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On lax transformations, adjunctions, and monads in $(\infty, 2)$-categories

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Abstract

We use the basic expected properties of the Gray tensor product of $(\infty, 2)$-categories to study (co)lax natural transformations. Using results of Riehl–Verity and Zaganidis we identify lax transformations between adjunctions and monads with commutative squares of (monadic) right adjoints. We also identify the colax transformations whose components are equivalences (generalizing the "icons" of Lack) with the 2-morphisms that arise from viewing $(\infty, 2)$-categories as simplicial ∞-categories. Using this characterization we identify the ∞-category of monads on a fixed object and colax morphisms between them with the ∞-category of associative algebras in endomorphisms.

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1. Introduction

Consider the following descriptions of *monads* on an ∞-category:

(A) A monad on $\mathcal{C}$ is an associative algebra in the monoidal ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{C})$ of endofunctors under composition.

(B) A monad is a functor of ∞-categories that is a monadic right adjoint.

(C) A monad is a functor of $(\infty, 2)$-categories $\mathfrak{mnd} \to \mathrm{CAT}_\infty$, where $\mathfrak{mnd}$ is the universal 2-category containing a monad and CAT_∞ is the $(\infty, 2)$-category of ∞-categories.

These three definitions are known to be equivalent by results of Lurie [22]*§4.7.3 and Riehl–Verity [27].[1] However, these comparisons only relate ∞-groupoids of monads. Our main goal in this paper is to enhance the comparisons to take into account *morphisms* of monads. For (A) the

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[1] Both prove versions of the monadicity theorem, which relate the first two and last two definitions, respectively.

obvious notion of morphism between monads on $\mathcal{C}$ is a homomorphism of algebras in $\mathrm{Fun}(\mathcal{C}, \mathcal{C})$, while for (B) it is a commutative triangle

$$\begin{tikzcd} \mathcal{A} \arrow[rr, "f"] \arrow[dr, "r"'] & & \mathcal{B} \arrow[dl, "r'"] \\ & \mathcal{C} & \end{tikzcd}$$

where r and r' are monadic right adjoints.[2] More generally, we can allow the ∞-category $\mathcal{C}$ to vary and consider commutative squares whose vertical morphisms are monadic right adjoints.

For ordinary 2-categories, Street [30] showed that such squares of monadic right adjoints correspond to what he called *monad functors*, which are the same thing as *lax natural transformations* between functors from $\mathfrak{mnd}$. To compare (B) and (C) we therefore start by studying lax transformations in the setting of $(\infty, 2)$-categories. These can be defined using the *(lax) Gray tensor product*. This has not yet been fully developed for $(\infty, 2)$-categories[3], and we do not do so here. Instead, we assume it has certain basic expected properties (see Assumption 3.5) and proceed from there to define $(\infty, 2)$-categories $\mathrm{FUN}(\mathcal{Y}, \mathcal{X})_{(\mathrm{co})\mathrm{lax}}$ of functors and (co)lax transformations between $(\infty, 2)$-categories $\mathcal{Y}$ and $\mathcal{X}$ in §3 after briefly reviewing some descriptions of $(\infty, 2)$-categories in §2. Specializing $\mathcal{Y}$ to the universal monad 2-category $\mathfrak{mnd}$ and the universal adjunction 2-category $\mathfrak{adj}$ we obtain $(\infty, 2)$-categories $\mathrm{MND}(\mathcal{X})_{(\mathrm{co})\mathrm{lax}}$ and $\mathrm{ADJ}(\mathcal{X})_{(\mathrm{co})\mathrm{lax}}$ of, respectively, monads and adjunctions, with (co)lax transformations as morphisms. Our comparison of (B) and (C) is then the combination of the following two results:

Theorem 1.1. *For any $(\infty, 2)$-category $\mathcal{X}$, restricting an adjunction to its right adjoint defines an equivalence of $(\infty, 2)$-categories*

$$\mathrm{ADJ}(\mathcal{X})_{\mathrm{lax}} \xrightarrow{\sim} \mathrm{FUN}(C_1, \mathcal{X})_{\mathrm{radj}},$$

where the latter is the $(\infty, 2)$-category of morphisms in $\mathcal{X}$ that are right adjoints, with commutative squares as morphisms.

Theorem 1.2. *The functor $\mathrm{ADJ}(\mathrm{CAT}_\infty)_{\mathrm{lax}} \to \mathrm{MND}(\mathrm{CAT}_\infty)_{\mathrm{lax}}$ taking an adjunction to its induced monad, has a fully faithful right adjoint with image the monadic adjunctions.*

We prove Theorem 1.1 in §4, using one of the main results of [27], which gives this equivalence on the level of underlying ∞-groupoids. Theorem 1.2 is then proved in §5 as a corollary of work of Zaganidis [32], whose thesis studied lax morphisms of adjunctions and monads in the framework of [27]. Combining these two theorems we get an equivalence of $(\infty, 2)$-categories

$$\mathrm{MND}(\mathrm{CAT}_\infty)_{\mathrm{lax}} \simeq \mathrm{FUN}(C_1, \mathrm{CAT}_\infty)_{\mathrm{mndradj}},$$

where the right-hand side is the $(\infty, 2)$-category of morphisms in CAT_∞ that are monadic right adjoints. (More generally, we can replace CAT_∞ by any $(\infty, 2)$-category that can be modelled by an ∞-*cosmos* in the sense of Riehl and Verity.)

We then turn to the relation between descriptions (A) and (C). To see that these give the same objects it is enough to observe that the one-object 2-category $\mathfrak{mnd}$ is the monoidal envelope of the non-symmetric associative operad, but to relate the morphisms we need to understand the

[2] On a fixed ∞-category $\mathcal{C}$, monads in sense (A) and (B) have already been compared by Heine [15].

[3] Though several constructions have recently appeared, and this is a topic of active research; see Remark 3.6.

connection between (co)lax transformations and 2-morphisms of monoidal ∞-categories. More generally, if we view $(\infty, 2)$-categories (in the guise of complete 2-fold Segal spaces) as cocartesian fibrations over $\mathbf{\Delta}^{\mathrm{op}}$, then for $(\infty, 2)$-categories $\mathfrak{X}$ and $\mathfrak{Y}$ we can define an ∞-category $\mathrm{Nat}(\mathfrak{X}, \mathfrak{Y})$ consisting of functors over $\mathbf{\Delta}^{\mathrm{op}}$ that preserve cocartesian morphisms and natural transformations between them. In §7 we prove the following characterization of these ∞-categories:

Theorem 1.3. *There is a functor*

$$\mathrm{Nat}(\mathfrak{X}, \mathfrak{Y}) \to \mathrm{Fun}(\mathfrak{X}, \mathfrak{Y})_{\mathrm{colax}}$$

that identifies the domain with the wide subcategory of the ∞-category $\mathrm{Fun}(\mathfrak{X}, \mathfrak{Y})_{\mathrm{colax}}$ underlying $\mathrm{FUN}(\mathfrak{X}, \mathfrak{Y})_{\mathrm{colax}}$ containing those colax transformations whose components are all equivalences.

The colax transformations in this subcategory are an $(\infty, 2)$-categorical analogue of the "icons" of Lack [18]. Combining this result with the non-symmetric analogues of the results on (symmetric) monoidal envelopes of ∞-operads from [22]*§2.2.4, we obtain the following comparison of descriptions (A) and (C) in §8:

Theorem 1.4. *For any object X of an $(\infty, 2)$-category $\mathfrak{X}$, there is an equivalence of ∞-categories*

$$\mathrm{Alg}(\mathrm{End}_{\mathfrak{X}}(X)) \xrightarrow{\sim} \mathrm{Mnd}(\mathfrak{X})_{\mathrm{colax}, X}$$

between the ∞-category of associative algebras in the monoidal ∞-category of endomorphisms of X under composition, and the fibre at X of the underlying ∞-category $\mathrm{Mnd}(\mathfrak{X})_{\mathrm{colax}}$ of $\mathrm{MND}(\mathfrak{X})_{\mathrm{colax}}$.

This equivalence is compatible with the forgetful functors to endomorphisms of X. Replacing lax by colax morphisms, we also obtain an equivalence between $\mathrm{Alg}(\mathrm{End}_{\mathfrak{X}}(X))^{\mathrm{op}}$ and $\mathrm{Mnd}(\mathfrak{X})_{\mathrm{lax}, X}$ and so combined with our first comparison we obtain for $\mathcal{C}$ an ∞-category equivalences

$$\mathrm{Alg}(\mathrm{Fun}(\mathcal{C}, \mathcal{C}))^{\mathrm{op}} \simeq \mathrm{Mnd}(\mathrm{CAT}_\infty)_{\mathrm{lax}, \mathcal{C}} \simeq \mathrm{Cat}_{\infty/\mathcal{C}}^{\mathrm{mndradj}}$$

where the right-hand side is the full subcategory of $\mathrm{Cat}_{\infty/\mathcal{C}}$ spanned by the monadic right adjoints.

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2. $(\infty, 2)$-Categories

In this section we fix some notation for various structures related to ∞-categories, and briefly review the different descriptions of $(\infty, 2)$-categories we make use of; we also make a few simple $(\infty, 2)$-categorical observations that will be useful later on.

Notation 2.1. We write $\mathcal{S}$ for the ∞-category of spaces (or ∞-groupoids), Cat_∞ for the ∞-category of ∞-categories, and $\mathrm{Cat}_{(\infty,2)}$ for the ∞-category of $(\infty,2)$-categories.

Notation 2.2. If $\mathcal{C}$ is an ∞-category, we write $\mathcal{C}^{\simeq}$ for the underlying ∞-groupoid of $\mathcal{C}$, which is the value at $\mathcal{C}$ of the right adjoint to the inclusion $\mathcal{S} \hookrightarrow \mathrm{Cat}_\infty$. This inclusion also has a left adjoint, which takes the ∞-category $\mathcal{C}$ to the ∞-groupoid obtained by inverting all morphisms in $\mathcal{C}$, which we denote by $\|\mathcal{C}\|$.

The ∞-category $\mathrm{Cat}_{(\infty,2)}$ admits several useful descriptions; in particular, we can view $(\infty,2)$-categories

- as complete 2-fold Segal spaces [2],
- as complete Segal $\mathbf{\Theta}_2$-spaces [26],
- as certain simplicial objects in Cat_∞ [21],
- or as ∞-categories enriched in Cat_∞ [9].

The first three of these descriptions are related through the following commutative diagram, where all functors except the lower right one are fully faithful:

$$\begin{tikzcd}
 & \mathrm{Seg}^{\mathcal{S}}_{\mathbf{\Delta}^{\mathrm{op}}}(\mathrm{Cat}_\infty) \arrow[r, hook] \arrow[d, hook] & \mathrm{Seg}_{\mathbf{\Delta}^{\mathrm{op}}}(\mathrm{Cat}_\infty) \arrow[r, hook] \arrow[d, hook] & \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Cat}_\infty) \arrow[d, hook] \\
\mathrm{Cat}_{(\infty,2)} \arrow[ur, hook] \arrow[r, hook] \arrow[dr, hook] & \mathrm{Seg}^{\text{2-fold}}_{\mathbf{\Delta}^{\mathrm{op}}\times\mathbf{\Delta}^{\mathrm{op}}}(\mathcal{S}) \arrow[r, hook] & \mathrm{Seg}_{\mathbf{\Delta}^{\mathrm{op}}\times\mathbf{\Delta}^{\mathrm{op}}}(\mathcal{S}) \arrow[r, hook] & \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}}, \mathcal{S}) \\
 & \mathrm{Seg}_{\mathbf{\Theta}_2^{\mathrm{op}}}(\mathcal{S}) \arrow[u, "\sim"] \arrow[rr, hook] & & \mathrm{Fun}(\mathbf{\Theta}_2^{\mathrm{op}}, \mathcal{S}). \arrow[u, "\tau^*"']
\end{tikzcd} \tag{1}$$

We now describe the ∞-categories and functors that appear in this diagram:

Definition 2.3. We write $\mathbf{\Delta}$ for the usual simplex category, consisting of the ordered sets $[n] := \{0, \ldots, n\}$ and order-preserving maps between them. A morphism $\phi\colon [m] \to [n]$ in $\mathbf{\Delta}$ is called *inert* if it is the inclusion of a subinterval, i.e. if $\phi(i) = \phi(0) + i$ for $i = 0, \ldots, m$, and *active* if it preserves the end points, i.e. $\phi(0) = 0$ and $\phi(m) = n$.

Definition 2.4. For an ∞-category $\mathcal{C}$ with finite limits, $\mathrm{Seg}_{\mathbf{\Delta}^{\mathrm{op}}}(\mathcal{C})$ denotes the full subcategory of $\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathcal{C})$ consisting of functors $X\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}$ satisfying the *Segal condition*, meaning that the natural map

$$X_n \to X_1 \times_{X_0} \cdots \times_{X_0} X_1,$$

induced by the inert maps $[0], [1] \to [n]$ in $\mathbf{\Delta}$, is an equivalence for all n. We also write $\mathrm{Seg}_{\mathbf{\Delta}^{\mathrm{op}}\times\mathbf{\Delta}^{\mathrm{op}}}(\mathcal{C})$ for the full subcategory $\mathrm{Seg}_{\mathbf{\Delta}^{\mathrm{op}}}(\mathrm{Seg}_{\mathbf{\Delta}^{\mathrm{op}}}(\mathcal{C}))$ of $\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathcal{C})) \simeq \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}}, \mathcal{C})$, consisting of functors $\mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}$ that satisfy the Segal condition in each variable.

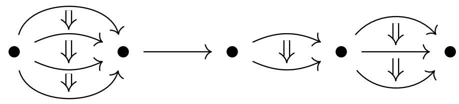
Definition 2.5. $\mathrm{Seg}^{\mathcal{S}}_{\mathbf{\Delta}^{\mathrm{op}}}(\mathrm{Cat}_\infty)$ denotes the full subcategory of $\mathrm{Seg}_{\mathbf{\Delta}^{\mathrm{op}}}(\mathrm{Cat}_\infty)$ consisting of Segal objects X such that X_0 is an ∞-groupoid. We can then define $\mathrm{Cat}_{(\infty,2)}$ to be the full subcategory of $\mathrm{Seg}^{\mathcal{S}}_{\mathbf{\Delta}^{\mathrm{op}}}(\mathrm{Cat}_\infty)$ consisting of functors X satisfying the *completeness condition*, namely that the underlying Segal space $X^{\simeq}$ is complete in the sense of [25].

Definition 2.6. $\mathrm{Seg}^{\text{2-fold}}_{\mathbf{\Delta}^{\mathrm{op}}\times\mathbf{\Delta}^{\mathrm{op}}}(\mathcal{S})$ denotes the full subcategory of $\mathrm{Seg}_{\mathbf{\Delta}^{\mathrm{op}}\times\mathbf{\Delta}^{\mathrm{op}}}(\mathcal{S})$ consisting of *2-fold Segal spaces*, meaning those objects X such that $X_{0,\bullet}\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{S}$ is constant.

Remark 2.7. The top right vertical morphism in (1) arises from the inclusion $\mathrm{Cat}_\infty \hookrightarrow \mathrm{Seg}_{\mathbf{\Delta}^{\mathrm{op}}}(\mathcal{S})$ of ∞-categories as the complete Segal objects, due to Rezk [25]. This also induces the other inclusions between the top two rows, and identifies $\mathrm{Cat}_{(\infty,2)}$ with the full subcategory of $\mathrm{Seg}^{2\text{-fold}}_{\mathbf{\Delta}^{\mathrm{op}}\times\mathbf{\Delta}^{\mathrm{op}}}(\mathcal{S})$ consisting of the complete 2-fold Segal spaces in the sense of Barwick [2].

Definition 2.8. The category $\mathbf{\Theta}_2$ has objects $[k](n_1,\ldots,n_k)$ for non-negative integers $k, n_1, \ldots, n_k$, with a morphism $[k](n_1,\ldots,n_k) \to [l](m_1,\ldots,m_l)$ given by a morphism $\phi\colon [k] \to [l]$ in $\mathbf{\Delta}$ together with a morphism $\psi_{ij}\colon [n_i] \to [m_j]$ in $\mathbf{\Delta}$ whenever $\phi(i-1) < j \leq \phi(i)$. Composition is defined in the obvious way, and we say this morphism is *inert* or *active* if ϕ and each of the maps ψ_{ij} is inert or active, respectively.

Remark 2.9. We can think of the objects of $\mathbf{\Theta}_2$ as globular pasting diagrams, such as



which corresponds to the object $[4](3,0,1,2)$. This leads to the equivalent definition of $\mathbf{\Theta}_2$ as a full subcategory of the category of strict 2-categories, by thinking of the object $[k](n_1,\ldots,n_k)$ as the strict 2-category with objects $0,\ldots,n$ whose category of morphisms $i \to j$ is $\prod_{i<k\leq j}[n_i]$ if $i \leq j$ and empty otherwise, and with composition given by taking products.

Notation 2.10. We shall use the following special notation for the most basic objects of $\mathbf{\Theta}_2$:

$$\begin{aligned} C_0 &:= [0](), \\ C_1 &:= [1](0), \\ C_2 &:= [1](1). \end{aligned}$$

They can be pictured, respectively, as

$$\bullet, \qquad \bullet \longrightarrow \bullet, \qquad \begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \bullet \end{tikzcd}.$$

We refer to the object C_n as the *n-cell*; it is the generic 2-category containing an n-morphism.

Definition 2.11. $\mathrm{Seg}_{\mathbf{\Theta}_2^{\mathrm{op}}}(\mathcal{S})$ denotes the full subcategory of $\mathrm{Fun}(\mathbf{\Theta}_2^{\mathrm{op}}, \mathcal{S})$ consisting of functors X that satisfy the following pair of Segal conditions:

- for every object $[k](n_1,\ldots,n_k)$, the morphism
$$X([k](n_1,\ldots,n_k)) \to X([1](n_1)) \times_{X(C_0)} \cdots \times_{X(C_0)} X([1](n_k))$$
is an equivalence,
- for every object $[1](n)$, the morphism
$$X([1](n)) \to X(C_2) \times_{X(C_1)} \cdots \times_{X(C_1)} X(C_2)$$
is an equivalence.

Remark 2.12. The bottom right vertical morphism in (1) is given by composition with the functor $\tau\colon \mathbf{\Delta} \times \mathbf{\Delta} \to \mathbf{\Theta}_2$, given on objects by $([k],[n]) \mapsto [k]([n],\ldots,[n])$. This restricts to an equivalence between $\mathrm{Seg}_{\mathbf{\Theta}_2^{\mathrm{op}}}(\mathcal{S})$ and $\mathrm{Seg}^{2\text{-fold}}_{\mathbf{\Delta}^{\mathrm{op}}\times\mathbf{\Delta}^{\mathrm{op}}}(\mathcal{S})$ and furthermore identifies $\mathrm{Cat}_{(\infty,2)}$ with the full subcategory of *complete* objects in $\mathrm{Seg}_{\mathbf{\Theta}_2^{\mathrm{op}}}(\mathcal{S})$ in the sense of Rezk [26]; this comparison was first proved by Barwick and Schommer-Pries [4] and in different ways by Bergner and Rezk [5] and the author [14].

Notation 2.13. We introduce some notation for various structures related to $(\infty, 2)$-categories:

(i) If $\mathcal{X}$ is an $(\infty, 2)$-category, we write $\iota_1\mathcal{X}$ for the underlying ∞-category of $\mathcal{X}$, and $\iota_0\mathcal{X}$ for the underlying ∞-groupoid. If we view $\mathcal{X}$ as an object $X_\bullet \in \mathrm{Seg}^{\mathcal{S}}_{\mathbf{\Delta}^{\mathrm{op}}}(\mathrm{Cat}_\infty)$, then $\iota_1\mathcal{X}$ is the complete Segal space obtained by taking the underlying ∞-groupoid levelwise, i.e. $X_\bullet^{\simeq}$, while $\iota_0\mathcal{X}$ is the ∞-groupoid X_0.

(ii) If $\mathcal{X}$ is an $(\infty, 2)$-category and x, y are objects of $\mathcal{X}$ then we write $\mathcal{X}(x, y)$ for the ∞-category of morphisms from x to y in $\mathcal{X}$. If we view $\mathcal{X}$ as a simplicial ∞-category X, then this is given by the pullback square

$$\begin{tikzcd} \mathcal{X}(x,y) \arrow[r] \arrow[d] & X_1 \arrow[d] \\ \{(x,y)\} \arrow[r] & X_0 \times X_0, \end{tikzcd}$$

where the right vertical map is the functor induced by the two maps $[0] \to [1]$.

(iii) If $\mathcal{X}$ and $\mathcal{Y}$ are $(\infty, 2)$-categories, we write $\mathrm{FUN}(\mathcal{X}, \mathcal{Y})$ for the $(\infty, 2)$-category of functors between them, i.e. the internal Hom in $\mathrm{Cat}_{(\infty,2)}$, and $\mathrm{Fun}(\mathcal{X}, \mathcal{Y}) := \iota_1\mathrm{FUN}(\mathcal{X}, \mathcal{Y})$ for its underlying ∞-category.

(iv) If $\mathcal{X}$ is an $(\infty, 2)$-category, we write $\mathcal{X}^{1\text{-op}}$ for the $(\infty, 2)$-category obtained from $\mathcal{X}$ by reversing the 1-morphisms, and $\mathcal{X}^{2\text{-op}}$ for that obtained by reversing the 2-morphisms. If $\mathcal{X}$ is represented by a simplicial ∞-category $X_\bullet$ then $\mathcal{X}^{2\text{-op}}$ corresponds to taking op levelwise to obtain $X_\bullet^{\mathrm{op}}$, while $\mathcal{X}^{1\text{-op}}$ is obtained by composing $X_\bullet$ with the order-reversing involution of $\mathbf{\Delta}$.

Remark 2.14. Another description of $(\infty, 2)$-categories is that they are precisely ∞-categories *enriched* in the symmetric monoidal ∞-category Cat_∞, in the sense of [9]. This definition is shown in [12] to be equivalent to $(\infty, 2)$-categories viewed as complete objects in $\mathrm{Seg}^{\mathcal{S}}_{\mathbf{\Delta}^{\mathrm{op}}}(\mathrm{Cat}_\infty)$, and hence is also equivalent to the other definitions we have considered thus far; the comparison also extends to an equivalence between $\mathrm{Seg}^{\mathcal{S}}_{\mathbf{\Delta}^{\mathrm{op}}}(\mathrm{Cat}_\infty)$ and *categorical algebras* in Cat_∞, defined in [9] as algebras for a family of (generalized non-symmetric) ∞-operads $\mathbf{\Delta}^{\mathrm{op}}_X$. This allows us to construct certain $(\infty, 2)$-categories as *free* algebras for these ∞-operads, as we will now explain:

Definition 2.15. A Cat_∞*-graph* on a space X is a functor $X \times X \to \mathrm{Cat}_\infty$; using the obvious naturality in X, these combine into an ∞-category $\mathrm{Graph}(\mathrm{Cat}_\infty)$. This can equivalently be viewed as the ∞-category $\mathrm{Fun}^{\mathcal{S}}(\mathbf{\Delta}^{\mathrm{el,op}}, \mathrm{Cat}_\infty)$ where $\mathbf{\Delta}^{\mathrm{el}}$ is the subcategory of $\mathbf{\Delta}$ containing the objects $[0], [1]$ and the two inert maps $d_0, d_1 \colon [0] \to [1]$, and $\mathrm{Fun}^{\mathcal{S}}(\mathbf{\Delta}^{\mathrm{el,op}}, \mathrm{Cat}_\infty)$ is the full subcategory of $\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{el,op}}, \mathrm{Cat}_\infty)$ consisting of functors Φ such that $\Phi_0 \in \mathcal{S}$. The forgetful functor from categorical algebras to graphs then corresponds to the functor $\mathrm{Seg}^{\mathcal{S}}_{\mathbf{\Delta}^{\mathrm{op}}}(\mathrm{Cat}_\infty) \to \mathrm{Graph}(\mathrm{Cat}_\infty)$ induced by composition with the inclusion $\mathbf{\Delta}^{\mathrm{el}} \to \mathbf{\Delta}$. This has a left adjoint $\mathrm{Free} \colon \mathrm{Graph}(\mathrm{Cat}_\infty) \to \mathrm{Seg}^{\mathcal{S}}_{\mathbf{\Delta}^{\mathrm{op}}}(\mathrm{Cat}_\infty)$, which can be described by an explicit formula (as it is given by free algebras for a family of ∞-operads).

Definition 2.16. In particular, given ∞-categories $\mathcal{C}_1, \ldots, \mathcal{C}_n$ we can define a Cat_∞-graph

$$[n](\mathcal{C}_1, \ldots, \mathcal{C}_n)_{\mathrm{graph}}$$

on the set $\{0, \ldots, n\}$ by

$$(i, j) \mapsto \begin{cases} \mathcal{C}_j, & i = j - 1, \\ \emptyset, & \text{otherwise.} \end{cases}$$

We write $[n](\mathcal{C}_1,\ldots,\mathcal{C}_n)$ for the free $(\infty,2)$-category on $[n](\mathcal{C}_1,\ldots,\mathcal{C}_n)_{\mathrm{graph}}$. The formula for free algebras implies that this $(\infty,2)$-category has objects $0,\ldots,n$, and the ∞-categories of maps are given by

$$[n](\mathcal{C}_1,\ldots,\mathcal{C}_n)(i,j) \simeq \begin{cases} \mathcal{C}_{i+1}\times\cdots\times\mathcal{C}_j, & i\leq j,\\ \emptyset, & i>j,\end{cases}$$

with composition given by the obvious equivalence

$$[n](\mathcal{C}_1,\ldots,\mathcal{C}_n)(i,j)\times[n](\mathcal{C}_1,\ldots,\mathcal{C}_n)(j,k) \xrightarrow{\sim} [n](\mathcal{C}_1,\ldots,\mathcal{C}_n)(i,k).$$

Note that any inert map $\phi\colon [m]\to[n]$ in $\mathbf{\Delta}$ induces a fully faithful functor

$$\bar{\phi}\colon [m](\mathcal{C}_{\phi(1)},\ldots,\mathcal{C}_{\phi(m)}) \to [n](\mathcal{C}_1,\ldots,\mathcal{C}_n),$$

as the free functor on the inclusion of graphs determined by ϕ.

Remark 2.17. In particular, we have a functor $[1](-)\colon \mathrm{Cat}_\infty \to \mathrm{Cat}_{(\infty,2)}$ with two natural morphisms $[0]\to[1](-)$. From the free-forgetful adjunction for graphs, we see that for any $(\infty,2)$-category $\mathcal{X}$ the fibre of

$$\mathrm{Map}([1](\mathcal{C}),\mathcal{X}) \to \mathrm{Map}([0],\mathcal{X})^{\times 2}$$

at objects $x,y\in\mathcal{X}$ is naturally equivalent to $\mathrm{Map}_{\mathrm{Cat}_\infty}(\mathcal{C},\mathcal{X}(x,y))$.

We can use these free $(\infty,2)$-categories to describe some colimits of $(\infty,2)$-categories that will be useful later on:

Lemma 2.18. *For any ∞-categories $\mathcal{C}_1,\ldots,\mathcal{C}_n$, the functor*

$$[1](\mathcal{C}_1)\amalg_{[0]}\cdots\amalg_{[0]}[1](\mathcal{C}_n)\to[n](\mathcal{C}_1,\ldots,\mathcal{C}_n),$$

induced by the inert maps $[0],[1]\to[n]$, is an equivalence.

Proof. Since taking free $(\infty,2)$-categories is a left adjoint, this is the free functor on a morphism of graphs

$$[1](\mathcal{C}_1)_{\mathrm{graph}}\amalg_{[0]_{\mathrm{graph}}}\cdots\amalg_{[0]_{\mathrm{graph}}}[1](\mathcal{C}_n)_{\mathrm{graph}}\to[n](\mathcal{C}_1,\ldots,\mathcal{C}_n)_{\mathrm{graph}},$$

which is obviously an equivalence. □

Lemma 2.19. *The functor $[1](-)\colon \mathrm{Cat}_\infty\to\mathrm{Cat}_{(\infty,2)}$ preserves weakly contractible colimits.*

Proof. Given a diagram $f\colon \mathcal{I}\to\mathrm{Cat}_\infty$ and $\mathcal{X}\in\mathrm{Cat}_{(\infty,2)}$, we have a natural commutative square

$$\begin{tikzcd} \mathrm{Map}([1](\mathrm{colim}_{\mathcal{I}} f),\mathcal{X}) \arrow[r]\arrow[d] & \lim_{\mathcal{I}^{\mathrm{op}}}\mathrm{Map}([1](f),\mathcal{X}) \arrow[d]\\ \mathrm{Map}([0],\mathcal{X})^{\times 2} \arrow[r] & \lim_{\mathcal{I}^{\mathrm{op}}}\mathrm{Map}([0],\mathcal{X})^{\times 2}. \end{tikzcd}$$

If $\mathcal{I}$ is weakly contractible then the bottom horizontal morphism is an equivalence, so to show the top horizontal morphism is an equivalence it suffices to show it is an equivalence on the fibre at any pair of objects $x,y\in\mathcal{C}$. Since limits commute, we can identify the map on fibres as

$$\mathrm{Map}_{\mathrm{Cat}_\infty}(\operatorname*{colim}_{\mathcal{I}} f,\mathcal{X}(x,y))\to\lim_{\mathcal{I}^{\mathrm{op}}}\mathrm{Map}_{\mathrm{Cat}_\infty}(f,\mathcal{X}(x,y)),$$

which is indeed an equivalence. □

Lemma 2.20. *If $\mathcal{I}$ is a weakly contractible ∞-category and $\mathcal{X}$ is an $(\infty, 2)$-category which corresponds to a simplicial ∞-category $X_\bullet$, then there is a natural equivalence*

$$\mathrm{Map}_{\mathrm{Cat}_{(\infty,2)}}([1](\mathcal{I}), \mathcal{X}) \simeq \mathrm{Map}_{\mathrm{Cat}_\infty}(\mathcal{I}, X_1).$$

Proof. We have a natural fibre sequence

$$\mathrm{Map}(\mathcal{I}, \mathcal{X}(x, y)) \to \mathrm{Map}(\mathcal{I}, X_1) \to \mathrm{Map}(\mathcal{I}, X_0)^{\times 2} \simeq \mathrm{Map}(\|\mathcal{I}\|, X_0)^{\times 2},$$

where the equivalence uses that X_0 is an ∞-groupoid. If $\mathcal{I}$ is weakly contractible, this is equivalent to the fibre sequence above for $\mathrm{Map}([1](\mathcal{I}), \mathcal{X})$. □

3. The Gray Tensor Product

For ordinary (strict) 2-categories, Gray [11] defined a (non-symmetric) tensor product $\otimes^{\mathrm{(co)lax}}$ (where $\mathbf{A} \otimes^{\mathrm{lax}} \mathbf{B} \cong \mathbf{B} \otimes^{\mathrm{colax}} \mathbf{A}$), colimit-preserving in each variable, such that the internal Homs are 2-categories of functors where morphisms are either lax or colax[4] natural transformations (depending on whether we take the adjoint in the first or second variable). In this section we first recall an explicit description of $I \otimes^{\mathrm{colax}} J$ for $I, J \in \mathbf{\Theta}_2$ and then discuss the (expected) extension of the Gray tensor product to $(\infty, 2)$-categories and its basic properties.

Notation 3.1. Recall that $[k]$ denotes the ordered set $\{0 < 1 < \cdots < n\}$. Viewing this as a poset, the product $[k] \times [m]$ of posets has the shape of a rectangular grid. This is a ranked poset; its maximal chains (i.e. the paths from $(0, 0)$ to (k, m)) all have length $k + m$ and form a poset denoted $\mathrm{MaxCh}([k] \times [m])$, whose partial order relation is generated by

$$↳ \leq ⌝$$

(Note that this poset is isomorphic to the poset $\mathrm{Sh}(k, m)$ of (k, m)-*shuffles*, ordered with $k + m$ as the least element and $m + k$ as the greatest element.)

Notation 3.2. For non-negative integers $i \leq j$ it is convenient to also introduce the notation $[i, j]$ for the ordered set $\{i < i + 1 < \cdots < j\}$, which is isomorphic to $[j - i]$. If $i > j$ it is convenient to take $[i, j] = \emptyset$.

Definition 3.3. If $I = [n](x_1, \ldots, x_n)$ and $J = [m](y_1, \ldots, y_m)$ are objects of $\mathbf{\Theta}_2$, then the Gray tensor product $I \otimes^{\mathrm{colax}} J$ is the 2-category with object set $\mathrm{ob}([n]) \times \mathrm{ob}([m])$ and Hom-categories (actually posets)

$$\begin{aligned}\mathrm{Hom}((i, j), (i', j')) &:= \begin{cases} \mathrm{MaxCh}([i, i'] \times [j, j']) \times \prod_{i < s \leq i'} [x_s] \times \prod_{j < t \leq j'} [y_t], & i \leq i', j \leq j' \\ \emptyset, & \text{otherwise} \end{cases} \\ &\cong \mathrm{MaxCh}([i, i'] \times [j, j']) \times I(i, i') \times J(j, j')\end{aligned}$$

The composition of morphisms $(i, j) \to (i', j')$ and $(i', j') \to (i'', j'')$ is defined by combining the composition in I and J with the natural inclusion

$$\mathrm{MaxCh}([i, i'] \times [j, j']) \times \mathrm{MaxCh}([i', i''] \times [j', j'']) \to \mathrm{MaxCh}([i, i''] \times [j, j''])$$

[4]We have tried to follow the convention that the prefix "co" refers to reversing the direction of 2-morphisms, while "op" refers to reversing that of 1-morphisms. Since the two types of lax natural transformations are related by reversing 2-morphisms, we call them lax and colax transformations (just as we would refer to lax and colax functors, though these do not appear in this paper). However, in the 2-categorical literature the term *oplax* natural transformation is also common.

that combines a path from (i, j) to (i', j') with a path from (i', j') to (i'', j'') to get the subset of paths from (i, j) to (i'', j'') that factor through (i', j'). With this definition there is also a canonical way to define functors between Gray tensor products from morphisms in $\mathbf{\Theta}_2$, so that we obtain a functor $\otimes^{\mathrm{colax}} \colon \mathbf{\Theta}_2 \times \mathbf{\Theta}_2 \to \mathrm{Cat}_{(\infty,2)}$.

Examples 3.4. $[1](0) \otimes^{\mathrm{colax}} [1](0)$ has 4 objects, $00, 01, 10, 11$, and $\mathrm{Hom}(00, 11) = (\Lsh \leq \urcorner)$. The remaining hom categories are discrete: contractible if the indices are non-decreasing, empty if some index decreases. The whole 2-category can therefore be depicted as a colax square:

$$\begin{tikzcd} 00 \arrow[r] \arrow[d] & 01 \arrow[d] \\ 10 \arrow[r] \arrow[ur, Rightarrow] & 11 \end{tikzcd}$$

Similarly, $C_2 \otimes^{\mathrm{colax}} C_1 = [1](1) \otimes^{\mathrm{colax}} [1](0)$ has the shape of a cylinder (with side squares colax):

$$\begin{tikzcd} 00 \arrow[r] \arrow[d, bend right] \arrow[d, bend left] \arrow[d, phantom, "\Rightarrow"] & 01 \arrow[d, bend right] \arrow[d, bend left] \arrow[d, phantom, "\Rightarrow"] \\ 10 \arrow[r] \arrow[ur, Rightarrow] & 11. \end{tikzcd}$$

This means that a diagram of shape $C_2 \otimes^{\mathrm{colax}} C_1$ in an $(\infty, 2)$-category $\mathcal{X}$ consists of the following data in $\mathcal{X}$:

- objects X, Y, X', Y',
- morphisms $f, g \colon X \to Y$, $f', g' \colon X' \to Y'$, $\xi \colon X \to X'$, $\eta \colon Y \to Y'$,
- 2-morphisms $\alpha \colon f \to g$, $\alpha' \colon f' \to g'$, $\phi \colon \eta f \to f'\xi$, $\psi \colon \eta g \to g'\xi$,
- an equivalence $\psi \circ (\eta\alpha) \simeq (\alpha'\xi) \circ \phi$ of 2-morphisms $\eta f \to g'\xi$.

Since we can view 2-categories as $(\infty, 2)$-categories, the classical Gray tensor product induces a functor

$$\otimes^{\mathrm{colax}} \colon \mathbf{\Theta}_2 \times \mathbf{\Theta}_2 \to \mathrm{Cat}_{(\infty,2)}.$$

We will make the following three assumptions about this functor:

Assumption 3.5.

(1) The functor $\otimes^{\mathrm{colax}}$ satisfies the co-Segal condition[5] in each variable. The unique extension to a functor $\mathcal{P}(\mathbf{\Theta}_2) \times \mathcal{P}(\mathbf{\Theta}_2) \to \mathrm{Cat}_{(\infty,2)}$ that preserves colimits in each variable therefore uniquely factors through a functor $\mathrm{Seg}_{\mathbf{\Theta}_2^{\mathrm{op}}}(\mathcal{S}) \times \mathrm{Seg}_{\mathbf{\Theta}_2^{\mathrm{op}}}(\mathcal{S}) \to \mathrm{Cat}_{(\infty,2)}$ that preserves colimits in each variable.

(2) The functor $\mathrm{Seg}_{\mathbf{\Theta}_2^{\mathrm{op}}}(\mathcal{S}) \times \mathrm{Seg}_{\mathbf{\Theta}_2^{\mathrm{op}}}(\mathcal{S}) \to \mathrm{Cat}_{(\infty,2)}$ takes fully faithful and essentially surjective morphisms in each variable to equivalences, and thus factors uniquely through a functor

$$\otimes^{\mathrm{lax}} \colon \mathrm{Cat}_{(\infty,2)} \times \mathrm{Cat}_{(\infty,2)} \to \mathrm{Cat}_{(\infty,2)}.$$

(3) The restriction of $\otimes^{\mathrm{lax}}$ to ordinary (strict) 2-categories agrees with the classical Gray tensor product.[6]

[5] By the co-Segal condition for a functor $\phi \colon \mathbf{\Theta}_2 \to \mathcal{C}$ we mean the Segal condition for $\phi^{\mathrm{op}} \colon \mathbf{\Theta}_2^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$.

[6] In fact, we only need this assumption in the case of *gaunt* 2-categories, meaning ones with no non-trivial invertible 1- or 2-morphisms, which may be more straightforward to prove than the general case.

Remark 3.6. Assumptions (1) and (2) have recently been proved by Y. Maehara [23], who shows that formally extending the ordinary Gray tensor product on $\mathbf{\Theta}_2$ gives a left Quillen bifunctor for $\mathbf{\Theta}_2$-sets. Several other constructions of Gray tensor products in various models of $(\infty,2)$-categories (some more generally in (∞,n)-categories) have also recently appeared, including [7, 24, 6].

Remark 3.7. As observed by Ayala–Francis [1], a colimit diagram in $\mathrm{Cat}_{(\infty,2)}$ whose underlying diagrams of ∞-categories and ∞-groupoids are also colimit diagrams is a colimit in $\mathrm{Seg}_{\mathbf{\Theta}_2^{\mathrm{op}}}(\mathcal{S})$. This is true for the diagrams exhibiting the co-Segal condition for $\otimes^{\mathrm{colax}}$, hence we can also take left Kan extensions to obtain a functor

$$\otimes^{\mathrm{colax}}_{\mathrm{Seg}} \colon \mathrm{Seg}_{\mathbf{\Theta}_2^{\mathrm{op}}}(\mathcal{S}) \times \mathrm{Seg}_{\mathbf{\Theta}_2^{\mathrm{op}}}(\mathcal{S}) \to \mathrm{Seg}_{\mathbf{\Theta}_2^{\mathrm{op}}}(\mathcal{S}),$$

colimit-preserving in each variable, such that there is a commutative diagram

$$\begin{tikzcd}
\mathrm{Seg}_{\mathbf{\Theta}_2^{\mathrm{op}}}(\mathcal{S}) \times \mathrm{Seg}_{\mathbf{\Theta}_2^{\mathrm{op}}}(\mathcal{S}) \arrow[r, "\otimes^{\mathrm{colax}}_{\mathrm{Seg}}"] \arrow[d] & \mathrm{Seg}_{\mathbf{\Theta}_2^{\mathrm{op}}}(\mathcal{S}) \arrow[d] \\
\mathrm{Cat}_{(\infty,2)} \times \mathrm{Cat}_{(\infty,2)} \arrow[r, "\otimes^{\mathrm{colax}}"] & \mathrm{Cat}_{(\infty,2)},
\end{tikzcd}$$

where the vertical morphisms are given by localization.

Definition 3.8. For $(\infty,2)$-categories $\mathcal{X}$ and $\mathcal{Y}$ we call $\mathcal{X} \otimes^{\mathrm{colax}} \mathcal{Y}$ the *colax Gray tensor product* of $\mathcal{X}$ and $\mathcal{Y}$. We will also write $\mathcal{X} \otimes^{\mathrm{lax}} \mathcal{Y} := \mathcal{Y} \otimes^{\mathrm{colax}} \mathcal{X}$, and call this the *lax Gray tensor product.*

Definition 3.9. The functor $\otimes^{(\mathrm{co})\mathrm{lax}}$ preserves colimits in each variable, and so has adjoints $\mathrm{FUN}(-,-)_{(\mathrm{co})\mathrm{lax}}$, which satisfy

$$\begin{aligned}
\mathrm{Map}_{\mathrm{Cat}_{(\infty,2)}}(\mathcal{X}, \mathrm{FUN}(\mathcal{Y},\mathcal{Z})_{\mathrm{colax}}) &\simeq \mathrm{Map}_{\mathrm{Cat}_{(\infty,2)}}(\mathcal{Y} \otimes^{\mathrm{colax}} \mathcal{X}, \mathcal{Z}) \\
&\simeq \mathrm{Map}_{\mathrm{Cat}_{(\infty,2)}}(\mathcal{X} \otimes^{\mathrm{lax}} \mathcal{Y}, \mathcal{Z}) \simeq \mathrm{Map}_{\mathrm{Cat}_{(\infty,2)}}(\mathcal{Y}, \mathrm{FUN}(\mathcal{X},\mathcal{Z})_{\mathrm{lax}}).
\end{aligned}$$

A *(co)lax natural transformation* (between functors $\mathcal{X} \to \mathcal{Y}$) is a functor of $(\infty,2)$-categories

$$\mathcal{X} \otimes^{(\mathrm{co})\mathrm{lax}} \Delta^1 \to \mathcal{Y}.$$

The $(\infty,2)$-category $\mathrm{FUN}(\mathcal{X},\mathcal{Y})_{(\mathrm{co})\mathrm{lax}}$ thus has usual functors of $(\infty,2)$-categories as objects, and (co)lax natural transformations as morphisms. Similarly, the 2-morphisms are functors of $(\infty,2)$-categories $\mathcal{X} \otimes^{(\mathrm{co})\mathrm{lax}} C_2 \to \mathcal{Y}$.

Remark 3.10. A lax natural transformation η between functors $F, G \colon \mathcal{X} \to \mathcal{Y}$ assigns to every morphism $f \colon X \to X'$ in $\mathcal{X}$ a lax square

$$\begin{tikzcd}
F(X) \arrow[r, "\eta_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \arrow[dl, Rightarrow, shorten=4mm] \\
F(X') \arrow[r, "\eta_{X'}"] & G(X'),
\end{tikzcd}$$

while a colax natural transformation assigns a colax square

$$\begin{tikzcd}
F(X) \arrow[r, "\eta_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\
F(X') \arrow[r, "\eta_{X'}"] \arrow[ur, Rightarrow, shorten=4mm] & G(X').
\end{tikzcd}$$

Remark 3.11. Note that if f in the previous remark is id_X then our definition requires the (co)lax square to be the identity of η_X, and then the compatibility with composition implies that if f is an equivalence then the (co)lax square commutes. This suggests that for functors from an ∞-groupoid (co)lax natural transformations should reduce to ordinary natural transformations. To see this more formally, first note that if X is an ∞-groupoid then the natural equivalence $X \simeq \mathrm{colim}_X C_0$ induces for any $(\infty,2)$-category $\mathcal{Y}$ an equivalence

$$X \otimes^{\mathrm{colax}} \mathcal{Y} \simeq \operatorname*{colim}_X (C_0 \otimes^{\mathrm{colax}} \mathcal{Y}) \simeq \operatorname*{colim}_X \mathcal{Y} \simeq X \times \mathcal{Y}.$$

Hence if $\mathcal{Z}$ is another $(\infty,2)$-category, we have natural equivalences

$$\mathrm{Map}(\mathcal{Y}, \mathrm{FUN}(X,\mathcal{Z})_{\mathrm{colax}}) \simeq \mathrm{Map}(X \otimes^{\mathrm{colax}} \mathcal{Y}, \mathcal{Z}) \simeq \mathrm{Map}(X \times \mathcal{Y}, \mathcal{Z}) \simeq \mathrm{Map}(\mathcal{Y}, \mathrm{FUN}(X,\mathcal{Z})),$$

which implies by the Yoneda lemma that we indeed have a natural equivalence

$$\mathrm{FUN}(X,\mathcal{Z})_{\mathrm{colax}} \simeq \mathrm{FUN}(X,\mathcal{Z}).$$

Remark 3.12. The paper [16] of Johnson-Freyd and Scheimbauer gives an alternative construction of the $(\infty,2)$-categories of $\mathrm{FUN}(\mathcal{X},\mathcal{Y})_{(\mathrm{co})\mathrm{lax}}$, without defining the Gray tensor product in general: in our notation they give explicit definitions of $(\infty,2)$-categories corresponding to our $\mathrm{FUN}(\tau([n],[m]),\mathcal{Y})_{(\mathrm{co})\mathrm{lax}}$ using the functor $\tau\colon \mathbf{\Delta} \times \mathbf{\Delta} \to \mathbf{\Theta}_2$, and then define $\mathrm{FUN}(\mathcal{X},\mathcal{Y})_{\mathrm{lax}}$ as the 2-fold Segal space

$$\mathrm{Map}(\mathcal{X}, \mathrm{FUN}(\tau(-,-),\mathcal{Y})_{\mathrm{colax}}).$$

Proposition 3.13. *There is a natural equivalence*

$$(\mathcal{X} \otimes^{\mathrm{lax}} \mathcal{Y})^{2\text{-op}} \simeq \mathcal{X}^{2\text{-op}} \otimes^{\mathrm{colax}} \mathcal{Y}^{2\text{-op}}.$$

Proof. There is such an equivalence for the tensor product of ordinary 2-categories, so there is a natural equivalence for $\mathcal{X},\mathcal{Y} \in \mathbf{\Theta}_2$, which extends by colimits to an equivalence for all $\mathcal{X},\mathcal{Y}$. □

Corollary 3.14. *There is a natural equivalence*

$$\mathrm{FUN}(\mathcal{X},\mathcal{Y})_{\mathrm{lax}}^{2\text{-op}} \simeq \mathrm{FUN}(\mathcal{X}^{2\text{-op}},\mathcal{Y}^{2\text{-op}})_{\mathrm{colax}}$$

for all $(\infty,2)$-categories $\mathcal{X},\mathcal{Y}$. □

Remark 3.15. Since $C_0 \otimes^{\mathrm{colax}} - \simeq \mathrm{id} \simeq - \otimes^{\mathrm{colax}} C_0$, we obtain natural morphisms $\mathcal{X} \otimes^{\mathrm{colax}} \mathcal{Y} \to \mathcal{X} \otimes^{\mathrm{colax}} C_0 \simeq \mathcal{X}$ and $\mathcal{X} \otimes^{\mathrm{colax}} \mathcal{Y} \to C_0 \otimes^{\mathrm{colax}} \mathcal{Y} \simeq \mathcal{Y}$, and so a natural morphism

$$\mathcal{X} \otimes^{\mathrm{colax}} \mathcal{Y} \to \mathcal{X} \times \mathcal{Y}.$$

We will now observe that this exhibits $\mathcal{X} \times \mathcal{Y}$ as a localization of $\mathcal{X} \otimes^{\mathrm{colax}} \mathcal{Y}$:

Proposition 3.16.

(i) If $\mathcal{C}$ and $\mathcal{D}$ are ∞-categories, the functor $\mathcal{C} \otimes^{\mathrm{colax}} \mathcal{D} \to \mathcal{C} \times \mathcal{D}$ exhibits $\mathcal{C} \times \mathcal{D}$ as the ∞-category $L_{(\infty,1)}(\mathcal{C} \otimes^{\mathrm{colax}} \mathcal{D})$ obtained by inverting all 2-morphisms in $\mathcal{C} \otimes^{\mathrm{colax}} \mathcal{D}$.

(ii) The natural commutative square

$$\begin{tikzcd} \iota_1\mathcal{X} \otimes^{\mathrm{colax}} \iota_1\mathcal{Y} \arrow[r] \arrow[d] & \iota_1\mathcal{X} \times \iota_1\mathcal{Y} \arrow[d] \\ \mathcal{X} \otimes^{\mathrm{colax}} \mathcal{Y} \arrow[r] & \mathcal{X} \times \mathcal{Y} \end{tikzcd}$$

is a pushout square for all $(\infty,2)$-categories $\mathcal{X},\mathcal{Y}$.

Proof. To prove (i) it suffices, since both sides preserve colimits in each variable, to show that this morphism is an equivalence for $\mathcal{C}$ and $\mathcal{D}$ either C_0 or C_1. The only non-trivial case is $C_1 \otimes^{\mathrm{colax}} C_1 \to C_1 \times C_1$, which indeed exhibits the commuting square $C_1 \times C_1$ as obtained by inverting the unique 2-morphism in $C_1 \otimes^{\mathrm{colax}} C_1$.

To prove (ii), it suffices to prove the analogue of (ii) for the pairing $\otimes^{\mathrm{colax}}_{\mathrm{Seg}}$ on $\mathrm{Seg}_{\mathbf{\Theta}_2^{\mathrm{op}}}(\mathcal{S})$, from which $\otimes^{\mathrm{colax}}$ is obtained by localization. This also preserves colimits in each variable, and ι_1 on $\mathrm{Seg}_{\mathbf{\Theta}_2^{\mathrm{op}}}(\mathcal{S})$ preserves colimits, so it suffices to check the square is a pushout for $\mathcal{X}, \mathcal{Y}$ being either C_0, C_1 or C_2. Here it follows from the description of the Gray tensor product in Definition 3.3 that

$$\begin{tikzcd}
\iota_1 C_i \otimes^{\mathrm{colax}} \iota_1 C_j \arrow[r] \arrow[d] & \iota_1 C_i \times \iota_1 C_j \arrow[d] \\
C_i \otimes^{\mathrm{colax}} C_j \arrow[r] & C_i \times C_j
\end{tikzcd}$$

is a pushout square in $\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathcal{S})$ and hence in the localization $\mathrm{Cat}_{(\infty,2)}$ since these are already local objects. □

Composing with the natural map from Remark 3.15 we get for any $(\infty, 2)$-categories $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ a natural map

$$\mathrm{Map}_{\mathrm{Cat}_{(\infty,2)}}(\mathcal{X} \times \mathcal{Z}, \mathcal{Y}) \to \mathrm{Map}_{\mathrm{Cat}_{(\infty,2)}}(\mathcal{X}, \otimes^{\mathrm{colax}} \mathcal{Z}, \mathcal{Y}),$$

which by adjunction induces a natural map

$$\mathrm{FUN}(\mathcal{X}, \mathcal{Y}) \to \mathrm{FUN}(\mathcal{X}, \mathcal{Y})_{\mathrm{colax}}.$$

We will now show that this identifies $\mathrm{FUN}(\mathcal{X}, \mathcal{Y})$ with a subobject of $\mathrm{FUN}(\mathcal{X}, \mathcal{Y})_{(\mathrm{co})\mathrm{lax}}$:

Corollary 3.17. *There is a natural identification of* $\mathrm{FUN}(\mathcal{X}, \mathcal{Y})$ *with the sub-*$(\infty, 2)$*-category of* $\mathrm{FUN}(\mathcal{X}, \mathcal{Y})_{(\mathrm{co})\mathrm{lax}}$ *containing all objects, with 1-morphisms the (co)lax natural transformations all of whose (co)lax naturality squares commute, and all 2-morphisms between these.*

Proof. Let $\mathcal{X}$ and $\mathcal{Y}$ be $(\infty, 2)$-categories, and consider the commutative diagram

$$\begin{tikzcd}
\iota_1 \mathcal{X} \otimes^{\mathrm{colax}} \iota_1 \mathcal{Y} \arrow[r] \arrow[d] & \iota_1 \mathcal{X} \times \iota_1 \mathcal{Y} \arrow[d] \\
\iota_1 \mathcal{X} \otimes^{\mathrm{colax}} \mathcal{Y} \arrow[r] \arrow[d] & \iota_1 \mathcal{X} \times \mathcal{Y} \arrow[d] \\
\mathcal{X} \otimes^{\mathrm{colax}} \mathcal{Y} \arrow[r] & \mathcal{X} \times \mathcal{Y}.
\end{tikzcd}$$

Here the top square and the outer square are pushouts by Proposition 3.16, hence so is the bottom square. Given a third $(\infty, 2)$-category $\mathcal{Z}$ we obtain a commutative diagram

$$\begin{tikzcd}
\mathrm{Map}(\mathcal{X} \times \mathcal{Y}, \mathcal{Z}) \arrow[r] \arrow[d] & \mathrm{Map}(\mathcal{X} \otimes^{\mathrm{colax}} \mathcal{Y}, \mathcal{Z}) \arrow[d] \\
\mathrm{Map}(\iota_1 \mathcal{X} \times \mathcal{Y}, \mathcal{Z}) \arrow[r] \arrow[d] & \mathrm{Map}(\iota_1 \mathcal{X} \otimes^{\mathrm{colax}} \mathcal{Y}, \mathcal{Z}) \arrow[d] \\
\mathrm{Map}(\iota_1 \mathcal{X} \times \iota_1 \mathcal{Y}, \mathcal{Z}) \arrow[r] & \mathrm{Map}(\iota_1 \mathcal{X} \otimes^{\mathrm{colax}} \iota_1 \mathcal{Y}, \mathcal{Z}),
\end{tikzcd}$$

where all squares are cartesian. We can rewrite this as

$$\begin{tikzcd}
\mathrm{Map}(\mathcal{X}, \mathrm{FUN}(\mathcal{Y}, \mathcal{Z})) \arrow[r] \arrow[d] & \mathrm{Map}(\mathcal{X}, \mathrm{FUN}(\mathcal{Y}, \mathcal{Z})_{\mathrm{lax}}) \arrow[d] \\
\mathrm{Map}(\iota_1\mathcal{X}, \mathrm{FUN}(\mathcal{Y}, \mathcal{Z})) \arrow[r] \arrow[d] & \mathrm{Map}(\iota_1\mathcal{X}, \mathrm{FUN}(\mathcal{Y}, \mathcal{Z})_{\mathrm{lax}}) \arrow[d] \\
\mathrm{Map}(\iota_1\mathcal{X}, \mathrm{FUN}(\iota_1\mathcal{Y}, \mathcal{Z}) \arrow[r] & \mathrm{Map}(\iota_1\mathcal{X}, \mathrm{FUN}(\iota_1\mathcal{Y}, \mathcal{Z})_{\mathrm{lax}}).
\end{tikzcd}$$

This says, firstly, that a functor $\mathcal{X} \to \mathrm{FUN}(\mathcal{Y}, \mathcal{Z})_{\mathrm{lax}}$ factors through $\mathrm{FUN}(\mathcal{Y}, \mathcal{Z})$ if and only if its restriction to the underlying ∞-category $\iota_1\mathcal{X}$ does so. In other words, $\mathrm{FUN}(\mathcal{Y}, \mathcal{Z}) \to \mathrm{FUN}(\mathcal{Y}, \mathcal{Z})_{\mathrm{lax}}$ is locally fully faithful. Furthermore, a functor from $\iota_1\mathcal{X}$ factors through $\mathrm{FUN}(\mathcal{Y}, \mathcal{Z})$ if and only if the induced functor to $\mathrm{FUN}(\iota_1\mathcal{Y}, \mathcal{Z})_{\mathrm{lax}}$ factors through $\mathrm{FUN}(\iota_1\mathcal{Y}, \mathcal{Z})$, which we can interpret via Proposition 3.16(i) as saying that the adjoint functor $\iota_1\mathcal{X} \otimes^{\mathrm{colax}} \iota_1\mathcal{Y} \to \mathcal{Z}$ takes all 2-morphisms in $\iota_1\mathcal{X} \otimes^{\mathrm{colax}} \iota_1\mathcal{Y}$ to equivalences in $\mathcal{Z}$, as required. □

Finally, we note the following colimit decomposition of the Gray tensor product of the generators C_i:

Lemma 3.18. *We have the following colimit decompositions in* $\mathrm{Seg}_{\mathbf{\Theta}_2^{\mathrm{op}}}(\mathcal{S})$ *(and hence in* $\mathrm{Cat}_{(\infty,2)}$*):*

$$C_1 \otimes^{\mathrm{colax}} C_1 \simeq [2](0,0) \cup_{C_1} C_2 \cup_{C_1} [2](0,0),$$

$$C_2 \otimes^{\mathrm{colax}} C_1 \simeq [2](1,0) \cup_{C_2} [1]([1]^2) \cup_{C_2} [2](0,1),$$

$$C_2 \otimes^{\mathrm{colax}} C_2 \simeq [2](1,1) \cup_{[1]([1]^2)} [1]([1]^3) \cup_{[1]([1]^2)} [2](1,1).$$

where the maps in the colimits are the obvious ones.

Proof. It suffices to prove that these give colimit diagrams in $\mathrm{Seg}_{\mathbf{\Theta}_2^{\mathrm{op}}}(\mathcal{S})$. But in fact in all three cases it is easy to see that we have a colimit diagram already in the ∞-category $\mathrm{Fun}(\mathbf{\Theta}_2^{\mathrm{op}}, \mathcal{S})$ of presheaves. □

4. Lax Morphisms of Adjunctions

In this section we will study (co)lax morphisms of adjunctions in an $(\infty, 2)$-category, which arise as a special case of (co)lax natural transformations:

Notation 4.1. Let $\mathfrak{adj}$ denote the "walking adjunction" 2-category, i.e. the free 2-category containing an adjunction. Following [29], Riehl and Verity [27] give a combinatorial description of this 2-category; we will not recall this here, but for notational convenience we will name the lower-dimensional parts of the category: it has two objects, $-$ and $+$, and morphisms are generated by $l\colon - \to +$ (the left adjoint) and $r\colon + \to -$ (the right adjoint).

Definition 4.2. Let $\mathcal{X}$ be an $(\infty, 2)$-category. An *adjunction* in $\mathcal{X}$ is a functor of $(\infty, 2)$-categories $\mathfrak{adj} \to \mathcal{X}$, and a *(co)lax morphism* of adjunctions is a (co)lax natural transformation between adjunctions, i.e. a functor

$$\mathfrak{adj} \otimes^{(\mathrm{co})\mathrm{lax}} C_1 \to \mathcal{X}.$$

We write $\mathrm{ADJ}(\mathcal{X})_{(\mathrm{co})\mathrm{lax}} := \mathrm{FUN}(\mathfrak{adj}, \mathcal{X})_{(\mathrm{co})\mathrm{lax}}$ for the $(\infty, 2)$-category of adjunctions in $\mathcal{X}$ and (co)lax morphisms between them, and $\mathrm{Adj}(\mathcal{X})_{(\mathrm{co})\mathrm{lax}}$ for the underlying ∞-category.

Remark 4.3. The symmetry of the definition of $\mathfrak{adj}$ gives equivalences

- $\mathfrak{adj}^{2\text{-op}} \simeq \mathfrak{adj}$, interchanging $-$ and $+$ and swapping l and r,
- $\mathfrak{adj}^{\mathrm{op}} \simeq \mathfrak{adj}$, fixing the objects but interchanging l and r,

Combined with Corollary 3.14, the first gives a natural equivalence

$$\mathrm{ADJ}(\mathfrak{X})^{2\text{-op}}_{\mathrm{lax}} \simeq \mathrm{ADJ}(\mathfrak{X}^{2\text{-op}})_{\mathrm{colax}}.$$

For ordinary 2-categories, one can show that

- a lax morphism of adjunctions corresponds to a commutative square of right adjoints,
- a colax morphism of adjunctions corresponds to a commutative square of left adjoints.

Our goal in this section is to extend these equivalences to the $(\infty,2)$-categorical setting, i.e. to identify the $(\infty,2)$-categories $\mathrm{ADJ}(\mathfrak{X})_{(\mathrm{co})\mathrm{lax}}$ with the full subcategories of the arrow $(\infty,2)$-category $\mathrm{FUN}(C_1,\mathfrak{X})$ spanned by the morphisms that are right and left adjoints, respectively. Our starting point is the following result of Riehl and Verity:

Theorem 4.4 (Riehl–Verity [27])**.** *Let $\mathfrak{X}$ be an $(\infty,2)$-category, and denote by* $\mathrm{Map}(\Delta^1,\mathfrak{X})^{\mathrm{ladj}}$ *and* $\mathrm{Map}(\Delta^1,\mathfrak{X})^{\mathrm{radj}}$ *the subspaces of* $\mathrm{Map}(\Delta^1,\mathfrak{X})$ *consisting of those components that correspond to left and right adjoint 1-morphisms, respectively. Then the maps*

$$\mathrm{Map}(\mathfrak{adj},\mathfrak{X}) \to \mathrm{Map}(\Delta^1,\mathfrak{X})^{\mathrm{ladj}}, \quad \mathrm{Map}(\mathfrak{adj},\mathfrak{X}) \to \mathrm{Map}(\Delta^1,\mathfrak{X})^{\mathrm{radj}}$$

given by evaluation at the morphisms l and r, respectively, are both equivalences.

Our description of the $(\infty,2)$-categories $\mathrm{ADJ}(\mathfrak{X})_{(\mathrm{co})\mathrm{lax}}$ will follow from a description of certain adjoints in $(\infty,2)$-categories of the form $\mathrm{FUN}(\mathfrak{Y},\mathfrak{X})_{(\mathrm{co})\mathrm{lax}}$. To state this we need some terminology:

Remark 4.5. Given a colax square

$$\begin{tikzcd} A \arrow[r, "l"] \arrow[d, "a"'] \arrow[dr, phantom, "\Nearrow\phi"] & B \arrow[d, "b"] \\ A' \arrow[r, "l'"'] & B' \end{tikzcd} \tag{2}$$

in some $(\infty,2)$-category, where ϕ is a 2-morphism $l'a \to bl$ and the morphisms l and l' have right adjoints r and r', respectively, then the *mate* of ϕ is the transformation $ar \to r'b$ given by the composite

$$ar \to r'l'ar \xrightarrow{r'\phi r} r'blr \to r'b,$$

using the unit $\mathrm{id} \to r'l'$ and the counit $lr \to \mathrm{id}$. We can depict this as a *lax* square

$$\begin{tikzcd} B \arrow[r, "r"] \arrow[d, "b"'] \arrow[dr, phantom, "\Swarrow"] & A \arrow[d, "a"] \\ B' \arrow[r, "r'"'] & B, \end{tikzcd} \tag{3}$$

Similarly, given a lax square (3) whose horizontal morphisms are right adjoints, we can produce a colax mate square (2). The adjunction identities moreover imply that taking mates twice gives back the original square.

Theorem 4.6. *Let $\mathfrak{X}$ be an $(\infty,2)$-category.*

(i) A 1-morphism in $\mathrm{FUN}(\mathcal{Y},\mathcal{X})_{\mathrm{lax}}$*, i.e. a lax natural transformation* $\phi\colon F\to G$*, is a left adjoint if and only if for every* $Y\in\mathcal{Y}$ *the morphism* $\phi_Y\colon F(Y)\to G(Y)$ *is a left adjoint, and the lax square*

$$\begin{tikzcd} F(Y) \arrow[r] \arrow[d] & G(Y) \arrow[d] \arrow[dl, Rightarrow] \\ F(Y') \arrow[r] & G(Y') \end{tikzcd}$$

commutes for all morphisms $Y\to Y'$ *in* $\mathcal{Y}$*. In this case the right adjoint is given by the mate of this square, which is also a lax square.*

(ii) A 1-morphism in $\mathrm{FUN}(\mathcal{Y},\mathcal{X})_{\mathrm{colax}}$*, i.e. a colax natural transformation* $\phi\colon F\to G$*, is a right adjoint if and only if for every* $Y\in\mathcal{Y}$ *the morphism* $\phi_Y\colon F(Y)\to G(Y)$ *is a right adjoint, and the colax square*

$$\begin{tikzcd} F(Y) \arrow[r] \arrow[d] & G(Y) \arrow[d] \\ F(Y') \arrow[r] \arrow[ur, Rightarrow] & G(Y') \end{tikzcd}$$

commutes for all morphisms $Y\to Y'$ *in* $\mathcal{Y}$*. In this case the left adjoint is given by the mate of this square, which is also a colax square.*

Remark 4.7. Although we have not found a specific mention of the 2-categorical analogue of this statement, it can be seen as a special case of Kelly's theory of *doctrinal adjunctions* [17]*Theorem 1.4, at least if the target 2-category is cocomplete. A related result is [31]*Theorem 1, which shows that a lax transformation of functors from a category to Cat has a left adjoint colax transformation if and only if each component has a left adjoint.

Proof. Statement (i) for the $(\infty,2)$-category $\mathcal{X}$ is equivalent to statement (ii) for $\mathcal{X}^{2\text{-op}}$ using the equivalences of Corollary 3.14 and Remark 4.3. It thus suffices to prove (i).

We first suppose that ϕ is a left adjoint, so that there exists a lax natural transformation $\rho\colon G\to F$ that is its right adjoint, a unit $\eta\colon \mathrm{id}\to\rho\phi$ and a counit $\epsilon\colon\phi\rho\to\mathrm{id}$. Since any functor of $(\infty,2)$-categories preserves adjunctions, we then have that the component $\phi_Y\colon F(Y)\to G(Y)$ is a left adjoint in $\mathcal{X}$ with right adjoint ρ_Y, with unit and counit given by the components of η and ϵ at Y. For a 1-morphism $f\colon Y\to Y'$, the lax transformations ϕ and ρ supply lax squares

$$\begin{tikzcd} F(Y) \arrow[r, "\phi_Y"] \arrow[d, "F(f)"'] & G(Y) \arrow[d, "G(f)"] \arrow[dl, Rightarrow, "\phi(f)"] \\ F(Y') \arrow[r, "\phi_{Y'}"'] & G(Y'), \end{tikzcd} \qquad \begin{tikzcd} G(Y) \arrow[r, "\rho_Y"] \arrow[d, "G(f)"'] & F(Y) \arrow[d, "F(f)"] \arrow[dl, Rightarrow, "\rho(f)"] \\ G(Y') \arrow[r, "\rho_{Y'}"'] & F(Y'). \end{tikzcd}$$

and the 2-morphisms η and ϵ supply diagrams that amount to commutative diagrams of 1-morphisms

$$\begin{tikzcd} & F(f)\rho_Y\phi_Y \arrow[d, "\rho(f)\phi_Y"] \\ F(f) \arrow[ur, "F(f)\eta_Y"] \arrow[dr, "\eta_{Y'}F(f)"'] & \rho_{Y'}G(f)\phi_Y \arrow[d, "\rho_{Y'}\phi(f)"] \\ & \rho_{Y'}\phi_{Y'}F(f), \end{tikzcd} \qquad \begin{tikzcd} G(f)\phi_Y\rho_Y \arrow[d, "\phi(f)\rho_Y"'] \arrow[dr, "G(f)\epsilon_Y"] & \\ \phi_{Y'}F(f)\rho_Y \arrow[d, "\phi_{Y'}\rho(f)"'] & G(f) \\ \phi_{Y'}\rho_{Y'}F(f) \arrow[ur, "\epsilon_{Y'}F(f)"'] & \end{tikzcd}$$

The second lax square has a *mate square*, which is an oplax square

$$\begin{tikzcd}
F(Y) \arrow[r, "\phi_Y"] \arrow[d, "F(f)"'] & G(Y) \arrow[d, "G(f)"] \\
F(Y') \arrow[r, "\phi_{Y'}"'] \arrow[ur, phantom, "\Nearrow \psi(f)"] & G(Y'),
\end{tikzcd}$$

where $\psi(f) \colon \phi_{Y'}F(f) \to G(f)\phi_Y$ is the composite

$$\phi_{Y'}F(f) \xrightarrow{\phi_{Y'}F(f)\eta_Y} \phi_{Y'}F(f)\rho_Y\phi_Y \xrightarrow{\rho(f)} \phi_{Y'}\rho_{Y'}G(f)\phi_Y \xrightarrow{\epsilon_{Y'}G(f)\phi_Y} G(f)\phi_Y.$$

We claim that $\psi(f)$ is an inverse to the 2-morphism $\phi(f)$. Indeed, using the equivalences of 2-morphisms from the unit and counit we get commutative diagrams

$$\begin{tikzcd}
\phi_{Y'}F(f) \arrow[r, "\phi_{Y'}F(f)\eta_Y"] \arrow[rrr, bend left, "\phi_{Y'}\eta_{Y'}F(f)"] \arrow[rr, bend right, "\psi(f)"'] & \phi_{Y'}F(f)\rho_Y\phi_Y \arrow[r, "\phi_{Y'}\rho(f)\phi_Y"] & \phi_{Y'}\rho_{Y'}G(f)\phi_Y \arrow[r, "\phi_{Y'}\rho_{Y'}\phi(f)"] \arrow[d, "\epsilon_{Y'}G(f)\phi_Y"] & \phi_{Y'}\rho_{Y'}\phi_{Y'}F(f) \arrow[d, "\epsilon_{Y'}\phi_{Y'}F(f)"] \\
 & & G(f)\phi_Y \arrow[r, "\phi(f)"'] & \phi_{Y'}F(f),
\end{tikzcd}$$

$$\begin{tikzcd}
G(f)\phi_Y \arrow[r, "\phi(f)"] \arrow[d, "G(f)\phi_Y\eta_Y"] & \phi_{Y'}F(f) \arrow[d, "\phi_{Y'}F(f)\eta_Y"] \arrow[rrd, bend left, "\psi(f)"] & & \\
G(f)\phi_Y\rho_Y\phi_Y \arrow[r, "\phi(f)\rho_Y\phi_Y"] \arrow[rrr, bend right, "G(f)\phi_Y\epsilon_Y"'] & \phi_{Y'}F(f)\rho_Y\phi_Y \arrow[r, "\phi_{Y'}\rho(f)\phi_Y"] & \phi_{Y'}\rho_{Y'}G(f)\phi_Y \arrow[r, "\epsilon_{Y'}G(f)\phi_Y"] & G(f)\phi_Y.
\end{tikzcd}$$

Together with the adjunction equivalences these diagrams show that $\psi(f)$ is inverse to $\phi(f)$, and so $\phi(f)$ is invertible. Thus any left adjoint morphism in $\mathrm{FUN}(\mathcal{Y}, \mathcal{X})_{\mathrm{lax}}$ does indeed lie in $\mathrm{FUN}(\mathcal{Y}, \mathcal{X})$.

We now need to prove the converse, i.e. if we have a natural transformation $\phi \colon F \to G$ in $\mathrm{FUN}(\mathcal{Y}, \mathcal{X})$ such that ϕ_Y is a left adjoint for all Y, then ϕ is a left adjoint in $\mathrm{FUN}(\mathcal{Y}, \mathcal{X})_{\mathrm{lax}}$. Since the space of left adjoints in $\mathrm{FUN}(\mathcal{Y}, \mathcal{X})_{\mathrm{lax}}$ commutes with colimits in $\mathcal{Y}$, it suffices to show this for $\mathcal{Y}$ being C_0, C_1, and C_2 (with the case of C_0 being trivial). For the case of C_1 we have a commutative square

$$\begin{tikzcd}
A \arrow[r, "l"] \arrow[d, "a"] & B \arrow[d, "b"] \\
A' \arrow[r, "l'"] & B'
\end{tikzcd}$$

given by an equivalence $\iota \colon bl \xrightarrow{\sim} l'a$ and where l and l' are left adjoints, and we must show that this has a right adjoint in $\mathrm{FUN}(C_1, \mathcal{X})_{\mathrm{lax}}$. Let $r \colon b \to a$ and $r' \colon b' \to a'$ be right adjoints of l and l', and let

$$\eta \colon \mathrm{id}_A \to rl, \quad \epsilon \colon lr \to \mathrm{id}_B,$$

$$\eta' \colon \mathrm{id}_{A'} \to r'l', \quad \epsilon' \colon l'r' \to \mathrm{id}_{B'},$$

be unit and counit 2-morphisms. The right adjoint will be given by the mate square

$$
\begin{tikzcd}
B \arrow[r, "r"] \arrow[d, "b"'] & A \arrow[d, "a"] \arrow[dl, Rightarrow, "\psi"] \\
A' \arrow[r, "r'"] & B',
\end{tikzcd}
$$

where the 2-morphism $\psi\colon ar \to r'b$ is the composite

$$ar \xrightarrow{\eta' ar} r'l'ar \xrightarrow{r'\iota^{-1}r} r'blr \xrightarrow{r'b\epsilon} r'b.$$

Composing the original square with the mate we get lax squares

$$
\begin{tikzcd}
B \arrow[r, "lr"] \arrow[d, "b"'] & B \arrow[d, "b"] \arrow[dl, Rightarrow, "(l'\psi)(\iota r)"] \\
B' \arrow[r, "l'r'"'] & B',
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "rl"] \arrow[d, "a"'] & A \arrow[d, "a"] \arrow[dl, Rightarrow, "(r'\iota)(\psi l)"] \\
A' \arrow[r, "r'l'"'] & A'.
\end{tikzcd}
$$

Using Lemma 3.18, to define the unit and counit we must define diagrams of shape $[2](1,0) \cup_{C_2} [1]([1]^2) \cup_{C_2} [2](0,1)$ in $\mathfrak{X}$. These are given by using the units and counits of the two adjunctions together with commutative squares of 2-morphisms of the form

$$
\begin{tikzcd}
a \arrow[r, "a\eta"] \arrow[d, equal] & arl \arrow[d, "(r'\iota)(\psi l)"] \\
a \arrow[r, "\eta' a"] & r'l'a
\end{tikzcd}
\qquad
\begin{tikzcd}
blr \arrow[r, "b\epsilon"] \arrow[d, "(l'\psi)(\iota r)"'] & b \arrow[d, equal] \\
l'r'b \arrow[r, "\epsilon' b"] & b,
\end{tikzcd}
$$

which can be defined as the commutative diagrams

$$
\begin{tikzcd}
a \arrow[rrr, "a\eta"] \arrow[dr, "\eta' a"] \arrow[dddd, equal] & & & arl \arrow[d] \\
& r'l'a \arrow[rr] \arrow[dr, "r'\iota^{-1}"] \arrow[dddrr, equal, bend right] & & r'l'arl \arrow[d, "r'\iota^{-1}rl"] \\
& & r'bl \arrow[r] \arrow[dr, equal] \arrow[ddr, "r'\iota"'] & r'blrl \arrow[d] \\
& & & r'bl \arrow[d, "r'\iota"] \\
a \arrow[rrr, "\eta' a"] & & & r'l'a,
\end{tikzcd}
\qquad
\begin{tikzcd}
blr \arrow[rrr, "b\epsilon"] \arrow[d, "\iota r"] \arrow[ddr, "\iota r"] \arrow[dddrr, equal, bend left] & & & b \arrow[dddd, equal] \\
l'ar \arrow[d] \arrow[dr, equal] & & & \\
l'r'l'ar \arrow[r] \arrow[d, "l'r'\iota^{-1}r"] & l'ar \arrow[dr, "\iota^{-1}r"] & & \\
l'r'blr \arrow[rr] \arrow[d] & & blr \arrow[dr] & \\
l'r'b \arrow[rrr, "\epsilon' b"] & & & b,
\end{tikzcd}
$$

composed of naturality squares together with the adjunction equivalences for l and l' and the invertibility equivalence of ι.

To check the adjunction identities it is convenient to first give an alternative description of these diagrams: Recall that the unit and counit of the adjunction $l \dashv r$ can be described as mates:

$$
\begin{tikzcd}
A \arrow[r, equal] \arrow[d, equal] & A \arrow[d, "l"] \\
A \arrow[r, "l"] & B
\end{tikzcd}
\rightsquigarrow
\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "l"'] & A \arrow[d, equal] \arrow[dl, Rightarrow, "\eta"] \\
B \arrow[r, "r"] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "l"] \arrow[d, "l"'] & B \arrow[d, equal] \\
B \arrow[r, equal] & B
\end{tikzcd}
\rightsquigarrow
\begin{tikzcd}
B \arrow[r, "r"] \arrow[d, equal] & A \arrow[d, "l"] \arrow[dl, Rightarrow, "\epsilon"] \\
B \arrow[r, equal] & B
\end{tikzcd}
$$

The diagrams for the unit and counit above can then be obtained by taking mates horizontally in the following cubes:

$$
\begin{tikzcd}
A \arrow[rr, equal] \arrow[dr, equal] \arrow[dd, "a"'] & & A \arrow[dr, "l"] \arrow[dd, "a"] & \\
& A \arrow[rr, "l"] \arrow[dd, "a"'] & & B \arrow[dd, "b"] \\
A' \arrow[rr, equal] \arrow[dr, equal] & & A' \arrow[dr, "l'"] & \\
& A' \arrow[rr, "l'"] & & B'
\end{tikzcd}
\rightsquigarrow
\begin{tikzcd}
A \arrow[rr, equal] \arrow[dr, "l"] \arrow[dd, "a"'] & & A \arrow[dl, Rightarrow] \arrow[dr, equal] \arrow[dd, "a"] & \\
& B \arrow[rr, "r"] \arrow[dd, "b"'] & & A \arrow[dd, "a"] \arrow[dll, Rightarrow, bend left] \\
A' \arrow[rr, equal] \arrow[dr, "l'"'] & & A' \arrow[dl, Rightarrow] \arrow[dr, equal] & \\
& B' \arrow[rr, "r'"] & & A',
\end{tikzcd}
$$

$$
\begin{tikzcd}
A \arrow[rr, "l"] \arrow[dr, "l"] \arrow[dd, "a"'] & & B \arrow[dr, equal] \arrow[dd, "b"] & \\
& B \arrow[rr, equal] \arrow[dd, "b"'] & & B \arrow[dd, "b"] \\
A' \arrow[rr, "l'"] \arrow[dr, "l'"'] & & B' \arrow[dr, equal] & \\
& B' \arrow[rr, equal] & & B'
\end{tikzcd}
\rightsquigarrow
\begin{tikzcd}
B \arrow[rr, "r"] \arrow[dr, equal] \arrow[dd, "b"'] & & A \arrow[dl, Rightarrow] \arrow[ddll, Rightarrow] \arrow[dr, "l"] \arrow[dd, "a"] & \\
& B \arrow[rr, equal] \arrow[dd, "b"'] & & B \arrow[dd, "b"] \\
B' \arrow[rr, "r'"] \arrow[dr, equal] & & A' \arrow[dl, Rightarrow] \arrow[dr, "l'"] & \\
& B' \arrow[rr, equal] & & B'.
\end{tikzcd}
$$

With this description checking the adjunction identities amounts to showing that the following composite cubes are horizontal and vertical identities, respectively (here we have omitted the 2-morphisms to make the diagram legible):

$$
\begin{tikzcd}
A \arrow[rrr, equal] \arrow[dr, "l"] \arrow[ddd, "a"'] & & & A \arrow[dr, equal] \arrow[ddd, "a"] & & \\
& B \arrow[rrr, "r"] \arrow[dr, equal] \arrow[ddd, "b"'] & & & A \arrow[dr, "l"] \arrow[ddd, "a"] & \\
& & B \arrow[rrr, equal] \arrow[ddd, "b"'] & & & B \arrow[ddd, "b"] \\
A' \arrow[rrr, equal] \arrow[dr, "l'"'] & & & A' \arrow[dr, equal] & & \\
& B' \arrow[rrr, "r'"] \arrow[dr, equal] & & & A' \arrow[dr, "l'"] & \\
& & B' \arrow[rrr, equal] & & & B',
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[rr, "r"] \arrow[dr, equal] \arrow[dd, "b"'] & & A \arrow[rr, equal] \arrow[dr, "l"] \arrow[dd, "a"] & & A \arrow[dr, equal] \arrow[dd, "a"] & \\
& B \arrow[rr, equal] \arrow[dd, "b"] & & B \arrow[rr, "r"] \arrow[dd, "b"] & & A \arrow[dd, "a"] \\
B' \arrow[rr, "r'"] \arrow[dr, equal] & & A' \arrow[rr, equal] \arrow[dr, "l'"] & & A' \arrow[dr, equal] & \\
& B' \arrow[rr, equal] & & B' \arrow[rr, "r'"] & & A'.
\end{tikzcd}
$$

Since taking mates is compatible with horizontal and vertical composition of squares, these composite cubes are obtained by taking horizontal mates in the composite cubes

$$
\begin{tikzcd}
A \arrow[rrr, equal] \arrow[dr, equal] \arrow[ddd, "a"'] & & & A \arrow[dr, "l"] \arrow[ddd, "a"] & & \\
& A \arrow[rrr, "l"] \arrow[dr, "l"] \arrow[ddd, "a"'] & & & B \arrow[dr, equal] \arrow[ddd, "b"] & \\
& & B \arrow[rrr, equal] \arrow[ddd, "b"'] & & & B \arrow[ddd, "b"] \\
A' \arrow[rrr, equal] \arrow[dr, equal] & & & A' \arrow[dr, "l'"] & & \\
& A' \arrow[rrr, "l'"] \arrow[dr, "l'"'] & & & B' \arrow[dr, equal] & \\
& & B' \arrow[rrr, equal] & & & B',
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[rr, equal] \arrow[dr, equal] \arrow[dd, "a"'] & & A \arrow[rr, "l"] \arrow[dr, "l"] \arrow[dd, "a"] & & B \arrow[dr, equal] \arrow[dd, "b"] & \\
& A \arrow[rr, "l"] \arrow[dd, "a"] & & B \arrow[rr, equal] \arrow[dd, "b"] & & B \arrow[dd, "b"] \\
A' \arrow[rr, equal] \arrow[dr, equal] & & A' \arrow[rr, "l'"] \arrow[dr, "l'"] & & B' \arrow[dr, equal] & \\
& A' \arrow[rr, "l'"] & & B' \arrow[rr, equal] & & B'.
\end{tikzcd}
$$

Since these composites are clearly identities, we have proved the adjunction identities for C_1.

We now discuss the case C_2. Here the putative left adjoint is a commutative cylindrical diagram

$$
\begin{tikzcd}
A \arrow[r, "l"] \arrow[d, bend right, "a"'] \arrow[d, bend left, "a'"] \arrow[d, phantom, "\overset{\alpha}{\Longrightarrow}"] & B \arrow[d, bend right, "b"'] \arrow[d, bend left, "b'"] \arrow[d, phantom, "\overset{\beta}{\Longrightarrow}"] \\
A' \arrow[r, "l'"] & B'
\end{tikzcd}
$$

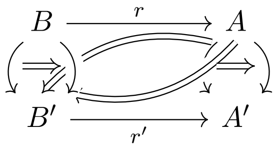
where l and l' are left adjoints. The commutativity data amounts to equivalences

$$bl \xrightarrow{\sim} l'a, \qquad b'l \xrightarrow{\sim} l'a'$$

together with a commutative square of morphisms $A \to B'$

$$\begin{tikzcd} bl \arrow[r, "\beta l"] \arrow[d, "\sim"] & b'l \arrow[d, "\sim"] \\ l'a \arrow[r, "l'\alpha"] & l'a'. \end{tikzcd}$$

The right adjoint is then given by a diagram of shape



where r and r' are the right adjoints of l and l', and the front and back lax squares are defined as above in the case C_1. The additional coherence data required amounts to a commutative square of morphisms, which we define as the composite of the diagram

$$\begin{tikzcd} ar \arrow[r, "\alpha r"] \arrow[d, "\eta' ar"] & a'r \arrow[d, "\eta' a'r"] \\ r'l'ar \arrow[r, "r'l'\alpha r"] \arrow[d, "\sim"] & r'l'a'r \arrow[d, "\sim"] \\ r'blr \arrow[r, "r'\beta lr"] \arrow[d, "r'b\epsilon"] & r'b'lr \arrow[d, "r'b'\epsilon"] \\ r'b \arrow[r, "r'\beta"] & r'b' \end{tikzcd}$$

using the specified square and two naturality squares.

The decomposition in Lemma 3.18 implies that the non-obvious part of defining the unit and counit (using the unit and counit for $l \dashv r$ and $l' \dashv r'$) is specifying two commutative cubes, which we can define using naturality data for the (co)units defined in the C_1-case; this can also be thought of as taking mates in one direction in a 4-dimensional cube. Naturality of taking mates then gives the adjunction identities by a 4-dimensional version of the argument above. □

Corollary 4.8. *Let* $\mathrm{FUN}(C_1, \mathcal{X})_{\mathrm{ladj}}$ *and* $\mathrm{FUN}(C_1, \mathcal{X})_{\mathrm{radj}}$ *denote the full sub-*$(\infty, 2)$*-categories of* $\mathrm{FUN}(C_1, \mathcal{X})$ *containing only the left and right adjoint morphisms in* $\mathcal{X}$*, respectively. There are equivalences*

$$\mathrm{ADJ}(\mathcal{X})_{\mathrm{lax}} \xrightarrow{\sim} \mathrm{FUN}(C_1, \mathcal{X})_{\mathrm{radj}},$$

$$\mathrm{ADJ}(\mathcal{X})_{\mathrm{colax}} \xrightarrow{\sim} \mathrm{FUN}(C_1, \mathcal{X})_{\mathrm{ladj}},$$

given by composition with the morphisms $C_1 \to \mathfrak{adj}$ *picking out the right and left adjoint 1-morphisms, respectively.*

Proof. We prove the first equivalence, the proof of the second is similar. For any $(\infty, 2)$-category $\mathcal{Z}$ we have natural equivalences

$$\mathrm{Map}(\mathcal{Z}, \mathrm{ADJ}(\mathcal{X})_{\mathrm{lax}}) \simeq \mathrm{Map}(\mathfrak{adj} \otimes^{\mathrm{lax}} \mathcal{Z}, \mathcal{X}) \simeq \mathrm{Map}(\mathfrak{adj}, \mathrm{FUN}(\mathcal{Z}, \mathcal{X})_{\mathrm{colax}}).$$

By Theorem 4.4 evaluation at the right adjoint gives an equivalence

$$\mathrm{Map}(\mathfrak{adj}, \mathrm{FUN}(\mathcal{Z}, \mathcal{X})_{\mathrm{colax}}) \xrightarrow{\sim} \mathrm{Map}(C_1, \mathrm{FUN}(\mathcal{Z}, \mathcal{X})_{\mathrm{colax}})^{\mathrm{radj}},$$

which by Theorem 4.6 is equivalent to the space of natural transformations, i.e. 1-morphisms in $\mathrm{FUN}(\mathcal{Z}, \mathcal{X})$, that are levelwise right adjoints. In other words, this is precisely $\mathrm{Map}(\mathcal{Z}, \mathrm{FUN}(C_1, \mathcal{X})_{\mathrm{radj}})$. □

As an immediate consequence, we get the following naturality statement for the process of taking mates of commutative squares:

Corollary 4.9. *There are natural functors of $(\infty, 2)$-categories*

$$\mathrm{FUN}(C_1, \mathcal{X})_{\mathrm{radj}} \xleftarrow{\sim} \mathrm{ADJ}(\mathcal{X})_{\mathrm{lax}} \to \mathrm{FUN}(C_1, \mathcal{X})_{\mathrm{ladj,lax}},$$

$$\mathrm{FUN}(C_1, \mathcal{X})_{\mathrm{ladj}} \xleftarrow{\sim} \mathrm{ADJ}(\mathcal{X})_{\mathrm{colax}} \to \mathrm{FUN}(C_1, \mathcal{X})_{\mathrm{radj,colax}}$$

given on objects by passing to the other adjoint and on morphisms by taking mates. □

Remark 4.10. The procedure of taking mates for functors of ∞-categories has previously been considered in [19] and [3] in the case where the mate is invertible, and more generally in the book [8].

Remark 4.11. Let $\mathfrak{adj}^n := \mathfrak{adj} \amalg_{[0]} \cdots \amalg_{[0]} \mathfrak{adj}$ denote the ∞-category of n composable adjunctions. Then Theorem 4.4 furnishes an equivalence

$$\mathrm{Map}(\mathfrak{adj}^n, \mathcal{X}) \simeq \mathrm{Map}(\Delta^n, \mathcal{X})^{\mathrm{radj}},$$

where the right-hand side denotes the subspace of $\mathrm{Map}(\Delta^n, \mathcal{X})$ of composable sequences of n right adjoints. Since right adjoints are closed under composition, we have a simplicial space $\mathrm{Map}(\Delta^\bullet, \mathcal{X})^{\mathrm{radj}}$ natural in the $(\infty, 2)$-category $\mathcal{X}$, which by the Yoneda lemma implies that the representing objects $\mathfrak{adj}^\bullet$ also form a simplicial object. We can then upgrade Corollary 4.8 to a natural equivalence

$$\mathrm{FUN}(\mathfrak{adj}^\bullet, \mathcal{X})_{\mathrm{lax}} \xrightarrow{\sim} \mathrm{FUN}(\Delta^\bullet, \mathcal{X})_{\mathrm{radj}}.$$

This implies that taking mates is compatible with composition, since we get a composite functor

$$\mathrm{FUN}(\Delta^\bullet, \mathcal{X})_{\mathrm{radj}} \xleftarrow{\sim} \mathrm{FUN}(\mathfrak{adj}^\bullet, \mathcal{X})_{\mathrm{lax}} \to \mathrm{FUN}((\Delta^\bullet)^{\mathrm{op}}, \mathcal{X})_{\mathrm{ladj,lax}},$$

where we write $(\Delta^\bullet)^{\mathrm{op}}$ to emphasize that the order of composition is reversed.

Remark 4.12. In his thesis [32], D. Zaganidis considers for $I \in \mathbf{\Theta}_2$ the universal 2-category $\mathfrak{adj}(I)_{\mathrm{lax}}$ with an I-shaped diagram of lax morphisms of adjunctions and shows that this also satisfies the universal property

$$\mathrm{Map}_{\mathrm{Cat}_{(\infty,2)}}(\mathfrak{adj}(I)_{\mathrm{lax}}, \mathcal{X}) \simeq \mathrm{Map}_{\mathrm{Cat}_{(\infty,2)}}(I, \mathrm{FUN}(C_1, \mathcal{X})_{\mathrm{radj}}).$$

It follows that $\mathfrak{adj}(I)_{\mathrm{lax}}$ is equivalent to the Gray tensor product $\mathfrak{adj} \otimes^{\mathrm{lax}} I$. In fact, it is not hard to see from the explicit definition of $\mathfrak{adj}(I)_{\mathrm{lax}}$ that this is the classical Gray tensor product of $\mathfrak{adj}$ and I, and so agrees with $\mathfrak{adj} \otimes^{\mathrm{lax}} I$ under Assumption 3.5.

5. Lax Morphisms of Monads

We now turn to monads and (co)lax morphisms between them, which again arise as (co)lax natural transformations.

Notation 5.1. Let $\mathfrak{mnd}$ denote the full subcategory of $\mathfrak{adj}$ on the object $-$; this is the "walking monad" 2-category.

Definition 5.2. Let $\mathfrak{X}$ be an $(\infty, 2)$-category. A *monad* in $\mathfrak{X}$ is a functor of $(\infty, 2)$-categories $\mathfrak{mnd} \to \mathfrak{X}$, and a *(co)lax morphism* of monads is a (co)lax natural transformation between monads, i.e. a functor

$$\mathfrak{mnd} \otimes^{(\mathrm{co})\mathrm{lax}} C_1 \to \mathfrak{X}.$$

We write $\mathrm{MND}(\mathfrak{X})_{(\mathrm{co})\mathrm{lax}} := \mathrm{FUN}(\mathfrak{mnd}, \mathfrak{X})_{(\mathrm{co})\mathrm{lax}}$ for the $(\infty, 2)$-category of monads in $\mathfrak{X}$ and (co)lax morphisms between them, and $\mathrm{Mnd}(\mathfrak{X})_{(\mathrm{co})\mathrm{lax}}$ for the underlying ∞-category.

Remark 5.3. For ordinary 2-categories, the notions of lax and colax morphisms of monads were first introduced by Street [30], who called them *monad functors* and *monad opfunctors*.

We will use results of Zaganidis to relate $\mathrm{MND}(\mathfrak{X})_{\mathrm{lax}}$ to $(\infty, 2)$-categories of monadic adjunctions and monadic right adjoints; as Zaganidis works in the framework of categories strictly enriched in quasicategories developed by Riehl and Verity, this requires the following additional assumption on the $(\infty, 2)$-category $\mathfrak{X}$:

Definition 5.4. We say an $(\infty, 2)$-category $\mathfrak{X}$ is *cosmifiable* if it can be modelled by a category strictly enriched in quasicategories that is an *∞-cosmos* in the sense of Riehl and Verity [28]*Definition 1.2.1.

Examples 5.5.

(i) The $(\infty, 2)$-category CAT_∞ is cosmifiable (it can be modelled by the simplicial category of quasicategories).

(ii) If $\mathfrak{X}$ is cosmifiable, then so is $\mathfrak{X}^{2\text{-op}}$ by [28]*Definition 1.2.25. In particular, $\mathrm{CAT}_\infty^{2\text{-op}}$ is cosmifiable.

In his thesis [32], Zaganidis considers the full sub-2-category $\mathfrak{mnd}(I)_{\mathrm{lax}}$ of $\mathfrak{adj}(I)_{\mathrm{lax}}$ (see Remark 4.12) consisting of an I-shaped diagram of lax morphisms of monads, which can be identified with the classical Gray tensor product of $\mathfrak{mnd}$ and I. Under Assumption 3.5, this means that $\mathfrak{mnd}(I)_{\mathrm{lax}}$ corresponds to $\mathfrak{mnd} \otimes^{\mathrm{lax}} I$. We can then state the main result of [32] as follows:

Theorem 5.6 (Zaganidis)**.** *Let $\mathfrak{X}$ be a cosmifiable $(\infty, 2)$-category. The restrictions*

$$\mathrm{Fun}(\mathfrak{adj} \otimes^{\mathrm{lax}} I, \mathfrak{X}) \to \mathrm{Fun}(\mathfrak{mnd} \otimes^{\mathrm{lax}} I, \mathfrak{X})$$

for $I \in \mathbf{\Theta}_2$ have fully faithful right adjoints, with image those functors $\mathfrak{adj} \otimes^{\mathrm{lax}} I \to \mathfrak{X}$ where the underlying adjunction at each object of I is monadic.

Remark 5.7. In the case $I = C_0$, the right adjoint

$$\mathrm{Fun}(\mathfrak{mnd}, \mathfrak{X}) \to \mathrm{Fun}(\mathfrak{adj}, \mathfrak{X})$$

is due to Riehl and Verity. Morally, this adjoint is given by an $(\infty, 2)$-categorical right Kan extension along the inclusion $\mathfrak{mnd} \hookrightarrow \mathfrak{adj}$ and should thus exist for any target $(\infty, 2)$-category where certain weighted limits exist. (Indeed, it seems plausible that the property of being "cosmifiable" precisely amounts to the existence of a class of such limits.) The ideal proof of Theorem 5.6 would then simply check that these weighted limits exist in $\mathrm{FUN}(I, \mathcal{X})_{\mathrm{colax}}$. However, the theory of weighted limits in $(\infty, 2)$-categories has not yet been set up. Riehl and Verity circumvent this by modelling $(\infty, 2)$-categories as categories enriched in quasicategories, and showing that certain ordinary weighted limits are homotopically meaningful, and Zaganidis applies the same technique to $\mathfrak{mnd}(I)_{\mathrm{lax}} \to \mathfrak{adj}(I)_{\mathrm{lax}}$.

This has the following consequence:

Corollary 5.8. *Let $\mathcal{X}$ be a cosmifiable $(\infty, 2)$-category. The forgetful functor*

$$\mathrm{ADJ}(\mathcal{X})_{\mathrm{lax}} \to \mathrm{MND}(\mathcal{X})_{\mathrm{lax}}$$

has a fully faithful right adjoint, with image the full sub-$(\infty, 2)$-category $\mathrm{ADJ}(\mathcal{X})_{\mathrm{lax,mnd}}$ of monadic adjunctions. In particular, there are equivalences of $(\infty, 2)$-categories

$$\mathrm{MND}(\mathcal{X})_{\mathrm{lax}} \xrightarrow{\sim} \mathrm{ADJ}(\mathcal{X})_{\mathrm{lax,mnd}} \xrightarrow{\sim} \mathrm{FUN}(\Delta^1, \mathcal{X})_{\mathrm{mndradj}},$$

where the latter denotes the full sub-$(\infty, 2)$-category of $\mathrm{FUN}(\Delta^1, \mathcal{X})$ whose objects are the monadic right adjoints.

Remark 5.9. For monads on a single, fixed ∞-category $\mathcal{C}$ (interpreted as associative algebras in endomorphisms of $\mathcal{C}$) this comparison (in the case where $\mathcal{X}$ is CAT_∞) has previously been obtained by Heine [15] by a different method.

Remark 5.10. By Examples 5.5(ii) if $\mathcal{X}$ is cosmifiable then so is $\mathcal{X}^{2\text{-op}}$. Here we have equivalences

$$\mathrm{ADJ}(\mathcal{X}^{2\text{-op}})_{\mathrm{lax}} \simeq \mathrm{FUN}(\mathfrak{adj}^{2\text{-op}}, \mathcal{X})^{2\text{-op}}_{\mathrm{colax}} \simeq \mathrm{ADJ}(\mathcal{X})^{2\text{-op}}_{\mathrm{colax}},$$

using the equivalence $\mathfrak{adj}^{2\text{-op}} \simeq \mathfrak{adj}$ of Remark 4.3 together with Corollary 3.14, and

$$\mathrm{MND}(\mathcal{X}^{2\text{-op}})_{\mathrm{lax}} \simeq \mathrm{FUN}(\mathfrak{mnd}^{2\text{-op}}, \mathcal{X})^{2\text{-op}}_{\mathrm{colax}} =: \mathrm{COMND}(\mathcal{X})^{2\text{-op}}_{\mathrm{colax}}.$$

Note that here $\mathfrak{mnd}^{2\text{-op}}$ is equivalent to the full subcategory of $\mathfrak{adj}$ on the object $+$. Applying Corollary 5.8 to $\mathcal{X}^{2\text{-op}}$ and reversing 2-morphisms we then see that the forgetful functor

$$\mathrm{ADJ}(\mathcal{X})_{\mathrm{colax}} \to \mathrm{COMND}(\mathcal{X})_{\mathrm{colax}}$$

given by restriction to the object $+$ has a fully faithful *left* adjoint with image the full sub-$(\infty, 2)$-category of *comonadic* adjunctions.

For the proof of Corollary 5.8 we use the following observation:

Lemma 5.11. *Suppose $\mathcal{X}$ is an $(\infty, 2)$-category and $\phi \colon \mathcal{A} \to \mathcal{B}$ is a functor of $(\infty, 2)$-categories such that for every $I \in \mathbf{\Theta}_2$ the induced functor*

$$(\phi \otimes^{\mathrm{lax}} I)^* \colon \mathrm{Fun}(\mathcal{B} \otimes^{\mathrm{lax}} I, \mathcal{X}) \to \mathrm{Fun}(\mathcal{A} \otimes^{\mathrm{lax}} I, \mathcal{X})$$

has a right adjoint R_I, and for every morphism in $\mathbf{\Theta}_2$ the mate square for these adjoints commutes. Then the functor

$$\phi^* \colon \mathrm{FUN}(\mathcal{B}, \mathcal{X})_{\mathrm{lax}} \to \mathrm{FUN}(\mathcal{A}, \mathcal{X})_{\mathrm{lax}}$$

has a right adjoint, given on objects by R_{C_0}, and with unit and counit transformations given objectwise by the unit and counit for R_{C_0} and on morphisms by the unit and counit for R_{C_1}.

Proof. The functors $(\phi \otimes^{\text{lax}} I)^*$ are natural in I, and so give a morphism in $\text{Fun}(\boldsymbol{\Theta}_2^{\text{op}}, \text{Cat}_\infty)$, with ϕ^* given as the induced morphism in $\text{Fun}(\boldsymbol{\Theta}_2^{\text{op}}, \mathcal{S})$. The right adjoints assemble to a morphism in $\text{FUN}(\boldsymbol{\Theta}_2^{\text{op}}, \text{CAT}_\infty)_{\text{lax}}$ by Theorem 4.6, but as the mate squares commute this a priori lax natural transformation is an ordinary natural transformation, and the units and counits determine an adjunction in $\text{FUN}(\boldsymbol{\Theta}_2^{\text{op}}, \text{CAT}_\infty)$. Passing to underlying ∞-groupoids, the right adjoints give a functor $R\colon \text{FUN}(\mathcal{A}, \mathcal{X})_{\text{lax}} \to \text{FUN}(\mathcal{B}, \mathcal{X})_{\text{lax}}$ given on objects by R_{C_0}, as required. To obtain the unit and counit transformations, observe that for $(\infty,2)$-categories $\mathcal{U}, \mathcal{V}, \mathcal{W}$ there is a natural map $\mathcal{U} \otimes^{\text{lax}} (\mathcal{V} \times \mathcal{W}) \to (\mathcal{U} \otimes^{\text{lax}} \mathcal{V}) \times \mathcal{W}$, determined by the natural maps $\mathcal{U} \otimes^{\text{lax}} (\mathcal{V} \times \mathcal{W}) \to \mathcal{U} \otimes^{\text{lax}} \mathcal{V}$ and $\mathcal{U} \otimes^{\text{lax}} (\mathcal{V} \times \mathcal{W}) \to \mathcal{U} \otimes^{\text{lax}} \mathcal{W} \to \mathcal{W}$. The levelwise unit gives maps

$$\text{Map}(I, \text{FUN}(\mathcal{B}, \mathcal{X})_{\text{lax}}) \simeq \text{Map}(\mathcal{B} \otimes^{\text{lax}} I, \mathcal{X}) \to \text{Map}([1], \text{Fun}(\mathcal{B} \otimes^{\text{lax}} I, \mathcal{X}) \simeq \text{Map}((\mathcal{B} \otimes^{\text{lax}} I) \times [1], \mathcal{X}),$$

natural in I, where we can now apply the map $\mathcal{B} \otimes^{\text{lax}} (I \times [1]) \to (\mathcal{B} \otimes^{\text{lax}} I) \times [1]$ to get a natural map

$$\text{Map}(I, \text{FUN}(\mathcal{B}, \mathcal{X})_{\text{lax}}) \to \text{Map}(\mathcal{B} \otimes^{\text{lax}} (I \times [1]), \mathcal{X}) \simeq \text{Map}(I, \text{FUN}([1], \text{FUN}(\mathcal{B}, \mathcal{X})_{\text{lax}})),$$

which corresponds to a functor of $(\infty,2)$-categories

$$\text{FUN}(\mathcal{B}, \mathcal{X})_{\text{lax}} \times [1] \to \text{FUN}(\mathcal{B}, \mathcal{X})_{\text{lax}},$$

as required. By naturality the two diagrams

$$\begin{tikzcd}
\mathcal{B} \otimes^{\text{lax}} (I \times [0]) \arrow[r, "\sim"] \arrow[d] & (\mathcal{B} \otimes^{\text{lax}} I) \times [0] \arrow[d] \\
\mathcal{B} \otimes^{\text{lax}} (I \times [1]) \arrow[r] & (\mathcal{B} \otimes^{\text{lax}} I) \times [1]
\end{tikzcd}$$

commute, which implies that this is a natural transformation from the identity to $R\phi^*$. Similarly the levelwise counits give a natural transformation

$$\text{FUN}(\mathcal{A}, \mathcal{X})_{\text{lax}} \times [1] \to \text{FUN}(\mathcal{A}, \mathcal{X})_{\text{lax}}$$

from $\phi^* R$ to the identity. To show that this gives an adjunction it suffices to check that the induced natural transformations $R \to R$ and $\phi^* \to \phi^*$ are given by equivalences for all objects and morphisms, which is clear since these are then induced by the adjunction equivalences for R_{C_0} and R_{C_1}. □

Proof of Corollary 5.8. Apply Lemma 5.11 to the adjunctions from Theorem 5.6. □

Corollary 5.12. *Let $\mathcal{X}$ be a cosmifiable $(\infty,2)$-category. The inclusion*

$$\text{Fun}([1], \iota_1\mathcal{X})_{\text{mndradj}} \hookrightarrow \text{Fun}([1], \iota_1\mathcal{X})_{\text{radj}}$$

has a left adjoint, which takes a right adjoint to the right adjoint of the associated monadic adjunction. □

Corollary 5.13. *Let $\mathcal{X}$ be a cosmifiable $(\infty,2)$-category. The functor $\text{Mnd}(\mathcal{X})_{\text{lax}} \to \iota_1\mathcal{X}$, taking a monad to the object it acts on, has cocartesian morphisms over morphisms in $\mathcal{X}$ that are right adjoints. If T is a monad on X and $\rho\colon X \to Y$ is a morphism with left adjoint λ then the cocartesian morphism over ρ has target $\rho T \lambda$ and is given by the transformation $\rho T \lambda \rho \to \rho T$ coming from the counit.*

Proof. We have a commutative diagram

$$\begin{tikzcd} \mathrm{Adj}(\mathcal{X})_{\mathrm{lax}} \arrow[r, "\sim"] \arrow[dr] & \mathrm{Fun}([1],\iota_1\mathcal{X})_{\mathrm{radj}} \arrow[r, hook] \arrow[d] & \mathrm{Fun}([1],\iota_1\mathcal{X}). \arrow[dl, "\mathrm{ev}_1"] \\ & \iota_1\mathcal{X}. & \end{tikzcd}$$

Here $\mathrm{ev}_1\colon \mathrm{Fun}([1],\iota_1\mathcal{X}) \to \iota_1\mathcal{X}$ is a cocartesian fibration, with the cocartesian morphisms given by composition. Since a composite of right adjoints is a right adjoint, the full subcategory $\mathrm{Fun}([1],\iota_1\mathcal{X})_{\mathrm{radj}}$ has cocartesian morphisms over maps in $\iota_1\mathcal{X}$ that are right adjoints, hence the same is true for the equivalent ∞-category $\mathrm{Adj}(\mathcal{X})_{\mathrm{lax}}$.

Now observe that we have a commutative triangle

$$\begin{tikzcd} \mathrm{Adj}(\mathcal{X})_{\mathrm{lax}} \arrow[rr, "L"] \arrow[dr] & & \mathrm{Mnd}(\mathcal{X})_{\mathrm{lax}} \arrow[dl] \\ & \iota_1\mathcal{X}, & \end{tikzcd}$$

and that the right adjoint $R\colon \mathrm{Mnd}(\mathcal{X})_{\mathrm{lax}} \hookrightarrow \mathrm{Adj}(\mathcal{X})_{\mathrm{lax}}$ also commutes with the functors to $\iota_1\mathcal{X}$. In this situation L necessarily takes a cocartesian morphism in $\mathrm{Adj}(\mathcal{X})_{\mathrm{lax}}$ to a cocartesian morphism in $\mathrm{Mnd}(\mathcal{X})_{\mathrm{lax}}$, hence $\mathrm{Mnd}(\mathcal{X})_{\mathrm{lax}}$ also has cocartesian morphisms over right adjoints in $\iota_1\mathcal{X}$. The description of the cocartesian morphisms in $\mathrm{Mnd}(\mathcal{X})_{\mathrm{lax}}$ now follows from the description of those in $\mathrm{Adj}(\mathcal{X})_{\mathrm{lax}}$. □

6. Lax Morphisms of Endofunctors

In this section we briefly consider endofunctors and (co)lax morphisms between them, and the forgetful functor from our $(\infty,2)$-categories of monads and (co)lax morphisms.

Definition 6.1. Let $\mathfrak{end}$ be the universal ∞-category with an endomorphism, given by the pushout square

$$\begin{tikzcd} \partial C_1 \arrow[r] \arrow[d] & C_0 \arrow[d] \\ C_1 \arrow[r] & \mathfrak{end}. \end{tikzcd}$$

Then $\mathfrak{end}$ can be identified with the 1-category $B\mathbb{N}$ corresponding to the free monoid $\mathbb{N}$. If $\mathcal{X}$ is an $(\infty,2)$-category, we define

$$\mathrm{END}(\mathcal{X})_{\mathrm{(co)lax}} := \mathrm{FUN}(\mathfrak{end},\mathcal{X})_{\mathrm{(co)lax}},$$

and write $\mathrm{End}(\mathcal{X})_{\mathrm{(co)lax}}$ for the underlying ∞-category.

Remark 6.2. Since the Gray tensor product preserves colimits in each variable by Assumption 3.5, the pushout square above induces a pullback square of $(\infty,2)$-categories

$$\begin{tikzcd} \mathrm{END}(\mathcal{X})_{\mathrm{(co)lax}} \arrow[r] \arrow[d] & \mathrm{FUN}(C_1,\mathcal{X})_{\mathrm{(co)lax}} \arrow[d] \\ \mathcal{X} \arrow[r, "\mathrm{diag}"] & \mathcal{X}\times\mathcal{X}, \end{tikzcd}$$

where the right vertical map is given by composition with the two inclusions $C_0 \hookrightarrow C_1$.

Remark 6.3. There is also a functor $\mathfrak{end} \to \mathfrak{mnd}$ picking out the underlying endofunctor of the universal monad, which induces a commutative triangle

$$\begin{tikzcd} \mathrm{MND}(\mathcal{X})_{\mathrm{(co)lax}} \arrow[rr] \arrow[dr] & & \mathrm{END}(\mathcal{X})_{\mathrm{(co)lax}} \arrow[dl] \\ & \mathcal{X}. & \end{tikzcd}$$

In the case where $\mathcal{X}$ is the $(\infty, 2)$-category of ∞-categories, we make some observations on how the cocartesian morphisms of Corollary 5.13 behave in this triangle:

Proposition 6.4.

(i) The projection $\mathrm{End}(\mathrm{CAT}_\infty)_{\mathrm{lax}} \to \mathrm{Cat}_\infty$ *has locally cocartesian morphisms and locally cartesian morphisms over functors that are right adjoints.*

(ii) The forgetful functor $\mathrm{Mnd}(\mathrm{CAT}_\infty)_{\mathrm{lax}} \to \mathrm{End}(\mathrm{CAT}_\infty)_{\mathrm{lax}}$ *preserves these locally cocartesian morphisms.*

Proof. We first prove (i). Suppose $(\mathcal{C}, P)$ and $(\mathcal{D}, Q)$ are objects of $\mathrm{End}(\mathrm{CAT}_\infty)_{\mathrm{lax}}$ and $R \colon \mathcal{C} \to \mathcal{D}$ is a morphism with a left adjoint $L \colon \mathcal{D} \to \mathcal{C}$. Then

$$\mathrm{Map}_{\mathrm{End}(\mathrm{CAT}_\infty)_{\mathrm{lax}}}((\mathcal{C}, P), (\mathcal{D}, Q))_R \simeq \mathrm{Map}_{\mathrm{Fun}(\mathcal{D},\mathcal{C})}(QR, RP).$$

From the adjunction identities it is immediate that this space is equivalent to $\mathrm{Map}_{\mathrm{End}(\mathcal{D})}(Q, RPL)$, so the natural transformation $RPLR \to RP$ coming from the counit $LR \to \mathrm{id}$ gives a locally cocartesian morphism over R from P to RPL. Similarly, the space is equivalent to $\mathrm{Map}_{\mathrm{End}(\mathcal{C})}(LQR, P)$, and the natural transformation $QR \to RLQR$ coming from the unit $\mathrm{id} \to RL$ gives a locally cartesian morphism.

(ii) is now clear from the description of the locally cocartesian morphisms in Corollary 5.13. □

Definition 6.5. Let $\mathrm{Cat}_\infty^{\mathrm{radj}}$ denote the subcategory of Cat_∞ containing only the morphisms that are right adjoints. Then we define $\mathrm{Mnd}(\mathrm{CAT}_\infty)^{\mathrm{radj}}_{\mathrm{lax}}$ and $\mathrm{End}(\mathrm{CAT}_\infty)^{\mathrm{radj}}_{\mathrm{lax}}$ by pulling back $\mathrm{Mnd}(\mathrm{CAT}_\infty)_{\mathrm{lax}}$ and $\mathrm{End}(\mathrm{CAT}_\infty)_{\mathrm{lax}}$ along the inclusion $\mathrm{Cat}_\infty^{\mathrm{radj}} \to \mathrm{Cat}_\infty$, i.e. we restrict to those lax morphisms between monads and endofunctors whose underlying morphism in Cat_∞ is a right adjoint.

Corollary 6.6. *There is a commuting triangle*

$$\begin{tikzcd} \mathrm{Mnd}(\mathrm{CAT}_\infty)^{\mathrm{radj}}_{\mathrm{lax}} \arrow[rr] \arrow[dr] & & \mathrm{End}(\mathrm{CAT}_\infty)^{\mathrm{radj}}_{\mathrm{lax}} \arrow[dl] \\ & \mathrm{Cat}_\infty^{\mathrm{radj}}, & \end{tikzcd}$$

where the two downward functors are cocartesian fibrations, and the horizontal functor preserves cocartesian morphisms. Moreover, the right-hand functor is also a cartesian fibration.

Proof. We know that the two downward functors are locally cocartesian fibrations, and that the horizontal functor preserves locally cocartesian morphisms. It then suffices by [20]*Proposition 2.4.2.8 to show that the locally cocartesian morphisms in $\mathrm{End}(\mathrm{CAT}_\infty)^{\mathrm{radj}}_{\mathrm{lax}}$ are closed under composition, which is clear from our description of these morphisms. Similarly, the right-hand functor is a cartesian fibration. □

7. Lax Transformations and Icons

Recall that, as we discussed in §2, we can view $\mathrm{Cat}_{(\infty,2)}$ as a full subcategory of $\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Cat}_\infty)$. The straightening–unstraightening equivalence identifies the latter ∞-category with the ∞-category $\mathrm{Cat}^{\mathrm{cocart}}_{\infty/\mathbf{\Delta}^{\mathrm{op}}}$ of cocartesian fibrations over $\mathbf{\Delta}^{\mathrm{op}}$ and functors over $\mathbf{\Delta}^{\mathrm{op}}$ that preserve cocartesian morphisms. This ∞-category has a natural enhancement to an $(\infty,2)$-category where the 2-morphisms are natural transformations over $\mathbf{\Delta}^{\mathrm{op}}$ between such functors, and this restricts to a notion of 2-morphism between functors of $(\infty,2)$-categories that is quite different from the usual notion of 2-morphisms as natural transformations.[7] For example, if we view monoidal ∞-categories $\mathcal{M}, \mathcal{M}'$ as (pointed) $(\infty,2)$-categories with one object, then a (pointed) functor of $(\infty,2)$-categories $\mathcal{M} \to \mathcal{M}'$ is a monoidal functor, but a natural transformation in the usual sense between two such functors $F, G\colon \mathcal{M} \to \mathcal{M}'$ amounts to specifying an object $x \in \mathcal{M}'$ and a natural equivalence

$$F(-) \otimes x \simeq x \otimes G(-),$$

while our new notion of 2-morphism gives precisely the monoidal natural transformations.

Our goal in this section is to identify these "new" 2-morphisms with certain colax natural transformations, namely those given at each object by an equivalence. This amounts to an ∞-categorical version of a result of Lack [18], who refers to this class of colax transformations as "icons".[8] In §8 we will use this result to identify two ∞-categories of monads on a fixed ∞-category.

It is convenient to view our new 2-morphisms in terms of a certain tensoring of $\mathrm{Cat}_{(\infty,2)}$ over Cat_∞, which we will now define:

Definition 7.1. The ∞-category $\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Cat}_\infty)$ is tensored over Cat_∞ by taking products with constant functors, i.e. for $X \in \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Cat}_\infty)$ and $\mathcal{C} \in \mathrm{Cat}_\infty$ we can define $X \times \mathcal{C}$ as the functor $[n] \mapsto X_n \times \mathcal{C}$. This preserves colimits in both variables, so we have a cotensoring $X^{\mathcal{C}}$ (given by $[n] \mapsto \mathrm{Fun}(\mathcal{C}, X_n)$) and an enrichment $\mathrm{Nat}(X,Y)$ in Cat_∞, satisfying

$$\mathrm{Map}_{\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Cat}_\infty)}(X \times \mathcal{C}, Y) \simeq \mathrm{Map}_{\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Cat}_\infty)}(X, Y^{\mathcal{C}}) \simeq \mathrm{Map}_{\mathrm{Cat}_\infty}(\mathcal{C}, \mathrm{Nat}(X,Y)).$$

Remark 7.2. If we view $\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Cat}_\infty)$ as cocartesian fibrations to $\mathbf{\Delta}^{\mathrm{op}}$, then morphisms in $\mathrm{Nat}(X,Y)$ indeed correspond to natural transformations between functors over $\mathbf{\Delta}^{\mathrm{op}}$ that preserve cocartesian morphisms.

If $\mathcal{X}$ and $\mathcal{Y}$ are $(\infty,2)$-categories, we obtain an ∞-category $\mathrm{Nat}(\mathcal{X},\mathcal{Y})$ whose objects are functors $\mathcal{X} \to \mathcal{Y}$. The precise result we will prove in this section is the following description of this ∞-category:

Theorem 7.3. *Given $(\infty,2)$-categories $\mathcal{X}$ and $\mathcal{Y}$ there are functors*

$$\mathrm{Nat}(\mathcal{X},\mathcal{Y}) \to \mathrm{Fun}(\mathcal{X},\mathcal{Y})_{\mathrm{colax}},$$

$$\mathrm{Nat}(\mathcal{X},\mathcal{Y})^{\mathrm{op}} \to \mathrm{Fun}(\mathcal{X},\mathcal{Y})_{\mathrm{lax}},$$

[7] For $(\infty,2)$-categories $\mathcal{X}, \mathcal{Y}$ a natural transformation is a morphism of $(\infty,2)$-categories $\mathcal{X} \times C_1 \to \mathcal{Y}$, or equivalently a morphism in $\mathrm{FUN}(\mathcal{X},\mathcal{Y})$.

[8] An acronym for "Identity Component Oplax Natural transformations".

where the first identifies $\mathrm{Nat}(\mathcal{X},\mathcal{Y})$ *with the subcategory of* $\mathrm{Fun}(\mathcal{X},\mathcal{Y})_{\mathrm{colax}}$ *whose morphisms are the colax natural transformations that are given by equivalences in* $\mathcal{Y}$*, and similarly in the lax case. More precisely, we have natural pullback squares*

$$\begin{tikzcd} \mathrm{Nat}(\mathcal{X},\mathcal{Y}) \arrow[r] \arrow[d] & \mathrm{Fun}(\mathcal{X},\mathcal{Y})_{\mathrm{colax}} \arrow[d] \\ \mathrm{Map}(\iota_0\mathcal{X},\iota_0\mathcal{Y}) \arrow[r] & \mathrm{Fun}(\iota_0\mathcal{X},\mathcal{Y}), \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Nat}(\mathcal{X},\mathcal{Y})^{\mathrm{op}} \arrow[r] \arrow[d] & \mathrm{Fun}(\mathcal{X},\mathcal{Y})_{\mathrm{lax}} \arrow[d] \\ \mathrm{Map}(\iota_0\mathcal{X},\iota_0\mathcal{Y}) \arrow[r] & \mathrm{Fun}(\iota_0\mathcal{X},\mathcal{Y}), \end{tikzcd}$$

where in the left-hand square the right vertical map is given by composition with the inclusion $\iota_0\mathcal{X} \to \mathcal{X}$ *(via the equivalence* $\mathrm{Fun}(\iota_0\mathcal{X},\mathcal{Y})_{\mathrm{colax}} \simeq \mathrm{Fun}(\iota_0\mathcal{X},\mathcal{Y})$ *of Remark 3.11) and the bottom horizontal map is given by composition with the inclusion* $\iota_0\mathcal{Y} \to \mathcal{Y}$ *(and the equivalence* $\mathrm{Fun}(\iota_0\mathcal{X},\iota_0\mathcal{Y}) \simeq \mathrm{Map}(\iota_0\mathcal{X},\iota_0\mathcal{Y})$*), and similarly in the right-hand square.*

To prove Theorem 7.3, we start with the following observation:

Proposition 7.4. *If* X *lies in one of the subcategories* $\mathrm{Cat}_{(\infty,2)}$, $\mathrm{Seg}^{\mathcal{S}}_{\mathbf{\Delta}^{\mathrm{op}}}(\mathrm{Cat}_\infty)$ *or* $\mathrm{Seg}_{\mathbf{\Delta}^{\mathrm{op}}}(\mathrm{Cat}_\infty)$*, then so does* $X^{\mathcal{C}}$ *for any* $\mathcal{C} \in \mathrm{Cat}_\infty$.

Proof. If $X \in \mathrm{Seg}_{\mathbf{\Delta}^{\mathrm{op}}}(\mathrm{Cat}_\infty)$, then the Segal map

$$X_n^{\mathcal{C}} \to X_1^{\mathcal{C}} \times_{X_0^{\mathcal{C}}} \cdots \times_{X_0^{\mathcal{C}}} X_1^{\mathcal{C}}$$

is just $\mathrm{Fun}(\mathcal{C},-)$ applied to the Segal map for X, and so is an equivalence since $\mathrm{Fun}(\mathcal{C},-)$ preserves limits. Moreover, for $X \in \mathrm{Seg}^{\mathcal{S}}_{\mathbf{\Delta}^{\mathrm{op}}}(\mathrm{Cat}_\infty)$ the ∞-category

$$X_0^{\mathcal{C}} \simeq \mathrm{Fun}(\mathcal{C}, X_0) \simeq \mathrm{Map}_{\mathcal{S}}(\|\mathcal{C}\|, X_0)$$

is an ∞-groupoid.

Now suppose $X \in \mathrm{Cat}_{(\infty,2)}$; we must show that the Segal space $(X^{\mathcal{C}})^{\simeq} \simeq \mathrm{Map}_{\mathrm{Cat}_\infty}(\mathcal{C}, X)$ is complete. Since complete Segal spaces are closed under limits, it suffices to consider the cases where $\mathcal{C}$ is Δ^0 (where we just get $X^{\Delta^0} \simeq X$) and Δ^1. Here $(X^{\Delta^1})^{\simeq}_0 \simeq X_0$ and $(X^{\Delta^1})^{\simeq}_1 \simeq \mathrm{Map}(\Delta^1, X_1)$. A morphism in X^{Δ^1} is thus a 2-morphism in X, i.e. a diagram of shape

$$\begin{tikzcd} \bullet \arrow[d, bend right=50] \arrow[d, bend left=50] \arrow[d, phantom, "\Rightarrow"] \\ \bullet \end{tikzcd}$$

in X, with composition in X^{Δ^1} given by composing two such diagrams vertically. For such a morphism to be an equivalence it follows that both the two 1-morphisms and the 2-morphism must be an equivalence in X, which proves completeness. □

Remark 7.5. If $\mathbf{X}$ is a strict 2-category, we can explicitly describe $\mathbf{X}^{[n]}$, which is again a strict 2-category:

- the objects are the objects of $\mathbf{X}$
- a morphism from x to y consists of $n+1$ morphisms $f_i\colon x \to y$ ($i = 0,\ldots,n$), and 2-morphisms $f_0 \to f_1 \to \cdots \to f_n$, i.e. a functor $[n] \to \mathbf{X}(x,y)$,
- a 2-morphism between two morphisms from x to y, given by $f_0 \to \cdots \to f_n$ and $g_0 \to \cdots \to g_n$, consists of a commutative diagram of 2-morphisms of the form

$$\begin{tikzcd} f_0 \arrow[r] \arrow[d] & f_1 \arrow[r] \arrow[d] & \cdots \arrow[r] & f_n \arrow[d] \\ g_0 \arrow[r] & g_1 \arrow[r] & \cdots \arrow[r] & g_n, \end{tikzcd}$$

i.e. a functor $[n]\times[1]\to \mathbf{X}(x,y)$,

- composition of morphisms and 2-morphisms is given in terms of composition in $\mathbf{X}$ in the evident way.

Corollary 7.6. *There exists a functor*

$$-\odot-\colon \mathrm{Cat}_{(\infty,2)}\times \mathrm{Cat}_\infty \to \mathrm{Cat}_{(\infty,2)},$$

which preserves colimits in each variable, and satisfies

$$\mathrm{Map}_{\mathrm{Cat}_{(\infty,2)}}(\mathcal{X}\odot\mathcal{C},\mathcal{Y})\simeq \mathrm{Map}_{\mathrm{Cat}_{(\infty,2)}}(\mathcal{X},\mathcal{Y}^{\mathcal{C}})\simeq \mathrm{Map}_{\mathrm{Cat}_\infty}(\mathcal{C},\mathrm{Nat}(\mathcal{X},\mathcal{Y}))$$

for all $(\infty,2)$-categories $\mathcal{X}$ and $\mathcal{Y}$ and ∞-categories $\mathcal{C}$. □

Remark 7.7. If we view an $(\infty,2)$-category $\mathcal{X}$ as an object of $\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}},\mathrm{Cat}_\infty)$, then $\mathcal{X}^{2\text{-op}}$ is obtained by taking opposite ∞-categories levelwise. For any ∞-category $\mathcal{C}$ we therefore have a natural equivalence

$$(\mathcal{X}^{\mathcal{C}})^{2\text{-op}}\simeq(\mathcal{X}^{2\text{-op}})^{\mathcal{C}^{\mathrm{op}}},$$

given levelwise by the equivalence

$$\mathrm{Fun}(\mathcal{C},\mathcal{X}_i)^{\mathrm{op}}\simeq \mathrm{Fun}(\mathcal{C}^{\mathrm{op}},\mathcal{X}_i^{\mathrm{op}}).$$

This translates into a natural equivalence

$$(\mathcal{X}\odot\mathcal{C})^{2\text{-op}}\simeq \mathcal{X}^{2\text{-op}}\odot\mathcal{C}^{\mathrm{op}}.$$

The morphisms in Theorem 7.3 will arise from natural transformations

$$\mathcal{X}\otimes^{\mathrm{colax}}\mathcal{C}\to\mathcal{X}\odot\mathcal{C},$$

$$\mathcal{X}\otimes^{\mathrm{lax}}\mathcal{C}\to\mathcal{X}\odot\mathcal{C}^{\mathrm{op}}.$$

In order to define these we require an explicit description of $I\odot[n]$ for $I\in\mathbf{\Theta}_2$ and $[n]\in\mathbf{\Delta}$. To obtain this we first define an explicit functor $\Phi\colon \mathbf{\Theta}_2\times\mathbf{\Delta}\to\mathrm{Cat}_{(\infty,2)}$ (in fact taking values in strict 2-categories) and check that this satisfies the co-Segal condition in each variable. Then we will use the universal property of $\odot$ to define a natural transformation $\odot\to\Phi$ and prove that this is an equivalence.

Definition 7.8. We define Φ on objects by

$$\Phi([k](n_1,\dots,n_k),[m]) := [k]([n_1]\times[m],\dots,[n_k]\times[m]),$$

in the notation of Definition 2.16. This has as objects $0,\dots,k$ and as Hom-categories ($0\le i\le j\le k$) the posets

$$\mathrm{Hom}(i,j) := \prod_{i<s\le j}([n_s]\times[m])\cong I(i,j)\times[m]^{j-i},$$

with composition given by isomorphism. In particular, we can view this as a strict 2-category. Recall that if $I=[k](n_1,\dots,n_k)$ and $J=[l](n'_1,\dots,n'_l)$ then a morphism $F\colon I\to J$ in $\mathbf{\Theta}_2$ is given by a morphism $\phi\colon[k]\to[l]$ in $\mathbf{\Delta}$ together with morphisms $\psi_{ij}\colon[n_i]\to[n'_j]$ in $\mathbf{\Delta}$ whenever $\phi(i-1)<j\le\phi(i)$. If we are also given a morphism $\mu\colon[m]\to[m']$, then the corresponding functor

$$\Phi(F,\mu)\colon \Phi(I,[m])\to\Phi(J,[m'])$$

is given on objects by $i \mapsto \phi(i)$ and on morphism categories by the functor

$$\prod_{i<s\leq j} [n_s] \times [m] \to \prod_{\phi(i)<t\leq\phi(j)} [n'_t] \times [m']$$

that in the component indexed by t is given by

$$\prod_{i<s\leq j} [n_s] \times [m] \to [n_r] \times [m] \xrightarrow{\psi_{rt}\times\mu} [n'_t] \times [m'],$$

where r is the unique index such that $\phi(r-1) < t \leq \phi(r)$. In other words, this is the functor

$$I(i,j) \times [m]^{j-i} \to J(\phi(i),\phi(j)) \times [m']^{\phi(j)-\phi(i)}$$

given by $F(i,j)$ in the first variable and in the second by

$$\begin{aligned} \mu_{i,j} \colon \prod_{i<s\leq j} [m] \to \prod_{i<s\leq j} [m]^{\phi(s)-\phi(s-1)} \xrightarrow{\prod_{i<s\leq j}\mu^{\phi(s)-\phi(s-1)}} & \prod_{i<s\leq j} [m']^{\phi(s)-\phi(s-1)} \\ \cong & \prod_{i<s\leq j} \prod_{\phi(s-1)<t\leq\phi(s)} [m'] \\ \cong & \prod_{\phi(i)<t\leq\phi(j)} [m'], \end{aligned}$$

where the first functor is given by the diagonals of $[m]$.

Example 7.9. For $\Phi(C_1,[1]) = \Phi([1](0),[1])$ we get $[1](1) = C_2$, and for $\Phi(C_2,[1])$ we get $[1]([1]\times[1])$, which we can think of as a "suspension" of the commutative square: it has two objects $0,1$ and a commutative square of morphisms from 0 to 1.

Lemma 7.10. *Φ satisfies the co-Segal condition in each variable.*

Proof. Lemma 2.18 immediately gives the first co-Segal condition in the $\mathbf{\Theta}_2$-variable:

$$\begin{aligned} \Phi([k](n_1,\ldots,n_k),[m]) &\simeq [k]([n_1]\times[m],\ldots,[n_k]\times[m]) \\ &\simeq [1]([n_1]\times[m]) \amalg_{[0]} \cdots \amalg_{[0]} [1]([n_k]\times[m]) \\ &\simeq \Phi([1](n_1),[m]) \amalg_{\Phi([0],[m])} \cdots \amalg_{\Phi([0],[m])} \Phi([1](n_k),[m]). \end{aligned}$$

Moreover, Lemma 2.19 gives the other co-Segal condition in the $\mathbf{\Theta}_2$-variable:

$$\begin{aligned} \Phi([1](n),[m]) \simeq [1]([n]\times[m]) &\simeq [1]([1]\times[m]) \amalg_{[1]([0]\times[m])} \cdots \amalg_{[1]([0]\times[m])} [1]([1]\times[m]) \\ &\simeq [1]([1]\times[m]) \amalg_{[1](m)} \cdots \amalg_{[1](m)} [1]([1]\times[m]) \\ &\simeq \Phi(C_2,[m]) \amalg_{\Phi(C_1,[m])} \cdots \amalg_{\Phi(C_1,[m])} \Phi(C_2,[m]). \end{aligned}$$

Similarly, Lemma 2.19 gives the co-Segal condition in the $\mathbf{\Delta}$-variable for $\Phi([1](n),-)$ from which the general case follows using the first decomposition. □

Construction 7.11. We will now define a natural transformation

$$\eta_{I,m} \colon I \odot [m] \to \Phi(I,[m])$$

(for $I \in \mathbf{\Theta}_2$ and $[m] \in \mathbf{\Delta}$). Since $I \odot [m]$ was defined by adjunction, to give this map is equivalent to giving

$$\eta'_{I,m} \colon I \to \Phi(I,[m])^{[m]}$$

where the latter is cotensoring with $[m]$. Applying the description of Remark 7.5 to $\Phi(I,[m])^{[m]}$ when $I = [k](n_1,\ldots,n_k)$, we see that

- its objects are $0, \dots, k$,
- a morphism from i to j is a functor

$$[m] \to [m]^{j-i} \times I(i,j),$$

(where $I(i,j) = \prod_{i<s\leq j}[n_s]$ if $i < j$ and $\emptyset$ if $i > j$)
- a 2-morphism of functors i to j is a functor

$$[m] \times [1] \to [m]^{j-i} \times I(i,j).$$

In other word, $\Phi(I,[m])^{[m]}(i,j)$ is the functor category $\mathrm{Fun}([m], \Phi(I,[m])(i,j))$. The functor $\eta'_{I,m}$ is defined to be the identity on objects, and for $i \leq j$ the functor of morphism categories

$$I(i,j) \to \Phi(I,[m])^{[m]}(i,j) = \mathrm{Fun}([m],[m]^{j-i} \times I(i,j))$$

is defined to be the one adjoint to the functor

$$[m] \times I(i,j) \xrightarrow{\eta''_{I,m}(i,j)} [m]^{j-i} \times I(i,j)$$

given by the product of the diagonal $[m] \to [m]^{j-i}$ and the identity of $I(i,j)$. Since the composition in $\Phi(I,[m])^{[m]}$ is defined in terms of composition in I, it is evident that this defines a functor of strict 2-categories. To show that η is natural, we must check that for $F\colon I \to J$ and $\alpha\colon [m] \to [k]$ the square

$$\begin{tikzcd}
I \odot [m] \arrow[r, "\eta_{I,m}"] \arrow[d, "F\odot\alpha"] & \Phi(I,[m]) \arrow[d, "\Phi(F,\alpha)"] \\
J \odot [k] \arrow[r, "\eta_{J,k}"] & \Phi(J,[k])
\end{tikzcd}$$

commutes, which is equivalent to the commutativity of the diagram

$$\begin{tikzcd}
I \arrow[r, "\eta'_{I,m}"] \arrow[dd, "F"'] & \Phi(I,[m])^{[m]} \arrow[dr, "\Phi(F,\alpha)^{[m]}"] & \\
 & & \Phi(J,[k])^{[m]}. \\
J \arrow[r, "\eta'_{J,k}"] & \Phi(J,[k])^{[k]} \arrow[ur, "\Phi(J,[k])^{\alpha}"'] &
\end{tikzcd}$$

This in turn amounts to the commutativity of the squares

$$\begin{tikzcd}
I(i,j) \times [m] \arrow[r, "\eta''_{I,m}(i,j)"] \arrow[d, "F(i,j)\times\alpha"] & I(i,j) \times [m]^{j-i} \arrow[d, "F(i,j)\times\alpha_{i,j}"] \\
J(Fi,Fj) \times [k] \arrow[r, "\eta''_{J,k}(Fi,Fj)"] & J(Fi,Fj) \times [k]^{Fj-Fi},
\end{tikzcd}$$

where $\alpha_{i,j}$ is defined as at the end of Definition 7.8, which is clear from the definitions of these functors.

Proposition 7.12. *The natural* 2*-functor*

$$\eta_{I,m}\colon I \odot \Delta^m \longrightarrow \Phi(I, \Delta^m)$$

is an equivalence.

Proof of Proposition 7.12. Since both $\odot$ and Φ are co-Segal, it is enough to establish the equivalence on generators $C_i \in \mathbf{\Theta}_2$, $i = 0,1,2$ and $[j] \in \mathbf{\Delta}$, $j = 0,1$. There are two non-trivial cases: it suffices to prove that the maps

$$\eta_{C_1,1}\colon C_1 \odot [1] \to \Phi(C_1,[1]) \cong [1](1) = C_2,$$

$$\eta_{C_2,1}\colon C_2 \odot [1] \to \Phi(C_2,[1]) \cong [1]([1]\times[1]),$$

are equivalences.

For the first case, note that by adjunction we have (for any 2-category $\mathcal{X}$)

$$\mathrm{Map}(C_1 \odot [1], \mathcal{X}) \simeq \mathrm{Map}(C_1, \mathcal{X}^{[1]}).$$

These are the 1-morphisms in $\mathcal{X}^{[1]}$, and if we think of $\mathcal{X}$ as an object of $\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Cat}_\infty)$ these are the objects of $\mathcal{X}_1^{[1]}$, which are the 2-morphisms in $\mathcal{X}$. Thus $C_1 \odot [1] \simeq C_2$, and the functor $C_1 \to C_2^{[1]}$ adjoint to the equivalence is the one picking out the non-trivial 2-morphism in C_2, which is indeed $\eta'_{C_1,1}$.

For the second case, we have

$$\mathrm{Map}(C_2 \odot [1], \mathcal{X}) \simeq \mathrm{Map}(C_2, \mathcal{X}^{[1]}) \simeq \mathrm{Map}([1], \mathcal{X}_1^{[1]}) \simeq \mathrm{Map}([1]\times[1], \mathcal{X}_1).$$

Since $[1]\times[1]$ is weakly contractible, by Lemma 2.20 this is the same thing as $\mathrm{Map}([1]([1]\times[1]), \mathcal{X})$. Thus $C_2 \odot [1] \simeq [1]([1]\times[1])$, and unwinding the definitions we see that the map

$$C_2 \to [1]([1]\times[1])^{[1]}$$

adjoint to the equivalence is the one picking out the non-trivial 2-morphism in $[1]([1]\times[1])^{[1]}$ (corresponding to the commutative square of non-trivial 2-morphisms in $[1]([1]\times[1])$), which is indeed $\eta'_{C_2,1}$. □

Construction 7.13. We now define a natural functor of strict 2-categories

$$\nu_{I,m}\colon I \otimes^{\mathrm{colax}} [m] \to \Phi(I,[m]),$$

for $I \in \mathbf{\Theta}_2$, using the explicit description of $I \otimes^{\mathrm{colax}} [m]$ as a strict 2-category from Definition 3.3. Let $I = [k](n_1,\ldots,n_k)$. On objects, the map $\nu_{I,m}$ is simply the projection $\mathrm{ob}([k]) \times \mathrm{ob}([m]) \to \mathrm{ob}([k])$. On hom categories, for fixed objects (x,i) and (x',i'), we need to give a map of posets

$$\mathrm{MaxCh}([x,x']\times[i,i']) \times I(x,x') \to [m]^{x'-x} \times I(x,x').$$

We define this map to send a maximal chain in $[x,x']\times[i,i']$ to the tuple of column indices where the vertical steps are taken. By unwinding the definitions involved we see that this indeed defines a natural functor of 2-categories.

Proposition 7.14. *There is a natural pushout square of $(\infty,2)$-categories*

$$\begin{tikzcd} \iota_0\mathcal{X} \times \mathcal{C} \arrow[r] \arrow[d] & \mathcal{X} \otimes^{\mathrm{colax}} \mathcal{C} \arrow[d] \\ \iota_0\mathcal{X} \times \|\mathcal{C}\| \arrow[r] & \mathcal{X} \odot \mathcal{C}. \end{tikzcd}$$

Proof. From the definition of $\nu_{I,m}$ we have for $I \in \boldsymbol{\Theta}_2$ and $[m] \in \boldsymbol{\Delta}$ a natural commutative square

$$\begin{tikzcd} \iota_0 I \times [m] \arrow[r] \arrow[d] & I \otimes^{\mathrm{colax}} [m] \arrow[d, "\nu_{I,m}"] \\ \iota_0 I \arrow[r] & \Phi(I,[m]). \end{tikzcd}$$

We can extend this by colimits to a commutative square in $\mathrm{Seg}_{\boldsymbol{\Theta}_2^{\mathrm{op}}}(\mathcal{S})$ of the correct form for $\mathcal{X} \in \mathrm{Seg}_{\boldsymbol{\Theta}_2^{\mathrm{op}}}(\mathcal{S})$ and $\mathcal{C} \in \mathrm{Seg}_{\boldsymbol{\Delta}^{\mathrm{op}}}(\mathcal{S})$, which induces the required square in $\mathrm{Cat}_{(\infty,2)}$ for $\mathcal{X} \in \mathrm{Cat}_{(\infty,2)}$, $\mathcal{C} \in \mathrm{Cat}_\infty$ after completion.

To see that this is a pushout square in $\mathrm{Cat}_{(\infty,2)}$ it suffices to check the original square is a pushout in $\mathrm{Seg}_{\boldsymbol{\Theta}_2^{\mathrm{op}}}(\mathcal{S})$ for all $I \in \boldsymbol{\Theta}_2$ and $[m] \in \boldsymbol{\Delta}$, and since the functors satisfy the co-Segal condition in each variable to this it is enough to check the cases where $I = C_0, C_1, C_2$ and $m = 0,1$. The cases involving C_0 and $[0]$ are trivial, so we are left with two non-trivial cases: $\Phi(C_1,[1]) \cong C_2$ and $\Phi(C_2,[1]) \cong [1]([1]\times[1])$. Using the colimit decomposition of Lemma 3.18 the square in the first case is

$$\begin{tikzcd} C_1 \amalg C_1 \arrow[r] \arrow[d] & {[2](0,0) \cup_{C_1} C_2 \cup_{C_1} [2](0,0)} \arrow[d] \\ C_0 \amalg C_0 \arrow[r] & C_2, \end{tikzcd}$$

which is a pushout since we have

$$[2](0,0) \cup_{C_1} C_0 \simeq (C_1 \cup_{C_0} C_1) \cup_{C_1} C_0 \simeq C_1.$$

In the second case the square is

$$\begin{tikzcd} C_1 \amalg C_1 \arrow[r] \arrow[d] & {[2](1,0) \cup_{C_2} [1]([1]^2) \cup_{C_2} [2](0,1)} \arrow[d] \\ C_0 \amalg C_0 \arrow[r] & {[1]([1]^2),} \end{tikzcd}$$

which is a pushout since we have

$$[2](1,0) \cup_{C_1} C_0 \simeq (C_2 \cup_{C_0} C_1) \cup_{C_1} C_0 \simeq C_2$$

and similarly for $[2](0,1)$. $\square$

Remark 7.15. Combining Proposition 7.14 with the equivalences of Remark 7.7 and Proposition 3.13, we also obtain (since the functor $(-)^{2\text{-op}}$ preserves colimits) natural pushout squares

$$\begin{tikzcd} \iota_0\mathcal{X} \times \mathcal{C} \arrow[r] \arrow[d] & \mathcal{X} \otimes^{\mathrm{lax}} \mathcal{C} \arrow[d] \\ \iota_0\mathcal{X} \times \|\mathcal{C}\| \arrow[r] & \mathcal{X} \odot \mathcal{C}^{\mathrm{op}}. \end{tikzcd}$$

The proof of Theorem 7.3 is now immediate:

Proof of Theorem 7.3. In the colax case the pushout square of Proposition 7.14 gives natural pullback squares

$$\begin{tikzcd} \mathrm{Map}(\mathcal{X} \odot \mathcal{C}, \mathcal{Y}) \arrow[r] \arrow[d] & \mathrm{Map}(\iota_0\mathcal{X} \times \|\mathcal{C}\|, \mathcal{Y}) \arrow[d] \\ \mathrm{Map}(\mathcal{X} \otimes^{\mathrm{colax}} \mathcal{C}, \mathcal{Y}) \arrow[r] & \mathrm{Map}(\iota_0\mathcal{X} \times \mathcal{C}, \mathcal{Y}), \end{tikzcd}$$

which we can rewrite using various adjunctions as

$$\begin{tikzcd} \mathrm{Map}(\mathcal{C}, \mathrm{Nat}(\mathcal{X}, \mathcal{Y})) \arrow[r] \arrow[d] & \mathrm{Map}(\mathcal{C}, \mathrm{Map}(\mathcal{X}, \mathcal{Y})) \arrow[d] \\ \mathrm{Map}(\mathcal{C}, \mathrm{Fun}(\mathcal{X}, \mathcal{Y})_{\mathrm{colax}}) \arrow[r] & \mathrm{Map}(\mathcal{C}, \mathrm{Fun}(\iota_0\mathcal{X}, \mathcal{Y})), \end{tikzcd}$$

which gives the desired pullback square in Cat_∞ by the Yoneda Lemma. In the lax case we proceed in the same way, using instead the variant pushout square of Remark 7.15. □

8. Monads as Algebras

In §4 we considered monads as functors of $(\infty, 2)$-categories from $\mathfrak{mnd}$. Alternatively, we can view monads in an $(\infty, 2)$-category $\mathcal{X}$ as associative algebras in the monoidal ∞-categories of endomorphisms of the objects of $\mathcal{X}$; this is the point of view taken by Lurie in [22]*§4.7. Our goal in this section is to compare these two approaches, and in particular use the results of §7 to relate the natural notions of morphisms in the two cases.

Remark 8.1. Applying Theorem 7.3 to monads and endofunctors, we get (as $\iota_0\mathfrak{mnd} \simeq \iota_0\mathfrak{end} \simeq C_0$) pullback squares of ∞-categories

$$\begin{tikzcd} \mathrm{Nat}(\mathfrak{mnd}, \mathcal{X})^{\mathrm{op}} \arrow[r] \arrow[d] & \mathrm{Mnd}(\mathcal{X})_{\mathrm{lax}} \arrow[d] \\ \iota_0\mathcal{X} \arrow[r] & \iota_1\mathcal{X}, \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Nat}(\mathfrak{mnd}, \mathcal{X}) \arrow[r] \arrow[d] & \mathrm{Mnd}(\mathcal{X})_{\mathrm{colax}} \arrow[d] \\ \iota_0\mathcal{X} \arrow[r] & \iota_1\mathcal{X}, \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{Nat}(\mathfrak{end}, \mathcal{X})^{\mathrm{op}} \arrow[r] \arrow[d] & \mathrm{End}(\mathcal{X})_{\mathrm{lax}} \arrow[d] \\ \iota_0\mathcal{X} \arrow[r] & \iota_1\mathcal{X}, \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Nat}(\mathfrak{end}, \mathcal{X}) \arrow[r] \arrow[d] & \mathrm{End}(\mathcal{X})_{\mathrm{colax}} \arrow[d] \\ \iota_0\mathcal{X} \arrow[r] & \iota_1\mathcal{X}. \end{tikzcd}$$

In particular, for an object $X \in \mathcal{X}$ we can identify the fibre $\mathrm{Mnd}(\mathcal{X})_{\mathrm{colax},X}$ with the fibre $\mathrm{Nat}(\mathfrak{mnd}, \mathcal{X})_X$, and similarly in the other three cases.

To describe these fibres, we therefore want to give an explicit description of the ∞-categories $\mathrm{Nat}(\mathfrak{mnd}, \mathcal{X})$ and $\mathrm{Nat}(\mathfrak{end}, \mathcal{X})$. By Remark 7.2, we can view $\mathrm{Nat}(\mathcal{X}, \mathcal{Y})$ in terms of the corresponding cocartesian fibrations $\int_{\mathbf{\Delta}^{\mathrm{op}}} \mathcal{X}$ and $\int_{\mathbf{\Delta}^{\mathrm{op}}} \mathcal{Y}$ as the ∞-category $\mathrm{Fun}^{\mathrm{cocart}}_{/\mathbf{\Delta}^{\mathrm{op}}}(\int_{\mathbf{\Delta}^{\mathrm{op}}} \mathcal{X}, \int_{\mathbf{\Delta}^{\mathrm{op}}} \mathcal{Y})$ of functors over $\mathbf{\Delta}^{\mathrm{op}}$ that preserve cocartesian morphisms and natural transformations between these. To describe this in the case of interest we recall some notions related to generalized non-symmetric ∞-operads (see for instance [9] for more details):

Definition 8.2. Recall that a double ∞-category is a functor $\mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ satisfying the Segal condition. (Thus an $(\infty, 2)$-category can be described as a double ∞-category whose value at

$[0]$ is an ∞-groupoid.) Equivalently, a double ∞-category is a cocartesian fibration over $\mathbf{\Delta}^{\mathrm{op}}$ corresponding to such a functor; we write Dbl_∞ for the ∞-category of double ∞-categories. We also write $\mathrm{Opd}_\infty^{\mathrm{ns,gen}}$ for the ∞-category of generalized non-symmetric ∞-operads, which are also certain ∞-categories over $\mathbf{\Delta}^{\mathrm{op}}$. In particular, a generalized non-symmetric ∞-operad has cocartesian morphisms over inert maps in $\mathbf{\Delta}^{\mathrm{op}}$, and the morphisms in $\mathrm{Opd}_\infty^{\mathrm{ns,gen}}$ are the morphisms over $\mathbf{\Delta}^{\mathrm{op}}$ that preserve these inert cocartesian morphisms. If $\mathcal{O}$ is a generalized non-symmetric ∞-operad, then so is $\mathcal{O} \times \mathcal{I}$ for any ∞-category $\mathcal{I}$, and we write $\mathrm{Alg}_{\mathcal{O}}(\mathcal{P})$ for the ∞-category given by the complete Segal space $\mathrm{Map}_{\mathrm{Opd}_\infty^{\mathrm{ns,gen}}}(\mathcal{O} \times \Delta^\bullet, \mathcal{P})$. (We abbreviate the ∞-category $\mathrm{Alg}_{\mathbf{\Delta}^{\mathrm{op}}}(\mathcal{P})$ of associative algebras to just $\mathrm{Alg}(\mathcal{P})$.)

Definition 8.3. Any double ∞-category is a generalized non-symmetric ∞-operad, so that there is a forgetful functor

$$\mathrm{Dbl}_\infty \to \mathrm{Opd}_\infty^{\mathrm{ns,gen}}.$$

This has a left adjoint Env, the *double envelope*, given by a simple explicit formula (see [13]*§A.8):

$$\mathrm{Env}(\mathcal{O}) \simeq \mathcal{O} \times_{\mathbf{\Delta}^{\mathrm{op}}} \mathrm{Act}(\mathbf{\Delta}^{\mathrm{op}})$$

where $\mathrm{Act}(\mathbf{\Delta}^{\mathrm{op}})$ is the full subcategory of $\mathrm{Fun}(C_1, \mathbf{\Delta}^{\mathrm{op}})$ spanned by the active maps, the fibre product uses the map to $\mathbf{\Delta}^{\mathrm{op}}$ given by evaluation at $0 \in C_1$, and the map $\mathrm{Env}(\mathcal{O}) \to \mathbf{\Delta}^{\mathrm{op}}$ is given by evaluation at $1 \in C_1$. From this formula we see that $\mathrm{Env}(\mathcal{O} \times \mathcal{I}) \simeq \mathrm{Env}(\mathcal{O}) \times \mathcal{I}$, so that if $\mathcal{M}$ is a double ∞-category and $\mathcal{O}$ is a generalized non-symmetric ∞-operad then the adjunction induces an equivalence

$$\mathrm{Alg}_{\mathcal{O}}(\mathcal{M}) \simeq \mathrm{Fun}^{\mathrm{cocart}}_{/\mathbf{\Delta}^{\mathrm{op}}}(\mathrm{Env}(\mathcal{O}), \mathcal{M}).$$

Remark 8.4. Note in particular that $\mathrm{Env}(\mathcal{O})_0 \simeq \mathcal{O}_0$ (as the only active map to $[0]$ in $\mathbf{\Delta}^{\mathrm{op}}$ is $\mathrm{id}_{[0]}$) while $\mathrm{Env}(\mathcal{O})_1 \simeq \mathcal{O}^{\mathrm{act}}$, the subcategory of $\mathcal{O}$ containing only the active maps (as every object in $\mathbf{\Delta}^{\mathrm{op}}$ has a unique active map to $[1]$). Thus if $\mathcal{O}_0 \simeq C_0$ (i.e. $\mathcal{O}$ is a non-symmetric ∞-operad) then $\mathrm{Env}(\mathcal{O})$ is a monoidal ∞-category given by a monoidal structure on $\mathcal{O}^{\mathrm{act}}$. If we think of objects of $\mathcal{O}$ as lists $(X_1, \ldots, X_n)$ of objects $X_i \in \mathcal{O}_1$, then this monoidal structure is given by concatenation,

$$(X_1, \ldots, X_n) \otimes (Y_1, \ldots, Y_m) \simeq (X_1, \ldots, X_n, Y_1, \ldots, Y_m).$$

Proposition 8.5. *There are equivalences* $\int_{\mathbf{\Delta}^{\mathrm{op}}} \mathfrak{mnd} \simeq \mathrm{Env}(\mathbf{\Delta}^{\mathrm{op}})$ *and* $\int_{\mathbf{\Delta}^{\mathrm{op}}} \mathfrak{end} \simeq \mathrm{Env}(\mathbf{\Delta}^{\mathrm{op}}_{\mathrm{int}})$*, and hence for* $\mathcal{X}$ *an* $(\infty, 2)$*-category there are natural equivalences*

$$\mathrm{Nat}(\mathfrak{mnd}, \mathcal{X}) \simeq \mathrm{Alg}_{\mathbf{\Delta}^{\mathrm{op}}}\Big(\int_{\mathbf{\Delta}^{\mathrm{op}}} \mathcal{X}\Big),$$

$$\mathrm{Nat}(\mathfrak{end}, \mathcal{X}) \simeq \mathrm{Alg}_{\mathbf{\Delta}^{\mathrm{op}}_{\mathrm{int}}}\Big(\int_{\mathbf{\Delta}^{\mathrm{op}}} \mathcal{X}\Big) \simeq \mathcal{X}_1 \times_{\mathcal{X}_0 \times \mathcal{X}_0} \mathcal{X}_0,$$

under which the functor $\mathrm{Nat}(\mathfrak{mnd}, \mathcal{X}) \to \mathrm{Nat}(\mathfrak{end}, \mathcal{X})$ *corresponds to that given by composition with* $\mathbf{\Delta}^{\mathrm{op}}_{\mathrm{int}} \to \mathbf{\Delta}^{\mathrm{op}}$.

Proof. By Remark 8.4 we know that $\mathrm{Env}(\mathbf{\Delta}^{\mathrm{op}})$ is a monoidal structure on $(\mathbf{\Delta}^{\mathrm{op}})^{\mathrm{act}} \simeq \mathbf{\Delta}_+$ given by concatenation, i.e. it is precisely $(\mathbf{\Delta}_+, \star)$ which is the monoidal category corresponding to the one-object 2-category $\mathfrak{mnd}$. Similarly, $\mathrm{Env}(\mathbf{\Delta}^{\mathrm{op}}_{\mathrm{int}})$ is a monoidal structure on $(\mathbf{\Delta}^{\mathrm{op}}_{\mathrm{int}})^{\mathrm{act}}$, which is (as only the identity maps are both active and inert) the set $\{0, 1, \ldots\}$, given by addition, which is precisely the monoidal category corresponding to the one-object 2-category $\mathfrak{end}$. $\square$

Definition 8.6. Let $i\colon C_0 \to \mathbf{\Delta}^{\mathrm{op}}$ denote the functor picking out the object $[0]$. Then right Kan extension along i gives a functor $i_*\colon \mathrm{Cat}_\infty \to \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Cat}_\infty)$ with $(i_*\mathcal{C})_n \simeq \mathcal{C}^{\times n+1}$. This is a double ∞-category, so we get an adjunction

$$i^* : \mathrm{Dbl}_\infty \rightleftarrows \mathrm{Cat}_\infty : i_*.$$

We write $\mathbf{\Delta}^{\mathrm{op}}_{\mathcal{C}} \to \mathbf{\Delta}^{\mathrm{op}}$ for the cocartesian fibration corresponding to $i_*\mathcal{C}$. Note that, since i^* preserves products, we get for any double ∞-category $\mathcal{M}$ an equivalence

$$\mathrm{Nat}(\mathcal{M}, i_*\mathcal{C}) \simeq \mathrm{Fun}(\mathcal{M}_0, \mathcal{C}).$$

Hence for any generalized non-symmetric ∞-operad $\mathcal{O}$ we get equivalences

$$\mathrm{Alg}_{\mathcal{O}}(\mathbf{\Delta}^{\mathrm{op}}_{\mathcal{C}}) \simeq \mathrm{Fun}^{\mathrm{cocart}}_{/\mathbf{\Delta}^{\mathrm{op}}}(\mathrm{Env}(\mathcal{O}), \mathbf{\Delta}^{\mathrm{op}}_{\mathcal{C}}) \simeq \mathrm{Fun}(\mathrm{Env}(\mathcal{O})_0, \mathcal{C}) \simeq \mathrm{Fun}(\mathcal{O}_0, \mathcal{C}).$$

Definition 8.7. Suppose $\mathcal{M}$ is a double ∞-category, viewed as a cocartesian fibration. The unit of the adjunction $i^* \dashv i_*$ corresponds to a functor $\mathcal{M} \to \mathbf{\Delta}^{\mathrm{op}}_{\mathcal{M}_0}$ that preserves cocartesian morphisms. For $X \in \mathcal{M}_0$ we define $\mathcal{M}^{\otimes}_X$ to be the pullback

$$\begin{tikzcd} \mathcal{M}^{\otimes}_X \arrow[r] \arrow[d] & \mathcal{M} \arrow[d] \\ \mathbf{\Delta}^{\mathrm{op}} \arrow[r] & \mathbf{\Delta}^{\mathrm{op}}_{\mathcal{M}_0}, \end{tikzcd}$$

where the bottom horizontal map corresponds to $(\mathbf{\Delta}^{\mathrm{op}})_0 \simeq \{X\} \to \mathcal{M}_0$. This is a pullback of cocartesian fibrations over $\mathbf{\Delta}^{\mathrm{op}}$, so the natural projection $\mathcal{M}^{\otimes}_X \to \mathbf{\Delta}^{\mathrm{op}}$ is a cocartesian fibration, which exhibits $\mathcal{M}^{\otimes}_X$ as a monoidal structure on $(\mathcal{M}^{\otimes}_X)_1$, which is the fibre of $\mathcal{M}_1 \to \mathcal{M}_0 \times \mathcal{M}_0$ at (X, X). For any generalized non-symmetric ∞-operad $\mathcal{O}$, the functor $\mathrm{Alg}_{\mathcal{O}}(-)$ preserves limits, and so gives a pullback square of ∞-categories

$$\begin{tikzcd} \mathrm{Alg}_{\mathcal{O}}(\mathcal{M}^{\otimes}_X) \arrow[r] \arrow[d] & \mathrm{Alg}_{\mathcal{O}}(\mathcal{M}) \arrow[d] \\ \{X\} \arrow[r] & \mathrm{Fun}(\mathcal{O}_0, \mathcal{M}_0). \end{tikzcd}$$

Remark 8.8. If $\mathcal{X}$ is an $(\infty, 2)$-category, then the monoidal ∞-category $\mathcal{X}^{\otimes}_X$ is the monoidal structure on the ∞-category $\mathcal{X}(X, X)$ of endomorphisms of X given by composition.

Applying this construction to monads and endofunctors via Remark 8.1 and Proposition 8.5, we get:

Corollary 8.9. *Let $\mathcal{X}$ be an $(\infty, 2)$-category and consider the commutative triangle*

$$\begin{tikzcd} \mathrm{Mnd}(\mathcal{X})^{(\mathrm{co})\mathrm{lax}} \arrow[rr] \arrow[dr] & & \mathrm{End}(\mathcal{X})^{(\mathrm{co})\mathrm{lax}} \arrow[dl] \\ & \iota_1\mathcal{X}. \end{tikzcd}$$

For any object $X \in \mathcal{X}$ we have equivalences of fibres

$$\mathrm{Mnd}(\mathcal{X})_{\mathrm{colax},X} \simeq \mathrm{Alg}(\mathcal{X}^{\otimes}_X), \qquad \mathrm{End}(\mathcal{X})_{\mathrm{colax},X} \simeq \mathcal{X}(X, X),$$

$$\mathrm{Mnd}(\mathcal{X})_{\mathrm{lax},X} \simeq \mathrm{Alg}(\mathcal{X}_X^{\otimes})^{\mathrm{op}}, \qquad \mathrm{End}(\mathcal{X})_{\mathrm{lax},X} \simeq \mathcal{X}(X,X)^{\mathrm{op}}.$$

Moreover, the morphisms on fibres at X in the triangle can be identified with the forgetful functors

$$\mathrm{Alg}(\mathcal{X}_X^{\otimes})^{(\mathrm{op})} \to \mathcal{X}(X,X)^{(\mathrm{op})}$$

from associative algebras to their underlying objects. □

In §4 we used results of Zaganidis and Riehl–Verity to construct a fully faithful functor

$$\mathrm{Mnd}(\mathrm{CAT}_\infty)_{\mathrm{lax}} \to \mathrm{Fun}(\Delta^1, \mathrm{Cat}_\infty)$$

with image the monadic right adjoints, and we just saw that the fibre of $\mathrm{Mnd}(\mathrm{CAT}_\infty)_{\mathrm{lax}}$ at a fixed ∞-category $\mathcal{C}$ is equivalent to the ∞-category $\mathrm{Alg}(\mathrm{End}(\mathcal{C}))$ of associative algebras in endofunctors of $\mathcal{C}$ under composition. Our functor thus restricts to a fully faithful functor $\mathrm{Alg}(\mathrm{End}(\mathcal{C})) \to \mathrm{Cat}_{\infty/\mathcal{C}}$ with image the monadic right adjoints. We denote the image of this functor at T by $\mathrm{Alg}_T(\mathcal{C})$.

On the other hand, the monoidal ∞-category $\mathrm{End}(\mathcal{C})$ acts on $\mathcal{C}$, so given a monoid $T \in \mathrm{Alg}(\mathrm{End}(\mathcal{C}))$ we can consider the ∞-category $\mathrm{Alg}_T^{\mathrm{Lur}}(\mathcal{C})$ of left T-modules in $\mathcal{C}$; the forgetful functor $\mathrm{Alg}_T^{\mathrm{Lur}}(\mathcal{C}) \to \mathcal{C}$ is proved by Lurie [22] to be a monadic right adjoint. The following proposition shows that these two monadic right adjoints associated to T are equivalent:

Proposition 8.10. *For $T \in \mathrm{Alg}(\mathrm{End}(\mathcal{C}))$ there is a canonical equivalence*

$$\mathrm{Alg}_T(\mathcal{C}) \simeq \mathrm{Alg}_T^{\mathrm{Lur}}(\mathcal{C}).$$

Proof. In the construction of Riehl and Verity the ∞-category $\mathrm{Alg}_T(\mathcal{C})$ and the monadic right adjoint $U_T \colon \mathrm{Alg}_T(\mathcal{C}) \to \mathcal{C}$ are obtained from the enriched right Kan extension of the functor $T \colon \mathfrak{mnd} \to \mathrm{CAT}_\infty$ along the inclusion $\mathfrak{mnd} \hookrightarrow \mathfrak{adj}$. From the structure of $\mathfrak{adj}$ we see that U_T is a left T-module in $\mathrm{Fun}(\mathrm{Alg}_T(\mathcal{C}), \mathcal{C})$. Now by [22]*Proposition 4.7.3.3 any right adjoint functor $G \colon \mathcal{D} \to \mathcal{C}$ has an *endomorphism monad*, meaning a terminal object in the ∞-category $\mathrm{LMod}(\mathrm{Fun}(\mathcal{D}, \mathcal{C}))_G$ of monads S on $\mathcal{C}$ together with an S-action on G. Moreover, such a monad S acting on G is the endomorphism monad of G if and only if the composite

$$S \to SGF \to GF$$

is an equivalence, where the first morphism uses the unit of the adjunction and the second the action of S on G.

The action of T on U_T certainly has this property, so T is the endomorphism monad of U_T. Now as U_T is a monadic right adjoint, Lurie's version of the Barr–Beck Theorem for ∞-categories [22]*Theorem 4.7.3.5 (together with [22]*Definition 4.7.3.4) furnishes an equivalence

$$\mathrm{Alg}_T(\mathcal{C}) \xrightarrow{\sim} \mathrm{Alg}_T^{\mathrm{Lur}}(\mathcal{C}) := \mathrm{LMod}_T(\mathcal{C})$$

over $\mathcal{C}$. □

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