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Cartesian factorization systems and pointed cartesian fibrations of ∞-categories

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Abstract

The goal of this paper is to prove an equivalence between the $(\infty,2)$-category of *cartesian* factorization systems on ∞-categories and that of *pointed* cartesian fibrations of ∞-categories. This generalizes a similar result known for ordinary categories and sheds some light on the interplay between these two seemingly distant concepts.

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Introduction

This paper is part of an ongoing project aimed at understanding fibrations in higher categories. The main source of inspiration was [7], in which the authors analyse, among other things, the relation between fibrations of categories and orthogonal factorization systems on the corresponding total categories. More precisely, they prove the following result.

Theorem (Thm. 3.9, [7]). *In a finitely complete category $\mathcal{C}$, $(\mathcal{E},\mathcal{M})$ is a simple reflective factorization system on $\mathcal{C}$ if and only if there exists a prefibration $p\colon \mathcal{C}\to\mathcal{B}$ preserving the terminal object with*

$$\mathcal{E} = p^{-1}(\mathrm{Iso}\mathcal{B}), \quad \mathcal{M} = \mathrm{Cart}(p)$$

where $\mathrm{Cart}(p)$ *denotes the class of p-cartesian morphisms.*

Our version of this for ∞-categories is given in Theorem 4.9, which we anticipate here.

Theorem. *There is an equivalence of $(\infty,2)$-categories between $\textsc{Cart}_*$ and $\textsc{Fact}_{cart}$, which sends an object $p\colon\mathcal{E}\to\mathcal{B}$ in $\textsc{Cart}_*$ to $(\mathcal{E},(S_L^p,S_R^p))$, where S_L^p is the class of maps inverted by p and S_R^p is the class of p-cartesian morphisms.*

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We have relaxed the completeness assumption but we instead consider the more natural notion of cartesian fibration rather than pre-fibration. The latter class is defined (see Section 3.7 in [7]) as that of functors $p: \mathcal{C} \to \mathcal{B}$ such that for every object C in $\mathcal{C}$, the induced funtor $p_C: \mathcal{C}_{/C} \to \mathcal{D}_{/pC}$ admits a right adjoint whose corresponding monad is *idempotent*. We will prove in Proposition 2.9 that cartesian fibrations of ∞-categories are characterized by a similar but stronger property, namely that such adjunctions exist and are localizations (in the sense that the right adjoint is fully faithful).

The paper is organized as follows. In Section 1 we recall the necessary background material that serves as foundation for what follows. In particular, we clarify what models for higher categories we use, and the relevant results available in the literature. We use marked simplicial categories, i.e. categories enriched over the category of marked simplicial sets, as our model for $(\infty, 2)$-categories.

Section 2 is devoted to cartesian fibrations and localizations. Here, we prove a useful lemma of independent interest (see Lemma 2.5) which will allow us to extend certain assignments to left adjoints. Next, we characterize cartesian fibrations as maps inducing localizations at the level of slice categories (Proposition 2.9).

In the next section, we introduce factorization systems and prove some facts which are stated without proof in the literature (see Propositions 3.3 and 3.5). Next, we consider localizations on ∞-categories with a terminal object which are induced by factorization systems, and we prove stability of factorization systems under the formation of slice categories.

In the fourth and final section, the $(\infty, 2)$-categories of cartesian factorization systems and pointed cartesian fibrations are introduced. We denote them, respectively, by FACT_{cart} and CART_*. Here, after a careful analysis of their mapping ∞-categories, we establish the equivalence in Theorem 4.9.

1. Preliminaries

In what follows, we will switch between ∞-categorical language and the ordinary one quite freely. If the context allows for ambiguity we will specify, for instance, if (co)limits are intended as ordinary ones or ∞-categorical ones (e.g. homotopy (co)limits in some model categorical presentation). Also, the term *∞-groupoid* will be used to mean an object in the ∞-category of homotopy types, without any precise model of this in mind (but rather just the ∞-category obtained as the free-cocompletion of the terminal one). When we actually mean Kan complex, this will be made clear.

1.1 ∞-groupoids and ∞-categories We will denote by $\mathcal{S}\mathrm{et}_\Delta$ the category of simplicial sets. We will employ the standard notation $\Delta^n \in \mathcal{S}\mathrm{et}_\Delta$ for the n-simplex, and for $\varnothing \neq S \subseteq [n]$ we write $\Delta^S \subseteq \Delta^n$ for the $(|S| - 1)$-dimensional face of Δ^n whose set of vertices is S. For $0 \leq i \leq n$, we will denote by $\Lambda^n_i \subseteq \Delta^n$ the i'th horn in Δ^n, that is, the subsimplicial set spanned by all $(n-1)$-dimensional faces containing the i'th vertex. By an ∞-category we will always mean a *quasi-category*, *i.e.*, a simplicial set X which admits extensions for all inclusions $\Lambda^n_i \hookrightarrow \Delta^n$, for all $n > 1$ and all $0 < i < n$ (known as *inner* horns). If an ∞-category X, in addition, admits extensions for $\Lambda^n_0 \hookrightarrow \Delta^n$ and $\Lambda^n_n \hookrightarrow \Delta^n$, then it is called a *Kan complex*. These will be our favourite models for ∞-categories and ∞-groupoids, respectively.

Given an ∞-category $\mathcal{C}$, we have several (equivalent) models for the ∞-groupoid of morphisms $\mathcal{C}(x, y)$ between a pair of objects x, y in $\mathcal{C}$. These exhibit $\mathcal{C}$ as *weakly enriched* over ∞-groupoids.

Proposition 1.1. *Let $\mathcal{C}$ be an ∞-category. The following simplicial sets are all equivalent Kan complexes:*

- *the simplicial set* $\mathrm{map}_{\mathcal{C}}(x,y)$ *defined by the following pullback square:*

$$\begin{array}{ccc} \mathrm{map}_{\mathcal{C}}(x,y) & \longrightarrow & \mathcal{C}^{\Delta^1} \\ \big\downarrow & & \big\downarrow {\scriptstyle (\pi_0,\pi_1)} \\ \Delta^0 & \xrightarrow{\{(x,y)\}} & \mathcal{C}\times\mathcal{C} \end{array}$$

  where (π_0,π_1) *is the map induced by the inclusion* $\partial\Delta^1 \to \Delta^1$.
- *the simplicial set* $\mathrm{map}^{\triangleleft}_{\mathcal{C}}(x,y)$, *whose set of* n*-simplices corresponds to*

$$\{\alpha\colon \Delta^{n+1} \to \mathcal{C} \mid \alpha(0) = x, \alpha_{|\Delta^{\{1,\dots,n+1\}}} = \sigma(y)\}$$

  i.e. their restrictions to the d^0*-face are degenerate at* y.
- *the simplicial set* $\mathrm{map}^{\triangleright}_{\mathcal{C}}(x,y)$, *whose set of* n*-simplices corresponds to*

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  i.e. their restrictions to the d^{n+1}*-face are degenerate at* x.

We will denote the ∞-groupoid represented (up to equivalence) by any of these Kan complexes by $\mathcal{C}(x,y)$.

Given an ∞-category X, we will denote its homotopy category by $\mathrm{ho}(X)$. This is the ordinary category having as objects the 0-simplices of X, and as morphisms $x \to y$ the set of equivalence classes of 1-simplices $f\colon x \to y$ of X under the equivalence relation generated by identifying f and f' if there is a 2-simplex H in X with $H|_{\Delta^{\{1,2\}}} = f$, $H|_{\Delta^{\{0,2\}}} = f'$ and $H|_{\Delta^{\{0,1\}}}$ degenerate on x. We recall that the functor of ∞-categories $\mathrm{ho}\colon \infty\text{-}\mathcal{C}at \to 1\text{-}\mathcal{C}at$ is left adjoint (in the ∞-categorical sense) to the ordinary nerve functor. This can of course be presented via a Quillen adjunction whose left adjoint has the form $\mathrm{ho}\colon \mathcal{S}et_{\Delta} \to 1\text{-}\mathcal{C}at$, where the domain is endowed with the Joyal model structure.

Definition 1.2. Let $f\colon \mathcal{C} \to \mathcal{D}$ be a map of ∞-categories. Then we say f is:

- *essentially surjective* if $\mathrm{ho}(f)\colon \mathrm{ho}(\mathcal{C}) \to \mathrm{ho}(\mathcal{D})$ is an essentially surjcetive functor between ordinary categories.
- *fully faithful* if the induced map $f_{x,y}\colon \mathcal{C}(x,y) \to \mathcal{D}(fx,fy)$ is an equivalence of ∞-groupoids for every pair of objects (x,y) in $\mathcal{C}$.

Just like for ordinary category theory, we have the following useful result.

Theorem 1.3 (Thm. 3.9.7, [1]). *A functor $f\colon \mathcal{C} \to \mathcal{D}$ between ∞-categories is an equivalence if and only if it is essentially surjective and fully faithful.*

As far as the basics of limits and colimits in ∞-categories that we use here, we refer the reader to Chapter 4 of [5] for the relevant terminology and results.

Finally, if $\mathcal{C}$ is an ∞-category, then we say that a subobject $\mathcal{A} \subset \mathcal{C}$ is a *full subcategory of* $\mathcal{C}$ *spanned by a set of objects* A if there exists a set of vertices $A \subset \mathcal{C}_0$ of $\mathcal{C}$ such that the simplicial set $\mathcal{A}$ consists of all the simplices in $\mathcal{C}$ whose vertices belong to A. When this is the case, it is clear that $\mathcal{A}$ is itself an ∞-category and the natural inclusion $\mathcal{A} \to \mathcal{C}$ is fully faithful.

1.2 Marked simplicial sets and marked-simplicial categories Our standard reference for marked simplicial sets is Chapter 3 of [5], and for scaled simplicial sets and marked simplicial categories we refer the reader to [4]. We will use such objects as models for higher categories, as precisely described in what follows.

Definition 1.4. A *marked simplicial set* is a pair (X, E) where X is simplicial set and M is a subset of the set of 1-simplices of X, called *marked* simplices, such that it contains the degenerate ones. A map of marked simplicial sets $f\colon (X, E_X) \to (Y, E_Y)$ is a map of simplicial sets $f\colon X \to Y$ satisfying $f(E_X) \subseteq E_Y$.

The category of marked simplicial sets will be denoted by $\mathcal{S}\mathrm{et}_\Delta^+$.

Notation 1.5. For simplicity, we will often speak only of the non-degenerate marked edges when considering a marked simplicial set. For example, if X is a simplicial set and E is any set of edges in X then we will denote by (X, E) the marked simplicial set whose underlying simplicial set is X and whose marked edges are E together with the degenerate edges. In addition, when there is no risk of ambiguity, we will omit the set of marked 1-simplices and just denote (X, E) by X.

Remark 1.6. The category $\mathcal{S}\mathrm{et}_\Delta^+$ of marked simplicial sets admits an alternative description, as the category of models of a limit sketch. In particular, it is a reflective localization of a presheaf category and it is a cartesian closed category.

Theorem 1.7 ([5])**.** *There exists a model category structure on the category $\mathcal{S}et_\Delta^+$ of marked simplicial sets in which cofibrations are exactly the monomorphisms and the fibrant objects are marked simplicial sets (X, E) in which X is an ∞-category and E is the set of equivalences of X,* i.e.*, 1-simplices $f\colon \Delta^1 \to X$ which are invertible in* $\mathrm{ho}(X)$*.*

Remark 1.8. Marked simplicial sets are a model for $(\infty, 1)$-categories. Because of the description of the fibrant objects in the model structure on $\mathcal{S}\mathrm{et}_\Delta^+$, we will often consider an ∞-category as a marked simplicial set, where we implicitly understand the marking as the one given by the equivalences.

Definition 1.9. We let $\mathcal{C}at_\Delta^+$ denote the category of categories enriched over marked simplicial sets. We will refer to these as *marked-simplicial categories.*

By virtue of Proposition A.3.2.4 and Theorem A.3.2.24 of [5], the category $\mathcal{C}at_\Delta^+$ is endowed with a model category structure in which the weak equivalences are the *Dwyer–Kan equivalences.* More explicitly, these are the maps $f\colon \mathcal{C} \to \mathcal{D}$ which are

- *fully-faithful:* the maps $f_{x,y}\colon \mathcal{C}(x, y) \to \mathcal{D}(f(x), f(y))$ are marked categorical equivalences;
- *essentially surjective:* the functor of ordinary categories given by $\mathbf{ho}(f)\colon \mathbf{ho}(\mathcal{C}) \to \mathbf{ho}(\mathcal{D})$ is essentially surjective, where for a marked-simplicial category $\mathcal{E}$ we denote by $\mathbf{ho}(\mathcal{E})$ the category whose objects are the objects of $\mathcal{E}$ and such that $\mathrm{Hom}_{\mathbf{ho}(\mathcal{E})}(x, y) := [\Delta^0, \mathcal{C}(x, y)]$ is the set of homotopy classes of maps from Δ^0 to $\mathcal{C}(x, y)$ with respect to the marked categorical model structure.

We also note that the trivial fibrations in $\mathcal{C}at_\Delta^+$ are the maps $f\colon \mathcal{C} \to \mathcal{D}$ which are surjective on objects and such that $f_{x,y}\colon \mathcal{C}(x, y) \to \mathcal{D}(f(x), f(y))$ is a trivial fibration of marked simplicial sets for every pair of objects (x, y) in $\mathcal{C}$.

Example 1.10. Our main example of a marked simplicial category is that of CAT_∞, the marked simplicial category of ∞-categories. It is defined by having as objects ∞-categories, and the marked simplicial set $\mathrm{CAT}_\infty(X,Y)$ between two such objects is defined to be the ∞-category Y^X with marking given by the equivalences. The rest of the structure is defined in the obvious way.

2. Cartesian fibrations

In this section we recall the notion of cartesian fibration of simplicial sets, and give a new characterization of it in terms of the existence of certain adjoint functors between ∞-categories. We fix a map $p\colon X \to Y$ of simplicial sets which we will refer to throughout the whole section.

Definition 2.1. A 1-simplex $f\colon x' \to x$ in X is said to be *p-cartesian* if for any given solid commutative square as depicted below, where $n \geq 2$, the dashed lifting exists provided $g(\Delta^{\{n-1,n\}}) = f$.

$$\begin{array}{ccc} \Lambda^n_n & \xrightarrow{g} & X \\ \downarrow & \nearrow & \downarrow p \\ \Delta^n & \xrightarrow{l} & Y \end{array}$$

Given the notion of cartesian 1-simplices, a cartesian fibration is essentially a map with enough cartesian lifts.

Definition 2.2. A map $p\colon X \to Y$ is a *cartesian fibration* if it is an inner fibration, and for every 1-simplex $h\colon y \to p(x)$ there exists a p-cartesian 1-simplex $\bar{h}\colon x' \to x$ with $p(\bar{h}) = h$

By dualizing the previous definitions we get the notion of *p-cocartesian* 1-simplices and *cocartesian fibrations.*

If the base of our map is an ∞-category we can give the following characterization.

Proposition 2.3 ([5], Prop.2.4.4.3)**.** *Let $p\colon X \to Y$ be an inner fibration of ∞-categories and $f\colon x' \to x$ a 1-simplex of X, then the following are equivalent:*

1. f is p-cartesian.

2. for every vertex $z \in X$ the following square is a pullback in the ∞-category of ∞-groupoids:

$$\begin{array}{ccc} X(z,x') & \xrightarrow{f\circ -} & X(z,x) \\ {\scriptstyle p_{z,x'}}\downarrow & & \downarrow{\scriptstyle p_{z,x}} \\ Y(pz,px') & \xrightarrow{p(f)\circ -} & Y(pz,px) \end{array} \tag{2.1}$$

We now recall the fibrational definition of adjunction of ∞-categories.

Definition 2.4 ([5], Def.5.2.2.1)**.** Let $\mathcal{C},\mathcal{D}$ be ∞-categories. An *adjunction* between $\mathcal{C}$ and $\mathcal{D}$ is a map $q\colon \mathcal{M} \to \Delta^1$ which is simultaneously a cartesian and a cocartesian fibration, satisfying $q^{-1}(0) \simeq \mathcal{C}$ and $q^{-1}(1) \simeq \mathcal{D}$. By Proposition 5.2.1.4 of [5], we can associate functors $f\colon \mathcal{C} \to \mathcal{D}$ and $g\colon \mathcal{D} \to \mathcal{C}$ to such q, and we say that f (resp. g) is *left adjoint* to g (resp. *right adjoint* to f). We denote this situation by $f \dashv g$.

The following result allows us to construct left adjoints given their assignments on objects, provided we have a suitable family of equivalences of hom-spaces.

Lemma 2.5. *Let $R: \mathcal{D} \to \mathcal{C}$ be a map of ∞-categories, and suppose given a map of vertices $L_0: \mathcal{C}_0 \to \mathcal{D}_0$. Given equivalences of ∞-groupoids $\Psi_{c,d}: \mathcal{D}(L_0c, d) \xrightarrow{\simeq} \mathcal{C}(c, Rd)$ natural in $d \in \mathcal{D}$, we can extend L_0 to a map $L: \mathcal{C} \to \mathcal{D}$ satisfying $L \dashv R$.*

Proof. Let $q: \mathcal{M} \to \Delta^1$ be the cartesian fibrations associated with R, so that $q^{-1}(0) \simeq \mathcal{C}$ and $q^{-1}(1) \simeq \mathcal{D}$. Our task is to show q is also cocartesian. To obtain this, we have to exhibit, for every $c \in \mathcal{C}$ a q-cocartesian morphism $\eta: c \to d$ in $\mathcal{M}$ with $d \in \mathcal{D}$. Let $d \overset{\text{def}}{=} L_0c$, and choose a q-cartesian lift of the unique morphism $0 \to 1 = q(x)$, which we denote by $h: Rx \to x$. Consider the following diagram for an arbitrary $x \in \mathcal{D}$.

$$\begin{array}{ccccccc} \mathcal{D}(L_0c, x) & \xrightarrow{\simeq} & \mathcal{M}(L_0c, x) & \dashrightarrow & \mathcal{M}(c, x) & \xleftarrow[\simeq]{} & \mathcal{C}(c, Rx) \\ & & \downarrow{\scriptstyle q_{L_0c,x}} & & \downarrow{\scriptstyle q_{c,x}} & & \downarrow{\scriptstyle q_{c,Rx}} \\ * & \xrightarrow{\simeq} & \Delta^1(1,1) & \xrightarrow{\simeq} & \Delta^1(0,1) & \xleftarrow[\simeq]{} & \Delta^1(0,0) \end{array}$$

where the right-hand square is the pullback of ∞-groupoids induced by postcomposing by h.

The naturality of the composite map $\mathcal{D}(L_0c, x) \xrightarrow{\Psi_{c,x}} \mathcal{C}(c, Rx) \xrightarrow{h\circ -} \mathcal{M}(c, x)$, together with the observation that $\mathcal{M}(L_0c, y) \cong \varnothing$ if $y \in \mathcal{C}$, gives us a morphism $f: c \to L_0c$ in $\mathcal{M}$ by the Yoneda lemma, which renders the left-hand square a pullback. In fact, since $- \circ f: \mathcal{M}(L_0c, x) \to \mathcal{M}(c, x)$ is a composite of equivalences, it must be an equivalence. □

2.1 Localizations We give here a brief recap on localizations of ∞-categories, and we suggest reading the relevant section in Chapter 5 of [5] to the interested reader.

Definition 2.6. A map $f: \mathcal{C} \to \mathcal{D}$ between ∞-categories is a *localization* if it admits a fully-faithful right adjoint.

Here is a useful characterization of localization functors.

Proposition 2.7 (Prop.5.2.7.4, [5])**.** *Let $\mathcal{C}$ be an ∞-category and let $L: \mathcal{C} \to \mathcal{C}$ be a functor with essential image $L\mathcal{C} \subset \mathcal{C}$. The following conditions are equivalent:*

1. *There exists a functor $f: \mathcal{C} \to \mathcal{D}$ with a fully faithful right adjoint $g: \mathcal{D} \to \mathcal{C}$ and an equivalence between $g \circ f$ and L.*
2. *When regarded as a functor from $\mathcal{C}$ to $L\mathcal{C}$, L is left adjoint to the inclusion $L\mathcal{C} \hookrightarrow \mathcal{C}$.*
3. *There exists a natural transformation $\alpha: \mathcal{C} \times \Delta^1 \to \mathcal{C}$ with $\alpha: \mathrm{Id}_{\mathcal{C}} \to L$ such that, for every object $C \in \mathcal{C}$, the morphisms $L(\alpha_C), \alpha_{LC}: LC \to LLC$ are equivalences in $\mathcal{C}$.*

The next result shows that this kind of localization is indeed an instance of a more general one, in which we formally invert a given family of morphisms.

Proposition 2.8 (Prop.5.2.7.12, [5])**.** *Let $\mathcal{C}$ be an ∞-category and let $L: \mathcal{C} \to \mathcal{C}$ be a localization functor with essential image $L\mathcal{C}$. Let $\mathcal{S}$ denote the collection of all morphisms f in $\mathcal{C}$ such that Lf is an equivalence. Then, for every ∞-category $\mathcal{D}$, composition with L induces a fully faithful functor*

$$\psi: \mathcal{D}^{L\mathcal{C}} \to \mathcal{D}^{\mathcal{C}}$$

whose essential image consists of all those map $F: \mathcal{C} \to \mathcal{D}$ such that $F(f)$ is an equivalence in $\mathcal{D}$ for every morphisms f in $\mathcal{S}$.

We can now give the following characterization of cartesian fibrations of ∞-categories in terms of localization functors.

Proposition 2.9. *Let p be an isofibration of ∞-categories. Then p is a cartesian fibration if and only if the map $p_x: X_{/x} \to Y_{/p(x)}$ admits a fully faithful right adjoint q_x for every object x in $\mathcal{C}$.*

Remark 2.10. In other words, this result is saying that an isofibration of ∞-categories is a cartesian fibration if and only if the induced maps $p_x: X_{/x} \to Y_{/p(x)}$ are localizations. Although the two statements are equivalent under the axiom of choice, it is clear from the proof that p is also endowed with a prescribed choice of p-cartesian lifts.

Proof. Suppose the right adjoints q_x exist for every vertex x in X. Firstly, we observe that p_x is also an isofibration of ∞-categories. Therefore, we can assume, without loss of generality, that $p_x \circ q_x = 1_{Y_{/p(x)}}$. Indeed, we can consider the following commutative square:

$$\begin{array}{ccc} \{0\} & \xrightarrow{q_x} & {X_{/x}}^{Y_{/p(x)}} \\ \big\downarrow & & \big\downarrow {\scriptstyle (p_{/x})^*} \\ J & \xrightarrow{\epsilon_x} & {Y_{/p(x)}}^{Y_{/p(x)}} \end{array}$$

where ϵ_x denotes the natural equivalence $q_x \circ p_x \simeq 1_{Y_{/p(x)}}$ which features as the counit of the adjunction $p_x \dashv q_x$, and $(p_{/x})^*$ is post-composition by p_x. Moreover, here J denotes the nerve of the free living groupoid on an arrow, which is well known to be an interval object for the Joyal model structure on $\mathcal{S}\mathrm{et}_\Delta$. A lift for this square provides a map $q'_x: Y_{/p(x)} \to X_{/x}$ which is equivalent to q_x and such that $p_x \circ q'_x = 1_{Y_{/p(x)}}$.

Now, given a 1-simplex $h: y \to p(x)$ in Y, set $\bar{h} \overset{\text{def}}{=} q_x(h): x' \to x$. By construction we have $p(\bar{h}) = h$, so let us show that $\bar{h}$ is also p-cartesian. We have to show that the following square is a pullback in the ∞-category of ∞-groupoids:

$$\begin{array}{ccc} X(z, x') & \xrightarrow{q_x(h)\circ -} & X(z, x) \\ {\scriptstyle p_{z,x'}} \big\downarrow & & \big\downarrow {\scriptstyle p_{z,x}} \\ Y(pz, px') & \xrightarrow{h\circ -} & Y(pz, px) \end{array}$$

this square can be modeled by a commutative square of Kan complexes where the vertical maps are Kan fibrations. Therefore, we have to show that the induced map at the level of fibers is a homotopy equivalence of Kan complexes. Equivalently, we can prove that there is an induced homotopy equivalence between the (homotopy) fibers of the horizontal maps. We have, by definition, the following (homotopy) pullback squares of simplicial sets, computed with respect to the Kan-Quillen model structure:

$$\begin{array}{ccc} X_{/x}(g, q_x(h)) & \longrightarrow & X(z, x') \\ \big\downarrow & & \big\downarrow {\scriptstyle q_x(h)\circ -} \\ \Delta^0 & \xrightarrow{\{g\}} & X(z, x) \end{array} \qquad \begin{array}{ccc} Y_{/p(x)}(p_x(g), h) & \longrightarrow & Y(pz, px') \\ \big\downarrow & & \big\downarrow {\scriptstyle h\circ -} \\ \Delta^0 & \xrightarrow{\{p(g)\}} & Y(pz, px) \end{array}$$

We observe that the map between these two fibers is the adjunction equivalence

$$Y_{/p(x)}(p_x(g), h) \simeq X_{/x}(g, q_x(h))$$

so we are done.

Conversely, assume p is a cartesian fibration. A choice of p-cartesian lifts for every 1-simplex $h: y \to p(x)$ as h, y and x vary provides an assignment on vertices of the form

$$(q_x)_0 : \left(Y_{/p(x)}\right)_0 \to \left(X_{/x}\right)_0$$

It follows from the previous paragraph that we also have homotopy equivalences of the form: $Y_{/p(x)}(p_x(g), h) \simeq X_{/x}(g, q_x(h))$ for every h. This implies, thanks to Lemma 2.5, that we can extend $(q_x)_0$ to a map $q_x: Y_{/p(x)} \to X_{/x}$ which, in addition, must be a fully faithful right adjoint of p_x. □

Corollary 2.11. *Let $p: X \to Y$ be a cartesian fibrations of ∞-categories. Suppose X and Y admit a terminal object and that p preserves it. Then p is a localization functor.*

Proof. Let $*_X$ (resp. $*_Y$) denote the terminal object of X (resp. Y). Then p is equivalent to the map $p_{*_X}: X_{/*_X} \to Y_{/p(*_Y)}$, since $p(*_X) \simeq *_Y$, and this map is a localization functor thanks to the previous result. □

3. Factorization systems

In this section we recall the fundamental definitions for factorizations systems on ∞-categories, we prove that the two definitions available in the literature are equivalent and we show that cartesian fibrations always induce a factorization system on the total category.

Definition 3.1 ([5])**.** Suppose given maps $f: a \to b$ and $g: x \to y$ in an ∞-category $\mathcal{C}$. Then we say that f is *left orthogonal* to g (and that g is *right orthogonal* to f) if the following square is a pullback in the ∞-category of ∞-groupoids.

$$\begin{array}{ccc} \mathcal{C}(b,x) & \xrightarrow{g\circ -} & \mathcal{C}(b,y) \\ {\scriptstyle -\circ f}\downarrow & & \downarrow{\scriptstyle -\circ f} \\ \mathcal{C}(a,x) & \xrightarrow{g\circ -} & \mathcal{C}(a,y) \end{array}$$

If this is the case, then we denote this relation by $f \perp g$.

Remark 3.2. Informally speaking, this definition is saying that for every given commutative square in $\mathcal{C}$ of the form:

$$\begin{array}{ccc} a & \longrightarrow & x \\ {\scriptstyle f}\downarrow & \overset{h}{\nearrow} & \downarrow{\scriptstyle g} \\ b & \longrightarrow & y \end{array}$$

there exists a lift as indicated by the dotted arrow h, which is unique up to a contractible space (i.e. ∞-groupoid) of choices.

To make this formal, let us introduce the next idea, which is due to Joyal (see [3]). We can view squares in $\mathcal{C}$ with a given lift in a coherent manner as maps $\Delta^3 \to \mathcal{C}$. Since $\Delta^3 \cong \Delta^1 \star \Delta^1$, we get a natural inclusion $\Delta^1 \times \Delta^1 \hookrightarrow \Delta^3$ which picks out the commutative square forgetting the lift. We thus get an induced isofibration $\mathcal{C}^{\Delta^3} \to \mathcal{C}^{\Delta^1 \times \Delta^1}$. Consider the following diagram, where both squares are pullbacks and the right-hand side vertical maps are obtained by restriction:

$$\begin{array}{ccc}
\mathrm{Lift}(f,g) & \longrightarrow & \mathcal{C}^{\Delta^3} \\
\downarrow & & \downarrow \\
\mathcal{C}^{\Delta^1}(f,g) & \longrightarrow & \mathcal{C}^{\Delta^1\times\Delta^1} \\
\downarrow & & \downarrow \\
\Delta^0 & \xrightarrow{\{(f,g)\}} & \mathcal{C}^{\{0\}\times\Delta^1}\times\mathcal{C}^{\{1\}\times\Delta^1}
\end{array}$$

Proposition 3.3. *The pair of maps (f,g) in $\mathcal{C}$ satisfies $f \perp g$ if and only if the induced map $\mathrm{Lift}(f,g) \to \mathcal{C}^{\Delta^1}(f,g)$ is a trivial fibration of simplicial sets.*

Proof. Since the map $\mathrm{Lift}(f,g) \to \mathcal{C}^{\Delta^1}(f,g)$ is a Joyal fibration of ∞-categories, it is enough to show it is an equivalence.

The spine inclusion $\mathsf{Sp}^3 \hookrightarrow \Delta^3$ is an inner anodyne map, so that we have an equivalence of ∞-categories $\mathcal{C}^{\Delta^3} \to \mathcal{C}^{\mathsf{Sp}^3}$ over $\mathcal{C}^{\{0\}\times\Delta^1}\times\mathcal{C}^{\{1\}\times\Delta^1}$. By pulling back along the map

$$\{(f,g)\}\colon \Delta^0 \to \mathcal{C}^{\{0\}\times\Delta^1}\times\mathcal{C}^{\{1\}\times\Delta^1}$$

we get an equivalence of ∞-categories of the form $\mathrm{Lift}(f,g) \simeq \mathcal{C}(b,x)$. Therefore, we are left with proving that $f \perp g$ if and only if $\mathcal{C}^{\Delta^1}(f,g) \simeq \mathcal{C}(b,x)$. Thanks to Proposition 5.1 of [2], we can express the term on the left-hand side by means of an end, as follows:

$$\mathcal{C}^{\Delta^1}(f,g) \simeq \int_{x\in\Delta^1} \mathcal{C}(fx,gx) \simeq \mathcal{C}(a,x)\times_{\mathcal{C}(a,y)}\mathcal{C}(b,y)$$

Hence, $\mathcal{C}^{\Delta^1}(f,g) \simeq \mathcal{C}(b,x)$ is equivalent to having the pullback square as in Definition 3.1. □

Let us now introduce the notion of factorization system.

Definition 3.4 ([5], Def.5.2.8.8 and [3], Section 24)**.** A factorization system on an ∞-category $\mathcal{C}$ consists of a pair $\mathcal{L},\mathcal{R}$ of collection of morphisms in $\mathcal{C}$ satisfying the following properties:

1. Both families $\mathcal{L}$ and $\mathcal{R}$ are closed under retracts.
2. Every morphism in $\mathcal{L}$ is left orthogonal to every morphism in $\mathcal{R}$ (a fact which we will denote by $\mathcal{L} \perp \mathcal{R}$).
3. For every morphism $h\colon x \to z$ in $\mathcal{C}$ there is a 2-simplex in $\mathcal{C}$ of the form:

$$\begin{array}{ccc}
x & \xrightarrow{\quad h \quad} & z \\
& {\scriptstyle f}\searrow \quad \nearrow{\scriptstyle g} & \\
& y &
\end{array}$$

with $f \in \mathcal{L}$ and $g \in \mathcal{R}$

For elements of the theory of factorization systems on ∞-categories we suggest reading Section 5.2.8 of [5].

The (dual of the) following result is stated without proof as Example 5.2.8.15 in [5].

Proposition 3.5. *Let $p\colon \mathcal{E} \to \mathcal{B}$ be a cartesian fibration of ∞-categories. Then we get an induced factorization system (p_L, p_R) on $\mathcal{E}$, where p_L is the class of morphisms which are sent to equivalences by p, and p_R is the class of p-cartesian morphisms.*

Proof. The fact that p_L is closed under retract is obvious, and for p_R it follows from Proposition 2.3 together with the fact that pullback squares are closed under retract. Indeed, suppose f is a retract of a p-cartesian morphism g. The square in (2.1) for f is a retract of the analogous square for g, so the same holds for the comparison map $\mathcal{E}(z,x') \to \mathcal{E}(z,x) \times_{\mathcal{B}(pz,px)} \mathcal{B}(pz,px')$, which must then be a retract of an equivalence. It follows that such map must be an equivalence as well, so f is p-cartesian.

Next, suppose $f \in p_L$ and $g \in p_R$. Consider the following cube (where we have omitted some of the obvious maps for sake of clarity):

$$\begin{array}{ccccccc}
 & & \mathcal{E}(b,c) & & \xrightarrow{\ -\circ f\ } & & \mathcal{E}(a,c) \\
 & \swarrow^{p_{b,c}} & \downarrow & & & \swarrow & \downarrow{\scriptstyle g\circ -} \\
\mathcal{B}(pb,pc) & & \xrightarrow{\ -\circ p(f)\ } & & \mathcal{B}(pa,pc) & & \\
 & & \mathcal{E}(b,d) & & \xrightarrow{\ -\circ f\ } & & \mathcal{E}(a,d) \\
\downarrow{\scriptstyle p(g)\circ -} & \swarrow^{p_{b,d}} & & & \downarrow & \swarrow^{p_{a,d}} & \\
\mathcal{B}(pb,pd) & & \xrightarrow{\ -\circ p(f)\ } & & \mathcal{B}(pa,pd) & &
\end{array}$$

The front face is a pullback since $p(f)$ is an equivalence by assumption. The left-hand side and right-hand side faces are pullbacks since g is p-cartesian, so it follows that the back face must be a pullback as well, i.e. $f \perp g$.

Finally, assume given a morphism $h\colon a \to b$ in $\mathcal{E}$. Let $\bar{h}\colon a' \to b$ be a p-cartesian lift of $p(h)$, and consider the following lifting problem:

$$\begin{array}{ccc}
\Lambda^2_2 & \xrightarrow{(h,\bar{h})} & \mathcal{E} \\
\big\downarrow & & \big\downarrow{\scriptstyle p} \\
\Delta^2 & \xrightarrow{s^0(p(h))} & \mathcal{B}
\end{array}$$

Since $\bar{h}$ is p-cartesian, we get a lift $H\colon \Delta^2 \to \mathcal{E}$ which is easily seen to be the factorization of h we are looking for. □

The following result describes a pretty common situation in which a factorization system induces a localization functor.

Proposition 3.6. *Let $\mathcal{E}$ be an ∞-category endowed with a factorization system $(\mathcal{L},\mathcal{R})$. Suppose $\mathcal{E}$ admits a terminal object $* \in \mathcal{E}$ and let $L\mathcal{E}$ be the full subcategory of $\mathcal{E}$ spanned by those objects x such that the map $!_x\colon x \to *$ belongs to $\mathcal{R}$. Then, we get a localization functor $L\colon \mathcal{E} \to \mathcal{E}$ which exhibits $L\mathcal{E}$ as a localization of $\mathcal{E}$. Moreover, if $\mathcal{L}$ has the two-out-of-three property, then this is the localization at the class of maps $\mathcal{L}$.*

Proof. We begin by factoring the map $!_x\colon x \to *$ as $x \to L_0x \to *$, where $l_x\colon x \to L_0x$ belongs to $\mathcal{L}$ and $L_0x \to *$ belongs to $\mathcal{R}$. In this manner, we obtain an assignment on objects of the form $L_0\colon \mathcal{E}_0 \to (L\mathcal{E})_0$. In order to apply Lemma 2.5, we have to exhibit equivalences of the form $\mathcal{E}(L_0x,y) \simeq \mathcal{E}(x,y)$ for every $y \in L\mathcal{E}$. Consider the following commutative square:

$$\begin{array}{ccc}
\mathcal{E}(L_0x,y) & \xrightarrow{-\circ l_x} & \mathcal{E}(x,y) \\
{\scriptstyle !_y\circ -}\big\downarrow & & \big\downarrow{\scriptstyle !_y\circ -} \\
\mathcal{E}(L_0x,*) & \xrightarrow{-\circ l_x} & \mathcal{E}(x,*)
\end{array}$$

By assumption, it is a pullback of ∞-groupoids, since $l_x \perp !_y$. Because $*$ is a terminal object in $\mathcal{E}$, the upper horizontal map must be an equivalence, which is natural in $y \in L\mathcal{E}$, so we can conclude by applying Lemma 2.5.

Turning to the second point, suppose $L(f)$ is an equivalence. This implies that we have a commutative diagram in $\mathcal{E}$ of the form:

$$\begin{array}{ccc} x & \xrightarrow{f} & y \\ {\scriptstyle l_x}\downarrow & & \downarrow{\scriptstyle l_y} \\ Lx & \xrightarrow{L(f)} & Ly \end{array}$$

Since $L(f)$ is an equivalence by assumption, and since $\mathcal{L}$ is assumed to have the two-out-of-three property, we get that f belongs to $\mathcal{L}$. Since the other implication is trivial, this concludes the proof. □

Definition 3.7. Let $(\mathcal{E}, (\mathcal{L}, \mathcal{R}))$ be a factorization system on an ∞-category $\mathcal{E}$ with terminal object. The localization functor $L\colon \mathcal{E} \to L\mathcal{E}$ described in Proposition 3.6 is called *simple* if a morphism $f\colon x \to y$ in $\mathcal{E}$ is in $\mathcal{R}$ if and only if the naturality square depicted below is a pullback in $\mathcal{E}$.

$$\begin{array}{ccc} x & \xrightarrow{f} & y \\ {\scriptstyle l_x}\downarrow & & \downarrow{\scriptstyle l_y} \\ Lx & \xrightarrow{L(f)} & Ly \end{array}$$

In case the factorization system is induced by a cartesian fibration we have the following corollary.

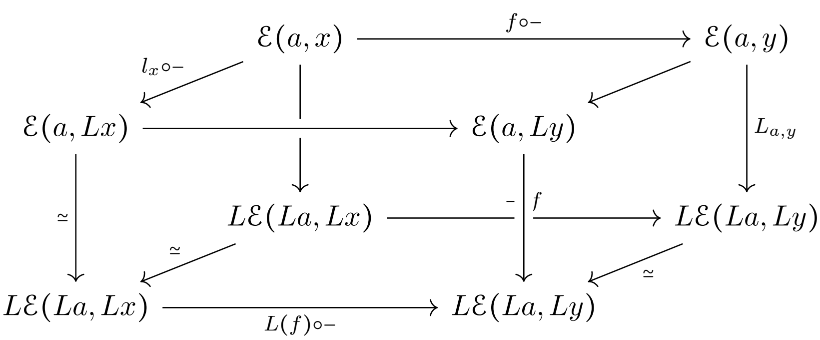
Corollary 3.8. *Let $p\colon \mathcal{E} \to \mathcal{B}$ be a cartesian fibration between ∞-categories admitting a terminal object. If p preserves the terminal object, then the localization functor $L\colon \mathcal{E} \to L\mathcal{E}$ (which exists thanks to Proposition 3.5 and Proposition 3.6) is simple.*

Proof. Firstly, let us observe that the localization functor coincides with p itself, thanks to the abovementioned propositions. We will use the notation $L\colon \mathcal{E} \to L\mathcal{E}$ to stress the fact that it is a localization.

Suppose the naturality square

$$\begin{array}{ccc} x & \xrightarrow{f} & y \\ {\scriptstyle l_x}\downarrow & & \downarrow{\scriptstyle l_y} \\ Lx & \xrightarrow{L(f)} & Ly \end{array} \tag{3.1}$$

is a pullback in $\mathcal{E}$. Let us now consider the following cube in $\mathcal{E}$ (where, for sake of clarity, we have omitted some of the arrows' labels), for any object a in $\mathcal{E}$:



The top face is obtained by applying $\mathcal{E}(a,-)$ to the square (3.1), and is thus a pullback. The front face and the bottom one are pullbacks since they both have a pair of parallel maps which are equivalences, so the back face must be a pullback as well. This proves f is p-cartesian.

Conversely, assume f is p-cartesian. Then the square

$$\begin{array}{ccc} \mathcal{E}(a,x) & \xrightarrow{f\circ-} & \mathcal{E}(a,y) \\ {\scriptstyle l_x\circ-}\downarrow & & \downarrow{\scriptstyle l_y\circ-} \\ \mathcal{E}(a,Lx) & \xrightarrow{L(f)\circ-} & \mathcal{E}(a,Ly) \end{array}$$

obtained by applying $\mathcal{E}(a,-)$ to the square (3.1) is equivalent to the square

$$\begin{array}{ccc} \mathcal{E}(a,x) & \xrightarrow{f\circ-} & \mathcal{E}(a,y) \\ {\scriptstyle l_x\circ-}\downarrow & & \downarrow{\scriptstyle l_y\circ-} \\ L\mathcal{E}(La,Lx) & \xrightarrow{L(f)\circ-} & L\mathcal{E}(La,Ly) \end{array}$$

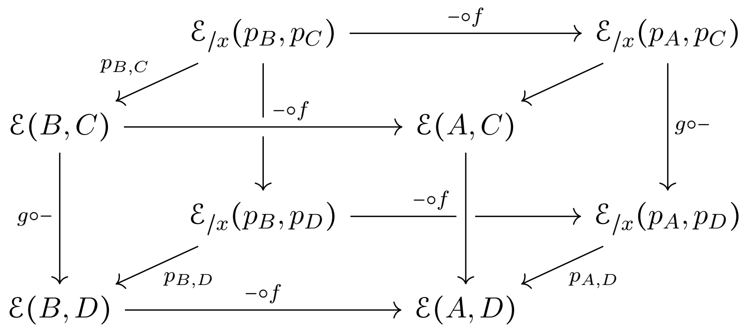
which is a pullback. It follows that the former must be a pullback as well, and since this holds for every a in $\mathcal{E}$ we conclude that the square (3.1) is indeed a pullback. □

We conclude this section with the following result, concerning the creation of factorization systems by suitable forgetful functors.

Lemma 3.9. *Let $\mathcal{E}$ be an ∞-category endowed with a factorization system $(\mathcal{L},\mathcal{R})$. Given any object $x \in \mathcal{E}$, we get a factorization system $(\mathcal{L}^x,\mathcal{R}^x)$ on $\mathcal{E}_{/x}$, defined by $\mathcal{L}^x = p^{-1}(\mathcal{L})$ and $\mathcal{R}^x = p^{-1}(\mathcal{R})$.*

In the previous situation, we say that $(\mathcal{L}^x,\mathcal{R}^x)$ is *created* by the projection $p\colon \mathcal{E}_{/x} \to \mathcal{E}$.

Proof. The stability under the formation of retracts and the factorization property are obvious consequences of the same properties for $(\mathcal{L},\mathcal{R})$. To show that $\mathcal{L}^x \perp \mathcal{R}^x$, we consider the following cube:



where $f\colon p_A \to p_B$ belongs to $\mathcal{L}^x$, $g\colon p_C \to p_D$ belongs to $\mathcal{R}^x$ and $p_I\colon I \to X$ are objects in $\mathcal{E}_{/x}$. It is easy to show that the front face, the left-hand side and the right-hand side ones are all pullbacks of ∞-groupoids, therefore the back face must be such as well. This implies $f \perp g$ and concludes the proof. □

4. The equivalence

In this section we will identify suitable subcategories of, respectively, the ∞-bicategory of cartesian fibrations (where we let the base vary) and that of ∞-bicategories endowed with a factorization system, and we will prove that they are equivalent.

Definition 4.1. Let FACT be the marked simplicial category whose objects are pairs $(\mathcal{E},(\mathcal{L},\mathcal{R}))$, where $\mathcal{E}$ is an ∞-category endowed with a factorization system $(\mathcal{L},\mathcal{R})$, and whose mapping ∞-category $\text{FACT}((\mathcal{E},(\mathcal{L},\mathcal{R})),(\mathcal{D},(S'_L,S'_R)))$ between two such objects is defined as the full subcategory of $\text{CAT}_\infty(\mathcal{E},\mathcal{D})$ spanned by those maps $f:\mathcal{E}\to\mathcal{D}$ satisfying $f(\mathcal{L})\subset S'_L$ and $f(\mathcal{R})\subset S'_R$.

We now want to isolate a specific subcategory of this $(\infty,2)$-category we have just defined. To achieve this, we have to introduce a condition on the localizations induced at the level of slice ∞-categories.

Suppose given an object $(\mathcal{E},(\mathcal{L},\mathcal{R}))$ in FACT, such that $\mathcal{E}$ admits a terminal object and $\mathcal{L}$ has the two-out-of-three property. Thanks to Proposition 3.6, we know that we get a localization functor $L:\mathcal{E}\to L\mathcal{E}$, which is the localization with respect to the class of maps $\mathcal{L}$. Moreover, $\mathcal{E}_{/x}$ also has a terminal object, and thanks to Lemma 3.9 we thus get a localization map $L_x:\mathcal{E}_{/x}\to(\mathcal{R})_{/x}$, where the ∞-category on the right is the full subcategory of $\mathcal{E}_{/x}$ spanned by those objects $f:e\to x$ with $f\in\mathcal{R}$.

Definition 4.2. An object $(\mathcal{E},(\mathcal{L},\mathcal{R}))$ in FACT is said to be *cartesian* if $\mathcal{E}$ admits a terminal object, $\mathcal{L}$ has the two-out-three property and the restriction

$$L_|:(\mathcal{R})_{/x}\to L\mathcal{E}_{/Lx}$$

of the functor $L:\mathcal{E}_{/x}\to L\mathcal{E}_{/Lx}$ is an equivalence of ∞-categories. We denote this last condition by $(*)$.

Remark 4.3. The last condition in the previous definition can be rephrased as follows. Firstly, we observe that thanks to Theorem 1.3, it is equivalent to the fact that $L_|$ is an essentially surjective and fully faithful functor. Essential surjectivity is easily shown to be equivalent to requiring that for every map $g:Le\to Lx$ in $\mathcal{R}$ we can find a map $p:e'\to x$ in $\mathcal{R}$ and an equivalence $g\simeq L(p)$ over Lx.

Fully-faithfulness translates into the fact that the map induced between the (homotopy) fibers of the vertical maps of the square depicted below at every element $f:e\to x$ in $\mathcal{R}$ (resp. $L(f)$) is an equivalence of ∞-groupoids, when g belongs to $\mathcal{R}$.

$$\begin{array}{ccc} \mathcal{E}(e,e') & \xrightarrow{L_{e,e'}} & L\mathcal{E}(Le,Le') \\ {\scriptstyle g\circ-}\downarrow & & \downarrow{\scriptstyle Lg\circ-} \\ \mathcal{E}(e,x) & \xrightarrow{L_{e,x}} & L\mathcal{E}(Le,Lx) \end{array} \tag{4.1}$$

We are now in a position to define one of the two ∞-bicategories of interest.

Definition 4.4. Let FACT_{cart} be the subcategory of FACT (as marked simplicial categories) spanned by cartesian objects and maps $f:(\mathcal{E},(\mathcal{L},\mathcal{R}))\to(\mathcal{D},(S'_L,S'_R))$ between cartesian objects which preserve the terminal object. By definition, this also implies $f(\mathcal{L})\subset S'_L$ and $f(\mathcal{R})\subset S'_R$.

It is not too hard to see that this is indeed a well-defined $(\infty,2)$-bicategory, corresponding to the subcategory of FACT on the cartesian objects, where we restrict the mapping ∞-categories to maps preserving the terminal objects.

We now define a suitable $(\infty,2)$-category of cartesian fibrations and isolate from it the subcategory of interest.

Definition 4.5. Let CART be the marked simplicial category whose objects are cartesian fibrations of ∞-categories and whose mapping ∞-category $\mathrm{CART}(p,q)$ is given by the subspace of $\mathrm{SQ}(p,q)$ (defined by the ordinary pullback depicted below) on those maps $X \to Y$ that send p-cartesian morphisms to q-cartesian morphisms.

$$\begin{array}{ccc} \mathrm{SQ}(p,q) & \longrightarrow & \mathrm{CAT}_\infty(W,Z) \\ \downarrow & & \downarrow {\scriptstyle -\circ p} \\ \mathrm{CAT}_\infty(X,Y) & \xrightarrow{q\circ -} & \mathrm{CAT}_\infty(X,Z) \end{array}$$

Since cartesian fibrations are, in particular, fibrations in the Joyal model structure, we have that CART is enriched over ∞-categories.

The cartesian fibration we will be interested in are introduced in the following definition.

Definition 4.6. A cartesian fibration $p: X \to W$ is said to be *pointed* if X and W have a terminal object and p preserves it.

The next result simplifies the description of the mapping ∞-category $\mathrm{CART}(p,q)$ in case p is a pointed cartesian fibration.

Lemma 4.7. *Given cartesian fibrations $p: X \to W$ and $q: Y \to Z$ between ∞-categories, with p pointed, we have an equivalence of ∞-categories between $\mathrm{CART}(p,q)$ and the full subcategory of $\mathrm{CAT}_\infty(X,Y)$ spanned by those maps $f: X \to Y$ that send p-cartesian morphisms to q-cartesian morphisms and such that if h is a morphism in X inverted by p then $f(h)$ is inverted by q.*

Proof. By Corollary 2.11, we get that p must be a localization functor. In particular, it is the localization at all the maps inverted by p, which means that we have an equivalence of ∞-categories of the form $\mathrm{CAT}_\infty(W,Z) \simeq \mathrm{CAT}'_\infty(X,Z)$, where $\mathrm{CAT}'_\infty(X,Z)$ denotes the full subcategory of $\mathrm{CAT}_\infty(X,Z)$ spanned by those functors $X \to Z$ that invert all the morphisms which are inverted by p. This implies that $\mathrm{SQ}(p,q)$ is equivalent to the full subcategory of $\mathrm{CAT}_\infty(X,Y)$ spanned by those functors $f: X \to Y$ such that $q \circ f$ inverts all the morphisms inverted by p. Under this identification, $\mathrm{CART}(p,q)$ corresponds to the one in the statement. □

We can now give the following definition, which isolates the subcategory of our interest.

Definition 4.8. Let CART_* be the (non-full) subcategory of the marked simplicial category CART spanned by pointed cartesian fibrations, with mapping ∞-category $\mathrm{CART}_*(p,q)$ defined as the full subcategory of $\mathrm{CART}(p,q)$ spanned (under the identification given by Lemma 4.7) by those maps $f: X \to Y$ which preserve the terminal object.

We can now prove the main result of this work.

Theorem 4.9. *There is an equivalence of $(\infty,2)$-categories between CART_* and FACT_{cart}, which sends an object $p: \mathcal{E} \to \mathcal{B}$ in CART_* to $(\mathcal{E}, (p_\mathcal{L}, p_\mathcal{R}))$, where $p_\mathcal{L}$ is the class of maps inverted by p and $p_\mathcal{R}$ is the class of p-cartesian morphisms.*

Proof. First, let us check that $(\mathcal{E}, (p_\mathcal{L}, p_\mathcal{R}))$ is indeed an object of FACT_{cart}. By hypothesis, $\mathcal{E}$ has a terminal object, and by Proposition 3.5 $(p_\mathcal{L}, p_\mathcal{R})$ is a factorization system on $\mathcal{E}$. The two-out-of-three property for $p_\mathcal{L}$ is trivially satisfied, so we are left with checking condition $(*)$ of Definition 4.2. The first thing we have to check is that given a morphism $g: pe \to px$, there is a

p-cartesian morphism $f: e' \to x$ and an equivalence $g \simeq p(f)$ over px. To get this, it is enough to pick f among the p-cartesian lifts of g with codomain x.

Thanks to Remark 4.3, the next thing we have to check is that the map between the (homotopy) fibers of the vertical maps in the square (4.1) over p-cartesian morphisms $e \to x$ is an equivalence. Since the morphism g appearing there is in $p_{\mathcal{R}}$, and is thus p-cartesian in the case at hand, that square is always a (homotopy) pullback, and therefore such comparison map between the fibers is necessarily an equivalence. This concludes the proof that $(\mathcal{E}, (p_{\mathcal{L}}, p_{\mathcal{R}}))$ is indeed an object in FACT_{cart}.

The assignments on objects we have just described is essentially surjective, since given $(\mathcal{E}, (\mathcal{L}, \mathcal{R}))$ in FACT_{cart}, we have that the associated localization functor $L: \mathcal{E} \to L\mathcal{E}$ is a cartesian fibration which we denote by p. Indeed, by assumption the induced maps $\mathcal{E}_{/x} \to L\mathcal{E}_{/Lx}$ are localizations for every x in $\mathcal{E}$, so p is a cartesian fibration thanks to Proposition 2.9, and it obviously preserves the terminal object so it is pointed. The cartesian factorization system on $\mathcal{E}$ induced by p, which we denote by $(p_{\mathcal{L}}, p_{\mathcal{R}})$, is equivalent to $(\mathcal{L}, \mathcal{R})$ since, by Proposition 3.6, we have that $\mathcal{L}$ coincides with the class of maps inverted by $L = p$, so $\mathcal{L} = p_{\mathcal{L}}$ by definition.

We can extend the assignment on objects to a map of marked simplicial categories thanks to the identification provided by Lemma 4.7, which provides a natural inclusion of the form $\text{CART}_*(p,q) \hookrightarrow \text{CAT}_\infty(\mathcal{E}, \mathcal{C})$ for every pair of cartesian fibrations $p: \mathcal{E} \to \mathcal{B}$ and $q: \mathcal{C} \to \mathcal{D}$. The image of this inclusion can be identified with the full subcategory spanned by those maps $f: \mathcal{E} \to \mathcal{C}$ which:

- preserve the terminal object, by assumption.
- are such that $f(p_{\mathcal{L}}) \subset q_{\mathcal{L}}$, since by Lemma 4.7 we have that $q \circ f$ inverts all the maps in $p_{\mathcal{L}}$.
- are such that $f(p_{\mathcal{R}}) \subset q_{\mathcal{R}}$, since morphisms of cartesian fibrations are required to preserve cartesian morphisms by definition.

Therefore, we have the desired equivalence $\text{CART}_* \simeq \text{FACT}_{Cart}$ of marked simplicial categories. □

Observation 4.10. By considering the fully faithful inclusion $\mathrm{N}: \mathbf{Cat} \hookrightarrow \mathbf{Cat}_\infty$ we get the analogous statement of the previous theorem for ordinary categories.

Finally, we record here the following corollary of the main theorem.

Corollary 4.11. *The reflection functor $L: \mathcal{E} \to L\mathcal{E}$ associated with a cartesian factorization system $(\mathcal{E}, (\mathcal{L}, \mathcal{R}))$ is simple.*

Proof. Thanks to Theorem 4.9, we know that $(\mathcal{L}, \mathcal{R})$ must be induced by a cartesian fibration $p: \mathcal{E} \to \mathcal{B}$. By Corollary 3.8, we get the desired result. □

We conclude this work with an example of a well-known factorization system that can be recovered from a cartesian fibration using the machinery developed here.

Example 4.12. Let $\mathcal{C}$ be an ∞-category with a terminal object, and denote by $\mathsf{iso}(\mathcal{C})$ the class of equivalences in $\mathcal{C}$. The inclusion $\{0\} \to \Delta^1$ induces the *domain projection* map $e_0: \mathcal{C}^{\Delta^1} \to \mathcal{C}$. Similarly, the inclusion $\{1\} \to \Delta^1$ induces the *codomain projection* map $e_1: \mathcal{C}^{\Delta^1} \to \mathcal{C}$. It is easy to see that e_0 is a cartesian fibration, whose cartesian morphisms have the following form:

$$\begin{array}{ccc} a & \longrightarrow & b \\ \downarrow & & \downarrow \\ c & \xrightarrow{\simeq} & d \end{array}$$

More precisely, the class of morphisms e_0-cart consists of those squares $\alpha\colon \Delta^1 \times \Delta^1 \to \mathcal{C}$ such that $\alpha(\Delta^1 \times \{1\}) \in \mathsf{iso}(\mathcal{C})$, i.e. $e_0\text{-cart} = {e_1}^{-1}(\mathsf{iso}(\mathcal{C}))$.

If $\mathcal{C}$ has finite limits, then e_1 is also a cartesian fibration, and e_1-cartesian morphisms are given by cartesian squares (see Example 4.1.19 in [6]). Since both e_0 and e_1 both preserve the terminal object (because limits are computed pointwise), we get two cartesian factorization systems on $\mathcal{C}^{\Delta^1}$. The one associated with e_0 consists of $({e_0}^{-1}(\mathsf{iso}(\mathcal{C})), {e_1}^{-1}(\mathsf{iso}(\mathcal{C})))$. Therefore, we get the factorization of a square in $\mathcal{C}$ of the form:

$$\begin{array}{ccc} a & \longrightarrow & b \\ \downarrow & & \downarrow \\ c & \longrightarrow & d \end{array}$$

into the following composite:

$$\begin{array}{ccccc} a & \xrightarrow{\simeq} & a & \longrightarrow & b \\ \downarrow & & \downarrow & & \downarrow \\ c & \longrightarrow & d & \xrightarrow{\simeq} & d \end{array}$$

The one associated with e_1 consists of $({e_1}^{-1}(\mathsf{iso}(\mathcal{C})), \text{cartesian squares})$, thus we get the factorization of a square in $\mathcal{C}$ of the form:

$$\begin{array}{ccc} a & \longrightarrow & b \\ \downarrow & & \downarrow \\ c & \longrightarrow & d \end{array}$$

into the following composite:

$$\begin{array}{ccccc} a & \xrightarrow{k} & p & \longrightarrow & b \\ \downarrow & & \downarrow & & \downarrow \\ c & \xrightarrow{\simeq} & c & \longrightarrow & d \end{array}$$

where the right-hand side square is cartesian and k is the essentially unique map induced by the original square.

The localization corresponding to e_0 is the one associated with the inclusion $\mathcal{C} \hookrightarrow \mathcal{C}^{\Delta^1}$ which sends an object $c \in \mathcal{C}$ to the unique map $!_c\colon c \to *$, seen as an object in $\mathcal{C}^{\Delta^1}$.

On the other hand, the localization corresponding to e_1 is the one associated with the inclusion $\mathcal{C} \hookrightarrow \mathcal{C}^{\Delta^1}$ which sends an object $c \in \mathcal{C}$ to the identity map $1_c\colon c \to c$, seen as an object in $\mathcal{C}^{\Delta^1}$.

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References

[1] Denis-Charles Cisinski, *Higher categories and homotopical algebra*, Cambridge Studies in Advanced Mathematics, vol. 180, Cambridge University Press, Cambridge, 2019.

[2] David Gepner, Rune Haugseng, and Thomas Nikolaus, *Lax colimits and free fibrations in ∞-categories*, Doc. Math. **22** (2017), 1225–1266.

[3] André Joyal, *Notes on quasi-categories*, 2008, Preprint.

[4] Jacob Lurie, $(\infty, 2)$*-categories and the Goodwillie Calculus I*, Preprint. Available at author's website.

[5] ______, *Higher Topos Theory*, Annals of Mathematics Studies, vol. 170, Princeton University Press, Princeton, NJ, 2009.

[6] E. Riehl and D. Verity, *Fibrations and yoneda lemma in an ∞-cosmos*, Journal of Pure and Applied Algebra **221** (2015), no. 3.

[7] J. Rosický and W. Tholen, *Factorization, fibration and torsion*, Journal of Homotopy and Related Structures **355** (2003), no. 9, 3611–3623.