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Homotopy Theory of Ultrametric Spaces

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Abstract

We introduce the notion of an edged space as an extension of that of a metric space, and study two model structures on edged spaces transferred through Quillen adjunctions given by Vietoris–Rips complexes. We show that a metric space is a fibrant-cofibrant object with respect to one of the model structures if and only if it is an ultrametric space. The two model categories give a new foundation of homotopy theories of ultrametric spaces and edged spaces.

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1. Introduction

We give a new foundation of ultrametric spaces in terms of model category theory. This study is motivated in global rigid geometry, which requires a natural connection between Archimedean analysis and non-Archimedean analysis. We put $\mathbb{R}_{\geq 0} := [0, \infty)$, $\mathbb{R}_{>0} := (0, \infty)$, and $\overline{\mathbb{R}}_{\geq 0} := [0, \infty]$. One natural way to connect those two branches of analysis is to perturb the triangle inequality through deformation of symmetric monoidal structures on the totally ordered set $\overline{\mathbb{R}}_{\geq 0}$ regarded as a complete cocomplete category. We first recall the relation between a symmetric monoidal structure on $\overline{\mathbb{R}}_{\geq 0}$ and the triangle inequality, which was originally studied in [6].

Let $\oplus$ denote a symmetric monoidal structure on $\overline{\mathbb{R}}_{\geq 0}$ for which 0 is the monoidal unit. An *extended* $\oplus$*-metric space* is a set M equipped with a map $d_M \colon M^2 \to \overline{\mathbb{R}}_{\geq 0}$ called an $\oplus$*-metric* satisfying the following axioms:

- (M1) For any $(m_0, m_1) \in M^2$, $d_M(m_0, m_1) = 0$ is equivalent to $m_0 = m_1$.
- (E2) For any $(m_0, m_1) \in M^2$, $d_M(m_0, m_1) = d_M(m_1, m_0)$.
- (E3) For any $(m_0, m_1, m_2) \in M^3$, $d_M(m_0, m_2) \leq d_M(m_0, m_1) \oplus d_M(m_1, m_2)$.

When the image of d_M is contained in $\mathbb{R}_{\geq 0}$, we call M an $\oplus$-metric space. For example, a $+$-metric space is precisely a metric space, and a max-metric space is precisely an ultrametric

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space. A new point of view given by Frédéric Paugam in private communication is to connect Archimedean analysis and non-Archimedean analysis by bridging $+$ and max by the action of the multiplicative group $\mathbb{R}_{>0}$. For $t \in \mathbb{R}_{>0}$, the map

$$\begin{array}{rcl} \oplus^t \colon \overline{\mathbb{R}}_{\geq 0}^2 & \to & \overline{\mathbb{R}}_{\geq 0} \\ (r_0, r_1) & \mapsto & (r_0^t + r_1^t)^{\frac{1}{t}} \end{array}$$

forms a symmetric monoidal structure on $\overline{\mathbb{R}}_{\geq 0}$ for which 0 is the monoidal unit. The multiplication $\mathbb{R}_{>0}^2 \to \mathbb{R}_{>0}$ naturally gives an action of $\mathbb{R}_{>0}$ on the flow $\{\oplus^t \mid t \in \mathbb{R}_{>0}\}$ of symmetric monoidal structures on $\overline{\mathbb{R}}_{\geq 0}$ connecting the symmetric monoidal structures $r_0 \oplus^1 r_1 = r_0 + r_1$ and $\lim_{t \to \infty} r_0 \oplus^t r_1 = \max\{r_0, r_1\}$. For an $(s,t) \in \mathbb{R}_{>0}^2$ and an $\oplus^s$-metric space M, we denote by M^t the $\oplus^{st}$-metric space which shares the underlying space with M and whose $\oplus^{st}$-metric is given by $d_{M^t}(m_0, m_1) \coloneqq d_M(m_0, m_1)^{\frac{1}{t}}$. In this way, the action of $\mathbb{R}_{>0}$ gives a flow in the set of $\oplus^t$-metrics with varying $t \in \overline{\mathbb{R}}_{\geq 0}$.

The reason why we allow ∞ as a value of an $\oplus$-metric is because it yields small limits and small colimits in the category of $\oplus$-metric spaces and short maps, as every extended $\oplus$-metric space is canonically isomorphic to a coproduct of non-empty $\oplus$-metric spaces. Also, it is natural to weaken (M1) so that the quotient in the corresponding category commutes with the forgetful functor to Set. One traditional candidate is to replace (M1) by the axiom "$d_M(m,m) = 0$ for any $m \in M$", but we adopt another weaker candidate

(E1) For any $m \in M$, $d_M(m,m) = \inf_{m_1 \in M} d_M(m, m_1) < \infty$.

because it is more suitable when we deal with ultrametric spaces in terms of model category theory, as we will explain at the end of §5. In this paper, we call the resulting extended notion an $\oplus$-*edged space*, and d_M an $\oplus$-*edge*. The benefit to allow the case $d_M(m_0, m_1) = 0$ and $m_0 \neq m_1$ is not just the categorical comfortability, but also the compatibility in the future study of global rigid geometry: For any prime number p, the action of $\mathbb{R}_{>0}$ gives a flow of p-adic valuations on $\mathbb{Z}$, and extends to an action of $\overline{\mathbb{R}}_{\geq 0}$ connecting the trivial valuation on $\mathbb{Z}$ and the pull-back of the trivial valuation on $\mathbb{F}_p$, which does not satisfy (M1).

Now we explain the most significant idea to relate the triangle inequality with a model structure: (E3) is equivalent to the right lifting property for the following diagram for any $(s,t) \in \mathbb{R}_{\geq 0}^2$:

$$\begin{array}{ccc} (s,t)\Lambda_1^2 & \longrightarrow & M \\ \downarrow & \nearrow & \downarrow \\ (s, s \oplus t, t)\Delta^2 & \longrightarrow & 0\Delta^0 \end{array}$$

In order to explain the diagram, we prepare a convention. For each $t \in \mathbb{R}_{\geq 0}$, we denote by $t\Delta^0$ (resp. $t\Delta^1$) the set $\{0\}$ (resp. $\{0,1\}$) equipped with the constant $\oplus$-edge whose value is t. For an $(s,t,u) \in \mathbb{R}_{\geq 0} \times \overline{\mathbb{R}}_{\geq 0} \times \mathbb{R}_{\geq 0}$, we denote by $(s,t,u)\Delta^2$ the set $\{0,1,2\}$ equipped with the $\oplus$-edge given as

$$(i,j) \mapsto \begin{cases} \min\{t,u\} & ((i,j) = (0,0)) \\ u & ((i,j) = (0,1)) \\ t & ((i,j) = (0,2)) \\ \min\{s,u\} & ((i,j) = (1,1)) \\ s & ((i,j) = (1,2)) \\ \min\{s,t\} & ((i,j) = (2,2)) \end{cases}.$$

For an $(s,t) \in \mathbb{R}^2_{\geq 0}$, we put $(s,t)\Lambda^2_1 \coloneqq (s,\infty,t)\Delta^2$. Under the viewpoint that a pair with a finite distance is an analogue of an edge, the map $\mathrm{id}_{\{0,1,2\}} \colon (s,t)\Lambda^2_1 \to (s, s \oplus t, t)\Delta^2$ can be regarded as an analogue of the anodyne morphism $\Lambda^2_1 \hookrightarrow \Delta^2$ in Kan model structure. In this way, (E3) can be interpreted as the right lifting property for the inner horn. Similarly, we can interpret (E2) as a right lifting property for outer horns under the assumption of (E1) and (E3), as we will show in Lemma 6.2.

One insightful idea in [2] is that the category of ultrametric spaces and short maps can be naturally embedded in the model category of simplicial $\mathbb{R}_{\geq 0}$-graded sets. Following the same strategy, we construct a "model of an $(\infty,1)$-category of max-edged spaces and short maps", which actually reflects the relation between the strong triangle inequality and the right lifting property. We also give a Quillen adjunction interpreting the completion functor restricted to the category of ultrametric spaces and short maps by using truncation of Vietoris–Rips complexes.

2. Preliminaries

Throughout this paper, we fix an uncountable Grothendieck universe $\mathscr{U}$ under the assumption of its existence. A set is said to be *$\mathscr{U}$-small* if it is an element of $\mathscr{U}$. We always assume that all objects and morphisms in a category are $\mathscr{U}$-small, and hence implicitly relativise to $\mathscr{U}$ predicates related to a category in a traditional way. For example, the cocompleteness of a category refers to all $\mathscr{U}$-small colimits instead of all small colimits, the κ-compactness of an object in a category for a $\mathscr{U}$-small regular cardinal κ refers to all $\mathscr{U}$-small κ-filtered colimits instead of all κ-filtered colimits, the κ-accessibility of a category for a $\mathscr{U}$-small regular cardinal refers to the $\mathscr{U}$-smallness of a generating set under $\mathscr{U}$-small κ-filtered colimits consisting of κ-compact objects instead of the smallness of a generating set under small κ-filtered colimits consisting of κ-compact objects. We denote by ω the least infinite ordinal, and by ω_1 the least uncountable ordinal.

We denote by Set the category of $\mathscr{U}$-small sets and maps, by $\Delta \subset \mathrm{Set}$ the $\mathscr{U}$-small subcategory of non-zero finite ordinals and order-preserving maps, and by Ab the category of $\mathscr{U}$-small Abelian groups and group homomorphisms. For an $n \in \omega$, we denote by $[n] \in \mathrm{ob}(\Delta)$ the finite ordinal $\{i \in \omega \mid i \leq n\}$. For a $\mathscr{U}$-small partially ordered set P, we also denote by P the $\mathscr{U}$-small category associated to P. In particular, we denote by $[n]$ the $\mathscr{U}$-small category associated to $[n] \in \mathrm{ob}(\Delta)$ for each $n \in \omega$. We put $\mathbb{R}_{\geq 0} \coloneqq [0,\infty)$, $\mathbb{R}_{>0} \coloneqq (0,\infty)$, and $\overline{\mathbb{R}}_{\geq 0} \coloneqq [0,\infty]$. Since they are $\mathscr{U}$-small totally ordered sets, they are regarded as $\mathscr{U}$-small categories. When we refer to the supremum and the infimum, we always consider those operations in $\overline{\mathbb{R}}_{\geq 0}$. In particular, we have $\sup \emptyset = 0$ and $\inf \emptyset = \infty$. For a $\mathscr{U}$-small category C and a category D, we denote by D^C the category of functors $C \to D$ and natural transformations. We put $\mathrm{sSet} \coloneqq \mathrm{Set}^{\Delta^{\mathrm{op}}}$.

3. Simplicial Graded Set

We recall the notion of a simplicial graded set originally invented by N. V. Durov in [1]. Let P be a $\mathscr{U}$-small partially ordered set. We put $\mathrm{sGSet}_P \coloneqq (\mathrm{Set}^P)^{\Delta^{\mathrm{op}}}$. A *$P$-graded set* is an object in Set^P, and a *simplicial P-graded set* is an object in sGSet_P. For an $(X,r) \in \mathrm{ob}(\mathrm{sGSet}_P) \times P$, we denote by $\mathrm{ev}_{P,r}(X) \in \mathrm{sSet}$ the simplicial set which assigns $X[n](r)$ to each $[n] \in \mathrm{ob}(\Delta)$. The correspondence $X \mapsto (\mathrm{ev}_{P,r}(X))_{r \in P}$ gives an isomorphism $\mathrm{ev}_P \colon \mathrm{sGSet}_P \to \mathrm{sSet}^P$ of categories.

We equip sSet with the proper combinatorial model structure called *Kan model structure* (cf. [7] pp. 827 – 828), sSet^P with the proper combinatorial model structure called *the projective model structure* (cf. [7] Proposition A.3.3.2), and sGSet_P with the combinatorial model structure for which a morphism in sGSet_P is a fibration (resp. cofibration, weak equivalence) if so is its image in sSet^P by ev_P.

One of the most interesting examples of a grade is given by a norm. A *P-normed set* is a set S equipped with a map $|\bullet|_S \colon S \to P$. Let S be a P-normed set. For a $p \in P$, we denote by $S_{<p}$ the subset $\{s \in S_p \mid |s|_S < p\}$. When S is $\mathscr{U}$-small, then the correspondence $p \mapsto S_{<p}$ gives a P-graded set with respect to the functoriality which converts the order into the inclusion. In this way, arguments on P-normed sets are naturally interpreted into those on P-graded sets. A map $f \colon M_0 \to M_1$ between P-normed sets M_0 and M_1 is said to be a *short map* if it satisfies $|f(m)|_{M_1} \leq |m|_{M_0}$ for any $m \in M_0$. We denote by NSet_P the category of $\mathscr{U}$-small P-normed sets and short maps. The correspondence $S \mapsto (S_{<p})_{p \in P}$ gives a functor $(\bullet_{<p})_{p \in P} \colon \mathrm{NSet}_P \to \mathrm{Set}^P$.

We say that P is *inf-complete* if every subset of P has an infimum. For example, $\mathbb{R}_{\geq 0}$ is inf-complete while $\mathbb{R}_{>0}$ is not. Suppose that P is inf-complete. For any P-graded set X, the set $\varinjlim X \coloneqq \varinjlim_{p \in P} X(p)$ equipped with the map which assigns to each $x \in \varinjlim X$ the infimum of the set of $p \in P$ such that x admits a representative in $X(p)$ forms a $\mathscr{U}$-small P-normed set. Since it is much easier to handle Set^P than NSet_P, this interpretation gives a new useful aspect of $R_{\geq 0}$-normed sets such as Banach spaces and Banach rings. Although N. V. Durov studied quite general settings, we only consider the case where P is $\mathbb{R}_{\geq 0}$ or $\mathbb{R}_{>0}$ in the rest of this paper.

4. Edged Space

Another one of the most interesting examples of a grade is given by an extension of the notion of a metric space. An *edged space* is a set M equipped with a map $d_M \colon M^2 \to \overline{\mathbb{R}}_{\geq 0}$ satisfying the following:

(E1) For any $m \in M$, $d_M(m, m) = \inf_{m_0 \in M} d_M(m_0, m) = \inf_{m_1 \in M} d_M(m, m_1) < \infty$.

For edged spaces M_0 and M_1, a map $f \colon M_0 \to M_1$ is said to be *short* if it satisfies $d_{M_1}(f(m_0), f(m_1)) \leq d_{M_0}(m_0, m_1)$ for any $(m_0, m_1) \in M_0^2$. We denote by ES the category of $\mathscr{U}$-small edged spaces and short maps. We say that M is *symmetric* if it satisfies the following:

(E2) For any $(m_0, m_1) \in M^2$, $d_M(m_0, m_1) = d_M(m_1, m_0)$.

We denote by $\mathrm{ES}_{\mathrm{sym}} \subset \mathrm{ES}$ the full subcategory of $\mathscr{U}$-small symmetric edged spaces. We note that we can use $\mathrm{ES}_{\mathrm{sym}}$ instead of ES in the rest part of this paper, because statements can be shown in a completely parallel way. We use ES only in order to explain the simple relation between (E2) and a certain right lifting property in Proposition 6.2 (2).

Proposition 4.1. *The category* ES *is complete and cocomplete.*

Proof. For any $\mathscr{U}$-small diagram in ES, the limit in ES is given as the subset of the limit of the underlying sets of values equipped with the supremum of distances of entries at each component consisting of points whose distance to themselves is finite, and the colimit in ES is given as the colimit of the underlying sets equipped with the infimum of distances of representatives at common components. □

For a symmetric monoidal structure $\oplus$ on $\overline{\mathbb{R}}_{\geq 0}$ for which 0 is the monoidal unit, an *$\oplus$-edged space* is a symmetric edged space M satisfying the following condition:

(E3) For any $(m_0, m_1, m_2) \in M^3$, $d_M(m_0, m_2) \leq d_M(m_0, m_1) \oplus d_M(m_1, m_2)$.

We denote by $\mathrm{ES}_{\oplus} \subset \mathrm{ES}_{\mathrm{sym}}$ the full-subcategory of $\mathscr{U}$-small $\oplus$-edged spaces. As we explained in §1, the notion of a $+$-edged (resp. a max-edged) space is an extension of that of a metric space (resp. an ultrametric space). Although we are interested in the connection between the $+$-monoidal structure and the max-monoidal structure through the deformation of $\oplus$, we only consider the max-monoidal structure in this paper.

Example 4.2. Every seminormed Abelian group M admits two canonical structures of an edged space: The maps

$$\begin{array}{rcl} M^2 & \to & \overline{\mathbb{R}}_{\geq 0} \\ (m_0, m_1) & \mapsto & \max \{|m_0|, |m_1|\} \end{array}$$

and

$$\begin{array}{rcl} M^2 & \to & \overline{\mathbb{R}}_{\geq 0} \\ (m_0, m_1) & \mapsto & |m_0 - m_1| \end{array}$$

The former structure gives a max-edged space which is not a metric space unless $M = \{0\}$, and the latter structure gives a symmetric edged space which is a max-edged space if and only if the seminorm of M is non-Archimedean.

Let M be an edged space. For an $[n] \in \mathrm{ob}(\Delta)$, we denote by $\| \bullet \|_{M,[n]}$ the map

$$\begin{array}{rcl} M^{[n]} & \to & \overline{\mathbb{R}}_{\geq 0} \\ (m_i)_{i \in [n]} & \mapsto & \max \left\{ d_M(m_i, m_j) \middle| (i,j) \in [n]^2, i \leq j \right\}. \end{array}$$

We denote by $\mathrm{VR}_{\mathbb{R}_{\geq 0}}(M)$ the Vietoris–Rips complex of M, i.e. the map which assigns to each $[n] \in \mathrm{ob}(\Delta)$ the $\mathbb{R}_{\geq 0}$-graded set $\mathrm{VR}_{\mathbb{R}_{\geq 0}}(M)[n]$ corresponding to the $\mathbb{R}_{\geq 0}$-normed set given as $\{x \in M^{[n]} \mid \|x\|_{M,[n]} < \infty\}$ equipped with the restriction of $\| \bullet \|_{M,[n]}$. When M is $\mathscr{U}$-small, then $\mathrm{VR}_{\mathbb{R}_{\geq 0}}(M)$ naturally forms a simplicial $\mathbb{R}_{\geq 0}$-graded set such that $VR(M)[0]$ is naturally identified with the underlying set of M and $\mathrm{VR}_{\mathbb{R}_{\geq 0}}(M)[1]$ is naturally identified with $\{(m_0, m_1) \in M^{[1]} \mid d_M(m_0, m_1) < \infty\}$ equipped with the restriction of d_M due to (E1). The correspondence $M \mapsto \mathrm{VR}_{\mathbb{R}_{\geq 0}}(M)$ gives a functor $\mathrm{VR}_{\mathbb{R}_{\geq 0}} \colon \mathrm{ES} \to \mathrm{sGSet}_{\mathbb{R}_{\geq 0}}$, which is fully faithful by the observation of $\mathrm{VR}_{\mathbb{R}_{\geq 0}}(M)[0]$ and $\mathrm{VR}_{\mathbb{R}_{\geq 0}}(M)[1]$. In this sense, the notion of a simplicial $\mathbb{R}_{\geq 0}$-graded space is an extension of that of an $\mathscr{U}$-small edged space, which is an extension of that of a $\mathscr{U}$-small metric space. We denote by $\mathrm{VR}_{\mathbb{R}_{>0}}$ the functor $\mathrm{ES} \to \mathrm{sGSet}_{\mathbb{R}_{>0}}$ given as the composite of $\mathrm{VR}_{\mathbb{R}_{\geq 0}}$ and the restriction

$$\mathrm{sGSet}_{\mathbb{R}_{\geq 0}} \overset{\cong}{\to} \mathrm{Set}^{\Delta^{\mathrm{op}} \times \mathbb{R}_{\geq 0}} \to \mathrm{Set}^{\Delta^{\mathrm{op}} \times \mathbb{R}_{> 0}} \overset{\cong}{\to} \mathrm{sGSet}_{\mathbb{R}_{>0}}.$$

Here, the adjunction $\mathrm{Set}^{\Delta^{\mathrm{op}} \times P} \overset{\cong}{\to} \mathrm{sGSet}_P$ is a categorical isomorphism, and hence we use its unique inverse instead of a quasi-inverse throughout this paper.

Remark 4.3. The Vietoris–Rips complex is deeply studied in the modern mathematics. We introduce two famous topics in terms of our formulation:

(1) Hausmann's theorem (cf. [4] Theorem 3.5) states that for any $\mathscr{U}$-small closed Riemannian manifold M regarded as a $+$-edged space with respect to to the induced extended metric, $\mathrm{ev}_{\mathbb{R}_{>0},r}(\mathrm{VR}_{\mathbb{R}_{>0}}(M))$ is homotopy equivalent to the singular complex of M for any sufficiently small $r \in \mathbb{R}_{>0}$. Therefore the Vietoris–Rips complex possess sufficient information of the homotopy theory of closed Riemannian manifolds.

(2) For any $\mathscr{U}$-small finite simplicial complex X equipped with an order-preserving norm, i.e. a map $|\bullet|_X\colon X \to \mathbb{R}_{>0}$ satisfying $|\sigma|_X \leq |\tau|_X$ for any $\tau \in X$ and any face σ of τ, regarded as an object of $(\mathrm{NSet}_{\mathbb{R}_{>0}})^{\Delta}$, *the persistent homology* of X is defined as the functor $\mathbb{R}_{>0} \to \mathrm{Ab}$ which assigns to each $r \in \mathbb{R}_{>0}$ the singular homology of $\mathrm{ev}_{\mathbb{R}_{>0},r}(\mathrm{VR}(M))$, where M denotes the simplicial $\mathbb{R}_{>0}$-graded set corresponding to X through $(\bullet_{<p})_{p \in \mathbb{R}_{>0}}$. The persistent homology possesses rich information on finite simplicial complex equipped with an order-preserving norm, and has recently been deeply studied in topological data analysis because of its ample applications.

For an $(r,[n]) \in \mathbb{R}_{\geq 0} \times \mathrm{ob}(\Delta)$, we denote by $[n]_r$ the $\mathscr{U}$-small max-edged space given as $[n]$ equipped with the constant map $[n]^2 \to \overline{\mathbb{R}}_{\geq 0}$ whose value is r. For a morphism $(f_0,f_1)\colon (r_0,[n_0]) \to (r_1,[n_1])$ in $\mathbb{R}_{\geq 0}^{\mathrm{op}} \times \mathrm{ob}(\Delta)$, we denote by $[f_1]_{f_0}$ the short map $f_1\colon [n_0]_{r_0} \to [n_1]_{r_1}$. The pair of the correspondence $(r,[n]) \mapsto [n]_r$ and $(f_0,f_1) \mapsto [f_1]_{f_0}$ gives a functor $[\bullet]_\bullet\colon \mathbb{R}_{\geq 0}^{\mathrm{op}} \times \Delta \to \mathrm{ES}$.

If M is $\mathscr{U}$-small, then the map $\mathrm{Hom}_{\mathrm{ES}}([n]_r, M) \to M^{[n]}$, $f \mapsto (f(ri))_{i \in [n]}$ is an injective map onto $\mathrm{VR}_{\mathbb{R}_{\geq 0}}(M)[n](r)$ for any $(r,[n]) \in \mathbb{R}_{\geq 0} \times \mathrm{ob}(\Delta)$ by the construction. Therefore through the adjunction, $\mathrm{VR}_{\mathbb{R}_{\geq 0}}$ corresponds to the functor $\Delta^{\mathrm{op}} \times \mathbb{R}_{\geq 0} \times \mathrm{ES} \to \mathrm{Set}$ given by the correspondence $(r,[n],M) \mapsto \mathrm{Hom}_{\mathrm{ES}}([n]_r, M)$. In particular, $\mathrm{VR}_{\mathbb{R}_{\geq 0}}$ is right adjoint to the left Kan extension $|\bullet|_{\mathrm{ES},\mathbb{R}_{\geq 0}}\colon \mathrm{sGSet}_{\mathbb{R}_{\geq 0}} \to \mathrm{ES}$ of $[\bullet]_\bullet$ through the composite of the Yoneda embedding $\mathbb{R}_{\geq 0}^{\mathrm{op}} \times \Delta \hookrightarrow \mathrm{Set}^{\Delta^{\mathrm{op}} \times \mathbb{R}_{\geq 0}} \overset{\cong}{\to} \mathrm{sGSet}_{\mathbb{R}_{\geq 0}}$. Similarly, $\mathrm{VR}_{\mathbb{R}_{>0}}$ is right adjoint to the left Kan extension $|\bullet|_{\mathrm{ES},\mathbb{R}_{>0}}\colon \mathrm{sGSet}_{\mathbb{R}_{>0}} \to \mathrm{ES}$ of $[\bullet]_\bullet$ restricted to $R_{>0}^{\mathrm{op}} \times \Delta$ through the composite of the Yoneda embedding $\mathbb{R}_{>0}^{\mathrm{op}} \times \Delta \hookrightarrow \mathrm{Set}^{\Delta^{\mathrm{op}} \times \mathbb{R}_{>0}} \overset{\cong}{\to} \mathrm{sGSet}_{\mathbb{R}_{>0}}$.

5. Model Structures on Edged Spaces

Let P denote either $\mathbb{R}_{\geq 0}$ or $\mathbb{R}_{>0}$. We transfer the model structure of sGSet_P to ES. For a $(w,X) \in P \times \mathrm{ob}(\mathrm{sSet})$, we denote by $\mu_P(w,X) \in \mathrm{ob}(\mathrm{sGSet}_P)$ the simplicial P-graded set corresponding to a functor $\Delta^{\mathrm{op}} \times P \to \mathrm{Set}$ which assigns $\mathrm{Hom}_P(r,w) \times X[n]$ to each $(r,[n]) \in P \times \mathrm{ob}(\Delta)$. By the definition of Hom_P, we have a natural identification

$$\mu_P(w,X)[n](r) \cong \begin{cases} X[n] & (r \leq w) \\ \emptyset & (w < r) \end{cases}$$

for each $(r,[n]) \in P \times \mathrm{ob}(\Delta)$. The correspondence $(w,X) \mapsto \mu_P(w,X)$ gives a functor $\mu_P\colon P \times \mathrm{sSet} \to \mathrm{sGSet}_P$. We denote by I (resp. J) the generating set of cofibrations (resp. trivial cofibrations) in sSet given as the $\mathscr{U}$-small set of sphere (resp. horn) embeddings, by $I_{\leq 1} \subset I$ (resp. $J_{\leq 2} \subset J$) the subset of morphisms whose targets are of dimension ≤ 1 (resp. ≤ 2), and by F (resp. W) the set of fibrations (weak equivalences) in sSet. We put $\mathscr{I}_P \coloneqq \{|\mu_P(\mathrm{id}_r, i)|_{\mathrm{ES},P} \mid (r,i) \in P \times I_{\leq 1}\}$ and $\mathscr{J}_P \coloneqq \{|\mu_P(\mathrm{id}_r, j)|_{\mathrm{ES},P} \mid (r,j) \in P \times J_{\leq 2}\}$. We denote by $\mathscr{F}_P$ the set of morphisms in ES satisfying the right lifting property for all morphisms in $\mathscr{J}_P$, by $\mathscr{W}_P$ the set of morphisms f in ES such that $\mathrm{VR}_P(f)$ is a weak equivalence in sGSet_P, and by $\mathscr{C}_P$ the set of morphisms in ES satisfying the left lifting property for all morphisms in $\mathscr{F}_P \cap \mathscr{W}_P$.

Theorem 5.1. *The category* ES *forms a combinatorial model category with respect to* $(\mathscr{C}_P, \mathscr{F}_P, \mathscr{W}_P)$*, and the adjoint pair* $(|\bullet|_{\mathrm{ES},P}, \mathrm{VR}_P)$ *forms a Quillen adjunction.*

In order to prove Theorem 5.1, we prepare conventions and lemmata. For a $\mathscr{U}$-small non-empty set S, we denote by $S^\bullet$ the $\mathscr{U}$-small contractible Kan complex realised as the functor $\Delta^{\mathrm{op}} \to \mathrm{Set}$ given as the composite of the inclusion $\Delta^{\mathrm{op}} \hookrightarrow \mathrm{Set}^{\mathrm{op}}$ and the functor $\mathrm{Set}^{\mathrm{op}} \to \mathrm{Set}$ represented by S.

Lemma 5.2. *For any $M \in \mathrm{ob}(\mathrm{ES})$, if the underlying set of M is countable, then M is ω_1-compact in* ES*.*

Proof. The assertion immediately follows from the description of $\mathscr{U}$-small colimits in ES in the proof of Proposition 4.1 and the non-existence of an injective anti-order preserving map $\omega_1 \hookrightarrow \overline{\mathbb{R}}_{\geq 0}$. □

Lemma 5.3. *The set of edged spaces whose underlying set is a subset of ω forms a $\mathscr{U}$-small set generating* ES *under $\mathscr{U}$-small ω_1-filtered colimits.*

Proof. The assertion immediately follows from the fact that every object is isomorphic to the $\mathscr{U}$-small ω_1-filtered colimit of countable subsets in ES. □

Lemma 5.4. *For any $(w, [n]) \in \mathbb{R}_{\geq 0} \times \mathrm{ob}(\Delta)$, the unique morphism*

$$\mathrm{VR}_P(|\mu_P(w, \Delta^n)|_{\mathrm{ES},P}) \to \left|\mu_P(w, \Delta^0)\right|_{\mathrm{ES},P}$$

in sGSet_P is a trivial fibration in sGSet_P.

Proof. For any $r \in R_{\geq 0}$, we have

$$\begin{aligned} &\mathrm{ev}_{P,r}(\mathrm{VR}_P(|\mu_P(w, \Delta^n)|_{\mathrm{ES},P})) \cong \mathrm{ev}_{P,r}(\mathrm{VR}_P([n]_w)) \cong \mathrm{Hom}_{\mathrm{ES}}([\bullet]_r, [n]_w) \\ \cong &\begin{cases} [n]^\bullet & (w \leq r) \\ \emptyset & (r < w) \end{cases}, \end{aligned}$$

and this implies the assertion because $[n]^\bullet$ is a contractible Kan complex and $\mathrm{id}_\emptyset$ is an isomorphism in sSet. □

We note that the presentation in the proof of Lemma 5.4 gives a natural identification $\mathrm{VR}_P(|\mu_P(w, \Delta^n)|_{\mathrm{ES},P}) \cong \mu_P(w, [n]^\bullet)$.

Proof of Theorem 5.1. By Proposition 4.1, Lemma 5.2, and Lemma 5.3, ES is a locally presentable category. Since $\mathscr{U}$-small colimits in $\mathrm{Set}^{P \times \Delta} \cong \mathrm{sGSet}_P$ are pointwise colimits and $[n]_r$ is ω_1-compact in ES for any $(r, [n]) \in P \times \mathrm{ob}(\Delta)$ by Lemma 5.2, VR_P commutes with all $\mathscr{U}$-small ω_1-filtered colimits. In particular, VR_P commutes with all ω_1-indexed transfinite composites. By the description of $\mathscr{U}$-small colimits in ES in the proof of Proposition 4.1, the distance of two points in a pushout (resp. a $\mathscr{U}$-small coproduct) is ∞ if they do not have representatives in a common component. It implies that VR_P commutes with all pushouts (resp. all $\mathscr{U}$-small coproducts), because of the finiteness of the distances of points in $[n]_r$ for any $(r, [n]) \in P \times \mathrm{ob}(\Delta)$.

The left adjoint functor $| \bullet |_{\mathrm{ES},P}$ sends the $\mathscr{U}$-small generating set

$$\begin{aligned} &\{\mu_P(\mathrm{id}_r, i) \mid (r, i) \in P \times I\} \\ (\text{resp.} \quad &\{\mu_P(\mathrm{id}_r, j) \mid (r, j) \in P \times J\} \quad) \end{aligned}$$

of cofibrations (resp. trivial cofibrations) in sGSet_P to the union of $\mathscr{I}_P$ (resp. $\mathscr{J}_P$) and a set of isomorphisms in ES. For any morphism f in ES, f belongs to $\mathscr{F}_P$ if and only if $\mathrm{VR}_P(f)$ is a fibration in sGSet_P, because for any $(r, j) \in P \times J$, the right lifting property of f for $|\mu_P(\mathrm{id}_r, j)|_{\mathrm{ES},P}$ is equivalent to the right lifting property of $\mathrm{ev}_{P,r}(\mathrm{VR}_P(f))$ for j by the definition of VR_P and μ_P. For any morphism f in ES, f belongs to $\mathscr{W}_P$ if and only if $\mathrm{VR}_P(f)$ is a weak equivalence in sGSet_P by the definition of $\mathscr{W}_P$.

Therefore by a well-known method to transfer a cofibrantly generated model structure through a left adjoint functor (cf. Theorem 3.6 in [3]), it suffices to show that the set $\mathscr{C}_{P,0}$ of morphisms in ES satisfying the left lifting property for $\mathscr{F}_P$ is contained in $\mathscr{W}_P$ in order to show that $(\mathscr{C}_P, \mathscr{F}_P, \mathscr{W}_P)$ is a combinatorial model structure. By the small object argument, $\mathscr{C}_{P,0}$ is the set of morphisms given by retracts of ω_1-indexed transfinite composites of pushouts of $\mathscr{U}$-small coproducts of morphisms in $\mathscr{J}_P$. Therefore it suffices to show that $\mathrm{VR}_P(j)$ is a weak equivalence in sGSet_P for any $j \in \mathscr{J}_P$, because VR_P commutes with all ω_1-indexed transfinite composites, all pushouts, and all $\mathscr{U}$-small coproduct. By the definition of $\mathscr{J}_P$, it suffices to show that $\mathrm{VR}_P(|\mu_P(w,j)|_{\mathrm{ES},P})$ is a weak equivalence for any $(w,j) \in P \times J_{\leq 2}$. We denote by $d \in \{1,2\}$ the dimension of the target of j.

Suppose $d = 1$. Since $\mathrm{VR}_P(|\mu_P(\mathrm{id}_w, j)|_{\mathrm{ES},P})$ is a section of the unique morphism in Lemma 5.4 applied to $[n] = [1]$, it is a weak equivalence in sGSet_P by Lemma 5.4 and the two-out-of-three property. We further show that $\mathrm{VR}_P(|\mu_P(\mathrm{id}_w, j)|_{\mathrm{ES},P})$ is a trivial cofibration in sGSet_P. We denote by ι the monomorphism $\Delta^0 \to [1]^\bullet$ in sSet given as the composite of j and the canonical embedding $\Delta^1 \hookrightarrow [1]^\bullet$. Through the identification $\mathrm{VR}_P(|\mu_P(w,\Delta^n)|_{\mathrm{ES},P}) \cong \mu_P(w,[n]^\bullet)$ natural on $[n] \in \mathrm{ob}(\Delta)$, we have

$$\mathrm{ev}_{P,r}(\mathrm{VR}_P(|\mu_P(\mathrm{id}_w, j)|_{\mathrm{ES},P})) \colon \begin{cases} [0]^\bullet \cong \Delta^0 \overset{\iota}{\to} [1]^\bullet & (w \leq r) \\ \emptyset \overset{\mathrm{id}_\emptyset}{\to} \emptyset & (r < w) \end{cases}$$

for any $r \in P$, and hence $\mathrm{VR}_P(|\mu_P(\mathrm{id}_w, j)|_{\mathrm{ES},P})$ is identified with the cofibration $\mu_P(\mathrm{id}_w, \iota)$ in sGSet_P. Therefore $\mathrm{VR}_P(|\mu_P(\mathrm{id}_w, j)|_{\mathrm{ES},P})$ is a trivial cofibration in sGSet_P.

Suppose $d = 2$. We denote by $\Lambda \in \mathrm{ob}(\mathrm{sSet})$ the source of j. We describe each i-dimensional face of Δ^2 with $i \in [1]$ as the image of the map $[i] \to [2]$ identified with the morphism $\Delta^i \to \Delta^2$ in sSet corresponding to the face. When Λ is the union of $\{0,1\}$ and $\{1,2\}$ (resp. $\{0,1\}$ and $\{0,2\}$, $\{0,2\}$ and $\{1,2\}$), we denote by $\iota_{0,1}$ the morphism $\Delta^0 \to \Delta^1$ in sSet corresponding to the inclusion $\{1\} \hookrightarrow \{1,2\}$ (resp. $\{0\} \hookrightarrow \{0,2\}$, $\{1\} \hookrightarrow \{1,2\}$), and by $\iota_{1,2}$ the morphism $\Delta^1 \to \Lambda$ in sSet corresponding to the inclusion from $\{0,1\}$ (resp. $\{0,1\}$, $\{0,2\}$) to Λ. Then $\iota_{1,2}$ is a pushout of $\iota_{0,1}$ in sSet. Since VR_P and the left adjoint functor $|\bullet|_{\mathrm{ES},P}$ commute with all pushouts, $\mathrm{VR}_P(|\mu_P(\mathrm{id}_w, \iota_{1,2})|_{\mathrm{ES},P})$ is a pushout of $\mathrm{VR}_P(|\mu_P(\mathrm{id}_w, \iota_{0,1})|_{\mathrm{ES},P})$ in sGSet_P. Since $\mathrm{VR}_P(|\mu_P(\mathrm{id}_w, \iota_{0,1})|_{\mathrm{ES},P})$ is a trivial cofibration in sGSet_P by the argument for the case $d = 1$, so is its pushout $\mathrm{VR}_P(|\mu_P(\mathrm{id}_w, \iota_{1,2})|_{\mathrm{ES},P})$. In particular, the composite $\mathrm{VR}_P(|\mu_P(\mathrm{id}_w, \iota_{1,2})|_{\mathrm{ES},P}) \circ \mathrm{VR}_P(|\mu_P(\mathrm{id}_w, \iota_{0,1})|_{\mathrm{ES},P})$ is a trivial cofibration $\mathrm{VR}_P(|\mu_P(w, \Delta^0)|_{\mathrm{ES},P}) \to \mathrm{VR}_P(|\mu_P(\mathrm{id}_w, \Lambda)|_{\mathrm{ES},P})$ in sGSet_P. By the two-out-of-three property, the unique morphism

$$\mathrm{VR}_P(|\mu_P(w,\Lambda)|_{\mathrm{ES},P}) \to \mathrm{VR}_P\left(\left|\mu_P(w,\Delta^0)\right|_{\mathrm{ES},P}\right) \cong \mu_P(w,\Delta^0)$$

in sGSet_P is a weak equivalence in sGSet_P. Therefore again by the two-out-of-three property and Lemma 5.4 applied to $[n] = [2]$, $\mathrm{VR}_P(|\mu_P(\mathrm{id}_w, j)|_{\mathrm{ES},P})$ is a weak equivalence in sGSet_P. □

We denote by ES_P the category ES equipped with the combinatorial model structure $(\mathscr{C}_P, \mathscr{F}_P, \mathscr{W}_P)$. We study other basic properties of ES_P.

Theorem 5.5. *The combinatorial model category $\mathrm{ES}_{\mathbb{R}_{\geq 0}}$ is a proper symmetric monoidal model category with respect to the Cartesian monoidal structure.*

In order to show Theorem 5.5, we prepare several lemmata. We denote by $A_{\max}$ the edged space given as $\overline{\mathbb{R}}_{\geq 0}$ equipped with $\max \colon \overline{\mathbb{R}}_{\geq 0}^2 \to \overline{\mathbb{R}}_{\geq 0}$. We denote by $P_{\max}$ the edged space given as P equipped with the restriction of $d_{A_{\max}}$. Then $P_{\max}$ and $A_{\max}$ form $\mathscr{U}$-small max-edged spaces.

Lemma 5.6. *The unique morphisms $P_{\max} \to [0]_0$ and $A_{\max} \to [0]_0$ in* ES *are trivial fibrations in* ES_P.

Proof. For any $w \in P$, $\mathrm{ev}_{P,w}(\mathrm{VR}_P(P_{\max}))$ (resp. $\mathrm{ev}_{P,w}(\mathrm{VR}_P(A_{\max}))$) is naturally identified with the contractible Kan complex $(P \cap [0,w])^{\bullet}$ (resp. $[0,w]^{\bullet}$), and hence the unique morphism $P_{\max} \to [0]_0$ (resp. $A_{\max} \to [0]_0$) in ES is a trivial fibration in ES_P. □

Lemma 5.7. *The morphism*

$$\begin{array}{rcl} p_{\max} \colon A_{\max} & \to & [1]_0 \\ r & \mapsto & \begin{cases} 0 & (r = 0) \\ 1 & (r \neq 0) \end{cases} \end{array}$$

in ES *is a trivial fibration in* $\mathrm{ES}_{\mathbb{R}_{>0}}$.

Proof. For any $w \in \mathbb{R}_{>0}$, $\mathrm{ev}_{P,w}(\mathrm{VR}_{\mathbb{R}_{>0}}(p_{\max}))$ is naturally identified with the morphism $p_w \colon [0,w]^{\bullet} \to [1]^{\bullet}$ in sSet induced by the map

$$\begin{array}{rcl} [0,w] & \to & [1] \\ r & \mapsto & \begin{cases} 0 & (r = 0) \\ 1 & (r \neq 0) \end{cases} \end{array},$$

which is surjective by $w > 0$, and is a trivial fibration in sSet, because it obviously satisfies the right lifting property for the generating set I of cofibrations in sSet. □

Lemma 5.8. *A morphism $f \colon M \to N$ in* ES *is a cofibration in* ES_P *if and only if it is injective, satisfies $d_N(f(m), f(m)) = d_M(m,m)$ for any $m \in M$, and satisfies $d_N(n,n) \in P$ for any $n \in N \setminus f(M)$.*

Proof. First, suppose that f is a cofibration. Let $(m_0, m_1) \in M^2$ with $m_0 \neq m_1$. We denote by s_{M,m_0,m_1} the morphism

$$\begin{array}{rcl} M & \to & \left|\mu_P(0, \Delta^1)\right|_{\mathrm{ES},P} \cong [1]_0 \\ m & \mapsto & \begin{cases} 0 & (m = m_0) \\ 1 & (m \neq m_0) \end{cases} \end{array}$$

in ES. Since f is a cofibration in ES_P, there exists an $\tilde{s}_{M,m_0,m_1} \colon N \to |\mu_P(0,\Delta^1)|_{\mathrm{ES},P}$ such that $\tilde{s}_{M,m_0,m_1} \circ f = s_{M,m_0,m_1}$ by Lemma 5.4 applied to $(w,[n]) = (0,[1])$. We have $\tilde{s}_{M,m_0,m_1}(f(m_0)) = s_{M,m_0,m_1}(m_0) = 0 \neq 1 = s_{M,m_0,m_1}(m_1) = \tilde{s}_{M,m_0,m_1}(f(m_1))$ by $m_0 \neq m_1$, and hence $f(m_0) \neq f(m_1)$.

We denote by v_M the morphism

$$\begin{array}{rcl} M & \to & A_{\max} \\ m & \mapsto & d_M(m,m) \end{array}$$

in ES. Since f is a cofibration in ES_P, there exists a $\tilde{v}_M \colon N \to A_{\max}$ such that $\tilde{v}_M \circ f = v_M$ by Lemma 5.6. For any $m \in M$, we have

$$d_N(f(m), f(m)) \geq d_{A_{\max}}(\tilde{v}_M(f(m)), \tilde{v}_M(f(m))) = d_{A_{\max}}(v_M(m), v_M(m)) = d_M(m,m)$$

by the shortness of $\tilde{v}_M$, and hence $d_N(f(m), f(m)) = d_M(m, m)$ by the shortness of f.

If $P = \mathbb{R}_{\geq 0}$, then we have $d_N(n, n) \in \mathbb{R}_{\geq 0} = P$ for any $n \in N \setminus f(M)$. Suppose $P = \mathbb{R}_{>0}$. We denote by w_f the morphism

$$\begin{array}{rcl} N & \to & [1]_0 \\ n & \mapsto & \begin{cases} p_{\max}(d_N(n,n)) & (n \in f(M)) \\ 1 & (n \notin f(M)) \end{cases} \end{array}$$

in ES. Then we have $w_f \circ f = p_{\max} \circ v_M$. Since f is a cofibration in $\mathrm{ES}_{\mathbb{R}_{>0}}$, there exists a $\tilde{w}_f \colon N \to A_{\max}$ such that $\tilde{w}_f \circ f = v_f$ and $p_{\max} \circ \tilde{w}_f = w_f$ by Lemma 5.7. For any $n \in N \setminus f(M)$, we have $d_N(n, n) \geq d_{A_{\max}}(\tilde{w}_f(n), \tilde{w}_f(n)) = \tilde{w}_f(n) > 0$ by the shortness of $\tilde{w}_f$ and $\tilde{w}_f(n) \in p_{\max}^{-1}(w_f(n)) = p_{\max}^{-1}(1) = \overline{\mathbb{R}}_{\geq 0} \setminus \{0\}$.

Next, suppose that f is injective, satisfies $d_N(f(m), f(m)) = d_M(m, m)$ for any $m \in M$, and satisfies $d_N(n, n) \in P$ for any $n \in N \setminus f(M)$. We denote by $\tilde{N}$ the $\mathscr{U}$-small edged space given as $(M \times \{0\}) \cup ((N \setminus f(M)) \times \{1\})$ equipped with the map

$$\begin{array}{rcl} ((M \times \{0\}) \cup ((N \setminus f(M)) \times \{1\}))^2 & \to & \overline{\mathbb{R}}_{\geq 0} \\ ((n_0, \lambda_0), (n_1, \lambda_1)) & \mapsto & \begin{cases} d_M(n_0, n_1) & (\lambda_0 = \lambda_1 = 0) \\ d_N(n_0, n_0) & (\lambda_0 = \lambda_1 = 1, n_0 = n_1) \\ \infty & (\text{otherwise}) \end{cases} \end{array}.$$

We denote by f_0 the canonical embedding $M \cong M \times \{0\} \hookrightarrow \tilde{N}$, and by f_1 the map

$$\begin{array}{rcl} (M \times \{0\}) \cup ((N \setminus f(M)) \times \{1\}) & \to & N \\ (n, \lambda) & \mapsto & \begin{cases} f(n) & (\lambda = 0) \\ n & (\lambda = 1) \end{cases} \end{array}.$$

Then f_0 and f_1 are morphisms in ES with $f_1 \circ f_0 = f$. Therefore it suffices to show that f_0 and f_1 are cofibrations in ES_P.

For each $d \in \mathbb{N}$, we denote by $i_d \in I$ the sphere embedding whose target is of dimension d. For any $n \in N \setminus f(M)$, $\mu_P(\mathrm{id}_{d_N(n,n)}, i_0)$ makes sense by the assumption $d_N(n, n) \in P$. The diagram

$$\begin{array}{ccc} \emptyset & \longrightarrow & M \\ \| & & \\ \coprod_{n \in N \setminus f(M)} |\mu_P(d_N(n,n), \partial\Delta^0)|_{\mathrm{ES},P} & & \\ \downarrow {\scriptstyle \coprod_{n \in N \setminus f(M)} |\mu_P(\mathrm{id}_{d_N(n,n)}, i_0)|_{\mathrm{ES},P}} & & \downarrow {\scriptstyle f_0} \\ \coprod_{n \in N \setminus f(M)} |\mu_P(d_N(n,n), \Delta^0)|_{\mathrm{ES},P} & & \\ \downarrow \cong & & \\ \coprod_{n \in N \setminus f(M)} [0]_{d_N(n,n)} & \xrightarrow{(n,1)_{n \in N \setminus f(M)}} & \tilde{N} \end{array}$$

is a pushout diagram in ES, because the pushout by $|\mu_P(\mathrm{id}_r, i_0)|_{\mathrm{ES},P}$ with $r \in P$ corresponds to the operation to add a new point n such that the distance between n and itself is r and the distance between n and another point is ∞. Therefore f_0 is a cofibration in ES_P. The diagram

$$\begin{array}{ccc}
\coprod_{(n_0,n_1)\in\tilde{N}^2}([0]_{d_N(f_1(n_0),f_1(n_1))} \sqcup [0]_{d_N(f_1(n_0),f_1(n_1))}) & \xrightarrow{(n_0,n_1)_{(n_0,n_1)\in\tilde{N}^2}} & \tilde{N} \\
\downarrow \cong & & \\
\coprod_{(n_0,n_1)\in\tilde{N}^2} |\mu_{\mathbb{R}_{\geq 0}}(d_N(f_1(n_0),f_1(n_1)),\partial\Delta^1)|_{\mathrm{ES},\mathbb{R}_{\geq 0}} & & \\
\downarrow \coprod_{(n_0,n_1)\in\tilde{N}^2} |\mu_{\mathbb{R}_{\geq 0}}(\mathrm{id}_{d_N(f_1(n_0),f_1(n_1))},i_1)|_{\mathrm{ES},\mathbb{R}_{\geq 0}} & & \downarrow f_1 \\
\coprod_{(n_0,n_1)\in\tilde{N}^2} |\mu_{\mathbb{R}_{\geq 0}}(d_N(f_1(n_0),f_1(n_1)),\Delta^1)|_{\mathrm{ES},\mathbb{R}_{\geq 0}} & & \\
\downarrow \cong & & \\
\coprod_{(n_0,n_1)\in\tilde{N}^2}[1]_{d_N(f_1(n_0),f_1(n_1))} & \xrightarrow{(f_1(n_0),f_1(n_1))_{(n_0,n_1)\in\tilde{N}^2}} & N
\end{array}$$

is a pushout diagram in ES, because the pushout by $|\mu_{\mathbb{R}_{\geq 0}}(\mathrm{id}_r, i_1)|_{\mathrm{ES},\mathbb{R}_{\geq 0}}$ with $r \in \mathbb{R}_{\geq 0}$ corresponds to the identity operation except for reducing the distance of the images n_0 and n_1 of the two distinct points of $|\mu_{\mathbb{R}_{\geq 0}}(\mathrm{id}_r, i_1)|_{\mathrm{ES},\mathbb{R}_{\geq 0}}$ to r. We note that for any $(n_0, n_1) \in \tilde{N}^2$, the distance between their images in the pushout is actually reduced to $d_N(f_1(n_0), f_1(n_1))$, because the inequality

$$\max_{i=0}^{1} d_{\tilde{N}}(n_i, n_i) = \max_{i=0}^{1} d_N(f_1(n_i), f_1(n_i)) \leq d_N(f_1(n_0), f_1(n_1))$$

implies that there is a morphism

$$\left|\mu_{\mathbb{R}_{\geq 0}}(d_N(f_1(n_0), f_1(n_1)), \partial\Delta^1)\right|_{\mathrm{ES},\mathbb{R}_{\geq 0}} \to \tilde{N}$$

such that the images of the two distinct points are n_0 and n_1. Therefore it suffices to show that $|\mu_{\mathbb{R}_{\geq 0}}(\mathrm{id}_w, i_1)|_{\mathrm{ES},\mathbb{R}_{\geq 0}}$ is a cofibration in ES_P for any $w \in \mathbb{R}_{\geq 0}$. If $w \in P$, then we have $|\mu_{\mathbb{R}_{\geq 0}}(\mathrm{id}_w, i_1)|_{\mathrm{ES},\mathbb{R}_{\geq 0}} = |\mu_P(\mathrm{id}_w, i_1)|_{\mathrm{ES},P} \in \mathscr{I}_P$. Therefore we may assume $(P, w) = (\mathbb{R}_{>0}, 0)$. For each $n \in \mathbb{N}$, we denote by M_n the $\mathscr{U}$-small edged space given as $[1]$ equipped with the map

$$\begin{array}{rcl}
[1]^2 & \to & \overline{\mathbb{R}}_{\geq 0} \\
(i_0, i_1) & \mapsto & \begin{cases} 0 & (i_0 = i_1) \\ n^{-1} & (i_0 \neq i_1) \end{cases}
\end{array}.$$

For any $n \in \mathbb{N}$, $\mathrm{id}_{[1]}$ gives a morphism $h_n \colon M_n \to M_{n+1}$ in ES, and $|\mu_{\mathbb{R}_{\geq 0}}(\mathrm{id}_0, i_1)|_{\mathrm{ES},\mathbb{R}_{\geq 0}}$ is naturally identified with the transfinite composite of $(h_n)_{n\in\mathbb{N}}$. Therefore it suffices to show that h_n is a cofibration in $\mathrm{ES}_{\mathbb{R}_{>0}}$ for any $n \in \mathbb{N}$. Since the diagram

$$\begin{array}{ccc}
[0]_{(n+1)^{-1}} \sqcup [0]_{(n+1)^{-1}} & \xrightarrow{(0,1)} & M_n \\
\downarrow \cong & & \\
|\mu_{\mathbb{R}_{>0}}((n+1)^{-1}, \partial\Delta^1)|_{\mathrm{ES},\mathbb{R}_{>0}} & & \\
\downarrow |\mu_{\mathbb{R}_{>0}}(\mathrm{id}_{(n+1)^{-1}}, i_1)|_{\mathrm{ES},\mathbb{R}_{>0}} & & \downarrow h_n \\
|\mu_{\mathbb{R}_{>0}}((n+1)^{-1}, \Delta^1)|_{\mathrm{ES},\mathbb{R}_{>0}} & & \\
\downarrow \cong & & \\
[1]_{(n+1)^{-1}} & \xrightarrow{\mathrm{id}_{[1]}} & M_{n+1}
\end{array}$$

is a pushout diagram in ES, h_n is a cofibration in $\mathrm{ES}_{\mathbb{R}_{>0}}$. □

Lemma 5.9. *(1) Every object in* ES *is a cofibrant object in* $\mathrm{ES}_{\mathbb{R}_{\geq 0}}$*.*
(2) An $M \in \mathrm{ob}(\mathrm{ES})$ *is a cofibrant object in* $\mathrm{ES}_{\mathbb{R}_{>0}}$ *if and only if* $d_M(m,m) > 0$ *for any* $m \in M$*.*

Proof. The assertion immediately follow from Lemma 5.8. □

Proof of Theorem 5.5. The left properness follows from Lemma 5.9 (1) and [7] Proposition A.2.4.2. The right properness follows from Theorem 5.1, because $\mathrm{sSet}^{\mathbb{R}_{\geq 0}}$ is right proper by [7] Remark A.2.8.4 and the right adjoint functor $\mathrm{VR}_{\mathbb{R}_{\geq 0}}$ preserves pull-backs. For an $(M,N) \in \mathrm{ob}(\mathrm{ES})^2$, we denote by $M \otimes N$ the $\mathscr{U}$-small edged space given as the set-theoretic direct product $M \times N$ equipped with the map

$$\begin{array}{rcl} (M \times N)^2 & \to & \overline{\mathbb{R}}_{\geq 0} \\ ((m_0,n_0),(m_1,n_1)) & \mapsto & \max\{d_M(m_0,m_1), d_N(n_0,n_1)\}, \end{array}$$

by $\overline{d}_{M,N}$ the map

$$\begin{array}{rcl} (N^M)^2 & \to & \overline{\mathbb{R}}_{\geq 0} \\ (f_0,f_1) & \mapsto & \inf\left\{C \in \overline{\mathbb{R}}_{\geq 0} \middle| \forall (m_0,m_1) \in M^2, d_N(f_0(m_0), f_1(m_1)) \leq \max\{C, d_M(m_0,m_1)\}\right\}, \end{array}$$

and by $\underline{\mathrm{Hom}}(M,N)$ the $\mathscr{U}$-small edged space given as N^M equipped with the map

$$\begin{array}{rcl} (N^M)^2 & \to & \overline{\mathbb{R}}_{\geq 0} \\ (f_0,f_1) & \mapsto & \max\left\{\overline{d}_{M,N}(f_0,f_0), \overline{d}_{M,N}(f_0,f_1), \overline{d}_{M,N}(f_1,f_1)\right\}. \end{array}$$

The correspondence $(M,N) \mapsto M \otimes N$ gives a direct product functor $\otimes \colon \mathrm{ES}^2 \to \mathrm{ES}$, and hence defines the Cartesian monoidal structure. The correspondence $(M,N) \mapsto \underline{\mathrm{Hom}}(M,N)$ gives an internal hom functor $\underline{\mathrm{Hom}} \colon \mathrm{ES}^{\mathrm{op}} \times \mathrm{ES} \to \mathrm{ES}$ right adjoint to $\otimes$, and hence $(\mathrm{ES}, \otimes)$ forms a closed symmetric monoidal category.

By Lemma 5.9 (1), it suffices to show the pushout product axiom (cf. [5] Definition 4.2.6/1). By the characterisation of cofibrations in $\mathrm{ES}_{\mathbb{R}_{\geq 0}}$ by Lemma 5.8, the pushout product preserves cofibrations in $\mathrm{ES}_{\mathbb{R}_{\geq 0}}$. By [5] Corollary 4.2.5, it suffices to show that the image of $\mathscr{I}_{\mathbb{R}_{\geq 0}} \times \mathscr{J}_{\mathbb{R}_{\geq 0}}$ by the pushout product consists of weak equivalences in $\mathrm{ES}_{\mathbb{R}_{\geq 0}}$. Let $(w_0, i) \in \mathbb{R}_{\geq 0} \times I_{\leq 1}$ and $(w_1, j) \in \mathbb{R}_{\geq 0} \times J_{\leq 2}$. We denote by k the pushout product of $|\mu_{\mathbb{R}_{\geq 0}}(\mathrm{id}_{w_0}, i)|_{\mathrm{ES},\mathbb{R}_{\geq 0}}$ and $|\mu_{\mathbb{R}_{\geq 0}}(\mathrm{id}_{w_1}, j)|_{\mathrm{ES},\mathbb{R}_{\geq 0}}$. We denote by $n_0 \in \{0,1\}$ the dimension of the target of i, $n_1 \in \{1,2\}$ the dimension of the target of j, and by $\Lambda \subset \Delta^{n_1}$ the source of j. Since $|\mu_{\mathbb{R}_{\geq 0}}(\mathrm{id}_{w_1}, j)|_{\mathrm{ES},\mathbb{R}_{\geq 0}}$ is stable up to isomorphism under permutations of $[n_0]$, we may assume that the edge of Δ^{n_1} which does not belong to Λ is presented as $\{0, n_1\}$. We denote by $\partial[0]_{w_0}$ the $\mathscr{U}$-small edged space $\emptyset$, by $\partial[1]_{w_0}$ the $\mathscr{U}$-small edged space given as [1] equipped with the map

$$\begin{array}{rcl} [1]^2 & \to & \overline{\mathbb{R}}_{\geq 0} \\ (a_0,a_1) & \mapsto & \begin{cases} w_0 & (a_0 = a_1) \\ \infty & (a_0 \neq a_1) \end{cases}, \end{array}$$

by $[1]^1_{w_1}$ the $\mathscr{U}$-small edged space given as [0] equipped with the constant map $[0]^2 \to \overline{\mathbb{R}}_{\geq 0}$ with value w_1, and by $[2]^1_{w_1}$ the $\mathscr{U}$-small edged space [2] equipped with the map

$$\begin{array}{rcl} [2]^2 & \to & \overline{\mathbb{R}}_{\geq 0} \\ (a_0,a_1) & \mapsto & \begin{cases} w_1 & ((a_0,a_1) \neq (0,2)) \\ \infty & ((a_0,a_1) = (0,2)) \end{cases}. \end{array}$$

Since we have assumed that the edge of Δ^{n_1} which does not belong to Λ is presented as $\{0, n_1\}$, k is naturally identified with the canonical embedding

$$([n_0]_{w_0} \otimes [n_1]^1_{w_1}) \sqcup_{\partial[n_0]_{w_0} \otimes [n_1]^1_{w_1}} (\partial[n_0]_{w_0} \otimes [n_1]_{w_1}) \hookrightarrow [n_0]_{w_0} \otimes [n_1]_{w_1}.$$

Let $r \in \mathbb{R}_{\geq 0}$. If $r < \max\{w_0, w_1\}$, then $\mathrm{ev}_{\mathbb{R}_{\geq 0},r}(\mathrm{VR}_{\mathbb{R}_{\geq 0}}(k))$ is $\mathrm{id}_\emptyset$. If $\max\{w_0, w_1\} \geq r$, then $\mathrm{ev}_{\mathbb{R}_{\geq 0},r}(\mathrm{VR}_{\mathbb{R}_{\geq 0}}(k))$ is naturally identified with the pushout product of i and j in sSet. Therefore $\mathrm{ev}_{\mathbb{R}_{\geq 0},r}(\mathrm{VR}_{\mathbb{R}_{\geq 0}}(k))$ is a trivial cofibration in sSet in both cases. This implies that k is a weak equivalence in $\mathrm{ES}_{\mathbb{R}_{\geq 0}}$. □

6. Homotopy Theories of Edged Spaces

Let P denote either $\mathbb{R}_{\geq 0}$ or $\mathbb{R}_{>0}$. First, we characterise a fibrant-cofibrant object in ES_P. Compared to the characterisation of a fibrant-cofibrant object in $\mathrm{sGSet}_P \cong \mathrm{sSet}^P$ studied in [2] 3.5.5, it is much easier to determine whether a given $\mathscr{U}$-small edged space is a fibrant-cofibrant in ES_P or not.

Theorem 6.1. *A $\mathscr{U}$-small edged space M is a fibrant-cofibrant object in ES_P if and only if M is a* max*-edged space and satisfies $d_M(m, m) \in P$ for any $m \in M$.*

In order to show Theorem 6.1, we prepare a lemma.

Lemma 6.2. *Let $M \in \mathrm{ob}(\mathrm{ES})$. The following hold:*

(1) The unique morphism $M \to [0]_0$ in ES satisfies the right lifting property for $|\mu_P(w, j)|_{\mathrm{ES},P}$ for any $(w, j) \in P \times J$ such that the target of j is of dimension 1.

(2) The unique morphism $M \to [0]_0$ in ES satisfies the right lifting property for $|\mu_P(w, j)|_{\mathrm{ES},P}$ for any $(w, j) \in P \times J$ such that the target of j is of dimension 2 if and only if M is a max*-edged space.*

(3) The $\mathscr{U}$-small edged space M is a fibrant object of ES_P if and only if M is a max*-edged space.*

Proof. The assertion (1) follows from the fact that any morphism $[0]_w \to [1]_w$ in ES admits the retraction $[1]_w \to [0]_w$. The assertion (3) immediately follows from the assertions (1) and (2), because $\mathscr{J}_P$ consists of morphisms in ES of the form $|\mu_P(w, j)|_{\mathrm{ES},P}$ for any $(w, j) \in P \times J$ such that the target of j is of dimension 1 or 2. We show the assertion (2). In order to simultaneously interpret the right lifting property for three 2-dimensional horn embeddings into inequalities related to the axiom of a max-edged space, we introduce a convention. Let $(w, k) \in P \times [2]$. Put

$$(k_0, k_1) := \begin{cases} (0,0) & (k = 0) \\ (1,0) & (k = 1) \\ (1,1) & (k = 2) \end{cases}.$$

The commutative diagram

$$\begin{array}{ccc} \mu_P(w, \Delta^1) & \longrightarrow & \mu_P(w, \Lambda^2_k) \\ \uparrow & & \uparrow \\ \mu_P(w, \Delta^0) & \longrightarrow & \mu_P(w, \Delta^1) \end{array}$$

in sGSet_P associated to the pushout diagram

$$\begin{array}{ccc} \Delta^1 & \longrightarrow & \Lambda^2_k \\ {\scriptstyle k_0}\uparrow & & \uparrow \\ \Delta^0 & \xrightarrow{k_1} & \Delta^1 \end{array}$$

in sSet is a pushout diagram in sGSet_P by the definition of μ_P, and hence its image

$$\begin{array}{ccc} |\mu_P(w,\Delta^1)|_{\mathrm{ES},P} & \longrightarrow & |\mu_P(w,\Lambda^2_k)|_{\mathrm{ES},P} \\ \uparrow & & \uparrow \\ |\mu_P(w,\Delta^0)|_{\mathrm{ES},P} & \longrightarrow & |\mu_P(w,\Delta^1)|_{\mathrm{ES},P} \end{array}$$

by the left adjoint functor $|\bullet|_{\mathrm{ES},P}$ forms a pushout diagram in ES. Let $(m_0, m_1, m_2) \in M^3$. Put

$$(m'_0, m'_1, m'_2, m'_3, m'_4, m'_5) := \begin{cases} (m_0, m_1, m_0, m_2, m_1, m_2) & (k = 0) \\ (m_0, m_1, m_1, m_2, m_0, m_2) & (k = 1) \\ (m_0, m_2, m_1, m_2, m_0, m_1) & (k = 2) \end{cases} .$$

and assume $w \geq \max\{d_M(m'_0, m'_1), d_M(m'_2, m'_3)\}$ so that $(m'_h, m'_{h+1})^2_{h=0}$ corresponds to the three edges of the set-theoretic map $[2] \to M$ corresponding to (m_0, m_1, m_2) and the first two edges define morphisms $|\mu_P(w,\Delta^1)|_{\mathrm{ES},P} \cong [1]_w \to M$ in ES. We denote by $i^{w,k}_{m_0,m_1,m_2}$ the morphism $|\mu_P(w,\Lambda^2_k)|_{\mathrm{ES},P} \to M$ associated to the two morphisms by the universality of the pushout in ES. Since the distance of points in $[2]_w$ is w, $i^{w,k}_{m_0,m_1,m_2}$ extends to a morphism $[2]_w \cong |\mu_P(w,\Delta^2)|_{\mathrm{ES},P} \to M$ in ES if and only if $d_M(m'_4, m'_5) \leq w$. In particular, the extension property for any $w \in P$ with $w \geq \max\{d_M(m'_0, m'_1), d_M(m'_2, m'_3)\}$ is equivalent to $d_M(m'_4, m'_5) \leq \max\{d_M(m'_0, m'_1), d_M(m'_2, m'_3)\}$. Therefore the unique morphism $M \to [0]_0$ satisfies the right lifting property for $|\mu_P(w, j)|_{\mathrm{ES},P}$ for any $(w, j) \in P \times J_{\leq 2}$ if and only if the inequalities

$$\begin{aligned} d_M(m_1, m_2) &\leq \max\{d_M(m_0, m_1), d_M(m_0, m_2)\} \\ d_M(m_0, m_2) &\leq \max\{d_M(m_0, m_1), d_M(m_1, m_2)\} \\ d_M(m_0, m_1) &\leq \max\{d_M(m_0, m_2), d_M(m_1, m_2)\} \end{aligned}$$

hold for any $(m_0, m_1, m_2) \in M^3$. The second inequality coincides with (E3). Under (E2), the other two inequalities are reduced to the second inequality. Due to (E1), the first inequality applied to the case $m_2 = m_0$ implies (E2). Thus the three conditions hold if and only if M is a max-edged space. □

Proof of Theorem 6.1. The assertion immediately follows from Lemma 5.9 and Lemma 6.2 (3). □

Theorem 6.1 immediately implies the following characterisation of an ultrametric space in terms of the model structures of ES, which is a geometric analogue of the simplicial result in [2] 4.3.11:

Corollary 6.3. *A $\mathscr{U}$-small metric space M is a fibrant-cofibrant object of $\mathrm{ES}_{\mathbb{R}_{\geq 0}}$ if and only if M is an ultrametric space.*

In this sense, the model structure of $\mathrm{ES}_{\mathbb{R}_{\geq 0}}$ gives a homotopy theory of ultrametric spaces in ES. We note that the same construction of the model structure does not work for the full subcategory $\mathrm{EMet} \subset \mathrm{ES}$ of $\mathscr{U}$-small extended metric spaces. Indeed, consider the modification of $[\bullet]_\bullet$ given by replacing the distance between every point and itself is 0 so that it gives a functor $\mathbb{R}_{\geq 0}^{\mathrm{op}} \times \Delta \to \mathrm{EMet}$. Then there is no model structure on EMet such that the geometric realisation $\mathrm{sGSet}_P \to \mathrm{EMet}$ associated to the modified $[\bullet]_\bullet$ gives a Quillen adjunction. The reflexivity of an extended metric space is so strong that if a short map between $\mathscr{U}$-small extended metric spaces satisfies the right lifting property for short maps between metric spaces analogous to $\mathscr{J}$, then it should be an isometric isomorphism. This is the technical reason why we adopt (E1) instead of the reflexivity. In this sense, the notion of an edged space is better than that of extended metric spaces when we consider the model category theoretic foundation to work with ultrametric spaces.

Next, we compare the homotopy theories of max-edged spaces through the observation of weak equivalences.

Theorem 6.4. *Let $f \colon M \to N$ be a short map between $\mathscr{U}$-small ultrametric spaces. Then the following hold:*

(1) The morphism f is a weak equivalence in $\mathrm{ES}_{\mathbb{R}_{\geq 0}}$ if and only if f is an isometric isomorphism.

(2) The morphism f is a weak equivalence in $\mathrm{ES}_{\mathbb{R}_{>0}}$ if and only if f induces an isometric isomorphism between the completions.

In order to show Theorem 6.4, we prepare a lemma. For a max-edged space M and an $r \in \mathbb{R}_{\geq 0}$, we put $D_r(M) := \{m \in M \mid d_M(m,m) \leq r\}$, and denote by $m_0 \sim_{M,r} m_1$ the relation on $(m_0, m_1) \in D_r(M)^2$ given as $d_M(m_0, m_1) \leq r$.

Lemma 6.5. *Let M be a $\mathscr{U}$-small* max-*edged space. For any $r \in \mathbb{R}_{\geq 0}$, the following hold:*

(1) The relation $\sim_{M,r}$ is an equivalence relation on $D_r(M)$.

(2) For any $m \in \mathrm{ev}_{\mathbb{R}_{\geq 0},r}(\mathrm{VR}_{\mathbb{R}_{\geq 0}}(M))[n] \subset M^{[n]}$ with $[n] \in \mathrm{ob}(\Delta)$, there exists a unique $N \in D_r(M)/{\sim_{M,r}}$ such that $m \in N^{[n]}$.

(3) For any subset $N \subset D_r(M)$ contained in a common equivalence class with respect to $\sim_{M,r}$, $N^\bullet$ is a subcomplex of $\mathrm{ev}_{\mathbb{R}_{\geq 0},r}(\mathrm{VR}_{\mathbb{R}_{\geq 0}}(M))$.

(4) The simplicial map $\coprod_{N \in D_r(M)/\sim_{M,r}} N^\bullet \to \mathrm{ev}_{\mathbb{R}_{\geq 0},r}(\mathrm{VR}_{\mathbb{R}_{\geq 0}}(M))$ given by the universality of the coproduct applied to the family of inclusions is an isomorphism in sSet.

(5) The simplicial map $\mathrm{ev}_{\mathbb{R}_{\geq 0},r}(\mathrm{VR}_{\mathbb{R}_{\geq 0}}(M)) \to \coprod_{N \in D_r(M)/\sim_{M,r}} \Delta^0$ which assigns to each element in $m \in \mathrm{ev}_{\mathbb{R}_{\geq 0},r}(\mathrm{VR}_{\mathbb{R}_{\geq 0}}(M))[n]$ the component corresponding to the equivalence class of any entry of m with respect to $\sim_{M,r}$ is a weak equivalence in sSet.

Proof. The reflexivity of $\sim_{M,r}$ follows from the definition of $D_r(M)$, the symmetry of $\sim_{M,r}$ follows from (E2), and the transitivity of $\sim_{M,r}$ follows from (E3). It implies the assertion (1). The assertions (2) and (3) follow from the equality

$$\mathrm{ev}_{\mathbb{R}_{\geq 0},r}(\mathrm{VR}_{\mathbb{R}_{\geq 0}}(M))[n] = \left\{ m \in M^{[n]} \middle| |m|_{M,[n]} \leq r \right\}.$$

The assertion (4) follows from the assertions (2) and (3). Since $N^\bullet$ is a contractible Kan complex for any $N \in M/{\sim_{M,r}}$, the canonical projection $\coprod_{N \in M/\sim_{M,r}} N^\bullet \twoheadrightarrow \coprod_{N \in M/\sim_{M,r}} \Delta^0$ is a weak equivalence in sSet. It implies the assertion (5). $\square$

Lemma 6.6. *A short map $f\colon M \to N$ between $\mathscr{U}$-small* max*-edged spaces is a weak equivalence in* ES_P *if and only if f induces a bijective map $M/{\sim_{M,r}} \to N/{\sim_{N,r}}$ for any $r \in P$.*

Proof. The assertion follows from Lemma 6.5 (5), because f is a weak equivalence in ES_P if and only if $\mathrm{ev}_{\mathbb{R}_{\geq 0},r}(\mathrm{VR}_P(f))$ is a weak equivalence in sSet for any $r \in P$ by the definition of $\mathscr{W}_P$. □

Proof of Theorem 6.4. By Lemma 6.6, f is a weak equivalence in ES_P if and only if the induced map $M/{\sim_{M,r}} \to N/{\sim_{N,r}}$ is bijective for any $r \in P$. The bijectivity for all $r > 0$ is equivalent to that f induces an isometric isomorphism between the completions. It implies the assertion (2). The bijectivity for $r = 0$ is equivalent to the bijectivity of f because of the reflexivity of metrics. Since the embeddings into the completions are isometric, it implies the assertion (1). □

We put $\mathrm{delim}_{\mathrm{ES}} \coloneqq \mathrm{id}_{\mathrm{ES}}\colon \mathrm{ES}_{\mathbb{R}_{>0}} \to \mathrm{ES}_{\mathbb{R}_{\geq 0}}$ and $\mathrm{lim}_{\mathrm{ES}} \coloneqq \mathrm{id}_{\mathrm{ES}}\colon \mathrm{ES}_{\mathbb{R}_{\geq 0}} \to \mathrm{ES}_{\mathbb{R}_{>0}}$. Using Lemma 6.6, we compare the two homotopy theories.

Corollary 6.7. *The trivial adjunction* $(\mathrm{delim}_{\mathrm{ES}}, \mathrm{lim}_{\mathrm{ES}})$ *forms a Quillen adjunction which is not a Quillen equivalence.*

Proof. The adjunction is a Quillen adjunction by $\mathscr{I}_{\mathbb{R}_{>0}} \subset \mathscr{I}_{\mathbb{R}_{\geq 0}}$ and $\mathscr{J}_{\mathbb{R}_{>0}} \subset \mathscr{J}_{\mathbb{R}_{\geq 0}}$. Since $P_{\max}$ and $A_{\max}$ are $\mathscr{U}$-small max-edged spaces, they are fibrant-cofibrant objects of $\mathrm{ES}_{\mathbb{R}_{>0}}$ and $\mathrm{ES}_{\mathbb{R}_{\geq 0}}$ respectively, while $A_{\max}$ is not a cofibrant object of $\mathrm{ES}_{\mathbb{R}_{>0}}$ by Lemma 5.9. The inclusion $i\colon P_{\max} \hookrightarrow A_{\max}$ is not a weak equivalence in $\mathrm{ES}_{\mathbb{R}_{\geq 0}}$, because $\mathrm{ev}_{\mathbb{R}_{\geq 0},0}(\mathrm{VR}_{\mathbb{R}_{\geq 0}}(i))$ is naturally identified with the inclusion $\emptyset \hookrightarrow \Delta^0$, which is not a weak equivalence in sSet. On the other hand, i induces a bijective map $P_{\max}/{\sim_{P_{\max},r}} \cong \{*\} \cong A_{\max}/{\sim_{A_{\max},r}}$ for any $r \in \mathbb{R}_{>0}$, and hence is a weak equivalence in $\mathrm{ES}_{\mathbb{R}_{>0}}$ by Lemma 6.6. □

By Theorem 6.1 and Theorem 6.4, it is reasonable to define the completeness of a max-edged space M as the property $d_M(m,m) > 0$ for any $m \in M$. Then $\mathrm{lim}_{\mathrm{ES}}$ can be seen as an analogue of the completion functor, although it is the right adjoint functor of the Quillen adjunction in Corollary 6.7 unlike the completion functor for ultrametric spaces. In this sense, $\mathrm{ES}_{\mathbb{R}_{\geq 0}}$ is a "model of the $(\infty,1)$-category of max-edged spaces and short maps", and $\mathrm{ES}_{\mathbb{R}_{>0}}$ is a "model of the $(\infty,1)$-category of complete max-edged spaces and short maps". We note that every ultrametric space M satisfies $d_M(m,m) = 0$ for any $m \in M$ and is not even a cofibrant object of $\mathrm{ES}_{\mathbb{R}_{>0}}$ unless $M = \emptyset$. Therefore this formulation of the completeness itself is just an analogue of that of an ultrametric space, but is not an extension of it.

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