is n-connective, it follows that

$$\begin{aligned} \mathbf{End}_*\left(K\left(A,n\right)\right) &= \mathrm{Hom}_{\mathcal{S}_*^{\geq n}}\left(K\left(A,n\right), K\left(A,n\right)\right) \\ &\simeq \mathrm{Hom}_{\mathrm{Mon}^{gp}_{\mathbb{E}_n}}\left(\Omega^n K\left(A,n\right), \Omega^n K\left(A,n\right)\right) \\ &\simeq \mathrm{Hom}_{\mathrm{Mon}^{gp}_{\mathbb{E}_n}}\left(A, A\right). \end{aligned}$$

Since A is a discrete group, we have finally that

$$\mathbf{End}_*\left(K\left(A,n\right)\right) \simeq \mathrm{Hom}_{\mathbf{Grp}}\left(A, A\right).$$

It follows that

$$\mathbf{Aut}_*\left(K\left(A,n\right)\right) \simeq \mathbf{Aut}\left(A\right).$$

Note also that we have a pullback diagram in $\mathcal{S}$:

$$\begin{tikzcd} \mathbf{Aut}\left(A\right) \simeq \mathbf{Aut}_*\left(K\left(A,n\right)\right) \arrow[r] \arrow[d] & \mathbf{Aut}\left(K\left(A,n\right)\right) \arrow[d, "ev_*"] \\ * \arrow[r] & K\left(A,n\right). \end{tikzcd}$$

Unwinding the definitions, since the inverse functor to Ω^n is B^n, we have that the map above

$$\mathbf{Aut}\left(A\right) \to \mathbf{Aut}\left(K\left(A,n\right)\right)$$

sends an automorphism $\varphi : A \xrightarrow{\sim} A$ to the automorphism

$$K\left(\varphi, n\right) : K\left(A,n\right) \xrightarrow{\sim} K\left(A,n\right).$$

Also, via the long exact sequence in homotopy groups from the fibration sequence associated to the above diagram, we conclude that

$$\pi_0\left(\mathbf{Aut}\left(K\left(A,n\right)\right)\right) \cong \mathbf{Aut}\left(A\right)$$

and the only other non-trivial homotopy group is

$$\pi_n\left(\mathbf{Aut}\left(K\left(A,n\right)\right)\right) \cong A.$$

In other words, we have an equivalence of spaces

$$\mathbf{Aut}\left(K\left(A,n\right)\right) \simeq \mathbf{Aut}\left(A\right) \times K\left(A,n\right). \tag{4}$$

In fact, we even have a semi-direct product decomposition

$$\mathbf{Aut}\left(K\left(A,n\right)\right)\simeq\mathbf{Aut}\left(A\right)\ltimes K\left(A,n\right).$$

Recall that $\theta_n : U_n^A \to B\mathbf{Aut}\left(K\left(A,n\right)\right)$ is the left fibration classified by the canonical functor

$$\Theta_n : B\mathbf{Aut}\left(K\left(A,n\right)\right)\to\mathcal{S}.$$

Let us compute what U_n^A is explicitly. As an ∞-category, this is the full subcategory of the slice category $\mathrm{Psh}_\infty\left(B\mathbf{Aut}\left(K\left(A,n\right)\right)\right)/\Theta_n$ on those maps $G\to\Theta_n$ with G a representable presheaf. But, $B\mathbf{Aut}\left(K\left(A,n\right)\right)$ only has one object, call it $\star$. Denote by $y\left(\star\right)$ its associated presheaf, which sends $\star$ to $\mathbf{Aut}\left(K\left(A,n\right)\right)$. By the Yoneda lemma, we have

$$\mathrm{Hom}\left(y\left(\star\right),\Theta_n\right)\simeq K\left(A,n\right).$$

It follows that the ∞-category U_n^A also has a single object, call it V. We can write

$$V : y\left(\star\right)\to\Theta_n.$$

The space of maps $\mathrm{Hom}_{U_n^A}\left(V,V\right)$ is the space of maps in the slice category

$$\mathrm{Psh}_\infty\left(B\mathbf{Aut}\left(K\left(A,n\right)\right)\right)/\Theta_n.$$

By [31, Proposition 5.5.5.12], we can identify $\mathrm{Hom}_{U_n^A}\left(V,V\right)$ with the fiber of the map

$$\mathrm{Hom}\left(y\left(\star\right),y\left(\star\right)\right)\to\mathrm{Hom}\left(y\left(\star\right),\Theta_n\right)$$

induced by composition with V. By the Yoneda lemma, this is equivalent to the fiber of the map

$$\mathbf{Aut}\left(K\left(A,n\right)\right)\to K\left(A,n\right).$$

Unwinding the definitions, we see that the above map

$$\mathbf{Aut}\left(K\left(A,n\right)\right)\simeq\mathbf{Aut}\left(A\right)\times K\left(A,n\right)\to K\left(A,n\right)$$

is just the first projection. It follows that

1) $\mathrm{Hom}_{U_n^A}\left(V,V\right)\simeq\mathbf{Aut}\left(A\right),$
2) The canonical map $\mathrm{Hom}_{U_n^A}\left(V,V\right)\to\mathbf{Aut}\left(K\left(A,n\right)\right)$ induced by the left fibration

$$\theta_n : U_n^A\to B\mathbf{Aut}\left(K\left(A,n\right)\right)$$

sends an automorphism

$$\varphi : A\xrightarrow{\sim}A$$

to the automorphism

$$K\left(\varphi,n\right) : K\left(A,n\right)\xrightarrow{\sim}K\left(A,n\right).$$

From 1) we conclude that $U_n^A=K\left(\mathbf{Aut}\left(A\right),1\right)$. Notice that $K\left(\mathbf{Aut}\left(A\right),1\right)$ is a 1-type, hence a groupoid. Viewing it as a groupoid, it is the groupoid with one object $\star$ such that

$$\mathrm{Hom}\left(\star,\star\right)=\mathbf{Aut}\left(A\right).$$

As such, there is a canonical functor into the category of abelian groups

$$\chi_A : K(\mathbf{Aut}(A), 1) \to Ab,$$

sending $\star$ to A and each automorphism of A to itself. Composition with the n^{th} Eilenberg-MacLane functor then yields a functor into spaces

$$K(\mathbf{Aut}(A), 1) \xrightarrow{\chi_A} Ab \xrightarrow{K(\,\cdot\,,n)} \mathcal{S}.$$

This functor sends the single object $\star$ to $K(A, n)$ and sends each automorphism

$$\varphi : A \xrightarrow{\sim} A$$

to

$$K(\varphi, n) : K(A, n) \xrightarrow{\sim} K(A, n),$$

hence by 2), there is a factorization

$$\begin{tikzcd} & K(\mathbf{Aut}(A), 1) \arrow[dl, dashed, "\theta_n"'] \arrow[d, "\chi_A"] \\ & Ab \arrow[d, "K(\,\cdot\,,n)"] \\ B\mathbf{Aut}(K(A, n)) \arrow[r] & \mathcal{S}. \end{tikzcd}$$

Definition A.1.4. The map $\theta_n : K(\mathbf{Aut}(A), 1) \to B\mathbf{Aut}(K(A, n))$ is the **universal** $K(A, n)$**-fibration**.

The following proposition justifies this terminology:

Proposition A.1.5. *Let $g : Y \to X$ be any map of spaces whose fibers are all equivalent to $K(A, n)$. Then there is a pullback diagram*

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "g"'] & K(\mathbf{Aut}(A), 1) \arrow[d, "\theta_n"] \\ X \arrow[r, "c_g"'] & B\mathbf{Aut}(K(A, n)). \end{tikzcd}$$

Proof. Under the equivalence $\mathcal{S}/X \simeq \mathrm{Fun}(X, \mathcal{S})$, $g : Y \to X$ corresponds to a functor

$$G : X \to \mathcal{S}$$

that factors as

$$X \xrightarrow{c_g} B\mathbf{Aut}(K(A, n)) \xrightarrow{\Theta_n} \mathcal{S}.$$

As

$$\int_{B\mathbf{Aut}(K(A,n))} \Theta_n \longrightarrow B\mathbf{Aut}(K(A, n))$$

can be canonically identified with

$$\theta_n : K(\mathbf{Aut}(A), 1) \to B\mathbf{Aut}(K(A, n)),$$

the result now follows from Lemma A.1.3. □

A **local system** on a space X with coefficients in an abelian group A is usually defined in the connected case as a group homomorphism

$$\pi_1(X) \to \mathbf{Aut}(A),$$

or in the non-connected case, as an action of the fundamental groupoid $\Pi_1(X)$ on A, or equivalently, a functor of groupoids

$$\Pi_1(X) \to K(\mathbf{Aut}(A), 1).$$

This is the same data as a map

$$\tau : X \to K(\mathbf{Aut}(A), 1).$$

Proposition A.1.6. *Let $n > 0$ be an integer. Given a local system τ as above, the n^{th}-cohomology group of X with coefficients in τ is in natural bijection with the set of homotopy classes of lifts*

$$\begin{tikzcd}
 & & K(\mathbf{Aut}(A), 1) \arrow[d, "\theta_n"] \\
X \arrow[r, "\tau"'] \arrow[urr, dashed] & K(\mathbf{Aut}(A), 1) \arrow[r, "\theta_n"'] & B\mathbf{Aut}(K(A, n)).
\end{tikzcd}$$

Proof. Let $\tau : X \to K(\mathbf{Aut}(A), 1)$ be a local system with coefficients in A. Notice that the composite

$$X \xrightarrow{\tau} K(\mathbf{Aut}(A), 1) \xrightarrow{\chi_A} Ab \xrightarrow{K(\,\cdot\,, n)} \mathcal{S} \tag{5}$$

has a factorization of the form

$$X \to \Pi_1(X) \xrightarrow{\tau'} Ab \xrightarrow{K(\,\cdot\,, n)} \mathcal{S}.$$

Denote by $L_X(\tau, n)$ the colimit of

$$K(\tau', n) : \Pi_1(X) \to \mathcal{S}.$$

There is a canonical map $L_X(\tau, n) \to \Pi_1(X)$ and by [5, cor 4.6], there is a natural bijection between the set of homotopy classes of lifts

$$\begin{tikzcd}
 & L_X(\tau, n) \arrow[d] \\
X \arrow[r] \arrow[ur, dashed] & \Pi_1(X)
\end{tikzcd}$$

and the n^{th} cohomology group of X with coefficients in τ. Note that the space of such lifts is canonically homotopy equivalent to the space of sections

$$\begin{tikzcd}
X \times_{\Pi_1(X)} L_X(\tau, n) \arrow[r] \arrow[d] & L_X(\tau, n) \arrow[d] \\
X \arrow[r] \arrow[u, dashed, bend left] & \Pi_1(X).
\end{tikzcd}$$

By Lemma A.1.3, it follows that

$$X \times_{\Pi_1(X)} L_X(\tau, n) \to X$$

can canonically be identified with the colimit of the composite (6). Recall that the composite (6) factors as

$$X \xrightarrow{\tau} K(\mathbf{Aut}(A), 1) \xrightarrow{\theta_n} B\mathbf{Aut}(K(A, n)) \xrightarrow{\Theta_n} \mathcal{S}.$$

Again by Lemma A.1.3, it follows that the colimit of (6) agrees with the left vertical arrow in the following pullback diagram

$$\begin{tikzcd} X \times_{B\mathbf{Aut}(K(A,n))} K(\mathbf{Aut}(A), 1) \arrow[r] \arrow[d] & K(\mathbf{Aut}(A), 1) \arrow[d, "\theta_n"] \\ X \arrow[r, "\theta_n \circ \tau"'] & B\mathbf{Aut}(K(A, n)). \end{tikzcd}$$

Finally, since this is a pullback diagram, the space of sections of this map is homotopy equivalent to the space of lifts as in the statement of the proposition. □

Denote by $\underline{A}$ the underlying set of the abelian group A, and denote by

$$\mathbf{Aut}(A) \ltimes \underline{A},$$

the action groupoid associated to the action of $\mathbf{Aut}(A)$ on $\underline{A}$, i.e. the groupoid whose set of objects is $\underline{A}$ and whose set of arrows is $\mathbf{Aut}(A) \times \underline{A}$, where a pair (φ, a) is an arrow from a to $\varphi(a)$. Denote by

$$\theta_0 : \mathbf{Aut}(A) \ltimes \underline{A} \to K(\mathbf{Aut}(A), 1)$$

the functor sending $\underline{A}$ to the unique object $\star$ and sending a pair (φ, a) to φ.

Proposition A.1.7. *Given a space X and a local system $\tau : X \to K(\mathbf{Aut}(A), 1)$, the 0^{th}-cohomology group of X with coefficients in τ is in natural bijection with the set of homotopy classes of lifts*

$$\begin{tikzcd} & \mathbf{Aut}(A) \ltimes \underline{A} \arrow[d, "\theta_0"] \\ X \arrow[r, "\tau"'] \arrow[ur, dashed] & K(\mathbf{Aut}(A), 1). \end{tikzcd}$$

Proof. Consider the composite

$$K(\mathbf{Aut}(A), 1) \xrightarrow{\chi_A} Ab \xrightarrow{K(\,\cdot\,,0)} \mathcal{S},$$

where the functor $K(\,\cdot\,, 0)$ sends an abelian group to its underlying set. It's easy to check by direct calculation that this functor classifies the left fibration θ_0. The local system τ has a factorization

$$X \to \Pi_1(X) \xrightarrow{\tau'} K(\mathbf{Aut}(A), 1),$$

and by Lemma A.1.3, the composite

$$\Pi_1(X) \xrightarrow{\tau'} K(\mathbf{Aut}(A), 1) \xrightarrow{\chi_A} Ab \xrightarrow{K(\,\cdot\,,0)} \mathcal{S} \tag{6}$$

classifies the left fibration

$$\Pi_1(X) \times_{K(\mathbf{Aut}(A),1)} \mathbf{Aut}(A) \ltimes \underline{A} \to \Pi_1(X).$$

By Proposition A.1.1, it follows that the colimit of the composite (6) is the fibered product $\Pi_1(X) \times_{K(\mathbf{Aut}(A),1)} \mathbf{Aut}(A) \ltimes \underline{A}$, and hence one has an identification

$$\Pi_1(X) \times_{K(\mathbf{Aut}(A),1)} \mathbf{Aut}(A) \ltimes \underline{A} \simeq L_X(\tau, 0),$$

using the notation from [5, Definition 3.1]. By [5, cor 4.6] there is a bijection between homotopy classes of lifts

$$\begin{tikzcd} & L_X(\tau,0) \arrow[d] \\ X \arrow[r] \arrow[ur, dashed] & \Pi_1(X) \end{tikzcd}$$

and degree 0 cohomology classes of X with coefficients in τ. However, the space of such lifts is naturally homotopy equivalent to the space of lifts

$$\begin{tikzcd} & \mathbf{Aut}(A) \ltimes \underline{A} \arrow[d, "\theta_0"] \\ X \arrow[r, "\tau"'] \arrow[ur, dashed] & K(\mathbf{Aut}(A),1). \end{tikzcd}$$

□

Lemma A.1.8. *Let $n \geq 2$ be an integer. The following is a pullback diagram*

$$\begin{tikzcd} B\mathbf{Aut}(K(A,n-1)) \arrow[r] \arrow[d] & K(\mathbf{Aut}(A),1) \arrow[d, "\theta_n"] \\ K(\mathbf{Aut}(A),1) \arrow[r, "\theta_n"] & B\mathbf{Aut}(K(A,n)), \end{tikzcd}$$

where $B\mathbf{Aut}(K(A,n)) \to K(\mathbf{Aut}(A),1) = \Pi_1(B\mathbf{Aut}(K(A,n)))$ is the canonical map from $B\mathbf{Aut}(K(A,n))$ to its 1-truncation. Also, the following diagram is a pullback square

$$\begin{tikzcd} B(\mathbf{Aut}(A) \ltimes A) \arrow[r] \arrow[d] & K(\mathbf{Aut}(A),1) \arrow[d, "\theta_1"] \\ K(\mathbf{Aut}(A),1) \arrow[r, "\theta_1"] & B\mathbf{Aut}(K(A,1)), \end{tikzcd}$$

where $\mathbf{Aut}(A) \ltimes A$ is the semi-direct product of groups.

Proof. Suppose that $n > 1$, then by (4), it follows that the canonical map from

$$B\mathbf{Aut}(K(A,n)) \to \Pi_1(B\mathbf{Aut}(K(A,n)))$$

is the map

$$B(pr_1) : B\mathbf{Aut}(K(A,n)) \simeq B(\mathbf{Aut}(A) \times K(A,n)) \to B(\mathbf{Aut}(A)).$$

The fiber of this map is $K(A, n+1)$, i.e. we have a pullback diagram

$$\begin{tikzcd} K(A,n+1) \arrow[r] \arrow[d] & B\mathbf{Aut}(K(A,n)) \arrow[d] \\ * \arrow[r] & B\mathbf{Aut}(A). \end{tikzcd}$$

This means that

$$K(A,n+1) \to B\mathbf{Aut}(K(A,n))$$

is an $\mathbf{Aut}(A)$-principal bundle, and therefore

$$B\mathbf{Aut}(K(A,n)) \simeq K(A,n+1)//\mathbf{Aut}(A).$$

Carefully tracing all the equivalences, this implies that the map

$$\theta_n : K(\mathbf{Aut}(A),1) \to B\mathbf{Aut}(K(A,n))$$

can be identified with the map

$$*//\mathbf{Aut}(A) \to K(A,n+1)//\mathbf{Aut}(A)$$

induced by the unique point of $K(A,n+1)$. Note that this basepoint inclusion is $\mathbf{Aut}(A)$-equivariant. It follows that

$$K(\mathbf{Aut}(A),1) \simeq \left(* \times_{K(A,n+1)} *\right)//\mathbf{Aut}(A),$$

which is furthermore equivalent to

$$K(A,n)//\mathbf{Aut}(A).$$

But we just argued that this is equivalent to

$$B\mathbf{Aut}(K(A,n-1))$$

Now suppose that $n=1$. Recall that the composite

$$K(\mathbf{Aut}(A),1) \xrightarrow{\theta_1} B\mathbf{Aut}(K(A,1)) \to \mathcal{S}$$

is canonically equivalent to the composite

$$K(\mathbf{Aut}(A),1) \xrightarrow{\chi_A} Ab \xrightarrow{K(\,\cdot\,,1)} \mathcal{S}.$$

Since additionally,

$$\theta_1 : K(\mathbf{Aut}(A),1) \to B\mathbf{Aut}(K(A,1))$$

is the left fibration associated with the canonical functor

$$B\mathbf{Aut}(K(A,1)) \to \mathcal{S},$$

it follows from Lemma A.1.3 that we can identify the map

$$K(\mathbf{Aut}(A),1) \times_{B\mathbf{Aut}(K(A,1))} K(\mathbf{Aut}(A),1) \to K(\mathbf{Aut}(A),1)$$

with the left fibration

$$\int_{K(\mathbf{Aut}(A),1)} K(\chi_A,1) \longrightarrow K(\mathbf{Aut}(A),1).$$

But

$$\int_{K(\mathbf{Aut}(A),1)} K(\chi_A,1)$$

can be identified with the Grothendieck construction of the functor $K(\chi_A,1)$ when the target is restricted to the 2-category of groupoids. Finally, a simple calculation identifies this Grothendieck construction with $B(\mathbf{Aut}(A) \ltimes A)$ (the semi-direct product structures arises from the natural composition formula for arrows in a Grothendieck construction). $\square$

Suppose that $n > 0$. Given a map $f : X \to B\mathbf{Aut}(K(A,n))$, we get an induced local system with coefficients in A by considering the composite

$$X \xrightarrow{f} B\mathbf{Aut}(K(A,n)) \to \Pi_1(B\mathbf{Aut}(K(A,n))) = K(\mathbf{Aut}(A),1).$$

Denoting by $\tau(f)$ the induced local system, f itself can be identified with a section

$$\begin{tikzcd} & B\mathbf{Aut}(K(A,n)) \arrow[d] \\ X \arrow[ur, dashed, "f"] \arrow[r, "\tau(f)"'] & K(\mathbf{Aut}(A),1). \end{tikzcd}$$

However, by Lemma A.1.8, f can be identified with a section

$$\begin{tikzcd} & K(\mathbf{Aut}(A),1) \arrow[d, "\theta_{n+1}"] \\ X \arrow[ur, dashed] \arrow[r, "\theta_{n+1}\circ\tau(f)"'] & B\mathbf{Aut}(K(A,n+1)). \end{tikzcd}$$

In light of this, the following two corollaries follows immediately from Proposition A.1.6:

Corollary A.1.9. *Let X be a space and let $n > 0$ be an integer. Then there is a natural bijection between the set of homotopy classes of maps*

$$[X, B\mathbf{Aut}(K(A,n))]$$

and the set of pairs (τ,α), with

$$\tau \in [X, K(\mathbf{Aut}(A),1)]$$

a local system on X and

$$\alpha \in H^{n+1}(X,\tau),$$

an $(n+1)^{st}$-cohomology class of X with values in τ. Moreover, there is a natural bijection between the set of homotopy classes of maps

$$[X, B(\mathbf{Aut}(A) \ltimes A)]$$

and the set of pairs (τ,α), with

$$\tau \in [X, K(\mathbf{Aut}(A),1)]$$

a local system on X and

$$\alpha \in H^1(X,\tau),$$

a degree 1 cohomology class of X with values in τ.

A.2 The ∞-topos case In this subsection we define local systems on an arbitrary ∞-topos with coefficients in an abelian group and their associated cohomology groups. This is closely connected with the definition of twisted cohomology in an ∞-topos; see e.g. [34, Section 4].

Definition A.2.1. Let $\mathcal{E}$ be an ∞-topos. Let A be an abelian group. Consider the groupoid $K(\mathbf{Aut}(A), 1)$, and its associated stack in $\mathcal{E}$, $\Delta(K(\mathbf{Aut}(A), 1))$. A **local system with coefficients in** A on $\mathcal{E}$ is a map

$$\tau : 1 \to \Delta(K(\mathbf{Aut}(A), 1))$$

in $\mathcal{E}$, where 1 is the terminal object.

Given a local system as above, there is an associated sheaf of abelian groups $\mathcal{F}_\tau$ classified by τ. (By a sheaf of abelian groups, we mean an abelian group object in $\operatorname{Disc}(\mathcal{E})$.) The main idea is that it is constructed by pulling back a canonical sheaf of abelian groups on $K(\mathbf{Aut}(A), 1)$. We now explain in detail.

An abelian sheaf on the space $K(\mathbf{Aut}(A), 1)$ is by definition a sheaf of abelian groups on the ∞-topos $\mathrm{Psh}_\infty(K(\mathbf{Aut}(A), 1)) \simeq \mathcal{S}/(\mathbf{Aut}(A), 1)$. Since $K(\mathbf{Aut}(A), 1)$ is a groupoid, this is the same as specifying a functor

$$K(\mathbf{Aut}(A), 1) \to Ab,$$

to the category of abelian groups. We have already discussed such a functor χ_A, namely the canonical functor sending $\star$ to A and each automorphism of A to itself. Let us denote this abelian sheaf by $\mathcal{F}_A$.

We will now show that τ corresponds canonically to a geometric morphism

$$\overline{\tau} : \mathcal{E} \to \mathcal{S}/(\mathbf{Aut}(A), 1),$$

and then we will define $\mathcal{F}_\tau$ as the pullback sheaf $\overline{\tau}^*\mathcal{F}_A$.

Indeed, by [31, Remark 6.3.5.10], for any ∞-topos $\mathcal{E}$, there is an equivalence of ∞-categories

$$\mathcal{E} \to \mathfrak{Top}^{\acute{e}t}_\infty/\mathcal{E}$$

between $\mathcal{E}$ and the ∞-category of étale geometric morphisms over $\mathcal{E}$, which sends an object $E \in \mathcal{E}$ to the canonical étale morphism $\mathcal{E}/E \to \mathcal{E}$. Hence τ corresponds to a section of the étale geometric morphism

$$\mathcal{E}/\Delta(K(\mathbf{Aut}(A), 1)) \to \mathcal{E}$$

corresponding to the object $\Delta(K(\mathbf{Aut}(A), 1))$ of $\mathcal{E}$. By [31, Proposition 6.3.5.8], there is a pullback diagram in the ∞-category of ∞-topoi

$$\begin{tikzcd}
\mathcal{E}/\Delta(K(\mathbf{Aut}(A), 1)) \arrow[r] \arrow[d] & \mathcal{S}/K(\mathbf{Aut}(A), 1) \arrow[d] \\
\mathcal{E} \arrow[r] & \mathcal{S},
\end{tikzcd}$$

so the aforementioned section can be identified with a lift

$$\begin{tikzcd}
 & \mathcal{S}/K(\mathbf{Aut}(A), 1) \arrow[d] \\
\mathcal{E} \arrow[r] \arrow[ur, dashed, "\overline{\tau}"] & \mathcal{S},
\end{tikzcd}$$

and since $\mathcal{S}$ is the terminal ∞-topos, we conclude that the data of the local system τ and the geometric morphism

$$\overline{\tau} : \mathcal{E} \to \mathcal{S}/K\left(\mathbf{Aut}\left(A\right), 1\right)$$

are equivalent, or more precisely:

Proposition A.2.2. *The construction just explained produces an equivalence of ∞-groupoids*

$$\mathrm{Hom}_{\mathcal{E}}\left(1, \Delta\left(K\left(\mathbf{Aut}\left(A\right), 1\right)\right)\right) \simeq \mathrm{Hom}_{\mathfrak{Top}_\infty}\left(\mathcal{E}, \mathcal{S}/K\left(\mathbf{Aut}\left(A\right), 1\right)\right)$$

between local systems with coefficients in A on $\mathcal{E}$ and geometric morphisms from $\mathcal{E}$ into $\mathcal{S}/K\left(\mathbf{Aut}\left(A\right), 1\right)$.

The following cor follows immediately from [31, Remark 7.1.6.15]:

Corollary A.2.3. *There is an equivalence of ∞-groupoids*

$$\mathrm{Hom}_{\mathcal{E}}\left(1, \Delta\left(K\left(\mathbf{Aut}\left(A\right), 1\right)\right)\right) \simeq \mathrm{Hom}_{Pro(\mathcal{S})}\left(Shape\left(\mathcal{E}\right), K\left(\mathbf{Aut}\left(A\right), 1\right)\right).$$

Example A.2.4. If X is a scheme, then a local system on its small étale ∞-topos $\mathrm{Sh}_\infty\left(X_{\acute{e}t}\right)$ is the same a morphism

$$\tau : \Pi_\infty^{\acute{e}t}\left(X\right) \to K\left(\mathbf{Aut}\left(A\right), 1\right)$$

from its étale fundamental ∞-groupoid to $K\left(\mathbf{Aut}\left(A\right), 1\right)$.

Definition A.2.5. Let $\tau : 1 \to K\left(\mathbf{Aut}\left(A\right), 1\right)$ be a local system with coefficients in A on $\mathcal{E}$. Then the abelian sheaf $\mathcal{F}_\tau := \overline{\tau}^*\mathcal{F}_A$ is the abelian sheaf **classified by the local system τ.**

Remark A.2.6. By the proof of Proposition A.1.7, the object in $\mathcal{S}/K\left(\mathbf{Aut}\left(A\right), 1\right)$ corresponding to the underlying sheaf of sets of $\mathcal{F}_A$ can be identified with the functor of groupoids

$$\theta_0 : \mathbf{Aut}\left(A\right) \ltimes \underline{A} \to K\left(\mathbf{Aut}\left(A\right), 1\right).$$

Moreover, by construction, there is a factorization of $\overline{\tau}$ of the form

$$\mathcal{E} \xrightarrow{\mathcal{E}/\tau} \mathcal{E}/\Delta\left(K\left(\mathbf{Aut}\left(A\right), 1\right)\right) \to \mathcal{S}/K\left(\mathbf{Aut}\left(A\right), 1\right).$$

Unwinding the definitions, one sees that the underlying sheaf of sets of $\mathcal{F}_\tau$, $\underline{\mathcal{F}_\tau}$ fits in a pullback diagram

$$\begin{tikzcd} \underline{\mathcal{F}_\tau} \arrow[r] \arrow[d] & \Delta\left(\mathbf{Aut}\left(A\right) \ltimes \underline{A}\right) \arrow[d, "\Delta(\theta_0)"] \\ 1 \arrow[r, "\tau"] & \Delta\left(K\left(\mathbf{Aut}\left(A\right), 1\right)\right). \end{tikzcd}$$

Definition A.2.7. Let A be an abelian group and $\mathcal{E}$ an ∞-topos. A **locally constant sheaf with values in** A on $\mathcal{E}$ is an abelian sheaf $\mathcal{F}$ on $\mathcal{E}$ such that there are objects $\left(U_i\right)_{i\in I}$ in $\mathcal{E}$ such that the canonical map

$$\coprod_{i\in I} U_i \to 1$$

is an epimorphism, and such that the pullback of $\mathcal{F}$ to each slice topos $\mathcal{E}/U_i$ is isomorphic to the constant abelian sheaf with value A.

Remark A.2.8. Let $\mathscr{C}$ be a small category equipped with a Grothendieck topology. Then any object E in $\mathrm{Sh}_\infty(\mathscr{C})$ admits an epimorphism from a coproduct of representables, hence a locally constant abelian sheaf in $\mathrm{Sh}_\infty(\mathscr{C})$ can be identified with a classical locally constant abelian sheaf on $\mathscr{C}$.

Proposition A.2.9. *Let $\tau : 1 \to K(\mathbf{Aut}(A), 1)$ be a local system in an ∞-topos $\mathcal{E}$. Then the abelian sheaf $\mathcal{F}_\tau$ classified by τ is a locally constant sheaf with values in A.*

Proof. Notice that $\Delta(K(\mathbf{Aut}(A), 1))$ is the classifying stack for $\mathbf{Aut}(A)$-torsors. In particular, the universal $\mathbf{Aut}(A)$-torsor

$$1 = \Delta(*) \xrightarrow{\Delta(\star)} \Delta(K(\mathbf{Aut}(A), 1))$$

is an epimorphism. Consider the following pullback diagram

$$\begin{tikzcd} P_\tau \arrow[r] \arrow[d] & 1 \arrow[d, "\Delta(\star)"] \\ 1 \arrow[r, "\tau"] & \Delta(K(\mathbf{Aut}(A), 1)). \end{tikzcd}$$

The map $P_\tau \to 1$ is an epimorphism and by Remark A.2.6, we can identify the underlying sheaf of sets of the pullback of $\mathcal{F}_\tau$ to $\mathcal{E}/P_\tau$ as the map $Q \to P_\tau$ in the following pullback diagram

$$\begin{tikzcd} Q \arrow[rr] \arrow[d] & & \Delta(\mathbf{Aut}(A) \ltimes \underline{A}) \arrow[d, "\theta_0"] \\ P_\tau \arrow[r] & 1 \arrow[r, "\tau"] & \Delta(K(\mathbf{Aut}(A), 1)). \end{tikzcd}$$

Note that the pullback diagram defining P_τ in particular commutes, so the above pullback diagram may also be computed as

$$\begin{tikzcd} Q \arrow[rr] \arrow[d] & & \Delta(\mathbf{Aut}(A) \ltimes \underline{A}) \arrow[d, "\theta_0"] \\ P_\tau \arrow[r] & 1 \arrow[r, "\Delta(\star)"] & \Delta(K(\mathbf{Aut}(A), 1)). \end{tikzcd}$$

By the proof of Proposition A.1.7, and the fact that Δ preserves finite limits, the following diagram is also a pullback

$$\begin{tikzcd} \Delta(\underline{A}) \arrow[r] \arrow[d] & \Delta(\mathbf{Aut}(A) \ltimes \underline{A}) \arrow[d, "\theta_0"] \\ \Delta(*) \arrow[r, "\Delta(\star)"] & \Delta(K(\mathbf{Aut}(A), 1)). \end{tikzcd}$$

It follows that $Q \simeq P_\tau \times \Delta(\underline{A}) \to P_\tau$, i.e. the pullback of $\underline{\mathcal{F}_\tau}$ to $\mathcal{E}/P_\tau$ is equivalent to the constant sheaf $\Delta^{P_\tau}(\underline{A})$. Hence the same is true for the abelian sheaf, i.e. the pullback of $\mathcal{F}_\tau$ to $\mathcal{E}/P_\tau$ is constant with value A, and hence $\mathcal{F}_\tau$ is locally constant. □

Remark A.2.10. For a general ∞-topos $\mathcal{E}$, it is not true that every locally constant sheaf with values in A is classified by a local system

$$\tau : 1 \to K(\mathbf{Aut}(A), 1),$$

however it is true if $\mathcal{E}$ is locally connected. The reason is as follows. Let $\mathcal{A}$ be any abelian sheaf. We say for an object E of $\mathcal{E}$, that an abelian sheaf $\mathcal{F}$ on $\mathcal{E}/E$ is *locally isomorphic to* $\mathcal{A}$ (or a twisted form of $\mathcal{A}$) if there is an epimorphism

$$\coprod_i E_i \to E,$$

such that the restriction of $\mathcal{F}$ to each E_i is isomorphic to the restriction of $\mathcal{A}$ to each E_i. It is a classical fact that the groupoid of abelian sheaves on E locally isomorphic to $\mathcal{A}$ is equivalent to the groupoid of morphisms $\mathrm{Hom}\left(E, B\left(\underline{\mathbf{Aut}}\left(\mathcal{A}\right)\right)\right)$, where $\underline{\mathbf{Aut}}\left(\mathcal{A}\right)$ is the automorphism sheaf of $\mathcal{A}$, c.f. [33, Chapter III, Section 4]. Now consider the constant sheaf $\Delta\left(A\right)$ for A an abelian group. To show that locally constant sheaves with values in A are classified by morphisms into

$$\Delta\left(K\left(\mathbf{Aut}\left(A\right),1\right)\right) \simeq \Delta\left(B\mathbf{Aut}\left(A\right)\right),$$

it suffices to show that

$$\Delta\left(B\mathbf{Aut}\left(A\right)\right) \simeq B\underline{\mathbf{Aut}}\left(\Delta\left(A\right)\right).$$

Notice that

$$B\mathbf{Aut}\left(A\right) \simeq \underset{n\in\Delta^{op}}{\underrightarrow{\mathrm{colim}}}\, \mathbf{Aut}\left(A\right)^n,$$

and since Δ preserves colimits and finite limits we have

$$\Delta\left(B\mathbf{Aut}\left(A\right)\right) \simeq \underset{n\in\Delta^{op}}{\underrightarrow{\mathrm{colim}}}\, \Delta\left(\mathbf{Aut}\left(A\right)\right)^n \simeq B\left(\Delta\left(\mathbf{Aut}\left(A\right)\right)\right).$$

So it suffices to show that

$$\Delta\left(\mathbf{Aut}\left(A\right)\right) \cong \underline{\mathbf{Aut}}\left(\Delta\left(A\right)\right),$$

when $\mathcal{E}$ is locally connected. This follows readily from the following observation: Let S be any set, and denote by Δ_{Disc} the inverse image functor of the essentially unique geometric morphism of 1-topoi

$$\mathrm{Disc}\left(\mathcal{E}\right) \to Set.$$

Then since Δ_{Disc} has a left adjoint Π_0, it preserves limits and we have

$$\begin{aligned} \Delta\left(\mathrm{Hom}\left(S,S\right)\right) &\cong \Delta\left(\prod_S S\right) \\ &\cong \prod_S \Delta\left(S\right) \\ &\cong \underline{\mathbf{End}}\left(\Delta\left(S\right)\right). \end{aligned}$$

Definition A.2.11. Let $\mathcal{E}$ be an ∞-topos, A an abelian group, and

$$\tau : 1 \to \Delta\left(K\left(\mathbf{Aut}\left(A\right),1\right)\right)$$

a local system on $\mathcal{E}$. The n^{th} **cohomology group of $\mathcal{E}$ with values in τ** is

$$\pi_0\mathrm{Hom}_{\mathcal{E}}\left(1, K\left(\mathcal{F}_\tau, n\right)\right),$$

where $K\left(\mathcal{F}_\tau, n\right)$ is the n^{th} Eilenberg-MacLane object of the abelian sheaf $\mathcal{F}_\tau$ classified by τ.

Remark A.2.12. Let $\mathscr{C}$ be a small category equipped with a Grothendieck topology. Let τ be a local system on $\mathrm{Sh}_\infty(\mathscr{C})$ with values in an abelian group A. By Remark A.2.8, we can identify the abelian sheaf classified by τ with a classical locally constant sheaf of abelian groups $\mathcal{F}_\tau$ on $\mathscr{C}$. Furthermore, by [31, Remark 7.2.2.17], we can identify the n^{th} cohomology group of $\mathrm{Sh}_\infty(\mathscr{C})$ with values in τ as just defined with the n^{th} cohomology group of $\mathcal{F}_\tau$ as computed using classical sheaf cohomology.

Theorem A.2.13. *Let $\mathcal{E}$ be an ∞-topos, A an abelian group and*

$$\tau : 1 \to \Delta(K(\mathbf{Aut}), 1)$$

a local system on $\mathcal{E}$ with values in A. The 0^{th} cohomology group of $\mathcal{E}$ with coefficients in τ is isomorphic to

$$\pi_0\left(\mathrm{Hom}_{\mathcal{E}/\Delta(K(\mathbf{Aut}(A),1))}(\tau, \Delta(\theta_0))\right),$$

i.e. π_0 of the space of lifts

$$\begin{tikzcd} & \Delta(\mathbf{Aut}(A) \ltimes \underline{A}) \arrow[d, "\Delta(\theta_0)"] \\ 1 \arrow[ur, dashed] \arrow[r, "\tau"'] & \Delta(K(\mathbf{Aut}(A), 1)) \end{tikzcd}$$

equipped with the group structure induced from that of A. Moreover, for $n > 0$, the n^{th} cohomology group of $\mathcal{E}$ with coefficients in τ can be identified with

$$\pi_0\left(\mathrm{Hom}_{\mathcal{E}/\Delta(B\mathbf{Aut}(K(A,n)))}(\Delta(\theta_n) \circ \tau, \Delta(\theta_n))\right),$$

i.e. π_0 of the space of lifts

$$\begin{tikzcd} & & \Delta(K(\mathbf{Aut}(A), 1)) \arrow[d, "\Delta(\theta_n)"] \\ 1 \arrow[urr, dashed] \arrow[r, "\tau"'] & \Delta(K(\mathbf{Aut}(A), 1)) \arrow[r, "\Delta(\theta_n)"'] & \Delta(B\mathbf{Aut}(K(A,n))). \end{tikzcd}$$

Proof. The statement about the 0^{th} cohomology group follows immediately from Remark A.2.6.

Now suppose that $n > 0$. Recall that $\mathcal{F}_A$ is the abelian sheaf on $\mathcal{S}/K(\mathbf{Aut}(A), 1)$ corresponding to the functor

$$\chi_A : K(\mathbf{Aut}(A), 1) \to Ab,$$

and $\mathcal{F}_\tau$ is by definition $\overline{\tau}^*\mathcal{F}_A$, where $\overline{\tau} : \mathcal{E} \to \mathcal{S}/K(\mathbf{Aut}(A), 1)$ is the geometric morphism induced by τ. Denote by $K(\mathcal{F}_A, n)$ the n^{th} Eilenberg-MacLane object of $\mathcal{F}_A$ in $\mathcal{S}/K(\mathbf{Aut}(A), 1)$. By [31, Remark 6.5.1.4], it follows that

$$\overline{\tau}^* K(\mathcal{F}_A, n) \simeq K(\mathcal{F}_\tau, n).$$

Under the equivalence

$$\mathcal{S}/K(\mathbf{Aut}(A), 1) \simeq \mathrm{Fun}(K(\mathbf{Aut}(A), 1), \mathcal{S}),$$

$K(\mathcal{F}_A, n)$ corresponds to the composite

$$K(\mathbf{Aut}(A), 1) \to Ab \xrightarrow{K(\,\cdot\,,n)} \mathcal{S},$$

which means that $K(\mathcal{F}_A, n)$ in $\mathcal{S}/K(\mathbf{Aut}(A), 1)$ is the left fibration classified by the above composite functor. Recall this functor also factors as the composite

$$K(\mathbf{Aut}(A), 1) \xrightarrow{\theta_n} B\mathbf{Aut}(K(A, n)) \xrightarrow{\Theta_n} \mathcal{S},$$

where $\Theta_n : B\mathbf{Aut}(K(A, n)) \to \mathcal{S}$ is the natural functor which in fact classifies the universal $K(A, n)$-fibration

$$\theta_n : K(\mathbf{Aut}(A), 1) \to B\mathbf{Aut}(K(A, n)).$$

Denote by

$$\mathcal{S}/\theta_n : \mathcal{S}/K(\mathbf{Aut}(A), 1) \to \mathcal{S}/B\mathbf{Aut}(K(A, n))$$

the geometric morphism induced by θ_n, then regarding θ_n as an object of $\mathcal{S}/B\mathbf{Aut}(K(A, n))$, we have a canonical identification

$$(\mathcal{S}/\theta_n)^* (\theta_n) \simeq K(\mathcal{F}_A, n).$$

And hence $K(\mathcal{F}_\tau, n)$ can be identified with the pullback of θ_n along the geometric morphism

$$\mathcal{E} \xrightarrow{\overline{\tau}} \mathcal{S}/K(\mathbf{Aut}(A), 1) \xrightarrow{\mathcal{S}/\theta_n} \mathcal{S}/B\mathbf{Aut}(K(A, n)).$$

Unwinding the definitions, this means that we have a pullback diagram in $\mathcal{E}$

$$\begin{tikzcd}
K(\mathcal{F}_\tau, n) \arrow[r] \arrow[d] & \Delta(K(\mathbf{Aut}(A), 1)) \arrow[d, "\Delta(\theta_n)"] \\
1 \arrow[r, "\Delta(\theta_n)\circ\tau"] & \Delta(B\mathbf{Aut}(K(A, n))).
\end{tikzcd}$$

The result now follows. $\square$

The following cor is proved in the same was as cor A.1.9:

Corollary A.2.14. *Let $\mathcal{E}$ be an ∞-topos, A an abelian group, and $n > 0$ be an integer. Then there is a natural bijection between the set of global sections*

$$\pi_0\Gamma(\Delta(B\mathbf{Aut}(K(A, n))))$$

and the set of pairs (τ, α), with

$$\tau \in \pi_0\Gamma(K(\mathbf{Aut}(A), 1))$$

a local system on $\mathcal{E}$ and

$$\alpha \in H^{n+1}(\mathcal{E}, \tau),$$

an $(n+1)^{st}$-cohomology class of $\mathcal{E}$ with values in τ. Moreover, there is a natural bijection between the set global sections

$$\pi_0\Gamma(\Delta(B(\mathbf{Aut}(A) \ltimes A)))$$

and the set of pairs (τ, α), *with*

$$\tau \in \pi_0 \Gamma \left(\Delta \left(K \left(\mathbf{Aut} \left(A \right), 1 \right) \right) \right)$$

a local system on $\mathcal{E}$ *and*

$$\alpha \in H^1 \left(\mathcal{E}, \tau \right),$$

a degree 1 *cohomology class of* $\mathcal{E}$ *with values in* τ.

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