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Controlled objects as a symmetric monoidal functor

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Abstract

The goal of this paper is to associate functorially to every symmetric monoidal additive category $\mathbf{A}$ with a strict G-action a lax symmetric monoidal functor $\mathbf{V}^G_{\mathbf{A}} : G\mathbf{BornCoarse} \to \mathbf{Add}_\infty$ from the symmetric monoidal category of G-bornological coarse spaces $G\mathbf{BornCoarse}$ to the symmetric monoidal ∞-category of additive categories $\mathbf{Add}_\infty$. Among others, this allows to refine equivariant coarse algebraic K-homology to a lax symmetric monoidal functor.

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1. Introduction

Given a group G, a G-bornological coarse space is a G-set with a G-coarse structure and a compatible G-bornology. A morphism between G-bornological coarse spaces is an equivariant map between G-sets which is proper and controlled. The category $G\mathbf{BornCoarse}$ of G-bornological coarse spaces has been introduced in [4, Sec. 2.1], see Section 3.2 for details. It provides an effective framework to study equivariant large scale geometry and assembly maps. We refer to [4] for further references and pointers to the applications.
Invariants of G-bornological coarse spaces are derived using equivariant coarse homology theories. Let $\mathcal{C}$ be a stable cocomplete ∞-category. A $\mathcal{C}$-valued equivariant coarse homology theory [4, Def. 3.10] is a functor

$$E : G\mathbf{BornCoarse} \to \mathcal{C} \tag{1.1}$$

with the following properties:

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1. coarsely invariant
2. excisive
3. vanishes on flasques
4. u-continuous.

We refer to [4, Sec. 3] for a detailed description of these axioms.
The category G**BornCoarse** has a symmetric monoidal structure $\otimes$, see [4, Ex. 2.17]. If also the ∞-category $\mathcal{C}$ has a symmetric monoidal structure, then we can ask whether the functor E can be refined to a lax symmetric monoidal functor.
The one-point space $*$ in G**BornCoarse** is the tensor unit for the structure $\otimes$ and hence a commutative algebra object. Every G-bornological coarse space is naturally a module over $*$, and every morphism $f\colon X \to X'$ between G-bornological coarse spaces is a morphism of modules. If the equivariant coarse homology theory E is lax symmetric monoidal, then $E(*)$ is a commutative algebra object in $\mathcal{C}$, and $E(X)$ is naturally an $E(*)$-module. Furthermore, the induced morphism $E(X) \to E(X')$ is a morphism of $E(*)$-modules. This additional information about the values of E can simplify calculations or can be applied to obtain localization results, see [1].
The examples of equivariant coarse homologies considered in the present paper depend on the choice of an additive category with a strict G-action $\mathbf{A}$. For example, it gives rise to the equivariant coarse algebraic K-theory $K\mathcal{X}^G_{\mathbf{A}}$ from [4, Sec. 8], the equivariant coarse topological Hochschild homology $\mathrm{THH}\mathcal{X}^G_{\mathbf{A}}$ of [2, Ex. 1.2], or to the equivariant coarse cyclic and Hochschild homology theories introduced in [9]. In fact, all these examples are obtained from a functor (see Definition 3.11)

$$\mathbf{V}^G_{\mathbf{A}} : G\mathbf{BornCoarse} \to \mathbf{Add}_1$$

(where $\mathbf{Add}_1$ denotes the 1-category of additive categories and additive functors) by postcomposing with suitable algebraic K-theory, cyclic, or (topological) Hochschild homology functors for additive categories. There is a universal such functor

$$\mathrm{UK} : \mathbf{Add}_1 \xrightarrow{\mathbf{Ch}^b(-)_\infty} \mathbf{Cat}^{\mathrm{ex}}_\infty \xrightarrow{\mathcal{U}_{loc}} \mathcal{M}_{loc} \ .$$

Here $\mathbf{Cat}^{\mathrm{ex}}_\infty$ is the large ∞-category of small stable ∞-categories and $\mathbf{Ch}^b(-)_\infty$ associates to an additive category the small stable ∞-category of bounded chain complexes localized at the homotopy equivalences. Furthermore $\mathcal{U}_{loc}$ is the universal localizing invariant for small stable ∞-categories of Blumberg-Gepner-Tabuada [6, Sec. 8.3]. Its target $\mathcal{M}_{loc}$ is cocomplete and stable, and it is called the ∞-category of non-commutative motives. The corresponding equivariant coarse homology theory introduced in [2, Def. 3.4]

$$\mathrm{UK}\mathcal{X}^G_{\mathbf{A}} : G\mathbf{BornCoarse} \to \mathcal{M}_{loc} \tag{1.2}$$

is called the universal coarse algebraic K-theory with coefficients.
The main result of the present paper is the following theorem:

Theorem 1.1. *A symmetric monoidal structure on* $\mathbf{A}$ *induces a lax symmetric monoidal refinement of the functor* $\mathrm{UK}\mathcal{X}^G_{\mathbf{A}}$.

In Proposition 3.32 we will further strengthen this result by providing a lax symmetric monoidal refinement of the functor $\underline{\mathrm{UK}\mathcal{X}}_{\mathbf{A}} : G\mathbf{BornCoarse} \to \mathbf{Fun}(G\mathbf{Orb}^{op}, \mathcal{M}_{loc})$ from [2, Thm 1.3], that packages the equivariant homology theories $\mathrm{UK}\mathcal{X}^H_{\mathbf{A}}$ for all subgroups H of G together in one object,.

Theorem 1.1 provides lax symmetric monoidal refinements for the equivariant coarse homology theories derived from $\mathrm{UK}\mathcal{X}^G_{\mathbf{A}}$.

Example 1.2. Let $\mathbf{1}_{\mathcal{M}_{loc}}$ be the tensor unit of $\mathcal{M}_{loc}$. Then the functor

$$\mathcal{K} := \mathtt{map}_{\mathcal{M}_{loc}}(\mathbf{1}_{\mathcal{M}_{loc}}, -) : \mathcal{M}_{loc} \to \mathbf{Sp}$$

is lax symmetric monoidal. Furthermore, since $\mathcal{K} \circ \mathrm{UK} : \mathbf{Add}_1 \to \mathbf{Sp}$ is equivalent to the algebraic K-theory functor for additive categories, we have an equivalence

$$K\mathcal{X}^G_{\mathbf{A}} \simeq \mathcal{K} \circ \mathrm{UK}\mathcal{X}^G_{\mathbf{A}} \ .$$

Corollary 1.3. *A symmetric monoidal structure on* $\mathbf{A}$ *induces a lax symmetric monoidal refinement of the functor* $K\mathcal{X}^G_{\mathbf{A}}$*.*

By replacing $\mathbf{1}_{\mathcal{M}_{loc}}$ by some other commutative coalgebra object in $\mathcal{M}_{loc}$ we get another $\mathbf{Sp}$-valued lax symmetric monoidal functor out of $\mathcal{M}_{loc}$. If the underlying object of the coalgebra is compact, then we get another example of a lax symmetric monoidal equivariant coarse homology theory. □

Example 1.4. In view of [6, Prop. 10.2] the topological Hochschild homology functor $\mathrm{THH} : \mathbf{Cat}^{\mathrm{ex}}_\infty \to \mathbf{Sp}$ has an essentially unique factorization

$$\mathbf{Cat}^{\mathrm{ex}}_\infty \xrightarrow{\mathcal{U}_{loc}} \mathcal{M}_{loc} \xrightarrow{\mathcal{THH}} \mathbf{Sp}$$

where $\mathcal{THH}$ preserves colimits. By [7, Cor. 6.9] there is a lax symmetric monoidal refinement of $\mathcal{THH}$. It can be composed with the lax symmetric monoidal refinement of $\mathrm{UK}\mathcal{X}^G_{\mathbf{A}}$ given by Theorem 1.1.

Corollary 1.5. *The coarse topological Hochschild homology functor* $\mathrm{THH}\mathcal{X}^G_{\mathbf{A}}$ *from [2, Ex. 1.2] has a lax symmetric monoidal refinement.*

□

Remark 1.6. The main ingredient of the lax symmetric monoidal structure on, e.g., $K\mathcal{X}^G_{\mathbf{A}}$, is the pairing

$$K\mathcal{X}^G_{\mathbf{A}}(X) \otimes K\mathcal{X}^G_{\mathbf{A}}(X') \to K\mathcal{X}^G_{\mathbf{A}}(X \times X') \ .$$

Note that $K\mathcal{X}^G_{\mathbf{A}}$ is a version of classical controlled algebraic K-theory. For controlled algebraic K-theory the presence of such pairings has been observed previously, and they were used in the study assembly maps, see, e.g., [14]. The main contribution of the present paper to this part of the story is the refinement to a full lax symmetric monoidal structure on the spectrum-valued functor with all higher coherences. □

In the remainder of this introduction we explain how Theorem 1.1 is proved. Recall that the functor $\mathrm{UK}\mathcal{X}^G_{\mathbf{A}}$ is defined as the composition

$$\mathrm{UK}\mathcal{X}^G_{\mathbf{A}} : G\mathbf{BornCoarse} \xrightarrow{\mathbf{V}^G_{\mathbf{A}}} \mathbf{Add}_1 \xrightarrow{\mathrm{UK}} \mathcal{M}_{loc} \ .$$

Since UK sends equivalences of additive categories to equivalences it has a factorization

$$\mathrm{UK} : \mathbf{Add}_1 \xrightarrow{\mathtt{Loc}} \mathbf{Add}_\infty \xrightarrow{\mathrm{UK}_\infty} \mathcal{M}_{loc} \ ,$$

where

$$\mathtt{Loc} : \mathbf{Add}_1 \to \mathbf{Add}_\infty := \mathbf{Add}_1[W_{\mathbf{Add}}^{-1}]$$

is the localization at the equivalences $W_{\mathbf{Add}}$ of additive categories, in the realm of ∞-categories. We refer to Definition 2.1 for the notion of a lax symmetric monoidal refinement of $\mathbf{V}^G_{\mathbf{A}}$. Theorem 1.1 now follows from the following two assertions:

Theorem 1.7 (Theorem 3.26)**.** *A symmetric monoidal structure on* $\mathbf{A}$ *induces a lax symmetric monoidal refinement*

$$\mathbf{V}^{G,\otimes}_{\mathbf{A}} : \mathtt{N}(G\mathbf{BornCoarse}^{\otimes}) \to \mathbf{Add}^{\otimes}_\infty$$

of the functor $\mathbf{V}^G_{\mathbf{A}}$*.*

Theorem 1.8 (Theorem 3.27)**.** *The functor* UK_∞ *admits a lax symmetric monoidal refinement*

$$\mathrm{UK}^{\otimes}_\infty : \mathbf{Add}^{\otimes}_\infty \to \mathcal{M}^{\otimes}_{loc} \ .$$

Example 1.9. If k is a field, then we can consider the category $\mathbf{Add}_{k,1}$ of k-linear additive categories and k-linear functors. There is a forgetful functor $\mathbf{Add}_{k,1} \to \mathbf{Add}_1$ forgetting the k-linear structure. Since the tensor product in $\mathbf{Add}_{k,1}$ involves forming tensor products of morphism spaces over k and not over $\mathbb{Z}$ as in $\mathbf{Add}_1$ (see Eq. (2.1) below), the forgetful functor is only lax symmetric monoidal.

If A is a symmetric monoidal k-linear additive category, then there is a version of Theorem 1.7 for functors taking values in $\mathbf{Add}^{\otimes}_{k,\infty}$ which has essentially the same proof (we do not see how to deduce it formally). The resulting symmetric monoidal refinement

$$\mathbf{V}^{G,\otimes}_{\mathbf{A}} : \mathtt{N}(G\mathbf{BornCoarse}^{\otimes}) \to \mathbf{Add}^{\otimes}_{k,\infty}$$

has been used to refine the equivariant coarse cyclic and Hochschild homology theories introduced in [9] to lax symmetric monoidal functors [9, Prop. 3.12]. □

The main difficulty in proving Theorem 1.7 is that the symmetric monoidal category of small additive categories is of a 2-categorical nature. A pedestrian approach to the proof of this theorem would thus require to work with symmetric monoidal structures on 2-categories and therefore tedious considerations of a large set of commuting diagrams. In this paper we prefer to use the language of symmetric monoidal ∞-categories. In Section 3.4, by using the Grothendieck construction, we encode the functor $\mathbf{V}^G_{\mathbf{A}} : G\mathbf{BornCoarse} \to \mathbf{Add}_1$ into a cocartesian fibration $\mathcal{V}^G_{\mathbf{A}} \to G\mathbf{BornCoarse}$ coming from an op-fibration of 1-categories. We then encode a symmetric monoidal refinement of the functor $\mathbf{V}^G_{\mathbf{A}}$ into a symmetric monoidal structure on $\mathcal{V}^G_{\mathbf{A}}$ and a symmetric monoidal refinement of the functor to $G\mathbf{BornCoarse}$. This only requires 1-categorical considerations. The machine of ∞-categories then produces, as explained in Section 2, the asserted symmetric monoidal refinement in Theorem 1.7.

The technical results Theorem 2.2 and Theorem 2.3 might be of independent interest in cases where one wants to construct symmetric monoidal refinements of functors from 1-categories to $\mathbf{Cat}_1$ or $\mathbf{Add}_1$.

Theorem 1.8 is shown in Section 3.5 by combining various results in the literature on dg-categories.

2. Symmetric monoidal functors to Cat and Add

In this section, we construct lax symmetric monoidal refinements of functors from symmetric monoidal 1-categories with values in the 2-categories **Cat** or **Add** of small categories or small additive categories.

2.1 From 2- to ∞-categories A symmetric monoidal structure on a 1-category $\mathbf{C}$ consists of the tensor functor

$$\otimes_{\mathbf{C}} : \mathbf{C} \times \mathbf{C} \to \mathbf{C} \ ,$$

the tensor unit $1_{\mathbf{C}}$ in $\mathbf{C}$, and the associator, symmetry and unit-transformations, which must satisfy various compatibility relations. If $\mathbf{C}$ and $\mathbf{D}$ are symmetric monoidal 1-categories, then we can consider lax symmetric monoidal functors from $\mathbf{C}$ to $\mathbf{D}$. Such a lax symmetric monoidal functor is given by a functor $F : \mathbf{C} \to \mathbf{D}$ together with a binatural transformation

$$F(C) \otimes_{\mathbf{D}} F(C') \to F(C \otimes_{\mathbf{C}} C') \ , \quad C, C' \in \mathbf{C}$$

that is compatible with the associators, symmetries and unit-transformations of $\mathbf{C}$ and $\mathbf{D}$ in a suitable way. We will list these structures and relations in Subsection 3.1 below.

The category **Cat** of small categories, functors and natural equivalences is naturally a 2-category. Similarly, the category **Add** of small additive categories, additive functors and natural equivalences is a 2-category. Furthermore, the category **Cat** is symmetric monoidal with respect to the Cartesian symmetric monoidal structure $\times := \otimes_{\mathbf{Cat}}$. Besides the cartesian monoidal structure $\times$ the category **Add** has a symmetric monoidal structure $\otimes_{\mathbf{Add}}$ classifying bi-additive functors: if $\mathbf{A}$ and $\mathbf{B}$ are two additive categories, then the objects of the tensor product $\mathbf{A} \otimes_{\mathbf{Add}} \mathbf{B}$ are pairs (A, B) of objects A in $\mathbf{A}$ and B in $\mathbf{B}$, and the morphisms are given by the tensor product

$$\mathtt{Hom}_{\mathbf{A} \otimes_{\mathbf{Add}} \mathbf{B}}((A, B), (A', B')) := \mathtt{Hom}_{\mathbf{A}}(A, A') \otimes_{\mathbb{Z}} \mathtt{Hom}_{\mathbf{B}}(B, B') \tag{2.1}$$

of abelian groups.

In the case of a symmetric monoidal structure on a 2-category, like **Cat** or **Add**, we have the same compatibility relations between the structures (tensor functor, tensor unit, etc.) as in the 1-categorical case, but they are satisfied up to 2-morphisms only, which in turn must satisfy higher compatibility relations. A similar remark applies to the notion of a (lax) symmetric monoidal functor.

In the present paper we consider the 1-categorical situation as explicitly manageable, and we will avoid to explicitly work with symmetric monoidal structures on 2-categories.

Let $\mathbf{C}$ be a symmetric monoidal 1-category. Our goal is to construct symmetric monoidal functors $F : \mathbf{C} \to \mathbf{Cat}$ or $F : \mathbf{C} \to \mathbf{Add}$ using 1-categorical data only. Instead of working with the symmetric monoidal 2-categories **Cat** or **Add** we will actually use the associated symmetric monoidal ∞-categories $\mathbf{Cat}_\infty$ or $\mathbf{Add}_\infty$.

We start with the ordinary 1-category $\mathbf{Cat}_1$ of small categories. Let $W_{\mathbf{Cat}}$ be the equivalences in $\mathbf{Cat}_1$. The localization in large ∞-categories

$$\mathbf{Cat}_\infty := \mathtt{N}(\mathbf{Cat}_1)[W_{\mathbf{Cat}}^{-1}]$$

is the large ∞-category of categories. It models the 2-category **Cat** in the following sense. The 2-category **Cat** can be considered as a category enriched in groupoids. Applying the nerve functor $\mathtt{N}$ to the $\mathtt{Hom}$-groupoids in **Cat** we get a fibrant[1] simplicially enriched category $\mathtt{N}(\mathbf{Cat})$. Applying the homotopy coherent nerve functor $\mathcal{N}$, we get an ∞-category

$$\mathtt{N}_2(\mathbf{Cat}) := \mathcal{N}(\mathtt{N}(\mathbf{Cat})) \ .$$

Then, we have an equivalence of ∞-categories

$$\mathtt{N}_2(\mathbf{Cat}) \simeq \mathbf{Cat}_\infty \ .$$

We refer to the appendix of [12] for more details about $\mathtt{N}_2$.

The category $\mathbf{Cat}_1$ is a symmetric monoidal category and therefore gives rise to an op-fibration of 1-categories [16, Constr. 2.0.01], and to a symmetric monoidal ∞-category [16, Def. 2.0.0.7 & Ex. 2.1.2.21]

$$\mathbf{Cat}_1^\otimes \to \mathbf{Fin}_* \ , \ \text{and} \quad \mathtt{N}(\mathbf{Cat}_1^\otimes) \to \mathtt{N}(\mathbf{Fin}_*) \ ,$$

respectively. The equivalences $W_{\mathbf{Cat}}$ are preserved by the cartesian product. Hence we can form a symmetric monoidal localization [13, Prop. 3.2.2]

$$\mathbf{Cat}_\infty^\otimes := \mathtt{N}(\mathbf{Cat}_1^\otimes)[W_{\mathbf{Cat}}^{\otimes,-1}] \to \mathtt{N}(\mathbf{Fin}_*)$$

whose underlying ∞-category is equivalent to $\mathbf{Cat}_\infty$. Conseqently, the symmetric monoidal ∞-category $\mathbf{Cat}_\infty^\otimes \to \mathtt{N}(\mathbf{Fin}_*)$ models the symmetric monoidal 2-category **Cat**. In this way we avoid to spell out the structures of a symmetric monoidal 2-category explicitly.

A similar reasoning applies to **Add**. We consider the large 1-category $\mathbf{Add}_1$ of small additive categories and exact functors with the equivalences $W_{\mathbf{Add}}$. Then we define the large ∞-category

$$\mathbf{Add}_\infty := \mathtt{N}(\mathbf{Add}_1)[W_{\mathbf{Add}}^{-1}]$$

and get an equivalence

$$\mathbf{Add}_\infty \simeq \mathtt{N}_2(\mathbf{Add}) \ .$$

We consider $\mathbf{Add}_1$ as a symmetric monoidal category with respect to $\otimes_{\mathbf{Add}}$ and get an op-fibration of 1-categories and a symmetric monoidal ∞-category

$$\mathbf{Add}_1^\otimes \to \mathbf{Fin}_* \ , \quad \mathtt{N}(\mathbf{Add}_1^\otimes) \to \mathtt{N}(\mathbf{Fin}_*) \ .$$

Since the equivalences $W_{\mathbf{Add}}$ are preserved by the tensor product $\otimes_{\mathbf{Add}}$, we get the symmetric monoidal localization

$$\mathbf{Add}_\infty^\otimes := \mathtt{N}(\mathbf{Add}_1^\otimes)[W_{\mathbf{Add}}^{\otimes,-1}] \to \mathtt{N}(\mathbf{Fin}_*)$$

whose underlying ∞-category is equivalent to $\mathbf{Add}_\infty$. Therefore $\mathbf{Add}_\infty^\otimes \to \mathtt{N}(\mathbf{Fin}_*)$ models the symmetric monoidal 2-category **Add**.

Let $\mathbf{C}$ be an ordinary category. A functor $F : \mathbf{C} \to \mathbf{Cat}_1$ (or $F : \mathbf{C} \to \mathbf{Add}_1$) gives rise to a functor between ∞-categories $F_\infty : \mathtt{N}(\mathbf{C}) \to \mathbf{Cat}_\infty$ (or $F_\infty : \mathtt{N}(\mathbf{C}) \to \mathbf{Add}_\infty$) in the natural way, e.g. as the composition

$$F_\infty : \mathtt{N}(\mathbf{C}) \xrightarrow{\mathtt{N}(F)} \mathtt{N}(\mathbf{Cat}_1) \xrightarrow{\mathtt{Loc}} \mathtt{N}(\mathbf{Cat}_1)[W_{\mathbf{Cat}}^{-1}] = \mathbf{Cat}_\infty \ .$$

[1] i.e., the $\mathtt{Hom}$-complexes are Kan complexes

A symmetric monoidal 1-category $\mathbf{C}$ gives rise to the symmetric monoidal ∞-category $\mathtt{N}(\mathbf{C}^{\otimes}) \to \mathtt{N}(\mathbf{Fin}_*)$ whose underlying ∞-category is equivalent to $\mathtt{N}(\mathbf{C})$. We now consider a functor $F\colon \mathbf{C} \to \mathbf{Cat}_1$ (or $F : \mathbf{C} \to \mathbf{Add}_1$). Recall that a map of ∞-operads [16, Def. 2.1.2.7] can be thought of as a (lax) symmetric monoidal functor [16, Def. 2.1.3.7] between the underlying categories.
Let F and F_∞ be as above.

Definition 2.1. *A lax symmetric monoidal refinement of F is a morphism of ∞-operads*

$$F^{\otimes} : \mathtt{N}(\mathbf{C}^{\otimes}) \to \mathbf{Cat}_\infty^{\otimes} \ , \quad (F^{\otimes} : \mathtt{N}(\mathbf{C}^{\otimes}) \to \mathbf{Add}_\infty^{\otimes})$$

that induces a functor equivalent to F_∞ on the underlying ∞-categories.

Using this definition we avoid to spell out the details of the notion of a lax-symmetric functor from $\mathbf{C}$ to the 2-category $\mathbf{Cat}$ or $\mathbf{Add}$.

2.2 Symmetric monoidal refinements of functors to $\mathbf{Cat}_1$ and $\mathbf{Add}_1$ In this subsection we state the technical results Theorem 2.2 and Theorem 2.3. Starting with a functor to the 1-category $\mathbf{Cat}_1$ or $\mathbf{Add}_1$ they provide a lax symmetric monoidal refinement of this functor which is now considered with values in the 2-category $\mathbf{Cat}$ or $\mathbf{Add}$.
Let $\mathbf{C}$ be a 1-category. A functor between 1-categories

$$F : \mathbf{C} \to \mathbf{Cat}_1$$

gives rise, via the Grothendieck construction, to an op-fibration

$$\pi_F : \mathcal{F} \to \mathbf{C} \ .$$

An object of the 1-category $\mathcal{F}$ is a pair (X, A) with X in $\mathbf{C}$ and A in $F(X)$. A morphism $(X, A) \to (Y, B)$ is a pair (f, ϕ) of a morphism $f : X \to Y$ in $\mathbf{C}$ and a morphim $\phi : F(f)(A) \to B$ in $F(Y)$.
Assume that the categories $\mathbf{C}$ and $\mathcal{F}$ have symmetric monoidal structures such that

$$\pi_F((X, A) \otimes_{\mathcal{F}} (X', A')) = X \otimes_{\mathbf{C}} X' \ , \tag{2.2}$$

i.e., π_F preserves the tensor product strictly. Then, we can write

$$(X, A) \otimes_{\mathcal{F}} (X', A') = (X \otimes_{\mathbf{C}} X', A \boxtimes_{X,X'} A')$$

in order to define the object $A \boxtimes_{X,X'} A'$ in $F(X \otimes_{\mathbf{C}} X')$. This construction gives for every pair of objects X, X' in $\mathbf{C}$ a bifunctor

$$\boxtimes_{X,X'} : F(X) \times F(X') \to F(X \otimes_{\mathbf{C}} X') \tag{2.3}$$

which is defined on morphisms in the canonical way. Let

$$f : X \to X' \ , \quad g : Y \to Y'$$

be morphisms in $\mathbf{C}$ and A in $F(X)$ and B in $F(Y)$. Then

$$(f, \mathtt{id}_{F(f)(A)}) : (X, A) \to (X', F(f)(A)) \ , \quad (g, \mathtt{id}_{F(g)(B)}) : (Y, B) \to (Y', F(g)(B))$$

are morphisms in $\mathcal{F}$ which are cocartesian lifts of f with domain (X, A), and of g with domain (Y, B), respectively. The second component of their tensor product

$$(f, \mathtt{id}_{F(f)(A)}) \otimes_{\mathcal{F}} (g, \mathtt{id}_{F(g)(B)}) : (X \otimes_{\mathbf{C}} Y, A \boxtimes_{X,Y} B) \to (X' \otimes_{\mathbf{C}} Y', F(f)(A) \boxtimes_{X',Y'} F(g)(B))$$

is a morphism

$$F(f \otimes_{\mathbf{C}} g)(A \boxtimes_{X,Y} B) \to F(f)(A) \boxtimes_{X',Y'} F(g)(B) \tag{2.4}$$

in $F(X' \otimes_{\mathbf{C}} Y')$. This morphism will appear in the assumptions of the two theorems below.

We now consider the following data:

1. a symmetric monoidal 1-category $\mathbf{C}$,
2. a functor $F : \mathbf{C} \to \mathbf{Cat}_1$,
3. a symmetric monoidal structure on the Grothendieck construction $\mathcal{F}$ of F.

Let $\pi_F : \mathcal{F} \to \mathbf{C}$ denote the associated projection.

Theorem 2.2. *Assume:*

1. *The functor π_F strictly preserves the tensor product, the tensor unit as well as the associator, unit, and symmetry transformations.*
2. *For every two objects (X, A) and (Y, B) in $\mathcal{F}$ and morphisms $f : X \to X'$ and $g : Y \to Y'$ in $\mathbf{C}$ the morphism (2.4)*
$$F(f \otimes_{\mathbf{C}} g)(A \boxtimes_{X,Y} B) \to F(f)(A) \boxtimes_{X',Y'} F(g)(B)\ .$$
is an isomorphism.

Then the data provide a lax symmetric monoidal refinement (Def. 2.1)

$$F^{\otimes} : \mathtt{N}(\mathbf{C}^{\otimes}) \to \mathbf{Cat}_{\infty}^{\otimes}$$

of the functor F.

Note that Condition 1 in the theorem implies Relation (2.2) so that the bifunctors $\boxtimes_{X,Y}$ appearing in Condition 2 are, in fact, defined.

The analoguous version for additive categories is the following.

Consider the following data:

1. a symmetric monoidal 1-category $\mathbf{C}$,
2. a functor $F : \mathbf{C} \to \mathbf{Add}_1$,
3. a symmetric monoidal structure on the Grothendieck construction $\mathcal{F}$ of F.

Let $\pi_F : \mathcal{F} \to \mathbf{C}$ denote the associated projection.

Theorem 2.3. *Assume:*

1. *The functor π_F strictly preserves the tensor product, the tensor unit as well as the associator, unit, and symmetry transformations.*
2. *The functors $\boxtimes_{X,X'}$ are bi-additive for all pairs X, X' of objects in $\mathbf{C}$.*
3. *For all pairs of objects (X, A) and (Y, B) in $\mathcal{F}$ and morphisms $f : X \to X'$ and $g : Y \to Y'$ in $\mathbf{C}$ the morphism (2.4)*
$$F(f \otimes_{\mathbf{C}} g)(A \boxtimes_{X,Y} B) \to F(f)(A) \boxtimes_{X',Y'} F(g)(B)\ .$$
is an isomorphism.

Then the data provide a lax symmetric monoidal refinement

$$F^{\otimes} : \mathtt{N}(\mathbf{C}^{\otimes}) \to \mathbf{Add}^{\otimes}_{\infty}$$

of the functor F.

Note that the last condition in both theorems requires that the tensor product of cocartesian lifts is again cocartesian.

2.3 Proofs of Theorem 2.2 and Theorem 2.3. We start with the proof of Theorem 2.2. Let $\mathtt{N}(\mathbf{C}^{\otimes}) \to \mathtt{N}(\mathbf{Fin}_*)$ denote the symmetric monoidal ∞-category corresponding to the symmetric monoidal category $\mathbf{C}$ [16, Ex. 2.1.2.21]. Let $\mathbf{Cat}^{\otimes}_{\infty} \to \mathtt{N}(\mathbf{Fin}_*)$ be the cocartesian fibration corresponding to the symmetric monoidal category of small categories. Then the ∞-category

$$\mathbf{Alg}_{\mathtt{N}(\mathbf{C})}(\mathbf{Cat}_{\infty}) := \{\text{operad maps } \mathtt{N}(\mathbf{C}^{\otimes}) \to \mathbf{Cat}^{\otimes}_{\infty}\}$$

corresponds to the ∞-category of lax symmetric monoidal functors $\mathtt{N}(\mathbf{C}) \to \mathbf{Cat}_{\infty}$, see the text after [16, Rem. 2.1.3.6]. So our task is to construct an object $F^{\otimes}$ of $\mathbf{Alg}_{\mathtt{N}(\mathbf{C})}(\mathbf{Cat}_{\infty})$.
By [16, Prop. 2.4.2.5] the ∞-category $\mathbf{Alg}_{\mathtt{N}(\mathbf{C})}(\mathbf{Cat}_{\infty})$ is identified with the ∞-category of $\mathtt{N}(\mathbf{C})$-monoids in $\mathbf{Cat}_{\infty}$ (denoted by $\mathbf{Mon}_{\mathtt{N}(\mathbf{C})}(\mathbf{Cat}_{\infty})$) [16, Def. 2.4.2.1]. By [16, Rem. 2.4.2.4], in order to provide an $\mathtt{N}(\mathbf{C})$-monoid in $\mathbf{Cat}_{\infty}$ and hence an object of $\mathbf{Alg}_{\mathtt{N}(\mathbf{C})}(\mathbf{Cat}_{\infty})$ it suffices to provide a cocartesian fibration

$$p : \mathcal{C}^{\otimes} \to \mathtt{N}(\mathbf{C}^{\otimes}) \tag{2.5}$$

which exhibits $\mathcal{C}^{\otimes}$ as a $\mathtt{N}(\mathbf{C})$-monoidal category [16, Rem. 2.1.2.13]. To this end we must show that the composition

$$\mathcal{C}^{\otimes} \xrightarrow{p} \mathtt{N}(\mathbf{C}^{\otimes}) \to \mathtt{N}(\mathbf{Fin}_*) \tag{2.6}$$

exhibits $\mathcal{C}^{\otimes}$ as an ∞-operad [16, Prop. 2.1.2.12]. In our applications below it is clear from the constructions that (2.6) exhibits $\mathcal{C}^{\otimes}$ as an ∞-operad so it suffices to provide the cocartesian fibration (2.5).
Let

$$\pi_F : \mathcal{F} \to \mathbf{C}$$

be a symmetric monoidal functor between 1-categories as in Theorem 2.2. We get an induced functor of symmetric monoidal categories

$$\pi_F^{\otimes} : \mathcal{F}^{\otimes} \to \mathbf{C}^{\otimes}$$

and thus a morphism of ∞-operads

$$\mathtt{N}(\pi_F^{\otimes}) : \mathtt{N}(\mathcal{F}^{\otimes}) \to \mathtt{N}(\mathbf{C}^{\otimes}) \ .$$

Our task is then to show that $\mathtt{N}(\pi_F^{\otimes})$ exhibits $\mathtt{N}(\mathcal{F}^{\otimes})$ as an $\mathtt{N}(\mathbf{C})$-monoidal category. It suffices to check that $\mathtt{N}(\pi_F^{\otimes})$ is a cocartesian fibration. To this end we check that $\pi_F^{\otimes}$ is an op-fibration of 1-categories.
By assumption, the underlying functor of π_F (after forgetting the symmetric monoidal structures) arose from a Grothendieck construction for a functor

$$F : \mathbf{C} \to \mathbf{Cat}_1 \ .$$

Recall from [16, Constr. 2.0.0.1] that the objects of $\mathbf{C}^{\otimes}$ in the fibre $\mathbf{C}\langle n\rangle$ of $\mathbf{C}^{\otimes}$ over $\langle n\rangle$ in $\mathbf{Fin}_*$ are n-tuples of objects of $\mathbf{C}$. Consider two objects

$$(X_1,\dots,X_n)\ ,\quad (Y_1,\dots,Y_m)$$

in $\mathbf{C}^{\otimes}\langle n\rangle$ and $\mathbf{C}^{\otimes}\langle m\rangle$ and an object

$$((X_1,A_1),\dots,(X_n,A_n))$$

in $\mathcal{F}^{\otimes}\langle n\rangle$, where A_i belongs to $F(X_i)$ for $i=1,\dots,n$. Let $\alpha\colon\langle n\rangle\to\langle m\rangle$ be a morphism in $\mathbf{Fin}_*$ and

$$f:(X_1,\dots,X_n)\to(Y_1,\dots,Y_m)$$

be a morphism in $\mathbf{C}^{\otimes}$ over α. Then f is given by a collection of morphisms $f:=(f_j)_{j\in\langle m\rangle}$ with

$$f_j:\otimes_{i\in\alpha^{-1}(j)}X_i\to Y_j\ .$$

We must provide a cocartesian lift of f. For j in $\langle m\rangle$ we have a morphism

$$g_j{:=}(f_j,\mathtt{id}_{\boxtimes_{i\in\alpha^{-1}(j)}A_i}):(\otimes_{i\in\alpha^{-1}(j)}X_i,\boxtimes_{i\in\alpha^{-1}(j)}A_i)\to(Y_j,F(f_j)(\boxtimes_{i\in\alpha^{-1}(j)}A_i))$$

in $\mathcal{F}$. Repeatedly using the Condition 2.2.2 one now checks in a straightforward manner that the collection $g:=(g_j)_{j\in\langle m\rangle}$ is the cocartesian lift of f. This finishes the proof of Theorem 2.2.
We now turn to the proof of Theorem 2.3. We want to consider $\mathbf{Add}_\infty$ as a symmetric monoidal subcategory of $\mathbf{Cat}_\infty$. To this end we first consider the subcategory $\mathbf{Cat}_\infty(\coprod)$ of ∞-categories which admit finite coproducts and coproduct preserving functors. By [16, Cor. 4.8.1.4] (applied to the collection $\mathcal{K}$ of finite sets) we get a symmetric monoidal subcategory

$$\mathbf{Cat}_\infty(\coprod)^{\otimes}\to\mathbf{Cat}_\infty^{\otimes}\ .$$

In the next step we view $\mathbf{Add}_\infty$ as a full subcategory of $\mathbf{Cat}_\infty(\coprod)$ of pointed 1-categories in which products and coproducts coincide. Using [16, Cor. 2.2.1.1] one then shows that

$$\mathbf{Add}_\infty^{\otimes}\to\mathbf{Cat}_\infty(\coprod)^{\otimes}$$

is again a suboperad.
We now consider the diagram

$$\begin{array}{ccc}\mathbf{Add}_\infty^{\otimes} & \longrightarrow & \mathbf{Cat}_\infty(\coprod)^{\otimes}\\ \uparrow & \nearrow & \downarrow\\ \mathtt{N}(\mathbf{C}^{\otimes}) & \longrightarrow & \mathbf{Cat}_\infty^{\otimes}\end{array}\ .$$

The lower horizontal map is a morphism of ∞-operads by Theorem 2.2. We first argue that the dotted lift exists. To this end we use [16, Notation 4.8.1.2]. One must check that F takes values in categories admitting finite coproducts (clear), and that the functors

$$\boxtimes_{X,Y}:F(X)\times F(Y)\to F(X\times Y)$$

preserves sums in both variables separately. This is indeed ensured by Assumption 2.3.2. Finally, for the dashed arrow we use that F takes values in $\mathbf{Add}_1$. Using the same explicit calculations as for Theorem 2.2 based on Condition 2.3.3 we show that $\mathtt{N}(\mathbf{C}^{\otimes})\to\mathbf{Add}_\infty^{\otimes}$ is a cocartesian fibration.

3. The symmetric monoidal functor of controlled objects

3.1 Symmetric monoidal structures In this subsection we write out, for later reference, the structures of a symmetric monoidal category and of a (lax) symmetric monoidal functor. Let $\mathbf{C}$ be a 1-category:

Definition 3.1. *[17, Sec. VII. 1. & 7.] A symmetric monoidal structure on* $\mathbf{C}$ *is given by the following data:*

1. *a bifunctor* $(- \otimes_{\mathbf{C}} -) : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$,
2. *an object* $1_{\mathbf{C}}$ *(the tensor unit),*
3. *a natural isomorphism (the associativity constraint)*

$$\alpha^{\mathbf{C}} : (- \otimes_{\mathbf{C}} -) \circ ((- \otimes_{\mathbf{C}} -) \times \mathtt{id}_{\mathbf{C}}) \to (- \otimes_{\mathbf{C}} -) \circ (\mathtt{id}_{\mathbf{C}} \times (- \otimes_{\mathbf{C}} -)) ,$$

4. *a natural isomorphism* $\eta^{\mathbf{C}} : 1_{\mathbf{C}} \otimes_{\mathbf{C}} - \to \mathtt{id}_{\mathbf{C}}$ *(the unit constraint),*
5. *a natural isomorphism (the symmetry)* $\sigma^{\mathbf{C}} : (- \otimes_{\mathbf{C}} -) \circ T \to (- \otimes_{\mathbf{C}} -)$*, where* $T : \mathbf{C} \times \mathbf{C} \to \mathbf{C} \times \mathbf{C}$ *is the flip functor.*

This data have to satisfy the following relations:

1. *the pentagon relation,*
2. *the triangle relation,*
3. *the inverse relation,*
4. *the associativity coherence.*

A symmetric monoidal category is a category equipped with a symmetric monoidal structure. We will use the name of the category as a superscript for the constraints, but if we evaluate e.g. the symmetry constraint $\sigma^{\mathbf{C}}$ at the objects C, C' of $\mathbf{C}$, then we write shortly $\sigma_{C,C'}$ instead of $\sigma^{\mathbf{C}}_{C,C'}$ since the type of objects in the subscript already determines the category in question. Let $\mathbf{C}$ and $\mathbf{D}$ be symmetric monoidal categories, and let $F : \mathbf{C} \to \mathbf{D}$ be a functor.

Definition 3.2. *[17, Sec. XI. 2.] A symmetric monoidal structure on* F *is given by the following data:*

1. *an isomorphism* $\epsilon^F : 1_{\mathbf{D}} \to F(1_{\mathbf{C}})$,
2. *a natural isomorphism* $\mu^F : (- \otimes_{\mathbf{D}} -) \circ (F \times F) \to F \circ (- \otimes_{\mathbf{C}} -)$.

This data have to satisfy the following relations:

1. *associativity relation,*
2. *unitality relation,*
3. *symmetry relation.*

Remark 3.3. If we weaken the assumptions and only require that ϵ^F and μ^F are natural transformations, then we get the definition of a lax symmetric monoidal functor.

3.2 Bornological coarse spaces In this subsection we recall the definition of the symmetric monoidal category $G\mathbf{BornCoarse}$ of G-bornological coarse spaces [3, Sec. 2], [4, Sec. 2.1]. In the definitions below we will use the following notation:

1. For a set Z we let $\mathcal{P}(Z)$ denote the power set of Z.

2. If a group G acts on a set X, then it acts diagonally on $X \times X$ and therefore on $\mathcal{P}(X \times X)$. For U in $\mathcal{P}(X \times X)$ we set
$$GU := \bigcup_{g \in G} gU \ .$$
3. For U in $\mathcal{P}(X \times X)$ and B in $\mathcal{P}(X)$ we define the U-thickening $U[B]$ by
$$U[B] := \{x \in X \mid \exists y \in B : (x, y) \in U\} \ . \tag{3.1}$$
4. For U in $\mathcal{P}(X \times X)$ we define its inverse by
$$U^{-1} := \{(y, x) \mid (x, y) \in U\} \ .$$
5. For U, V in $\mathcal{P}(X \times X)$ we define their composition by
$$U \circ V := \{(x, z) \mid \exists y \in X : (x, y) \in U \wedge (y, z) \in V\} \ . \tag{3.2}$$

Let G be a group and let X be a G-set.

Definition 3.4. *A G-coarse structure $\mathcal{C}$ on X is a subset of $\mathcal{P}(X \times X)$ with the following properties:*
1. *$\mathcal{C}$ is closed under composition, inversion, and forming finite unions or subsets.*
2. $\mathtt{diag}(X) \in \mathcal{C}$
3. *For every U in $\mathcal{C}$ we have $GU \in \mathcal{C}$.*

The pair $(X, \mathcal{C})$ is called a G-coarse space, and the members of $\mathcal{C}$ are called (coarse) entourages of X.

Let $(X, \mathcal{C})$ and $(X', \mathcal{C}')$ be G-coarse spaces and let $f \colon X \to X'$ be an equivariant map between the underlying sets.

Definition 3.5. *The map f is controlled if for every U in $\mathcal{C}$ we have $(f \times f)(U) \in \mathcal{C}'$.*

We obtain a category G**Coarse** of G-coarse spaces and controlled equivariant maps.
Let G be a group and let X be a G-set.

Definition 3.6. *A G-bornology $\mathcal{B}$ on X is a subset of $\mathcal{P}(X)$ with the following properties:*
1. *$\mathcal{B}$ is closed under forming finite unions and subsets.*
2. *$\mathcal{B}$ contains all finite subsets of X.*
3. *$\mathcal{B}$ is G-invariant.*

The pair $(X, \mathcal{B})$ is called a G-bornological space, and the members of $\mathcal{B}$ are called bounded subsets of X.

Let $(X, \mathcal{B})$ and $(X', \mathcal{B}')$ be G-bornological spaces and let $f \colon X \to X'$ be an equivariant map between the underlying sets.

Definition 3.7. *The map f is proper if for every B' in $\mathcal{B}'$ we have $f^{-1}(B') \in \mathcal{B}$.*

We obtain a category G**Born** of G-bornological spaces and proper equivariant maps.
Let X be a G-set equipped with a G-coarse structure $\mathcal{C}$ and a G-bornology $\mathcal{B}$.

Definition 3.8. *The coarse structure $\mathcal{C}$ and the bornology $\mathcal{B}$ are said to be compatible if for every B in $\mathcal{B}$ and U in $\mathcal{C}$ we have $U[B] \in \mathcal{B}$.*

Definition 3.9. *A G-bornological coarse space is a triple $(X, \mathcal{C}, \mathcal{B})$ consisting of a G-set X, a G-coarse structure $\mathcal{C}$ and a G-bornology $\mathcal{B}$ on X, such that $\mathcal{C}$ and $\mathcal{B}$ are compatible.*

Usually we will denote a G-bornological coarse space by the symbol X and write $\mathcal{B}(X)$ and $\mathcal{C}(X)$ for its bornology and coarse structures.

Definition 3.10. *A morphism $f : X \to X'$ between G-bornological coarse spaces is an equivariant map of the underlying G-sets that is controlled and proper.*

We obtain a category $G\mathbf{BornCoarse}$ of G-bornological coarse spaces and morphisms.
Next we describe the symmetric monoidal structure on $G\mathbf{BornCoarse}$ [4, Ex. 2.17]. We have a forgetful functor

$$U : G\mathbf{BornCoarse} \to G\mathbf{Set}$$

which associates to every G-bornological coarse space X its underlying G-set. This functor is faithful. The category $G\mathbf{Set}$ is endowed with the cartesian symmetric monoidal structure. The symmetric monoidal structure on $G\mathbf{BornCoarse}$ will be defined in such a way that the functor U preserves the unit and the tensor product strictly, i.e., the morphisms 1 and 2 in Definition 3.2 are identities. In other words, the associator, unit and symmetry constraints are imported from $G\mathbf{Set}$ and satisfy the relations required in Definition 3.1 automatically.
We start with the description of the bifunctor

$$- \otimes_{G\mathbf{BornCoarse}} - : G\mathbf{BornCoarse} \times G\mathbf{BornCoarse} \to G\mathbf{BornCoarse} .$$

Let X and X' be two G-bornological coarse spaces. Then their tensor product

$$X \otimes_{G\mathbf{BornCoarse}} X'$$

is the G-bornological coarse spaces defined as follows:

1. The underlying G-set of $X \otimes_{G\mathbf{BornCoarse}} X'$ is the cartesian product of the underlying G-sets $X \times X'$.
2. The G-bornology on $X \times X'$ is generated by the subsets $B \times B'$ for all B in $\mathcal{B}(X)$ and B' in $\mathcal{B}(X')$.
3. The G-coarse structure on $X \times X'$ is generated by the entourages $U \times U'$ for U in $\mathcal{C}(X)$ and U' in $\mathcal{C}(X')$.

Here a G-bornological (or coarse, respectivley) structure generated by a family of subsets (or entourages) is the minimal G-bornological (or G-coarse) structure containing these subsets (or entourages). Note that the underlying G-coarse space of the tensor product represents the cartesian product of the underlying G-coarse spaces of the factors in $G\mathbf{Coarse}$, but the tensor product is not the cartesian product in $G\mathbf{BornCoarse}$ in general.
From now on we will use the shorter notation $X \otimes X'$ for the tensor product of G-bornological coarse spaces, i.e., we omit the subscript $G\mathbf{BornCoarse}$.
If $f : X \to Y$ and $f' : X' \to Y'$ are morphisms of G-bornological coarse spaces, then their tensor product

$$f \otimes f' : X \otimes Y \to X' \otimes Y'$$

is induced by the equivariant map of underlying G-sets $(x, y) \mapsto (f(x), f(y))$. This finishes the description of the bifunctor 3.1.1
The tensor unit $1_{G\mathbf{BornCoarse}}$ 3.1.2 is given by the one-point space $*$.
As explained above, the associativity, unit and symmetry constraints are imported from $G\mathbf{Set}$. It is straightforward to check that they are implemented by morphisms of G-bornological coarse spaces.
This finishes the description of the symmetric monoidal structure $\otimes$ on the category $G\mathbf{BornCoarse}$.

3.3 Controlled objects In this section, for every additive category $\mathbf{A}$ with a strict G-action, we describe the functor

$$\mathbf{V}^G_{\mathbf{A}} : G\mathbf{BornCoarse} \to \mathbf{Add}_1$$

which sends a G-bornological coarse space X to its additive category $\mathbf{V}^G_{\mathbf{A}}(X)$ of equivariant X-controlled $\mathbf{A}$-objects [4, Sec. 8.2].
For a group G, let BG be the category with one object $*$ and $\mathtt{End}_{BG}(*) \cong G$. Then $\mathbf{Fun}(BG, \mathbf{Add}_1)$ is the category of additive categories with a strict G-action. Explicitly, an additive category with a strict G-action is an additive category $\mathbf{A}$ (the evaluation of the functor at the object $*$ in BG) together with an action of G on $\mathbf{A}$ by exact functors, which is strictly associative. Our notation for the action of g in G on objects A of $\mathbf{A}$ and morphisms f is

$$(g, A) \mapsto gA \ , \quad (g, f : A \to A') \mapsto (gf : gA \to gA') \ .$$

Let $\mathbf{A}$ be an additive category with a strict G-action and X be a G-bornological coarse space. We consider the bornology $\mathcal{B}(X)$ of X as a poset with a G-action $(g, B) \mapsto gB$, hence as a category with a strict G-action, i.e., an object of $\mathbf{Fun}(BG, \mathbf{Cat}_1)$.
The category $\mathbf{Fun}(\mathcal{B}(X), \mathbf{A})$ has an induced G-action which can explicitly be described as follows. If $M : \mathcal{B} \to \mathbf{A}$ is a functor and g is an element of G, then $gM \colon \mathcal{B}(X) \to \mathbf{A}$ is the functor which sends a bounded set B in $\mathcal{B}(X)$ to the object $gM(g^{-1}(B))$ of $\mathbf{A}$. If $\rho : M \to M'$ is a natural transformation between two such functors, then we let $g\rho : gM \to gM'$ denote the canonically induced natural transformation.

Definition 3.11. *[4, Def. 8.3] An equivariant X-controlled $\mathbf{A}$-object is a pair (M, ρ) consisting of a functor $M \colon \mathcal{B}(X) \to \mathbf{A}$ and a family $\rho = (\rho(g))_{g\in G}$ of natural isomorphisms $\rho(g) \colon M \to gM$ satisfying the following conditions:*

1. *$M(\emptyset) \cong 0$.*
2. *For all B, B' in $\mathcal{B}(X)$, the commutative square*

$$\begin{array}{ccc} M(B \cap B') & \longrightarrow & M(B) \\ \downarrow & & \downarrow \\ M(B') & \longrightarrow & M(B \cup B') \end{array}$$

   is a pushout square.
3. *For all B in $\mathcal{B}(X)$ there exists a finite subset F of B such that the inclusion $F \to B$ induces an isomorphism $M(F) \xrightarrow{\cong} M(B)$.*
4. *For all pairs of elements g, g' of G we have the relation $\rho(gg') = g\rho(g') \circ \rho(g)$.*

If U is an invariant coarse entourage of X, i.e., an element of $\mathcal{C}(X)^G$, then we get a G-equivariant functor

$$U[-] : \mathcal{B}(X) \to \mathcal{B}(X)$$

which sends a bounded subset B of X to its U-thickening $U[B]$ as defined in Eq. (3.1). Indeed, the U-thickening $U[B]$ of a bounded subset B is again bounded by the compatibility of the coarse structure $\mathcal{C}(X)$ and the bornology $\mathcal{B}(X)$, and $U[-]$ preserves the inclusion relation. Since U is G-invariant we have the equality $U[gB] = g(U[B])$. It implies that $U[-]$ is G-equivariant. If $M : \mathcal{B}(X) \to \mathbf{A}$ is a functor, then we write $U[-]^*M := M \circ U[-]$ for the pull-back of M along $U[-]$.

Let $(M, \rho), (M', \rho')$ be two equivariant X-controlled $\mathbf{A}$-objects and U be an invariant coarse entourage of X.

Definition 3.12. *An equivariant U-controlled morphism $\phi\colon (M, \rho) \to (M', \rho')$ is a natural transformation*

$$\phi\colon M \to U[-]^*M' \ ,$$

such that $\rho'(g) \circ \phi = (g\phi) \circ \rho(g)$ for all elements g of G.

We let $\mathrm{Mor}_U((M, \rho), (M', \rho'))$ denote the abelian group of equivariant U-controlled morphisms. If U' is in $\mathcal{C}(X)^G$ such that $U \subseteq U'$, then for every B in $\mathcal{B}(X)$ we have $U[B] \subseteq U'[B]$. These inclusions induce a transformation between functors $U[-]^*M' \to U'[-]^*M'$ and therefore a map

$$\mathrm{Mor}_U((M, \rho), (M', \rho')) \to \mathrm{Mor}_{U'}((M, \rho), (M', \rho'))$$

by postcomposition. Using these maps in the interpretation of the colimit we define the abelian group of equivariant controlled morphisms from (M, ρ) to (M', ρ') by

$$\mathtt{Hom}_{\mathbf{V}^G_{\mathbf{A}}(X)}((M, \rho), (M', \rho')) := \mathop{\mathtt{colim}}_{U \in \mathcal{C}(X)^G} \mathrm{Mor}_U((M, \rho), (M', \rho')) \ .$$

We now consider a morphism in $\mathtt{Hom}_{\mathbf{V}^G_{\mathbf{A}}(X)}((M, \rho), (M', \rho'))$ represented by $\phi : M \to U[-]^*M'$, and a morphism in $\mathtt{Hom}_{\mathbf{V}^G_{\mathbf{A}}(X)}((M', \rho'), (M'', \rho''))$ represented by $\phi' : M' \to U'[-]^*M''$. Their composition in $\mathtt{Hom}_{\mathbf{V}^G_{\mathbf{A}}(X)}((M, \rho), (M'', \rho''))$ is then represented by the morphism

$$U[-]^*\phi' \circ \phi : M \to (U' \circ U)[-]^*M'' \ ,$$

where

$$U[-]^*\phi' : U[-]^*M' \to (U' \circ U)[-]^*M''$$

is defined in the canonical manner. We denote the resulting category of equivariant X-controlled $\mathbf{A}$-objects and equivariant controlled morphisms by $\mathbf{V}^G_{\mathbf{A}}(X)$. This category is additive [4, Lemma 8.7].

Let $f\colon X \to X'$ be a morphism of G-bornological coarse spaces, and let (M, ρ) be an equivariant X-controlled $\mathbf{A}$-object. Since f is proper, it induces an equivariant functor $f^{-1} : \mathcal{B}(X') \to \mathcal{B}(X)$, and we can define a functor $f_*M\colon \mathcal{B}(X') \to \mathbf{A}$ by

$$f_*M := M \circ f^{-1} \ .$$

Furthermore, we define

$$f_*\rho(g) := \rho(g) \circ f^{-1} \ .$$

Let U be in $\mathcal{C}(X)^G$ and let $\phi\colon (M,\rho) \to (M',\rho')$ be an equivariant U-controlled morphism. Then $V := (f \times f)(U)$ belongs to $\mathcal{C}(X')^G$ and we have $U[f^{-1}(B')] \subseteq f^{-1}(V[B'])$ for all bounded subsets B' of X'. Therefore, we obtain an induced V-controlled morphism

$$f_*\phi = \{f_*M(B') \xrightarrow{\phi_{f^{-1}(B')}} M(U[f^{-1}(B')]) \to f_*M(V[B'])\}_{B' \in \mathcal{B}(X')} \ .$$

One checks that this construction defines an additive functor

$$f_*\colon \mathbf{V}^G_{\mathbf{A}}(X) \to \mathbf{V}^G_{\mathbf{A}}(X') \ .$$

This completes the construction of the functor

$$\mathbf{V}^G_{\mathbf{A}}\colon G\mathbf{BornCoarse} \to \mathbf{Add}_1 \ . \tag{3.3}$$

In the following we give a more explicit description of the objects and morphisms in $\mathbf{V}^G_{\mathbf{A}}(X)$ which will be used in the description of the symmetric monoidal structure on the Grothendieck construction associated to the functor $\mathbf{V}^G_{\mathbf{A}}$ in Section 3.4.

Convention 3.13. We consider an additive category $\mathbf{A}$. If $(A_i)_{i\in I}$ is a family of objects of $\mathbf{A}$ with at most finitely many non-zero members, then we use the symbol $\bigoplus_{i\in I} A_i$ in order to denote a choice of an object of $\mathbf{A}$ together with a family of morphisms $(A_j \to \bigoplus_{i\in I} A_i)_{j\in I}$ representing the coproduct of the family.
Since in an additive category coproducts and products coincide, for every j in I we furthermore have a canonical projection

$$\bigoplus_{i\in I} A_i \to A_j$$

such that the diagram

$$\begin{tikzcd} A_j \arrow[r] \arrow[rr, bend left, "\mathtt{id}_{A_j}"] & \bigoplus_{i\in I} A_i \arrow[r] & A_j \end{tikzcd}$$

commutes.
If $(A'_{i'})_{i'\in I'}$ is a second family of this type and $(\phi_{i,i'} : A'_{i'} \to A_i)_{i'\in I', i\in I}$ is a family of morphisms in $\mathbf{A}$, then we have a unique morphism $\oplus\phi_{i,i'}$ such that the squares

$$\begin{tikzcd} A'_{i'} \arrow[r, "\phi_{i,i'}"] \arrow[d] & A_i \arrow[d] \\ \bigoplus_{i'\in I'} A'_{i'} \arrow[r, "\oplus\phi_{i,i'}"] & \bigoplus_{i\in I} A_i \end{tikzcd} \tag{3.4}$$

commute for every i' in I' and i in I. □

Let $\mathbf{A}$ be a small additive category with strict G-action. Let X be a G-bornological coarse space (see Definition 3.9), and let (M,ρ) be an equivariant X-controlled $\mathbf{A}$-object (see Definition 3.11). Let B be in $\mathcal{B}(X)$ and x be a point in B. The inclusion $\{x\} \to B$ induces a morphism

$M(\{x\}) \to M(B)$ in $\mathbf{A}$. The conditions 3.11.1, 3.11.2 and 3.11.3 together imply that $M(\{x\}) = 0$ for all but finitely many points of B, and that the canonical morphism (induced by the universal property of the coproduct in $\mathbf{A}$)

$$\bigoplus_{x\in B} M(\{x\}) \xrightarrow{\cong} M(B) \tag{3.5}$$

is an isomorphism.
Let now U be in $\mathcal{C}(X)^G$, and let $\phi : (M,\rho) \to (M',\rho')$ be an equivariant U-controlled morphism. By Definition 3.12, the morphism ϕ is given by a natural transformation of functors $\phi : M \to U[-]^*M'$ satisfying an equivariance condition. For every point x in X we get a morphism

$$M(\{x\}) \to M'(U[\{x\}]) \overset{(3.5)}{\cong} \bigoplus_{x'\in U[\{x\}]} M(\{x'\}) \tag{3.6}$$

in $\mathbf{A}$. We let

$$\phi_{x',x} : M(\{x\}) \to M'(\{x'\}) \tag{3.7}$$

denote the composition of (3.6) with the projection onto the summand corresponding to x'. In this way we get a family of morphisms $(\phi_{x',x})_{x',x\in X}$ in $\mathbf{A}$. In a similar manner, for g in G, the transformation $\rho(g) : M \to gM$ gives rise to a family of morphisms

$$\big(\rho(g)_x : M(\{x\}) \to gM(\{g^{-1}x\})\big)_{x\in X}\ . \tag{3.8}$$

By construction the family $(\phi_{x',x})_{x',x\in X}$ satisfies the following conditions:

1. For all x, x' in X the condition $\phi_{x',x} \neq 0$ implies that $(x',x) \in U$.
2. We have $\rho'(g)_{x'} \circ \phi_{x',x} = (g\phi)_{g^{-1}x',g^{-1}x} \circ \rho(g)_x$ for all x, x' in X and g in G.

Lemma 3.14. *We have a bijection between equivariant U-controlled morphisms $\phi : (M,\rho) \to (M',\rho')$ and families $(\phi_{x',x})_{x',x\in X}$ of morphisms as in (3.7) satisfying Conditions 1 and 2.*

Proof. Let (M,ρ) and (M',ρ') be in $\mathbf{V}^G_{\mathbf{A}}(X)$. We must show that a matrix $(\phi_{x',x})_{x',x\in X}$ of morphisms as in (3.7) which satisfies Conditions 1 and 2 gives rise to an equivariant controlled morphism $\phi : (M,\rho) \to (M',\rho')$. Let U be in $\mathcal{C}^G(X)$ such that Condition 1 holds true. We must construct an equivariant natural transformation $\phi : M \to U[-]^*M'$.
We consider B in $\mathcal{B}(X)$. Then $(M(\{x\})_{x\in B}$ and $(M'(\{x'\}))_{x'\in U[B]}$ are families of objects in $\mathbf{A}$ with at most finitely many non-zero members. Using Convention 3.13, and in particular the notation from (3.4), we can define the morphism $\phi_B : M(B) \to M'(U[B])$ such that the diagram

$$\begin{array}{ccc} \bigoplus_{x\in B} M(\{x\}) & \xrightarrow{\oplus\phi_{x',x}} & \bigoplus_{x'\in U[B]} M'(\{x'\}) \\ \cong\big\downarrow (3.5) & & \cong\big\downarrow (3.5) \\ M(B) & \xrightarrow{\phi_B} & M'(U[B]) \end{array}$$

commutes. It is now straightforward to check that the family $(\phi_B)_{B\in\mathcal{B}(X)}$ assembles to a natural transformation $\phi : M \to U[-]^*M'$ as required. By construction the morphism ϕ is U-controlled. Furthermore, Condition 2 implies that ϕ satisfies the equivariance condition stated in Definition 3.12. □

Let $f : X_0 \to X_1$ be a morphism of G-bornological coarse spaces and (M_i, ρ_i) be objects of $\mathbf{V}^G_{\mathbf{A}}(X_i)$ for $i = 0, 1$. Then a morphism

$$\phi : f_*(M_0, \rho_0) \to (M_1, \rho_0) \tag{3.9}$$

induces a matrix

$$\left(\phi^f_{x_1,x_0} : M_0(\{x_0\}) \to M_1(\{x_1\})\right)_{x_0 \in X_0, x_1 \in X_1} . \tag{3.10}$$

To this end we observe that

$$(f_* M_0)(\{x_1'\}) = M_0(f^{-1}(\{x_1'\})) \overset{(3.5)}{\cong} \bigoplus_{x_0 \in f^{-1}(\{x_1'\})} M_0(\{x_0\})$$

so that

$$\left(\phi_{x_1,x_1'} := \oplus_{x_0 \in f^{-1}(\{x_1'\})} \phi^f_{x_0,x_1} : M_0(f^{-1}(\{x_1'\})) \cong \bigoplus_{x_0 \in f^{-1}(\{x_1'\})} M_0(\{x_0\}) \to M_1(\{x_1\})\right)_{x_1', x_1 \in X_1}$$

is the matrix representing ϕ according to Lemma 3.14. As a consequence of Lemma 3.14 we obtain:

Corollary 3.15. *A matrix (3.10) represents a morphism (3.9) iff the following conditions are satisfied:*

1. *There exists an entourage U_1 in $\mathcal{C}(X_1)$ such that for every x_0 in X_0 and x_1 in X_1 the condition $\phi^f_{x_1,x_0} \neq 0$ implies that $(x_1, f(x_0)) \in U_1$.*
2. *For every g in G we have the equality*

$$\rho_1(g)_{x_1} \circ \phi^f_{x_1,x_0} = (g\phi^f)_{g^{-1}x_1, g^{-1}x_0} \circ \rho(g)_{x_0} .$$

3.4 The symmetric monoidal refinement of $\mathbf{V}^G_{\mathbf{A}}$ Let $\mathbf{A}$ be a small additive category with a strict G-action. Then we let

$$\pi : \mathcal{V}^G_{\mathbf{A}} \to G\mathbf{BornCoarse}$$

denote the Grothendieck construction associated to the functor $\mathbf{V}^G_{\mathbf{A}}$ in Eq. (3.3) viewed as a functor from $G\mathbf{BornCoarse}$ to $\mathbf{Cat}_1$. The goal of this section is the construction of a symmetric monoidal structure (see Definition 3.1) on $\mathcal{V}^G_{\mathbf{A}}$ that satisfies the assumptions of Theorem 2.3.

Assumption 3.16. *We assume that $\mathbf{A}$ has a symmetric monoidal structure and that the strict action of G on $\mathbf{A}$ has a refinement to an action by symmetric monoidal functors.*

In order to introduce the notation for later arguments, we spell out the Assumption 3.16 explicitly. According to Definition 3.1 the category $\mathbf{A}$ comes with the following data:

1. a bifunctor $- \otimes_{\mathbf{A}} -$,
2. a tensor unit $1_{\mathbf{A}}$,
3. an associativity constraint $\alpha^{\mathbf{A}}$,
4. a unit constraint $\eta^{\mathbf{A}}$,
5. a symmetry constraint $\sigma^{\mathbf{A}}$.

This data satisfy the relations named in Definition 3.1.
The strict action of G on $\mathbf{A}$ by symmetric monoidal functors is implemented by the following data. For every g in G we have:

1. an additive functor $g : \mathbf{A} \to \mathbf{A}$,
2. an isomorphism $\epsilon^g : 1_{\mathbf{A}} \to g1_{\mathbf{A}}$,
3. a natural isomorphism $\mu^g : (g - \otimes_{\mathbf{A}} g-) \to g(- \otimes_{\mathbf{A}} -)$,

satisfying the relations named in Definition 3.2. We require that for all g and h in G the following relation between the composition of symmetric monoidal functors and multiplication in G holds true:

$$(g, \epsilon^g, \mu^g) \circ (h, \epsilon^h, \mu^h) = (gh, \epsilon^{gh}, \mu^{gh}) \ . \tag{3.11}$$

Note that the word *strict* refers to the associativity of the G-action reflected by the equality sign in Eq. (3.11). We do not require that that symmetric monoidal structure is preserved strictly.
We now describe the category $\mathcal{V}^G_{\mathbf{A}}$ explicitly.

1. The objects of $\mathcal{V}^G_{\mathbf{A}}$ are pairs $(X, (M, \rho))$ of objects X in $G\mathbf{BornCoarse}$ and (M, ρ) in $\mathbf{V}^G_{\mathbf{A}}(X)$.
2. A morphism $(f, \phi) : (X, (M, \rho)) \to (X', (M', \rho'))$ consists of a morphism $f : X \to X'$ in $G\mathbf{BornCoarse}$ and a morphism $\phi : f_*(M, \rho) \to (M', \rho')$ in $\mathbf{V}^G_{\mathbf{A}}(X')$.
3. The composition of morphisms is given by
$$(f', \phi') \circ (f, \phi) := (f' \circ f, \phi' \circ f'_*(\phi)) \ .$$

The functor

$$\pi : \mathcal{V}^G_{\mathbf{A}} \to G\mathbf{BornCoarse} \ , \quad (X, (M, \rho)) \mapsto X$$

is the obvious functor which forgets the second component.
We now start with the description of the symmetric monoidal structure on $\mathcal{V}^G_{\mathbf{A}}$.
Let $*$ denote the one-point space. Then we can consider the equivariant $*$-controlled $\mathbf{A}$-object $1_* = (M^{unit}, \rho^{unit})$ in $\mathbf{V}^G_A(*)$ defined as follows:

1. The functor $M^{unit} : \mathcal{B}(*) \to \mathbf{A}$ is uniquely determined by $M^{unit}(\{*\}) := 1_{\mathbf{A}}$.
2. $\rho^{unit}(g) := \epsilon^g$ for all g in G.

Definition 3.17. *The tensor unit of $\mathcal{V}^G_{\mathbf{A}}$ is defined to be the object $1_{\mathcal{V}^G_{\mathbf{A}}} := (*, 1_*)$.*

In order to construct the bifunctor

$$- \otimes_{\mathcal{V}^G_{\mathbf{A}}} - : \mathcal{V}^G_{\mathbf{A}} \times \mathcal{V}^G_{\mathbf{A}} \to \mathcal{V}^G_{\mathbf{A}} \tag{3.12}$$

we start with its definition on objects. We consider two objects $(X, (M, \rho))$ and $(X', (M', \rho'))$ in $\mathcal{V}^G_{\mathbf{A}}$. Then we define the functor

$$M \boxtimes M' : \mathcal{B}(X \otimes X') \to \mathbf{A}$$

as follows:

1. For every B in $\mathcal{B}(X \otimes X')$ we set (see Convention 3.13)
$$(M \boxtimes M')(B) := \bigoplus_{(x,x') \in B} M(\{x\}) \otimes_{\mathbf{A}} M'(\{x'\}) \ .$$

   Note that the sum has finitely many non-zero summands because of Definition 3.11 (3).

2. If B' is in $\mathcal{B}(X \otimes X')$ such that $B' \subseteq B$, then the morphism

$$(M \boxtimes M')(B' \subseteq B) \colon (M \boxtimes M')(B') \to (M \boxtimes M')(B)$$

is given by the canonical map

$$\bigoplus_{(x,x') \in B'} M(\{x\}) \otimes_{\mathbf{A}} M'(\{x'\}) \to \bigoplus_{(x,x') \in B} M(\{x\}) \otimes_{\mathbf{A}} M'(\{x'\})$$

as described in Convention 3.13.

By using our Convention 3.13 and the universal property of the direct sum, one easily checks that this describes a functor satisfying the first three conditions of Definition 3.11.
We now define the family $\rho \boxtimes \rho'$, by defining $(\rho \boxtimes \rho')(g)_B$ as follows:

$$\bigoplus_{(x,x') \in B} M(\{x\}) \otimes_{\mathbf{A}} M'(\{x'\}) \xrightarrow{\oplus_{(x,x') \in B} \rho(g)_x \otimes \rho'(g)_{x'}} \bigoplus_{(x,x') \in B} gM(\{g^{-1}x\}) \otimes_{\mathbf{A}} gM'(\{g^{-1}x'\})$$

using the notation (3.8). One checks using Eq. (3.11) that $(M \boxtimes M', \rho \boxtimes \rho')$ satisfies the remaining condition of Definition 3.11 and therefore belongs to $\mathbf{V}^G_{\mathbf{A}}(X \otimes X')$.

Definition 3.18. *We define the bifunctor* (3.12) *on objects by*

$$(X, (M, \rho)) \otimes_{\mathcal{V}^G_{\mathbf{A}}} (X', (M', \rho')) := (X \otimes X', (M \boxtimes M', \rho \boxtimes \rho')) \ .$$

Let $(f, \phi) : (X_0, (M_0, \rho_0)) \to (X_1, (M_1, \rho_1))$ be a morphism in $\mathcal{V}^G_{\mathbf{A}}$. Then we define the morphism

$$(g, \psi) := (f, \phi) \otimes_{\mathcal{V}^G_{\mathbf{A}}} (X', (M', \rho')) : (X_0 \otimes X', (M_0 \boxtimes M', \rho_0 \boxtimes \rho')) \to (X_1 \otimes X', (M_1 \boxtimes M', \rho_1 \boxtimes \rho'))$$

as follows.

1. We set $g := f \otimes \mathtt{id}_{X'} : X_0 \otimes X' \to X_1 \otimes X'$ using the tensor product in $G\mathbf{BornCoarse}$.
2. In order to describe the morphism

$$\psi : (f \otimes \mathtt{id}_{X'})_*(M_0 \boxtimes M', \rho_0 \boxtimes \rho') \to (M_1 \boxtimes M', \rho_1 \boxtimes \rho')$$

we use Corollary 3.15. We must describe the matrix

$$(\psi^{f \otimes \mathtt{id}_{X'}}_{(x_1,y'),(x_0,x')})_{(x_0,x') \in X_0 \times X', (x_1,y') \in X_1 \times X'} \ .$$

Now note that by definition

$$(M_0 \boxtimes M')(\{x_0, x'\}) \cong M_0(\{x_0\}) \otimes_{\mathbf{A}} M'(\{x'\})$$

so that we can set

$$\psi^{f \otimes \mathtt{id}_{X'}}_{(x_1,y'),(x_0,x')} := \phi^f_{x_1,x_0} \otimes_{\mathbf{A}} (\mathtt{id}_{(M',\rho')})_{y',x'} : M_0(\{x_0\}) \otimes_{\mathbf{A}} M'(\{x'\}) \to M_1(\{x_1\}) \otimes_{\mathbf{A}} M'(\{y'\}) \ . \tag{3.13}$$

One easily checks that this matrix satisfies the conditions listed in Corollary 3.15 and therefore represents the desired morphism.

In a similar manner we define $(X, (M, \rho)) \otimes (f', \phi')$ for a morphism $(f', \phi') : (X'_0, (M'_0, \rho'_0)) \to (X'_1, (M'_1, \rho_1))$.

Definition 3.19. *We define the bifunctor* $(- \otimes_{\mathcal{V}^G_{\mathbf{A}}} -)$ (3.12) *on morphisms by the preceding description.*

It is straightforward to check that $(- \otimes_{\mathcal{V}^G_{\mathbf{A}}} -)$ (3.12) is a bifunctor, i.e., that its description on morphisms is compatible with composition.
Next we define the associativity constraint $\alpha^{\mathcal{V}^G_{\mathbf{A}}}$. We consider three objects $(X,(M,\rho))$, $(X',(M',\rho'))$, and $(X'',(M'',\rho''))$. Then

$$(f,\phi) := \alpha_{(X,(M,\rho)),(X',(M',\rho')),(X'',(M'',\rho''))}$$

must be a morphism

$$((X\otimes X')\otimes X'',((M\boxtimes M')\boxtimes M'',(\rho\boxtimes\rho')\boxtimes\rho'')) \to (X\otimes(X'\otimes X''),(M\boxtimes(M'\boxtimes M''),\rho\boxtimes(\rho'\boxtimes\rho'')))$$

We set

$$f := \alpha_{X,X',X''}$$

using the associativity constraint of G**BornCoarse**. The second component ϕ is given via Corollary 3.15 by the matrix whose only non-trivial entries are

$$\phi^f_{(x,(x',x'')),((x,x'),x'')} := \alpha_{M(\{x\}),M'(\{x'\}),M''(\{x''\})}$$

using the associativity constraint of $\mathbf{A}$. The first condition of Corollary 3.15 is satisfied for the diagonal entourage of $X\times(X'\times X'')$, and for the second condition we use that G acts on $\mathbf{A}$ by symmetric monoidal functors, in particular the relation 3.2.1 for μ^g for all g in G.

Definition 3.20. *We define the associativity constraint* $\alpha^{\mathcal{V}^G_{\mathbf{A}}}$ *by the description above.*

It is straightforward to check that $\alpha^{\mathcal{V}^G_{\mathbf{A}}}$ is a natural transformation.
Following Definition 3.17 the unit constraint $\eta^{\mathcal{V}^G_{\mathbf{A}}}$ of $\mathcal{V}^G_{\mathbf{A}}$ is implemented by morphisms

$$(f,\phi) : (*\otimes X,(M^{unit}\boxtimes M,\rho^{unit}\boxtimes\rho)) \to (X,(M,\rho))$$

for all objects $(X,(M,\rho))$ of $\mathcal{V}^G_{\mathbf{A}}$. We set

$$f := \eta_X$$

using the unit constraint of G**BornCoarse**. Note that

$$(M^{unit}\boxtimes M)(\{(*,x)\}) \cong 1_{\mathbf{A}} \otimes_{\mathbf{A}} M(\{x\}) \ .$$

Hence, using Corollary 3.15, we can define morphism ϕ such that the non-trivial entries of its matrix are

$$\phi^f_{x,(*,x)} := \eta_{M(\{x\})}$$

using the unit constraint of $\mathbf{A}$. It is easy to check that this matrix satisfies the first condition of Corollary 3.15 for the diagonal of X and the second condition since the morphisms ϵ^g satisfy the relation 3.2.2 for all g in G.

Definition 3.21. *We define the unit constraint* $\eta^{\mathcal{V}^G_{\mathbf{A}}}$ *by the description above.*

It is straightforward to check that $\eta^{\mathcal{V}^G_{\mathbf{A}}}$ is a natural transformation.
As last, we define the symmetry constraint $\sigma^{\mathcal{V}^G_{\mathbf{A}}}$. We consider two objects $(X,(M,\rho))$ and $(X',(M',\rho'))$ of $\mathcal{V}^G_{\mathbf{A}}$. Then we must define a morphism

$$(f,\phi):(X\otimes X',(M\boxtimes M',\rho\boxtimes\rho'))\to(X'\otimes X,(M'\boxtimes M,\rho'\boxtimes\rho))\ .$$

We set

$$f:=\sigma_{X,X'}$$

using the symmetry contraint for $G\mathbf{BornCoarse}$. The morphism ϕ is the given, using Corollary 3.15, by the matrix whose only non-trivial entries are

$$\phi^f_{(x',x),(x,x')}:=\sigma_{M(\{x\}),M'(\{x'\})}$$

using the symmetry constraint of $\mathbf{A}$. One easily checks that the first condition of Corollary 3.15 is satisfied for the diagonal entourage of $X'\times X$. In order to verify the second condition we use that the transformations μ^g satisfy the relation 3.2.3 for every g in G.

Definition 3.22. *We define the symmetry constraint $\sigma^{\mathcal{V}^G_{\mathbf{A}}}$ of $\mathcal{V}^G_{\mathbf{A}}$ by the description above.*

It is straightforward to check that $\sigma^{\mathcal{V}^G_{\mathbf{A}}}$ is a natural transformation.

Proposition 3.23. *The functor $-\otimes_{\mathcal{V}^G_{\mathbf{A}}}-$ and the object $1_{\mathcal{V}^G_{\mathbf{A}}}$ together with the natural isomorphisms $\alpha^{\mathcal{V}^G_{\mathbf{A}}}$, $\eta^{\mathcal{V}^G_{\mathbf{A}}}$ and $\sigma^{\mathcal{V}^G_{\mathbf{A}}}$ define a symmetric monoidal structure on $\mathcal{V}^G_{\mathbf{A}}$.*
The functor $\pi:\mathcal{V}^G_{\mathbf{A}}\to G\mathbf{BornCoarse}$ preserves the tensor product and the tensor unit as well as the associator, unit, and symmetry transformations.

Proof. One verifies the relations listed in Definition 3.1 in a straightforward manner by inserting the definitions and using that the corresponding relations are satisfied for the symmetric monoidal structures on $\mathbf{A}$ and $G\mathbf{BornCoarse}$. □

Let X and X' be G-bornological coarse spaces.

Proposition 3.24. *The functor*

$$\boxtimes_{X,X'}:\mathbf{V}^G_{\mathbf{A}}(X)\times\mathbf{V}^G_{\mathbf{A}}(X')\to\mathbf{V}^G_{\mathbf{A}}(X\otimes X')$$

obtained in (2.3) is additive in both variables.

Proof. Let (M_i,ρ_i) be in $\mathbf{V}^G_{\mathbf{A}}(X)$ for $i=0,1$ and (M',ρ) be in $\mathbf{V}^G_{\mathbf{A}}(X')$. In view of the symmetry it suffices to show that the canonical morphism

$$(M_0\boxtimes_{X,X'}M')\oplus(M_1\boxtimes_{X,X'}M')\to(M_0\oplus M_1)\boxtimes_{X,X'}M'$$

is an isomorphism. In view of Conditions 3.11.2 and 3.11.3 it suffices to show that

$$\left[(M_0\boxtimes_{X,X'}M')\oplus(M_1\boxtimes_{X,X'}M')\right](\{(x,x')\})\to\left[(M_0\oplus M_1)\boxtimes_{X,X'}M'\right](\{(x,x')\})$$

is an isomorphism for every point (x,x') in $X\times X'$. By inserting the definitions we see that this morphism is the same as

$$(M_0(\{x\})\otimes_{\mathbf{A}}M'(\{x'\}))\oplus(M_1(\{x\})\otimes_{\mathbf{A}}M'(\{x'\}))\to(M_0(\{x\})\oplus M_1(\{x\}))\otimes_{\mathbf{A}}M'(\{x'\})\ .$$

But this last morphism is an isomorphism since the tensor product in $\mathbf{A}$ is additive in the first argument. □

Let $f\colon X \to X'$ and $f'\colon Y \to Y'$ be two morphisms of G-bornolgical coarse spaces. Let (M,ρ) be in $\mathbf{V}^G_{\mathbf{A}}(X)$ and (N,η) be in $\mathbf{V}^G_{A}(Y)$.

Lemma 3.25. *The morphism*

$$(f \otimes f')_*((M,\rho) \boxtimes_{X,Y} (N,\eta)) \to f_*(M,\rho) \boxtimes_{X',Y'} f'_*(N,\eta)$$

in $\mathbf{V}^G_{\mathbf{A}}(X' \otimes Y')$ *(see (2.4)) is an isomorphism.*

Proof. In view of Conditions 3.11.2 and 3.11.3 it suffices to show that

$$[(f \otimes f')_*((M,\rho) \boxtimes_{X,Y} (N,\eta))](\{(x',y')\}) \to [f_*(M,\rho) \boxtimes_{X',Y'} f'_*(N,\eta)](\{(x',y')\})$$

is an isomorphism for every point (x',y') in $X' \times Y'$. Inserting the definitions this morphism is given by

$$\bigoplus_{(x,y)\in(f\times f')^{-1}(\{x',y'\})} M(\{x\}) \otimes_{\mathbf{A}} N(\{y\}) \to \left(\bigoplus_{x\in f^{-1}(\{x'\})} M(\{x\})\right) \otimes_{\mathbf{A}} \left(\bigoplus_{y\in(f')^{-1}(\{y'\})} N(\{y\})\right) \tag{3.14}$$

which for every (x,y) in $(f\times f')^{-1}(\{x',y'\})$ is the morphism

$$M(\{x\}) \otimes_{\mathbf{A}} N(\{y\}) \to \left(\bigoplus_{x\in f^{-1}(\{x'\})} M(\{x\})\right) \otimes_{\mathbf{A}} \left(\bigoplus_{y\in(f')^{-1}(\{y'\})} N(\{y\})\right)$$

induced by the inclusions of the respective summands of the tensor factors. Since the tensor product in $\mathbf{A}$ preserves sums in both arguments we conclude that the morphism in Eq. (3.14) is an isomorphism. □

In view of Theorem 2.3 the Propositions 3.23 and 3.24 and Lemma 3.25 now imply:

Theorem 3.26. *If* $\mathbf{A}$ *is a symmetric monoidal additive category with a strict action of* G *by symmetric monoidal functors, then the functor* $\mathbf{V}^G_{\mathbf{A}} : G\mathbf{BornCoarse} \to \mathbf{Add}_\infty$ *admits a refinement to a lax symmetric monoidal functor*

$$\mathbf{V}^{G,\otimes}_{\mathbf{A}} : \mathtt{N}(G\mathbf{BornCoarse}^{\otimes}) \to \mathbf{Add}^{\otimes}_\infty \ .$$

3.5 The symmetric monoidal K-theory functor for additive categories

In [2] a universal K-theory functor

$$\mathrm{UK} : \mathtt{N}(\mathbf{Add}_1) \to \mathcal{M}_{loc}$$

was considered, where $\mathcal{M}_{loc}$ is the category of non-commutative motives of Blumberg-Gepner-Tabuada [6]. This functor was defined as the upper horizontal composition in the diagram

$$\begin{array}{ccccc} \mathtt{N}(\mathbf{Add}_1) & \xrightarrow{\mathbf{Ch}^b(-)_\infty} & \mathbf{Cat}^{\mathrm{ex}}_\infty & \xrightarrow{\mathcal{U}_{loc}} & \mathcal{M}_{loc} \\ \downarrow{\scriptstyle \mathtt{Loc}} & & & \nearrow{\scriptstyle \mathrm{UK}_\infty} & \\ \mathbf{Add}_\infty & & & & \end{array} ,$$

where $\mathcal{U}_{loc}$ is the universal localizing invariant, and $\mathbf{Ch}^b(-)_\infty$ sends an additive category $\mathbf{A}$ to the stable ∞-category of bounded chain complexes over $\mathbf{A}$ with homotopy equivalences inverted. Since the functor UK preserves equivalences of additive categories we have the indicated factorization UK_∞.

Theorem 3.27. *The functor* UK_∞ *admits a lax symmetric monoidal refinement*

$$\mathrm{UK}_\infty^\otimes : \mathbf{Add}_\infty^\otimes \to \mathcal{M}_{loc}^\otimes \ .$$

Proof. The proof of this theorem will be finished at the end of the present section.

Let $\mathbf{Cat}_\infty^{\mathrm{perf}}$ be the full subcategory of $\mathbf{Cat}_\infty^{\mathrm{ex}}$ of idempotent complete small stable ∞-categories. By [7, Thm. 5.8] the restriction of the universal localizing invariant to $\mathbf{Cat}_\infty^{\mathrm{perf}}$ has a natural symmetric monoidal refinement. By precomposition with the symmetric monoidal refinement of the idempotent completion functor $\mathrm{Idem} : \mathbf{Cat}_\infty^{\mathrm{ex}} \to \mathbf{Cat}_\infty^{\mathrm{perf}}$ we obtain a symmetric monoidal refinement

$$\mathcal{U}_{loc}^\otimes : \mathbf{Cat}_\infty^{\mathrm{ex},\otimes} \to \mathcal{M}_{loc}^\otimes \ .$$

of the universal localizing invariant $\mathcal{U}_{loc} : \mathbf{Cat}_\infty^{\mathrm{ex}} \to \mathcal{M}_{loc}$ itself. It therefore remains to produce a lax symmetric monoidal functor

$$\mathrm{St}^\otimes : \mathtt{N}(\mathbf{Add}_1)^\otimes \to \mathbf{Cat}_\infty^{\mathrm{ex},\otimes}$$

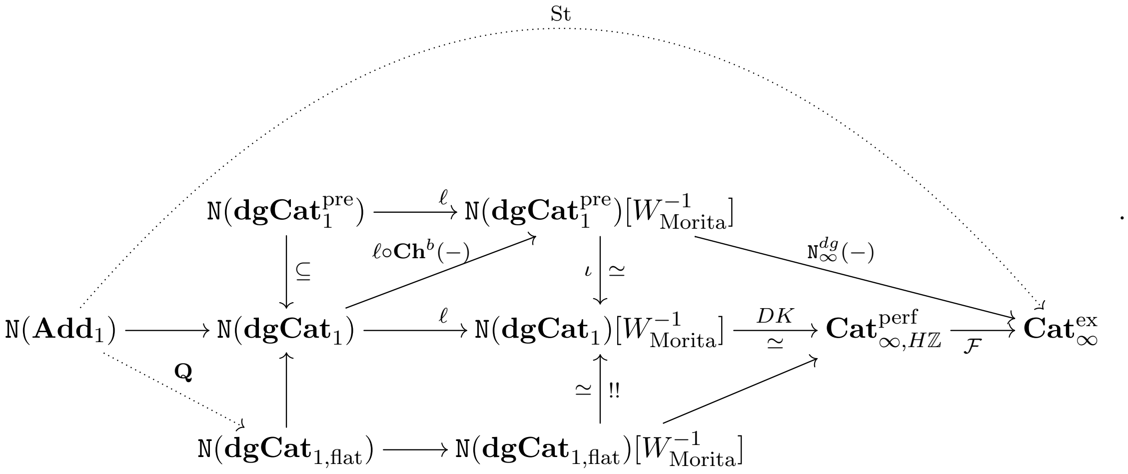
refining $\mathbf{Ch}^b(-)_\infty$. We use the symbol St in order to indicate that this functor is related with stabilization.

We are going to use the following notation. The category $\mathbf{dgCat}_1$ is the 1-category of small dg-categories. The set W_{Morita} is the set of Morita equivalences, i.e., functors between dg-categories $\mathcal{C} \to \mathcal{D}$ which induce an equivalence of derived categories [15, Sec. 4.6], [10, Def. 2.29].

The category $\mathbf{dgCat}_1$ contains the full subcategory $\mathbf{dgCat}_{1,\mathrm{flat}}$ of locally flat dg-categories, i.e., dg-categories $\mathbf{C}$ with the property that for every two objects C, C' in $\mathbf{C}$ the complex $\mathtt{Hom}_{\mathbf{C}}(C, C')$ consists of flat $\mathbb{Z}$-modules. It furthermore contains the full subcategory $\mathbf{dgCat}_1^{\mathrm{pre}}$ of pre-triangulated dg-categories [15, Sec. 4.5], [8, Sec. 3].

By $\mathbf{Cat}_{\infty,H\mathbb{Z}}^{\mathrm{perf}}$ we denote the ∞-category of $H\mathbb{Z}$-linear stable idempotent complete ∞-categories and $H\mathbb{Z}$-linear exact functors, and $\mathcal{F}$ forgets the $H\mathbb{Z}$-linear structure. For the equivalence marked by DK (for Dold-Kan) we refer to [10, Cor. 5.5].

Proposition 3.28. *We have the bold part of the following commuting diagram:*



(3.15)

Proof. 1. The functor $\mathbf{Ch}^b$ extends from additive categories to dg-categories, see e.g., [8, Sec. 1], where it is denoted by Pre$-$Tr. For every dg-category the canonical inclusion $\mathcal{C} \to \mathbf{Ch}^b(\mathcal{C})$ represents the pretriangulated hull [15, Sec. 4.5], [8, Sec. 3]. In particular, the functor $\mathbf{Ch}^b$ has values in pretriangulated dg-categories.

2. The two triangles in the corresponding square commute since for every dg-category $\mathcal{C}$ the canonical inclusion induces a Morita equivalence $\mathcal{C} \to \mathbf{Ch}^b(\mathcal{C})$. To this end we use that the inclusion of $\mathcal{C}$ into its triangulated hull is a Morita equivalence [15, Sec. 4.6].

3. The dg-nerve $\mathtt{N}^{dg} : \mathtt{N}(\mathbf{dgCat}_1^{\mathrm{pre}}) \to \mathbf{Cat}_\infty^{\mathrm{ex}}$ sends Morita equivalences to equivalences and therefore descends to $\mathtt{N}_\infty^{dg}$ as indicated.

4. We have an equivalence $\mathcal{F} \circ DK \circ \iota \simeq \mathtt{N}_\infty^{dg}$, [11, Prop. 3.18], [18, Rem. 3.1]. In order to provide more details we consider the functor $Z^0 : \mathbf{dgCat}_1^{\mathrm{pre}} \to \mathbf{Cat}_\infty$ which associates to a dg-category its underlying category (with $\mathtt{Hom}_{Z^0(\mathcal{C})}(A,B) = Z^0(\mathtt{Hom}_{\mathcal{C}}(A,B))$) considered as an ∞-category. We furthermore let $W_{\mathcal{C}}$ be the morphisms in $Z^0(\mathcal{C})$ which become isomorphisms in the homotopy category $H^0(\mathcal{C})$ (with $\mathtt{Hom}_{H^0(\mathcal{C})}(A,B) = H^0(\mathtt{Hom}_{\mathcal{C}}(A,B))$). Then both functors

$$Z^0(\mathcal{C}) \to \mathtt{N}_\infty^{dg}(\ell(\mathcal{C}))\ , \quad Z^0(\mathcal{C}) \to \mathcal{F}(DK(\iota(\ell(\mathcal{C}))))$$

present the localization $Z^0(\mathcal{C}) \to Z^0(\mathcal{C})[W_{\mathcal{C}}^{-1}]$.

□

The horizontal composition given by the middle row in (3.15) defines a functor St.

Lemma 3.29. *The functor* St *is equivalent to the functor* $\mathbf{Ch}^b(-)_\infty$ *constructed in [2, Prop. 2.11].*

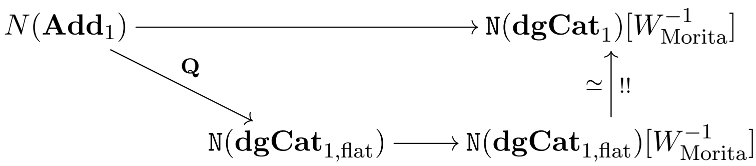
Proof. By [2, Rem. 2.9] we have the first equivalence of functors in the chain

$$\mathbf{Ch}^b(-)_\infty \simeq \mathtt{N}^{dg} \circ \mathbf{Ch}^b(-) \simeq \mathtt{N}_\infty^{dg} \circ \ell \circ \mathbf{Ch}^b$$

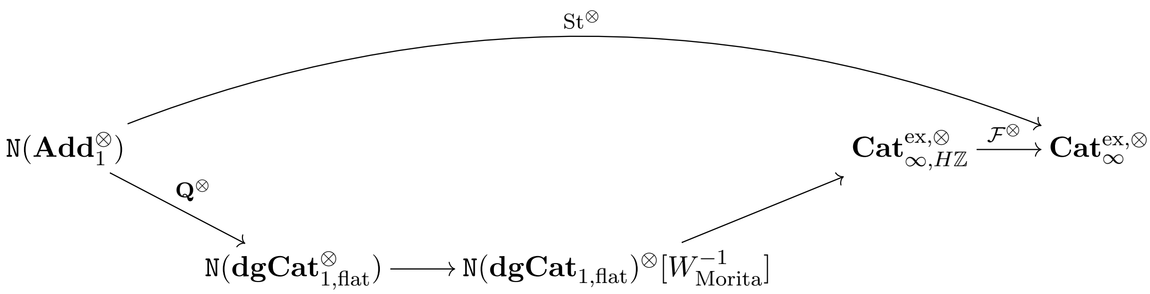
from $\mathbf{Add}_1$ to $\mathbf{Cat}_\infty^{\mathrm{ex}}$. This implies the Lemma in view of the commutativity of (3.15). □

Proposition 3.30. *The functor* St *has a lax symmetric monoidal refinement* $\mathrm{St}^{\otimes}$.

Proof. All 1-categories in the lower two lines of the diagram (3.15) have symmetric monoidal structures and the functors connecting them have canonical lax symmetric monoidal refinements. The same is true for the ∞-categories and the remaining functors except for $\mathtt{N}(\mathbf{dgCat}_1)[W_{\mathrm{Morita}}^{-1}]$ and the corresponding functors. The problem is that the tensor product of dg-categories is not compatible with Morita equivalences and therefore does not descend to the localization directly. For this reason one considers the subcategory of locally flat dg-categories and uses the equivalence !! in order to transfer the symmetric monoidal structures. So in order to construct the lax symmetric monoidal refinement of St we must bypass this node of the diagram. To this end we use a lax symmetric monoidal flat resolution functor $\mathbf{Q}$ as indicated. The left triangle in the diagram (3.15) is filled by a natural transformation (not an isomorphism), but the square



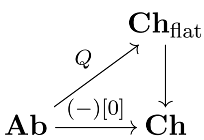
does commute. We then get the following commuting diagram of lax symmetric monoidal functors



defining the lax symmetric monoidal refinement $\mathrm{St}^{\otimes}$ of St.

It remains to argue that a lax symmetric monoidal flat resolution functor $\mathbf{Q}$ exists. We start with the following well-known fact.

Lemma 3.31. *1. There exists a functor Q fitting into the commuting diagram*



such that the filler is a quasi-isomorphism.

2. The functor Q has a lax symmetric monoidal structure.

Proof. The natural idea works. The functor Q sends A in $\mathbf{Ab}$ to

$$Q(A) := (F_1(A) \stackrel{d_A}{\to} F_0(A))$$

in $\mathbf{Ch}$, where $F_0(A) := \mathbb{Z}[A]$ is the free abelian group generated by the underlying set of A, $F_1(A)$ is the kernel of the canonical homomorphism $F_0(A) \to A$, and d_A is the inclusion.

The lax symmetric monoidal refinement of Q can then be defined in a straightforward manner. □

We define the flat resolution functor for additive categories by

$$\begin{array}{ccccc} & & & & \mathbf{dgCat}_{1,\mathrm{flat}} \\ & \overset{\mathbf{Q}}{\nearrow} & & \nearrow & \downarrow \\ \mathbf{Add}_1 & \longrightarrow & \mathbf{Cat}_{\mathbf{Ab}} & \longrightarrow & \mathbf{dgCat}_1 \end{array}$$

where the dotted arrow is the natural functor induced from the lax symmetric monoidal functor Q which provides a functor from **Ab**-enriched categories to **Ch**-enriched categories with flat `Hom`-complexes. Furthermore, the lax symmetric monoidal structure on Q induces naturally a lax symmetric monoidal structure on $\mathbf{Q}$. This concludes the proof of Proposition 3.30. □

This finishes the proof of Theorem 3.27. □

3.6 Functors on the orbit category For a group G, denote by $G\mathbf{Orb}$ the category of transitive G-sets and G-equivariant maps. Each object of $G\mathbf{Orb}$ is isomorphic to G/H for H subgroup of G, where G/H is endowed with the action of G by left multiplication.
Let $\mathbf{A}$ be an additive category. In view of [2, Def. 3.14] we have a universal K-theory functor

$$\underline{\mathrm{UK}\mathcal{X}_{\mathbf{A}}} : G\mathbf{BornCoarse} \to \mathbf{Fun}(G\mathbf{Orb}^{op}, \mathcal{M}_{loc}) \tag{3.16}$$

whose evaluation at G/H is equivalent to the composition

$$G\mathbf{BornCoarse} \overset{\mathrm{Res}^G_H}{\to} H\mathbf{BornCoarse} \xrightarrow{\mathrm{UK}\mathcal{X}^H_{\mathbf{A}}} \mathcal{M}_{loc} \ . \tag{3.17}$$

Proposition 3.32. *If $\mathbf{A}$ is symmetric monoidal, then the functor $\underline{\mathrm{UK}\mathcal{X}_{\mathbf{A}}}$ from (3.16) has a lax symmetric monoidal refinement such that its evaluations at the G/H for any subgroup H of G is equivalent to the lax symmetric refinement of the functor in (3.17) given by Theorem 1.1 applied to $\mathrm{UK}\mathcal{X}^H_{\mathbf{A}}$.*

Proof. The definition of the functor $\underline{\mathrm{UK}\mathcal{X}_{\mathbf{A}}}$ given in [2, Def. 3.14] uses marked additive categories, hence a first approach would be to extend the theory of the present paper from additive to marked additive categories as considered in [5]. Then the formula for $\underline{\mathrm{UK}\mathcal{X}_{\mathbf{A}}}$ given in [2, Def. 3.14] would yield the symmetric monoidal refinement in an obvious manner. In the following we apply a different idea which avoids markings and applies Theorem 3.26.
Let $j^G : BG \to G\mathbf{Orb}^{op}$ be the fully faithful inclusion sending the unique object of BG to the G-set G with the G-action by left multiplication. Note that right-multiplication identifies G with $\mathtt{Aut}_{G\mathbf{Orb}^{op}}(G)$. We furthermore have a fully faithful inclusion $G\mathbf{BornCoarse} \to \mathbf{Fun}(BG, \mathbf{BornCoarse})$. We post-compose with the functor $\mathbf{V}_{\mathbf{A}} : \mathbf{BornCoarse} \to \mathbf{Add}_\infty$ and then apply the right Kan-extension functor j^G_* along j^G. In this way we get the functor

$$\begin{aligned} \underline{\mathbf{V}_{\mathbf{A}}} : G\mathbf{BornCoarse} \to \mathbf{Fun}(BG, \mathbf{BornCoarse}) &\overset{\mathbf{V}_{\mathbf{A}}}{\to} \mathbf{Fun}(BG, \mathbf{Add}_\infty) \\ &\overset{j^G_*}{\to} \mathbf{Fun}(G\mathbf{Orb}^{op}, \mathbf{Add}_\infty) \ . \end{aligned}$$

Using the pointwise formula for the right Kan-extension, for every S in $G\mathbf{Orb}$ we get an equivalence

$$\underline{\mathbf{V}_{\mathbf{A}}}(X)(S) \simeq \lim_{BG_{/S}} \mathbf{V}_{\mathbf{A}}(X) \ ,$$

where G acts on $\mathbf{V_A}(X)$ by functoriality via its action on X. We fix a base point s in S and let G_s denote the stabilizer of s in G. The functor $BG_s \to BG_{/S}$ sending the unique object of BG_s to the unique map $G \to S$ with $e \mapsto s$, and given by the inclusion $G_s \to G$ on the level of morphisms, is an equivalence of categories. Hence we get an equivalence

$$\underline{\mathbf{V_A}}(X)(S) \simeq \lim_{BG_s} \mathbf{V_A}(\mathrm{Res}^G_{G_s} X) \ . \tag{3.18}$$

By [5, Def. 3.4.1, Cor. 3.4.8] the objects of the right-hand side in (3.18) are pairs (M, ρ) of an object of M in $\mathbf{V_A}(X)$ and a cocycle $\rho = (\rho_h)_{h \in G_s}$ of morphisms $\rho_h : M \to hM$ in $\mathbf{V_A}(X)$. The only difference to Definition 3.11 is that the cocyles in the latter consists of $\mathtt{diag}(X)$-controlled morphisms.
We define $\underline{\mathbf{V}^G_\mathbf{A}}(X)(S)$ as the full subcategory of those objects (M, ρ) such that $\rho = (\rho_h)_{h \in G_s}$ consists of $\mathtt{diag}(X)$-controlled morphisms. This subcategory is independent of the choice of the base point s in S and equal to $\mathbf{V}^{G_s}_\mathbf{A}(\mathrm{Res}^G_{G_s} X)$. Moreover if $S \to S'$ is a morphism in $G\mathbf{Orb}$, then the induced functor $\underline{\mathbf{V_A}}(X)(S') \to \underline{\mathbf{V_A}}(X)(S)$ preserves these subcategories. Similarly, if $X \to X'$ is a morphism in $G\mathbf{BornCoarse}$, then the induced functor $\underline{\mathbf{V_A}}(X)(S) \to \underline{\mathbf{V_A}}(X')(S)$ preserves the subcategories. We therefore get a subfunctor

$$\underline{\mathbf{V}^G_\mathbf{A}} : G\mathbf{BornCoarse} \to \mathbf{Fun}(G\mathbf{Orb}^{op}, \mathbf{Add}_\infty) \tag{3.19}$$

of $\underline{\mathbf{V_A}}$. This functor is equivalent to the composition appearing in the formula for $\underline{\mathrm{UK}\mathcal{X}_\mathbf{A}}$ in [2, Def. 3.14] before the application of UK_∞.
We now use that $\mathbf{A}$ is symmetric monoidal. Note that the restriction functor

$$j^{G,*} : \mathbf{Fun}(G\mathbf{Orb}^{op}, \mathbf{Add}_\infty) \to \mathbf{Fun}(BG, \mathbf{Add}_\infty)$$

has a canonical symmetric monoidal refinement. Hence its right adjoint j^G_* has a lax symmetric monoidal refinement $j^{G,\otimes}_*$. Therefore using the lax symmetric monoidal refinement of $\mathbf{V_A}$ given by Theorem 3.26 (applied to the trivial group) the functor in (3.19) admits the lax symmetric monoidal refinement

$$\underline{\mathbf{V_A}}^\otimes : G\mathbf{BornCoarse}^\otimes \to \mathbf{Fun}(BG, \mathbf{BornCoarse}^\otimes) \xrightarrow{\mathbf{V}^\otimes_\mathbf{A}} \mathbf{Fun}(BG, \mathbf{Add}^\otimes_\infty) \xrightarrow{j^{G,\otimes}_*} \mathbf{Fun}(G\mathbf{Orb}^{op}, \mathbf{Add}^\otimes_\infty) \ . \tag{3.20}$$

We now check that this structure induces a lax symmetric monoidal refinement $\underline{\mathbf{V}^{G}_\mathbf{A}}^\otimes$ on the subfunctor $\underline{\mathbf{V}^G_\mathbf{A}}$. It suffices to show that the unit constraint factorizes over the subcategory, and that the tensor product of objects preserves the subcategories.
The case of the unit constraint is clear since the inclusion $\underline{\mathbf{V}^G_\mathbf{A}}(*) \to \underline{\mathbf{V_A}}(*)$ is an equivalence of functors on $G\mathbf{Orb}^{op}$. Let $\mathcal{M}$ be in $\underline{\mathbf{V}^G_\mathbf{A}}(X)$ and $\mathcal{M}'$ be in $\underline{\mathbf{V}^G_\mathbf{A}}(X)$. Let S be in $G\mathbf{Orb}^{op}$ and fix a base point s in S. Let (M, ρ) and (M', ρ') represent the evaluations $\mathcal{M}(S)$ and $\mathcal{M}(S')$. Then ρ and ρ' are implemented by diag-controlled morphisms. The evaluation of $\mathcal{M} \otimes \mathcal{M}'$ in $\underline{\mathbf{V}^G_\mathbf{A}}(X \otimes X')$ at S is given by $(M \boxtimes M', \rho \boxtimes \rho')$. It follows from the explicit formula for the tensor product of morphisms Eq. (3.13) that $\rho \boxtimes \rho$ is again implemented by $\mathtt{diag}(X \times X')$-controlled morphisms.
We thus get a lax symmetric monoidal subfunctor

$$\underline{\mathbf{V}^{G\otimes}_A} : G\mathbf{BornCoarse}^\otimes \to \mathbf{Fun}(G\mathbf{Orb}, \mathbf{Add}^\otimes_\infty) \ .$$

We finally postcompose with the lax symmetric monoidal functor $\mathrm{UK}^{\otimes}_{\infty}$ from Theorem 3.27 in order to get the desired lax symmetric monoidal functor

$$\underline{\mathrm{UK}\mathcal{X}}^{\otimes}_{\mathbf{A}} : G\mathbf{BornCoarse}^{\otimes} \to \mathbf{Fun}(G\mathbf{Orb}, \mathbf{Add}^{\otimes}_{\infty}) \stackrel{\mathrm{UK}^{\otimes}_{\infty}}{\to} \mathbf{Fun}(G\mathbf{Orb}^{op}, \mathcal{M}^{\otimes}_{loc})$$

refining the functor $\underline{\mathrm{UK}\mathcal{X}}_{\mathbf{A}}$ from (3.16). □

Remark 3.33. In Proposition 3.32 we assume that $\mathbf{A}$ has a trivial G-action. The case of additive categories with non-trivial G-action is more complicated and would require a more general version of Theorem 3.26 whose proof would go beyond the main methods of the present paper.

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