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Opetopic algebras I: Algebraic structures on opetopic sets

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Abstract

We define a family of structures called “opetopic algebras”, which are algebraic structures with an underlying opetopic set. Examples of such are categories, planar operads, and Loday’s combinads over planar trees. Opetopic algebras can be defined in two ways, either as the algebras of a “free pasting diagram” parametric right adjoint monad, or as models of a small projective sketch over the category of opetopes. We define an opetopic nerve functor that fully embeds each category of opetopic algebras into the category of opetopic sets. In particular, we obtain fully faithful opetopic nerve functors for categories and for planar coloured $\mathcal{S}$et-operads.

This paper is the first in a series aimed at using opetopic spaces as models for higher algebraic structures.

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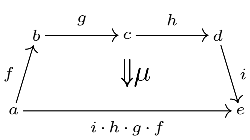
1. Introduction

This paper deals with algebraic structures whose operations have *higher dimensional “tree-like” arities*. As an example in lieu of a definition, a category $\mathcal{C}$ is an algebraic structure whose operation of *composition* has as its inputs, or “arities”, sequences of composable morphisms of the category. These sequences can can be seen as *filiform* or *linear* trees. Thus, the operation of composition itself can be given a geometric interpretation on the right:



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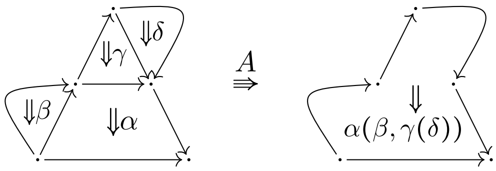
Here, γ is the *compositor* of f, g, h, and i, and points towards the actual composition $i \cdot h \cdot g \cdot f$. A second example, one dimension higher, is that of a planar coloured $\mathcal{S}$et-operad $\mathcal{P}$ (a.k.a. a non-symmetric multicategory), whose compositors have planar trees of composable multimorphisms of $\mathcal{P}$ as arities.



Here, A is the compositor of the pasting of α, β, γ, and δ as on the left, and points towards the actual composition $\alpha(\beta, \gamma(\delta))$.

Heuristically extending this pattern, one infers that such an algebraic structure one dimension above that of planar coloured operads should have an operation of composition whose arities are suitably "planar" trees of "operations" whose inputs are *planar trees*. Indeed, such algebraic structures are precisely the (coloured) $\mathbb{PT}$-*combinads in* $\mathcal{S}$et (combinads over the combinatorial pattern of planar trees) of Loday [15].

The goal of this article is to give a precise definition and extend the previous hierarchy of algebraic structures.

Structure	Sets =0-algebras	Categories =1-algebras	Operads =2-algebras	$\mathbb{PT}$−combinads =3-algebras	⋯
Arity	$\{*\}$ =1-opetopes	Lists =2-opetopes	Trees =3-opetopes	4-opetopes	⋯

1.1 Context It is well-known that the recursive sequence of higher dimensional operations with tree-like arities are encoded by the opetopes (*ope*ration poly*topes*) of Baez and Dolan [2], which were originally introduced in order to give a definition of weak n-categories and a precise formulation of the "microcosm" principle.

The fundamental definitions of [2] are those of $\mathcal{P}$-*opetopic sets* and n-*coherent* $\mathcal{P}$-*algebras* (for a coloured symmetric $\mathcal{S}$et-operad $\mathcal{P}$), the latter being $\mathcal{P}$-opetopic sets along with certain "horn-filling" operations that are "universal" in a suitable sense. When $\mathcal{P}$ is the identity monad on $\mathcal{S}$et (i.e. the unicolour symmetric $\mathcal{S}$et-operad with a single unary operation), $\mathcal{P}$-opetopic sets are simply called *opetopic sets*, and n-coherent $\mathcal{P}$-algebras are the authors' proposed definition of *weak* n-*categories*.

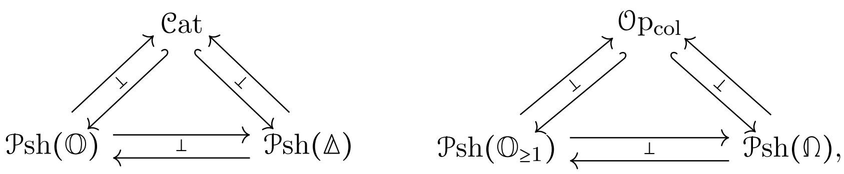
While the coinductive definitions in [2] of $\mathcal{P}$-opetopic sets and n-coherent $\mathcal{P}$-algebras are straightforward and general, they have the disadvantage of not defining a *category* of $\mathcal{P}$-opetopes such that presheaves over it are precisely $\mathcal{P}$-opetopic sets, even though this is ostensibly the case. Directly defining the category of $\mathcal{P}$-opetopes turns out to be a tedious and non-trivial task, and was worked out explicitly by Cheng in [4, 6] for the particular case of the identity monad on $\mathcal{S}$et, giving the category $\mathbb{O}$ of opetopes.

The complexity in the definition of a category $\mathbb{O}_{\mathcal{P}}$ of $\mathcal{P}$-opetopes has its origin in the difficulty of working with a suitable notion of *symmetric tree*. Indeed, the objects of $\mathbb{O}_{\mathcal{P}}$ are trees of trees of ... of trees of operations of the symmetric operad $\mathcal{P}$, and their automorphism groups are determined by the action of the ("coloured") symmetric groups on the sets of operations of $\mathcal{P}$.

However, when the action of the symmetric groups on the sets of operations of $\mathcal{P}$ is free, it turns out that the objects of the category $\mathbb{O}_{\mathcal{P}}$ are rigid, i.e. have no non-trivial automorphisms (this follows from [6, proposition 3.2]). The identity monad on $\mathcal{S}$et is of course such an operad, and this vastly simplifies the definition of $\mathbb{O}$. Indeed, such operads are precisely the (finitary) *polynomial monads* in $\mathcal{S}\mathrm{et}$, and the machinery of polynomial endofunctors and polynomial monads developed in [12, 13, 9] gives a very satisfactory definition of $\mathbb{O}$ [11, 7] which we review in sections 2 and 3.

1.2 Contributions The main contribution of the present article is to show how the polynomial definition of $\mathbb{O}$ allows, for all $k, n \in \mathbb{N}$ with $k \leq n$, a definition of (k,n)*-opetopic algebras*, which constitute a full subcategory of the category $\mathcal{P}\mathrm{sh}(\mathbb{O})$ of opetopic sets. More precisely, we show that the polynomial monad whose set of operations is the set $\mathbb{O}_{n+1}$ of $(n+1)$-dimensional opetopes can be extended to a parametric right adjoint monad whose algebras are the (k,n)-opetopic algebras. Important particular cases are the categories of $(1,1)$- and $(1,2)$-opetopic algebras, which are the categories $\mathcal{C}\mathrm{at}$ and $\mathcal{O}\mathrm{p}_{\mathrm{col}}$ of small categories and coloured planar $\mathcal{S}\mathrm{et}$-operads respectively. Loday's combinads over the combinatorial pattern of planar trees [15] are also recovered as $(1,3)$-opetopic algebras.

We further show that each category of (k,n)-opetopic algebras admits a fully faithful *opetopic nerve functor* to $\mathcal{P}\mathrm{sh}(\mathbb{O})$. As a direct consequence of this framework, we obtain commutative triangles of adjunctions



(1.1)

where $\mathbb{\Delta}$ is the category of simplices, Ω is the planar version of Moerdijk and Weiss's category of *dendrices*, the labelled arrows are fully faithful *nerve* functors, and $\mathbb{O}_{\geq 1}$ is the full subcategory of $\mathbb{O}$ on opetopes of dimension ≥ 1. This gives a direct comparison between the opetopic nerve of a category (resp. a planar operad) and its corresponding well-known simplicial (resp. dendroidal) nerve.

This formalism seems to provide infinitely many types of (k,n)-opetopic algebras. However this is not really the case, as the notion stabilises at the level of combinads. Specifically, we show a phenomenon we call *algebraic trompe-l'œil*, where an (k,n)-opetopic algebra is entirely specified by its underlying opetopic set and by a $(1,3)$-opetopic algebra. In other words, its algebraic data can be "compressed" into a $(1,3)$-algebra (a combinad). The intuition behind this is that fundamentally, opetopes are just trees whose nodes are themselves trees, and that once this is obtained at the level of combinads, opetopic algebras can encode no further useful information.

1.3 Outline We begin by recalling elements of the theory of polynomial functors and polynomial monads in section 2. This formalism is the basis for the modern definition of opetopes and of the category of opetopes [13, 7] that we survey in section 3. Section 4.4 contains the central constructions of this article, namely those of opetopic algebras and coloured opetopic algebras, as well as the definition of the opetopic nerve functor, which is a fully faithful functor from (coloured) opetopic algebras into opetopic sets. Section 5 is devoted to showing how the algebraic information carried by opetopes turns out to be limited.

1.4 Related work The (k,n)-opetopic algebras that we obtain are related to the n-coherent $\mathcal{P}$-algebras of [2] as follows: for $n \geq 1$, $(1,n)$-opetopic algebras are precisely 1-coherent $\mathcal{P}$-algebras for $\mathcal{P}$ the polynomial monad $\mathbb{O}_{n-1} \longleftarrow E_n \longrightarrow \mathbb{O}_n \longrightarrow \mathbb{O}_{n-1}$. We therefore do not obtain all n-coherent $\mathcal{P}$-algebras with our framework, and this means in particular that we cannot capture all the weak n-categories of [2] (except for $n = 1$, which are just usual 1-categories). This is not unexpected, as weak n-categories are not defined just by *equations* on the opetopes of an opetopic set, but by its more subtle *universal* opetopes.

However, we are able to promote the triangles of (1.1) to Quillen equivalences of simplicial model categories. This, along with a proof that opetopic spaces (i.e. simplicial presheaves on $\mathbb{O}$) model $(\infty, 1)$-categories and planar ∞-operads, will be the subject of the second paper of this series.

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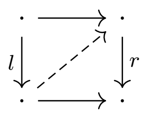
1.6 Preliminary category theory We review relevant notions and basic results from the theory of locally presentable categories. Original references are [8, 17], and most of this material can be found in [1].

1.6.1 Presheaves and nerve functors For $\mathcal{C} \in \mathcal{C}\mathrm{at}$ (the category of small categories), we write $\mathcal{P}\mathrm{sh}(\mathcal{C})$ for the category of $\mathcal{S}\mathrm{et}$-valued presheaves over $\mathcal{C}$, i.e. the category of functors $\mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{S}\mathrm{et}$ and natural transformations between them. If $X \in \mathcal{P}\mathrm{sh}(\mathcal{C})$ is a presheaf, then its *category of elements* $\mathcal{C}/X$ is the comma category y/X, where $\mathsf{y} : \mathcal{C} \hookrightarrow \mathcal{P}\mathrm{sh}(\mathcal{C})$ is the Yoneda embedding.

Let $\mathcal{C} \in \mathcal{C}\mathrm{at}$ and $F : \mathcal{C} \longrightarrow \mathcal{D}$ be a functor to a (not necessarily small) category $\mathcal{D}$. Then the *nerve functor associated to* F (also called the *nerve of* F) is the functor $N_F : \mathcal{D} \longrightarrow \mathcal{P}\mathrm{sh}(\mathcal{C})$ mapping $d \in \mathcal{D}$ to $\mathcal{D}(F-, d)$. The functor F is said to be *dense* if for all $d \in \mathcal{D}$, the colimit of $F/d \longrightarrow \mathcal{C} \xrightarrow{F} \mathcal{D}$ exists in $\mathcal{D}$, and is canonically isomorphic to d. Equivalently, F is dense if and only if N_F is fully faithful.

Let $i : \mathcal{C} \longrightarrow \mathcal{D}$ be a functor between small categories. Then the precomposition functor $i^* : \mathcal{P}\mathrm{sh}(\mathcal{D}) \longrightarrow \mathcal{P}\mathrm{sh}(\mathcal{C})$ has a left adjoint $i_!$ and a right adjoint i_*, given by left and right Kan extension along i^{op} respectively. If i has a right adjoint j, then $i^* \dashv j^*$, or equivalently, $i^* \cong j_!$. Note that the nerve of i is the functor $N_i = i^* \mathsf{y}_{\mathcal{D}}$, where $\mathsf{y}_{\mathcal{D}} : \mathcal{D} \hookrightarrow \mathcal{P}\mathrm{sh}(\mathcal{D})$ is the Yoneda embedding. Recall that i^* is the nerve of the functor $\mathsf{y}_{\mathcal{D}} i$ and that i_* is the nerve of the functor $N_i = i^* \mathsf{y}_{\mathcal{D}}$, i.e. it is the nerve of the nerve of i.

1.6.2 Orthogonality Let $\mathcal{C}$ be a category, and $l, r \in \mathcal{C}^{\rightarrow}$. We say that l is *left orthogonal to* r (equivalently, r is *right orthogonal* to l), written $l \perp r$, if for any solid commutative square as follows, there exists a *unique* dashed arrow making the two triangles commute (the relation $\perp$ is also known as the *unique lifting property*):



If $\mathcal{C}$ has a terminal object 1, then for all $X \in \mathcal{C}$, we write $l \perp X$ if l is left orthogonal to the unique map $X \longrightarrow 1$. Let L and R be two classes of morphisms of $\mathcal{C}$. We write $\mathsf{L} \perp \mathsf{R}$ if for all $l \in \mathsf{L}$ and $r \in \mathsf{R}$ we have $l \perp r$. The class of all morphisms f such that $\mathsf{L} \perp f$ (resp. $f \perp \mathsf{R}$) is denoted $\mathsf{L}^\perp$ (resp. ${}^\perp\mathsf{R}$).

1.6.3 Cocontinous localisations Let J be a class of morphisms of a cocomplete category $\mathcal{C}$. Recall that the *cocontinuous localisation of* $\mathcal{C}$ *at* J is a cocontinuous functor $\gamma_\mathsf{J} : \mathcal{C} \longrightarrow \mathsf{J}^{-1}\mathcal{C}$ such that $\gamma_\mathsf{J} f$ is an isomorphism for every $f \in \mathsf{J}$, and such that γ_J is universal (among cocontinuous functors) for this property. We say that J has the *3-for-2* property when for every composable pair $\cdot \xrightarrow{f} \cdot \xrightarrow{g} \cdot$ of morphisms in $\mathcal{C}$, if any two of f, g and gf are in J, then so is the third. Since we will only be interested in cocontinuous localisations, we will drop the adjective "cocontinuous" and simply use "localisation" henceforth.

Assume now that $\mathcal{C}$ is a small category, and that J is a *set* (rather than a proper class) of morphisms of $\mathcal{P}\mathrm{sh}(\mathcal{C})$, and consider the full subcategory $\mathcal{C}_\mathsf{J} \hookrightarrow \mathcal{P}\mathrm{sh}(\mathcal{C})$ of all those $X \in \mathcal{P}\mathrm{sh}(\mathcal{C})$ such that $\mathsf{J} \perp X$. A category is *locally presentable* if and only if it is equivalent to one of the form $\mathcal{C}_\mathsf{J}$. The pair $({}^\perp(\mathsf{J}^\perp), \mathsf{J}^\perp)$ forms an *orthogonal factorisation system*, meaning that any morphism f in $\mathcal{P}\mathrm{sh}(\mathcal{C})$ can be factored as $f = pi$, where $p \in \mathsf{J}^\perp$ and $i \in {}^\perp(\mathsf{J}^\perp)$. Applied to the unique arrow $X \longrightarrow 1$, this factorisation provides a left adjoint (i.e. a reflection) $a_\mathsf{J} : \mathcal{P}\mathrm{sh}(\mathcal{C}) \longrightarrow \mathcal{C}_\mathsf{J}$ to the inclusion $\mathcal{C}_\mathsf{J} \hookrightarrow \mathcal{P}\mathrm{sh}(\mathcal{C})$. Furthermore, a_J is the localisation of $\mathcal{P}\mathrm{sh}(\mathcal{C})$ at J. [1]

With $\mathcal{C}$ and J as in the previous paragraph, the class of J-*local isomorphisms* W_J is the class of all morphisms $f \in \mathcal{P}\mathrm{sh}(\mathcal{C})^\to$ such that for all $X \in \mathcal{C}_\mathsf{J}$, $f \perp X$, that is, $\mathsf{W}_\mathsf{J} = {}^\perp\mathcal{C}_\mathsf{J}$. [2] It is the smallest class of morphisms that contains J, that satisfies the 3-for-2 property, and that is closed under colimits in $\mathcal{P}\mathrm{sh}(\mathcal{C})^\to$ [8, theorem 8.5]. Thus the localisation a_J is also the localisation of $\mathcal{P}\mathrm{sh}(\mathcal{C})$ at W_J. Furthermore, W_J is closed under pushout along any morphism in $\mathcal{P}\mathrm{sh}(\mathcal{C})$ (this follows since every pushout can be expressed as a colimit in $\mathcal{P}\mathrm{sh}(\mathcal{C})^\to$).

2. Polynomial functors and polynomial monads

We survey elements of the theory of polynomial functors, trees, and monads. For more comprehensive references, see [12, 9].

2.1 Polynomial functors

Definition 2.1 (Polynomial functor)**.** A *polynomial (endo)functor P over I* is a diagram in $\mathcal{S}\mathrm{et}$ of the form

$$I \xleftarrow{s} E \xrightarrow{p} B \xrightarrow{t} I. \tag{2.2}$$

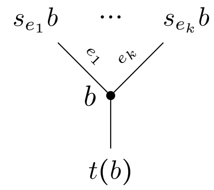
P is said to be *finitary* if the fibres of $p : E \longrightarrow B$ are finite sets. We will always assume polynomial functors to be finitary.

We use the following terminology for a polynomial functor P as in equation (2.2), which is motivated by the intuition that a polynomial functor encodes a multi-sorted signature of function symbols. The elements of B are called the *nodes* or *operations* of P, and for every node b, the

[1]The results of this paragraph still hold when $\mathcal{P}\mathrm{sh}(\mathcal{C})$ is replaced by a locally κ-presentable category $\mathcal{E}$ and a set K of κ-small morphisms of $\mathcal{E}$. We call a localisation of the form $a_\mathsf{K} : \mathcal{E} \longrightarrow \mathcal{E}_\mathsf{K}$ the *Gabriel–Ulmer localisation* of $\mathcal{E}$ at K.

[2]This is *not* ${}^\perp(\mathsf{J}^\perp)$; the inclusion ${}^\perp(\mathsf{J}^\perp) \subseteq \mathsf{W}_\mathsf{J}$ is in general *strict*.

elements of the fibre $E(b) := p^{-1}(b)$ are called the *inputs* of b. The elements of I are called the *colours* or *sorts* of P. For every input e of a node b, we denote its colour by $s_e(b) := s(e)$.



Definition 2.3 (Morphism of polynomial functor)**.** A morphism from a polynomial functor P over I (as in equation (2.2)) to a polynomial functor P' over I' (on the second row) is a commutative diagram of the form

$$\begin{array}{ccccccc} I & \xleftarrow{s} & E & \xrightarrow{p} & B & \xrightarrow{t} & I \\ {\scriptstyle f_0}\downarrow & & {\scriptstyle f_2}\downarrow & \lrcorner & {\scriptstyle f_1}\downarrow & & {\scriptstyle f_0}\downarrow \\ I' & \xleftarrow{s'} & E' & \xrightarrow{p'} & B' & \xrightarrow{t} & I' \end{array}$$

where the middle square is cartesian (i.e. is a pullback square). If P and P' are both polynomial functors over I, then a morphism from P to P' *over* I is a commutative diagram as above, but where f_0 is required to be the identity. Let $\mathcal{P}\mathrm{oly}\mathcal{E}\mathrm{nd}$ denote the category of polynomial functors and morphisms of polynomial functors, and $\mathcal{P}\mathrm{oly}\mathcal{E}\mathrm{nd}(I)$ the category of polynomial functors over I and morphisms of polynomial functors over I.

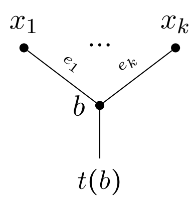
Remark 2.4 (Polynomial functors really are functors!). The term "polynomial (endo)functor" is due to the association of P to the composite endofunctor

$$\mathcal{S}\mathrm{et}/I \xrightarrow{s^*} \mathcal{S}\mathrm{et}/E \xrightarrow{p_*} \mathcal{S}\mathrm{et}/B \xrightarrow{t_!} \mathcal{S}\mathrm{et}/I$$

where we have denoted $a_!$ and a_* the left and right adjoints to the pullback functor a^* along a map of sets a. Explicitly, for $(X_i \mid i \in I) \in \mathcal{S}\mathrm{et}/I$, $P(X)$ is given by the "polynomial"

$$P(X) = \left(\sum_{b \in B(j)} \prod_{e \in E(b)} X_{s(e)} \mid j \in I \right), \tag{2.5}$$

where $B(j) := t^{-1}(j)$ and $E(b) = p^{-1}(b)$. Visually, elements of $P(X)_j$ are nodes $b \in B$ such that $tb = j$, and whose inputs are decorated by elements of $(X_i \mid i \in I)$ in a manner compatible with their colours. Graphically, an element of PX_i can be represented as



with $b \in B$ such that $t(b) = i$, and $x_j \in X_{s_{e_j}b}$ for $1 \le j \le k$. Moreover, the endofunctor $P : \mathcal{S}\mathrm{et}/I \longrightarrow \mathcal{S}\mathrm{et}/I$ preserves connected limits: s^* and p_* preserve all limits (as right adjoints), and $t_!$ preserves and reflects connected limits.

This construction extends to a fully faithful functor $\mathcal{P}\mathrm{oly}\mathcal{E}\mathrm{nd}(I) \longrightarrow \mathcal{C}\mathrm{art}(\mathcal{S}\mathrm{et}/I)$, the latter being the category of endofunctors of $\mathcal{S}\mathrm{et}/I$ and cartesian natural transformations[3]. In fact, the

[3] We recall that a natural transformation is *cartesian* if all its naturality squares are cartesian.

image of this full embedding consists precisely of those endofunctors that preserve connected limits [9, section 1.18]. The composition of endofunctors gives $\mathcal{C}\mathrm{art}(\mathcal{S}\mathrm{et}/I)$ the structure of a monoidal category, and $\mathcal{P}\mathrm{oly}\mathcal{E}\mathrm{nd}(I)$ is stable under this monoidal product [9, proposition 1.12]. The identity polynomial functor $I \longleftarrow I \longrightarrow I \longrightarrow I$ is associated to the identity endofunctor; thus $\mathcal{P}\mathrm{oly}\mathcal{E}\mathrm{nd}(I)$ is a monoidal subcategory of $\mathcal{C}\mathrm{art}(\mathcal{S}\mathrm{et}/I)$.

Lastly, a polynomial functor is finitary if and only if its associated endofunctor is finitary (preserves filtered colimits).

2.2 Trees

Definition 2.6 (Polynomial tree)**.** A polynomial functor T given by

$$T_0 \xleftarrow{s} T_2 \xrightarrow{p} T_1 \xrightarrow{t} T_0$$

is a *(polynomial) tree* [12, section 1.0.3] if

(1) the sets T_0, T_1 and T_2 are finite (in particular, each node has finitely many inputs);
(2) the map t is injective;
(3) the map s is injective, and the complement of its image $T_0 - \operatorname{im} s$ has a single element, called the *root*;
(4) let $T_0 = T_2 + \{r\}$, with r the root, and define the *walk-to-root* function σ by $\sigma(r) = r$, and otherwise $\sigma(e) = tp(e)$; then we ask that for all $x \in T_0$, there exists $k \in \mathbb{N}$ such that $\sigma^k(x) = r$.

We call the colours of a tree its *edges* and the inputs of a node the *input edges* of that node.

Let $\mathcal{T}\mathrm{ree}$ be the full subcategory of $\mathcal{P}\mathrm{oly}\mathcal{E}\mathrm{nd}$ whose objects are trees. Note that it is the category of *symmetric* or *non-planar* trees (the automorphism group of a tree is in general non-trivial) and that its morphisms correspond to inclusions of non-planar subtrees. An *elementary tree* is a tree with at most one node. Let $\mathit{el}\mathcal{T}\mathrm{ree}$ be the full subcategory of $\mathcal{T}\mathrm{ree}$ spanned by elementary trees.

Definition 2.7 (P-tree)**.** For $P \in \mathcal{P}\mathrm{oly}\mathcal{E}\mathrm{nd}$, the category $\operatorname{tr} P$ of *P-trees* is the slice category $\mathcal{T}\mathrm{ree}/P$. The fundamental difference between $\mathcal{T}\mathrm{ree}$ and $\operatorname{tr} P$ is that the latter is always rigid i.e. it has no non-trivial automorphisms [12, proposition 1.2.3]. In particular, this implies that $\mathcal{P}\mathrm{oly}\mathcal{E}\mathrm{nd}$ does not have a terminal object.

Notation 2.8. Every P-tree $T \in \operatorname{tr} P$ corresponds to a morphism from a tree (which we shall denote by $\langle T \rangle$) to P, so that $T : \langle T \rangle \longrightarrow P$. We point out that $\langle T \rangle_1$ is the set of nodes of $\langle T \rangle$, while $T_1 : \langle T \rangle_1 \longrightarrow P_1$ is a decoration of the nodes of $\langle T \rangle$ by nodes of P, and likewise for edges.

Definition 2.9 (Category of elements)**.** For $P \in \mathcal{P}\mathrm{oly}\mathcal{E}\mathrm{nd}$, its *category of elements*[4] $\operatorname{elt} P$ is the slice $\mathit{el}\mathcal{T}\mathrm{ree}/P$. Explicitly, for P as in equation (2.2), the set of objects of $\operatorname{elt} P$ is $I + B$, and for each $b \in B$, there is a morphism $\mathsf{t} : t(b) \longrightarrow b$, and a morphism $\mathsf{s}_e : s_e(b) \longrightarrow b$ for each $e \in E(b)$. Remark that there is no non-trivial composition of arrows in $\operatorname{elt} P$.

Proposition 2.10 ([12, proposition 2.1.3])**.** *There is an equivalence of categories* $\mathcal{P}\mathrm{sh}(\operatorname{elt} P) \simeq \mathcal{P}\mathrm{oly}\mathcal{E}\mathrm{nd}/P$.

[4]Not to be confused with the category of elements of a presheaf over some category.

2.3 Addresses

Definition 2.11 (Address)**.** Let $T \in \mathfrak{T}\mathrm{ree}$ be a polynomial tree and σ be its walk-to-root function (definition 2.6). We define the *address* function & on edges inductively as follows:

(1) if r is the root edge, let $\&r := []$,

(2) if $e \in T_0 - \{r\}$ and if $\&\sigma(e) = [x]$, define $\&e := [xe]$.

The address of a node $b \in T_1$ is defined as $\&b := \&t(b)$. Note that this function is injective since t is. Let $T^\bullet$ denote its image, the set of *node addresses* of T, and let $T^|$ be the set of addresses of leaf edges, i.e. those not in the image of t.

Assume now that $T : \langle T \rangle \longrightarrow P$ is a P-tree. If $b \in \langle T \rangle_1$ has address $\&b = [p]$, write $\mathsf{s}_{[p]} T := T_1(b)$. For convenience, we let $T^\bullet := \langle T \rangle^\bullet$, and $T^| := \langle T \rangle^|$.

Remark 2.12. The formalism of addresses is a useful bookkeeping syntax for the operations of grafting and substitution on trees. The syntax of addresses will extend to the category of opetopes and will allow us to give a precise description of the composition of morphisms in the category of opetopes (see definition 3.10) as well as certain constructions on opetopic sets.

Notation 2.13. We denote by $\mathrm{tr}^| P$ the set of P-trees with a marked leaf, i.e. endowed with the address of one of its leaves. Similarly, we denote by $\mathrm{tr}^\bullet P$ the set of P-trees with a marked node.

2.4 Grafting

Definition 2.14 (Elementary P-trees)**.** Let P be a polynomial endofunctor as in equation (2.2). For $i \in I$, define $\mathsf{I}_i \in \mathrm{tr}\, P$ as having underlying tree

$$\{i\} \longleftarrow \varnothing \longrightarrow \varnothing \longrightarrow \{i\}, \tag{2.15}$$

along with the obvious morphism to P, that which maps i to $i \in I$. This corresponds to a tree with no nodes and a unique edge decorated by i. Define $\mathsf{Y}_b \in \mathrm{tr}\, P$, the *corolla* at b, as having underlying tree

$$s(E(b)) + \{*\} \xleftarrow{s} E(b) \longrightarrow \{b\} \longrightarrow s(E(b)) + \{*\}, \tag{2.16}$$

where the right map sends b to $*$, and where the morphism $\mathsf{Y}_b \longrightarrow P$ is the identity on $s(E(b)) \subseteq I$, maps $*$ to $t(b) \in I$, is the identity on $E(b) \subseteq E$, and maps b to $b \in B$. This corresponds to a P-tree with a unique node, decorated by b. Observe that for $T \in \mathrm{tr}\, P$, giving a morphism $\mathsf{I}_i \longrightarrow T$ is equivalent to specifying the address $[p]$ of an edge of T decorated by i. Likewise, morphisms of the form $\mathsf{Y}_b \longrightarrow T$ are in bijection with addresses of nodes of T decorated by b.

Definition 2.17 (Grafting)**.** For $S, T \in \mathrm{tr}\, P$, $[l] \in S^|$ such that the leaf of S at $[l]$ and the root edge of T are decorated by the same $i \in I$, define the *grafting* $S \circ_{[l]} T$ of S and T on $[l]$ by the following pushout (in $\mathrm{tr}\, P$):

$$\begin{array}{ccc} \mathsf{I}_i & \xrightarrow{[]} & T \\ {\scriptstyle [l]}\downarrow & \ulcorner & \downarrow \\ S & \longrightarrow & S \circ_{[l]} T. \end{array} \tag{2.18}$$

Note that if S (resp. T) is a trivial tree, then $S \circ_{[l]} T = T$ (resp. S). We assume, by convention, that the grafting operator $\circ$ associates to the right.

Proposition 2.19 ([12, proposition 1.1.21])**.** *Every P-tree is either of the form I_i, for some $i \in I$, or obtained by iterated graftings of corollas (i.e. P-trees of the form Y_b for $b \in B$).*

Notation 2.20 (Total grafting). Take $T, U_1, \ldots, U_k \in \operatorname{tr} P$, where $T^{|} = \{[l_1], \ldots, [l_k]\}$, and assume the grafting $T \circ_{[l_i]} U_i$ is defined for all i. Then the *total grafting* will be denoted concisely by

$$T \bigcirc_{[l_i]} U_i = (\cdots(T \underset{[l_1]}{\circ} U_1) \underset{[l_2]}{\circ} U_2 \cdots) \underset{[l_k]}{\circ} U_k. \tag{2.21}$$

It is easy to see that the result does not depend on the order in which the graftings are performed.

Definition 2.22 (Substitution)**.** Let $[p] \in T^{\bullet}$ and $b = \mathsf{s}_{[p]} T$. Then T can be decomposed as

$$T = A \underset{[p]}{\circ} \mathsf{Y}_b \bigcirc_{[e_i]} B_i, \tag{2.23}$$

where $E(b) = \{e_1, \ldots, e_k\}$, and $A, B_1, \ldots, B_k \in \operatorname{tr} P$. For U a P-tree with a bijection $\wp : U^{|} \longrightarrow E(b)$ over I, we define the *substitution* $T \square_{[p]} U$ as

$$T \underset{[p]}{\square} U := A \underset{[p]}{\circ} U \bigcirc_{\wp^{-1} e_i} B_i. \tag{2.24}$$

In other words, the node at address $[p]$ in T has been replaced by U, and the map $\wp$ provides "rewiring instructions" to connect the leaves of U to the rest of T.

2.5 Polynomial monads

Definition 2.25 (Polynomial monad)**.** A *polynomial monad over I* is a monoid in $\mathcal{P}\mathrm{oly}\mathcal{E}\mathrm{nd}(I)$. Note that a polynomial monad over I is thus necessarily a cartesian monad on $\mathcal{S}\mathrm{et}/I$.[5] Let $\mathcal{P}\mathrm{oly}\mathcal{M}\mathrm{nd}(I)$ be the category of monoids in $\mathcal{P}\mathrm{oly}\mathcal{E}\mathrm{nd}(I)$. That is, $\mathcal{P}\mathrm{oly}\mathcal{M}\mathrm{nd}(I)$ is the category of polynomial monads over I and morphisms of polynomial functors over I that are also monoid morphisms.

If M is in $\mathcal{P}\mathrm{oly}\mathcal{M}\mathrm{nd}(I)$ and M' is in $\mathcal{P}\mathrm{oly}\mathcal{M}\mathrm{nd}(J)$ then a morphism of polynomial monads $M \longrightarrow M'$ is a morphism of polynomial functors that respects the monoid structure. The category of all polynomial monads is denoted $\mathcal{P}\mathrm{oly}\mathcal{M}\mathrm{nd}$.

Definition 2.26 ($(-)^{\star}$ construction)**.** Given a polynomial endofunctor P as in equation (2.2), we define a new polynomial endofunctor $P^{\star}$ as

$$I \xleftarrow{s} \operatorname{tr}^{|} P \xrightarrow{p} \operatorname{tr} P \xrightarrow{t} I \tag{2.27}$$

where s maps a P-tree with a marked leaf to the decoration of that leaf, p forgets the marking, and t maps a tree to the decoration of its root. Remark that for $T \in \operatorname{tr} P$ we have $p^{-1} T = T^{|}$, and in particular, $P^{\star}$ is finitary.

Theorem 2.28 ([12, section 1.2.7], [13, sections 2.7 to 2.9], [10, corollary 5.1.5])**.** *The polynomial functor $P^{\star}$ has a canonical structure of a polynomial monad. Furthermore, the functor $(-)^{\star}$ is left adjoint to the forgetful functor $\mathcal{P}\mathrm{oly}\mathcal{M}\mathrm{nd}(I) \longrightarrow \mathcal{P}\mathrm{oly}\mathcal{E}\mathrm{nd}(I)$, and the adjunction is monadic.*

[5] We recall that a monad is *cartesian* if its endofunctor preserves pullbacks and its unit and multiplication are cartesian natural transformations.

Corollary 2.29 ([9, corollary 4.7]). *The $(-)^\star$ construction defines a left adjoint to the forgetful functor* $\mathcal{P}\mathrm{oly}\mathcal{M}\mathrm{nd} \longrightarrow \mathcal{P}\mathrm{oly}\mathcal{E}\mathrm{nd}$.

Definition 2.30 (Readdressing function). We abuse notation slightly by letting $(-)^\star$ denote the associated monad on $\mathcal{P}\mathrm{oly}\mathcal{E}\mathrm{nd}(I)$. Let M be a polynomial monad as on the left below. By theorem 2.28, M is a $(-)^\star$-algebra, and we will write its structure map $M^\star \longrightarrow M$ as on the right:

$$I \xleftarrow{s} E \xrightarrow{p} B \xrightarrow{t} I, \qquad \begin{array}{ccccccc} I & \longleftarrow & \mathrm{tr}^| M & \longrightarrow & \mathrm{tr}\, M & \longrightarrow & I \\ \| & & \downarrow \wp & \lrcorner & \downarrow \mathsf{t} & & \| \\ I & \longleftarrow & E & \longrightarrow & B & \longrightarrow & I. \end{array} \tag{2.31}$$

We call $\wp_T : T^| \xrightarrow{\cong} E(\mathsf{t}T)$ the *readdressing* function of T, and $\mathsf{t}T \in B$ is called the *target* of T. If we think of any $b \in B$ as the corolla Y_b, then the target map t "contracts" a tree to a corolla, and since the middle square is a pullback, the number of leaves is preserved. The map $\wp_T$ establishes a coherent correspondence between the set $T^|$ of leaf addresses of a tree T and the set $E(\mathsf{t}T)$ of inputs of $\mathsf{t}T$.

2.6 The Baez–Dolan construction

Definition 2.32 (Baez–Dolan $(-)^+$ construction). Let M be a polynomial monad as in equation (2.2), and define its *Baez–Dolan construction* M^+ to be

$$B \xleftarrow{\mathsf{s}} \mathrm{tr}^\bullet M \xrightarrow{p} \mathrm{tr}\, M \xrightarrow{\mathsf{t}} B \tag{2.33}$$

where s maps an M-tree with a marked node to the label of that node, p forgets the marking, and t is the target map. If $T \in \mathrm{tr}\, M$, remark that $p^{-1}T = T^\bullet$ is the set of node addresses of T, and in particular, M^+ is finitary. If $[p] \in T^\bullet$, then $\mathsf{s}[p] := \mathsf{s}_{[p]} T$.

Theorem 2.34 ([13, section 3.2]). *The polynomial functor M^+ has a canonical structure of a polynomial monad.*

Remark 2.35. The $(-)^+$ construction is an endofunctor on $\mathcal{P}\mathrm{oly}\mathcal{M}\mathrm{nd}$ whose definition is motivated as follows. If we begin with a polynomial monad M, then the colours of M^+ are the operations of M. The operations of M^+, along with their output colour, are given by the monad multiplication of M: they are the *relations* of M, i.e. the reductions of trees of M to operations of M. The monad multiplication on M^+ is given as follows: the reduction of a tree of M^+ to an operation of M^+ (which is a tree of M) is obtained by *substituting* trees of M into nodes of trees of M.

Let M be a finitary polynomial monad whose underlying polynomial functor is

$$I \xleftarrow{s} E \xrightarrow{p} B \xrightarrow{t} I.$$

The Baez–Dolan construction gives the polynomial monad M^+ whose underlying polynomial functor is

$$B \xleftarrow{\mathsf{s}} \mathrm{tr}^\bullet M \xrightarrow{p} \mathrm{tr}\, M \xrightarrow{\mathsf{t}} B.$$

Recall also from theorem 2.28 that the category $\mathcal{P}\mathrm{oly}\mathcal{M}\mathrm{nd}(I)$ is the category of $(-)^\star$-algebras. The following fact is analogous to proposition 2.10 and is at the heart of the Baez–Dolan construction (indeed, it is even the original *definition* of the construction, see [2, definition 15]).

Proposition 2.36. *For M a polynomial monad, there is an equivalence of categories $\mathcal{A}\mathrm{lg}(M^+) \simeq \mathcal{P}\mathrm{oly}\mathcal{M}\mathrm{nd}(I)/M$.*

Proof. Given a M^+-algebra $M^+X \xrightarrow{x} X$ in $\mathcal{S}\mathrm{et}/B$, define $\Phi X \in \mathcal{P}\mathrm{oly}\mathcal{E}\mathrm{nd}(I)/M$ as

$$\begin{array}{ccccccc} I & \longleftarrow & E_X & \longrightarrow & X & \longrightarrow & I \\ \| & & \downarrow & \lrcorner & \downarrow & & \| \\ I & \longleftarrow & E & \longrightarrow & B & \longrightarrow & I. \end{array}$$

There is an evident bijection $\mathrm{tr}\,\Phi X \cong M^+X$ in $\mathcal{S}\mathrm{et}/I$, and the structure map x extends by pullback along $E_X \longrightarrow X$ to a map $(\Phi X)^\star \longrightarrow \Phi X$ in $\mathcal{P}\mathrm{oly}\mathcal{E}\mathrm{nd}(I)$. It is easy to verify that this determines a $(-)^\star$-algebra structure on ΦX, and that the map $\Phi X \longrightarrow M$ in $\mathcal{P}\mathrm{oly}\mathcal{E}\mathrm{nd}(I)$ is a morphism of $(-)^\star$-algebras. Conversely, given an $N \in \mathcal{P}\mathrm{oly}\mathcal{M}\mathrm{nd}(I)/M$ whose underlying polynomial functor is

$$I \longleftarrow E' \longrightarrow B' \longrightarrow I,$$

then the bijection $\mathrm{tr}\,N \cong M^+B'$ in $\mathcal{S}\mathrm{et}/I$ and the $(-)^\star$-algebra map $N^\star \longrightarrow N$ provide a map $M^+B' \xrightarrow{\Psi N} B'$ in $\mathcal{S}\mathrm{et}/I$. It is easy to verify that ΨN is the structure map of a M^+-algebra and that the constructions Φ and Ψ are functorial and mutually inverse. □

Remark 2.37. The previous result provides an equivalence between $\mathcal{P}\mathrm{oly}\mathcal{M}\mathrm{nd}(I)/M$ and the category of M^+-algebras. A "coloured" version of this result can (informally) be stated as follows: for $\mathcal{A}\mathrm{lg}^{\mathrm{col}}(M^+)$ a suitable category of coloured M^+-algebras, there is an equivalence $\mathcal{A}\mathrm{lg}^{\mathrm{col}}(M^+) \simeq \mathcal{P}\mathrm{oly}\mathcal{M}\mathrm{nd}/M$.

3. Opetopes

In this section, we use the formalism of polynomial functors and polynomial monads of section 2 to define opetopes and morphisms between them. This gives us a category $\mathbb{O}$ of opetopes and a category $\mathcal{P}\mathrm{sh}(\mathbb{O})$ of opetopic sets. Our construction of opetopes is precisely that of [13], and by [13, theorem 3.16], also that of [14], and by [5, corollary 2.6], also that of [4]. As we will see, the category $\mathbb{O}$ is rigid, i.e. it has no non-trivial automorphisms (it is in fact a *direct* category).

3.1 Polynomial definition of opetopes

Definition 3.1 (The $\mathfrak{Z}^n$ monad)**.** Let $\mathfrak{Z}^0$ be the identity polynomial monad on $\mathcal{S}\mathrm{et}$, as depicted on the left below, and let $\mathfrak{Z}^n := (\mathfrak{Z}^{n-1})^+$. Write $\mathfrak{Z}^n$ as on right:

$$\{*\} \longleftarrow \{*\} \longrightarrow \{*\} \longrightarrow \{*\}, \qquad\qquad \mathbb{O}_n \xleftarrow{\mathsf{s}} E_{n+1} \xrightarrow{p} \mathbb{O}_{n+1} \xrightarrow{\mathsf{t}} \mathbb{O}_n. \tag{3.2}$$

Definition 3.3 (Opetope)**.** An *n-dimensional opetope* (or simply *n-opetope*) ω is by definition an element of $\mathbb{O}_n$, and we write $\dim \omega = n$. An opetope $\omega \in \mathbb{O}_n$ with $n \geq 2$ is called *degenerate* if its underlying tree has no nodes (thus consists of a unique edge); it is *non degenerate* otherwise.

Following (2.31), for $\omega \in \mathbb{O}_{n+2}$, the structure of polynomial monad $(\mathfrak{Z}^n)^\star \longrightarrow \mathfrak{Z}^n$ gives a bijection $\wp_\omega : \omega^| \longrightarrow (\mathsf{t}\,\omega)^\bullet$ between the leaves of ω and the nodes of $\mathsf{t}\,\omega$, preserving the decoration by n-opetopes.

Example 3.4. (1) The unique 0-opetope is denoted ♦ and called the *point*.

(2) The unique 1-opetope is denoted ■ and called the *arrow*.
(3) If $n \geq 2$, then $\omega \in \mathbb{O}_n$ is a $\mathfrak{Z}^{n-2}$-tree, i.e. a tree whose nodes are labeled with $(n-1)$-opetopes, and edges are labeled with $(n-2)$-opetopes. In particular, 2-opetopes are $\mathfrak{Z}^0$-trees, i.e. linear trees, and thus in bijection with $\mathbb{N}$. We will refer to them as *opetopic integers*, and write $\mathbf{n}$ for the 2-opetope having exactly n nodes.

Proposition 3.5. *Let $\omega \in \mathbb{O}_n$ with $n \geq 2$. We have the following.*
(1) If ω is degenerate, say $\omega = \mathsf{I}_\phi$ for some $\phi \in \mathbb{O}_{n-2}$, then $\mathsf{t}\omega = \mathsf{Y}_\phi$, and $\wp_\omega : \omega^| = \{[]\} \longrightarrow \mathsf{Y}^\bullet_\phi = \{[]\}$ obviously maps $[]$ to $[]$.
(2) If $\omega = \mathsf{Y}_\psi$ for some $\psi \in \mathbb{O}_{n-1}$, then $\mathsf{t}\omega = \psi$. Further, $\omega^| = \{[[q]] \mid [q] \in \psi^\bullet\}$, and $\wp_\omega$ maps $[[q]]$ to $[q]$.
(3) Otherwise, ω decomposes as $\omega = \nu \circ_{[l]} \mathsf{Y}_\psi$, for some $\nu \in \mathbb{O}_n$, $\psi \in \mathbb{O}_{n-1}$, and $[l] \in \nu^|$, and

$$\mathsf{t}\omega = (\mathsf{t}\nu) \underset{\wp_\nu[l]}{\square} \psi.$$

The readdressing function $\wp_\omega : \omega^| \longrightarrow (\mathsf{t}\omega)^\bullet$ is given as follows. Let $[j] \in \omega^|$.
(a) If $[l] \sqsubseteq [j]$, then $[j] = [l[q]]$ for some $[q] \in \psi^\bullet$, and $\wp_\omega[l[q]] = (\wp_\nu[l]) \cdot [q]$.
(b) If $[l] \not\sqsubseteq [j]$, then $[j] \in \nu^|$. Assume $\wp_\nu[l] \sqsubseteq \wp_\nu[j]$. Then $\wp_\nu[j] = (\wp_\nu[l]) \cdot [[q]] \cdot [a]$, for some $[q] \in (\mathsf{s}_{\wp_\nu[l]} \mathsf{t}\nu)^\bullet = (\mathsf{t}\psi)^\bullet$, and let $\wp_\omega[j] = (\wp_\nu[l]) \cdot (\wp_\psi^{-1}[q]) \cdot [a]$.
(c) If $\wp_\nu[l] \not\sqsubseteq \wp_\nu[j]$, then $\wp_\omega[j] = \wp_\nu[j]$.

Proof. Follows from the polynomial monad structure on the $\mathfrak{Z}^n$'s. □

3.2 Higher addresses By definition, an opetope ω of dimension $n \geq 2$ is a $\mathfrak{Z}^{n-2}$-tree, and thus the formalism of tree addresses (definition 2.11) can be applied to designate nodes of ω, also called its *source faces* or simply *sources*. In this section, we iterate this formalism to give the concept of *higher dimensional address*, which turns out to be more convenient. This material is largely taken from [7] and [11].

Definition 3.6 (Higher address)**.** Start by defining the set $\mathbb{A}_n$ of *n-addresses* as follows:

$$\mathbb{A}_0 = \{*\}, \qquad \mathbb{A}_{n+1} = \text{lists}\,\mathbb{A}_n,$$

where lists X is the set of finite lists (or words) on the alphabet X.

Explicitly, the unique 0-address is $*$ (also written $[]$ by convention), while an $(n+1)$-address is a sequence of n-addresses. Such sequences are enclosed by brackets. Note that the address $[]$, associated to the empty word, is in $\mathbb{A}_n$ for all $n \geq 0$. However, the surrounding context will almost always make the notation unambiguous.

Here are examples of higher addresses:

$$[] \in \mathbb{A}_1, \qquad [* * * *] \in \mathbb{A}_1, \qquad [[][*][]] \in \mathbb{A}_2, \qquad [[[[*]]]] \in \mathbb{A}_4.$$

For $\omega \in \mathbb{O}$ an opetope, nodes of ω can be specified uniquely using higher addresses, as we now show. Recall that E_{n-1} is the set of inputs of $\mathfrak{Z}^{n-2}$. In $\mathfrak{Z}^0$, we set $E_1(■) = \{*\}$, so that the unique "node address"[6] of ■ is $* \in \mathbb{A}_0$.

Let $n \geq 2$, and assume by induction that that for all $k < n$ and all k-opetopes ψ, the nodes of ψ are assigned $(k-1)$-addresses, i.e. that we have an injective map $\& : \psi^\bullet \longrightarrow \mathbb{A}_{k-1}$. This allows

[6] Of course, ■ is not a tree, but this abuse of terminology is convenient, as it allows us to talk about higher addresses and opetopes in a more uniform manner.

us to consider $\psi^\bullet$ as a subset of $\mathbb{A}_k$. Recall that an opetope $\omega \in \mathbb{O}_n$ is a $\mathfrak{Z}^{n-2}$-tree $\omega : \langle\omega\rangle \longrightarrow \mathfrak{Z}^{n-2}$, and write $\langle\omega\rangle$ as

$$I_\omega \longleftarrow E_\omega \longrightarrow B_\omega \longrightarrow I_\omega.$$

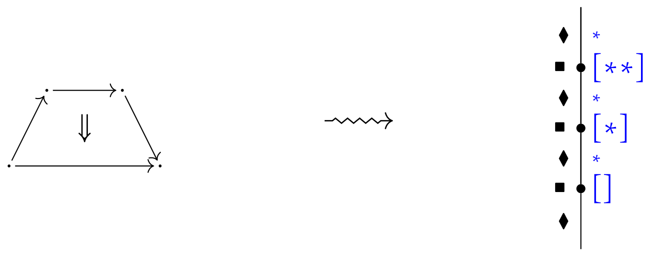
A node $b \in B_\omega$ has an address $\&b$, which is a list of edges of ω describing the path from the root to b. Write this list as $[e_1 \cdots e_k]$, where $e_1, \ldots, e_k \in E_\omega$. Now, the edge e_1 is an input edge of the root node b_1 of ω, and so it corresponds to a node c_1 of the $(n-1)$-opetope ψ_1 decorating b_1. By induction, c_1 has a higher address $[q_1] := \&b_1 \in \mathbb{A}_{n-2}$. Likewise, e_2 is an input edge of the node b_2 that sits just above e_1. If ψ_2 is the $(n-1)$-opetope decorating b_2, then e_2 corresponds to a node c_2 of ψ_2, that has an $(n-2)$-address $[q_2] := \&b_2 \in \mathbb{A}_{n-2}$. Repeating the argument, each e_i in the list $[e_1 \cdots e_k]$ gives rise to an $(n-2)$-address $[q_i]$.

The crucial part is the following: instead of considering the address of b to be the list of edges $[e_1 \cdots e_k]$, we amend the definition of address slightly, and say that $\&b := [[q_1] \cdots [q_k]]$. It is now a list of $(n-2)$-addresses, i.e. an $(n-1)$-address, and it uniquely identifies the node b in ω. This completes the induction process, which we illustrate bellow with some examples.

Henceforth, we write $\omega^\bullet$ for the set of *higher* addresses of the nodes of ω, and likewise for $\omega^|$. As in definition 2.11, if $[p] \in \omega^\bullet$ is a node higher address of ω, then $\mathsf{s}_{[p]}\,\omega$ is the decoration of the node at $[p]$, which is an $(n-1)$-opetope. Let $[l] = [p[q]] \in \mathbb{A}_{n-1}$ be an address such that $[p] \in \omega^\bullet$ and $[q] \in \left(\mathsf{s}_{[p]}\,\omega\right)^\bullet$. Then as a shorthand, we write

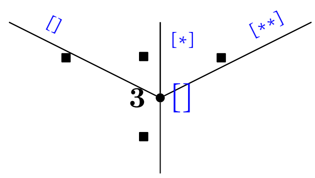
$$\mathsf{e}_{[l]}\,\omega := \mathsf{s}_{[q]}\,\mathsf{s}_{[p]}\,\omega. \tag{3.7}$$

Example 3.8. Consider the 2-opetope on the left, called **3**:

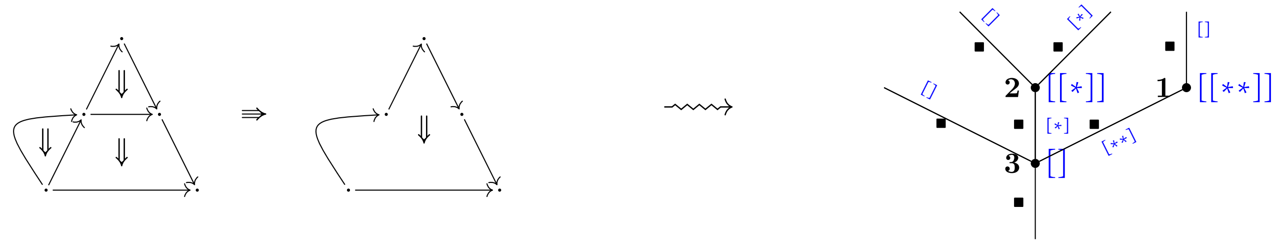


Its underlying pasting diagram consists of 3 arrows ▪ grafted linearly. Since the only node address of ▪ is $* \in \mathbb{A}_0$, the underlying tree of **3** can be depicted as on the right. On the left of this tree are the decorations: nodes are decorated with $▪ \in \mathbb{O}_1$, while the edges are decorated with $♦ \in \mathbb{O}_0$. For each node in the tree, the set of input edges of that node is in bijective correspondence with the node addresses of the decorating opetope, and this address is written on the right of each edge. In this low dimensional example, these addresses can only be $*$. Finally, on the right of each node of the tree is its 1-address, which is just a sequence of 0-addresses giving "walking instructions" to get from the root to that node.

The 2-opetope **3** can then be seen as a corolla in some 3-opetope as follows:



As previously mentioned, the set of input edges is in bijective correspondence with the set of node addresses of **3**.Here is now an example of a 3-opetope, with its annotated underlying tree on the right (the 2-opetopes **1** and **2** are analogous to **3**):



3.3 The category of opetopes In this subsection, we define the category $\mathbb{O}$ of planar opetopes introduced in [11], following the work of [4].

Proposition 3.9 (Opetopic identities, [11, theorem 4.1])**.** *Let $\omega \in \mathbb{O}_n$ with $n \geq 2$.*

(1) (Inner edge) For $[p[q]] \in \omega^\bullet$ (forcing ω to be non degenerate), we have $\mathsf{t}\,\mathsf{s}_{[p[q]]}\,\omega = \mathsf{s}_{[q]}\,\mathsf{s}_{[p]}\,\omega$.

(2) (Globularity 1) If ω is non degenerate, we have $\mathsf{t}\,\mathsf{s}_{[]}\,\omega = \mathsf{t}\,\mathsf{t}\,\omega$.

(3) (Globularity 2) If ω is non degenerate, and $[p[q]] \in \omega^|$, we have $\mathsf{s}_{[q]}\,\mathsf{s}_{[p]}\,\omega = \mathsf{s}_{\wp_\omega[p[q]]}\,\mathsf{t}\,\omega$.

(4) (Degeneracy) If ω is degenerate, we have $\mathsf{s}_{[]}\,\mathsf{t}\,\omega = \mathsf{t}\,\mathsf{t}\,\omega$.

Definition 3.10 ([11, section 4.2])**.** With these identities in mind, we define the category $\mathbb{O}$ of opetopes by generators and relations as follows.

(1) (Objects) We set $\operatorname{ob}\mathbb{O} = \sum_{n\in\mathbb{N}} \mathbb{O}_n$.

(2) (Generating morphisms) Let $\omega \in \mathbb{O}_n$ with $n \geq 1$. We introduce a generator $\mathsf{t}\,\omega \xrightarrow{\mathsf{t}} \omega$, called the *target map*. If $[p] \in \omega^\bullet$, then we introduce a generator $\mathsf{s}_{[p]}\,\omega \xrightarrow{\mathsf{s}_{[p]}} \omega$, called a *source map*. A *face map* is either a source or the target map.

(3) (Relations) We impose 4 relations described by the following commutative squares, that are well defined thanks to proposition 3.9. Let $\omega \in \mathbb{O}_n$ with $n \geq 2$

(a) **(Inner)** for $[p[q]] \in \omega^\bullet$ (forcing ω to be non degenerate), the following square must commute:

$$\begin{array}{ccc} \mathsf{s}_{[q]}\,\mathsf{s}_{[p]}\,\omega & \xrightarrow{\mathsf{s}_{[q]}} & \mathsf{s}_{[p]}\,\omega \\ {\scriptstyle \mathsf{t}}\downarrow & & \downarrow{\scriptstyle \mathsf{s}_{[p]}} \\ \mathsf{s}_{[p[q]]}\,\omega & \xrightarrow{\mathsf{s}_{[p[q]]}} & \omega \end{array}$$

(b) **(Glob1)** if ω is non degenerate, the following square must commute:

$$\begin{array}{ccc} \mathsf{t}\,\mathsf{t}\,\omega & \xrightarrow{\mathsf{t}} & \mathsf{t}\,\omega \\ {\scriptstyle \mathsf{t}}\downarrow & & \downarrow{\scriptstyle \mathsf{t}} \\ \mathsf{s}_{[]}\,\omega & \xrightarrow{\mathsf{s}_{[]}} & \omega. \end{array}$$

(c) **(Glob2)** if ω is non degenerate, and for $[p[q]] \in \omega^|$, the following square must commute:

$$\begin{array}{ccc} \mathsf{s}_{\wp_\omega[p[q]]}\,\mathsf{t}\,\omega & \xrightarrow{\mathsf{s}_{\wp_\omega[p[q]]}} & \mathsf{t}\,\omega \\ {\scriptstyle \mathsf{s}_{[q]}}\downarrow & & \downarrow{\scriptstyle \mathsf{t}} \\ \mathsf{s}_{[p]}\,\omega & \xrightarrow{\mathsf{s}_{[p]}} & \omega. \end{array}$$

(d) **(Degen)** if ω is degenerate, the following square must commute:

$$\begin{array}{ccc} \mathsf{t}\,\mathsf{t}\,\omega & \xrightarrow{\mathsf{t}} & \mathsf{t}\,\omega \\ {\scriptstyle \mathsf{s}_{[]}}\downarrow & & \downarrow{\scriptstyle \mathsf{t}} \\ \mathsf{t}\,\omega & \xrightarrow{\mathsf{t}} & \omega. \end{array}$$

See [11] for a graphical explanation of these relations.

Notation 3.11. For $n \in \mathbb{N}$, we let $\mathbb{O}_{\leq n}$ be the full subcategory of $\mathbb{O}$ spanned by opetopes of dimension at most n. The subcategories $\mathbb{O}_{<n}$, $\mathbb{O}_{\geq n}$, $\mathbb{O}_{>n}$, and $\mathbb{O}_{=n}$ are defined similarly. Note that the latter is simply the set $\mathbb{O}_n$.

3.4 Opetopic sets Recall from section 1.6 that $\mathcal{P}\mathrm{sh}(\mathbb{O})$ is the category of opetopic sets, i.e. $\mathcal{S}\mathrm{et}$-valued presheaves over $\mathbb{O}$. For $X \in \mathcal{P}\mathrm{sh}(\mathbb{O})$ and $\omega \in \mathbb{O}$, we will refer to the elements of the set X_ω as the *cells* of X of *shape* ω.

Definition 3.12. (1) The representable presheaf at $\omega \in \mathbb{O}_n$ is denoted $O[\omega]$. Its cells are morphisms of $\mathbb{O}$ of the form $\mathsf{f} : \psi \longrightarrow \omega$, for f a sequence of face maps, which we write $\mathsf{f}\omega \in O[\omega]_\psi$ for short. For instance, the cell of maximal dimension is simply ω (as the corresponding sequence of face maps is empty), its $(n-1)$-cells are $\left\{\mathsf{s}_{[p]}\,\omega \mid [p] \in \omega^\bullet\right\} \cup \{\mathsf{t}\,\omega\}$, and there is no cell of dimension $> n$.

(2) The *boundary* $\partial O[\omega]$ of ω is the maximal subpresheaf of $O[\omega]$ not containing the cell ω. We write $\mathsf{b}_\omega : \partial O[\omega] \hookrightarrow O[\omega]$ for the *boundary inclusion*. The set of boundary inclusions is denoted by B.

(3) The *spine* $S[\omega]$ is the maximal subpresheaf of $\partial O[\omega]$ not containing the cell $\mathsf{t}\,\omega$, and we write $\mathsf{s}_\omega : S[\omega] \hookrightarrow O[\omega]$ for the *spine inclusion* of ω. The set of spine inclusions is denoted by S.

Lemma 3.13. *For $\omega \in \mathbb{O}$, with $\dim \omega \geq 1$ the following square is a pushout and a pullback*[7]*, where all arrows are canonical inclusions:*

$$\begin{array}{ccc} \partial O[\mathsf{t}\,\omega] & \longrightarrow & S[\omega] \\ \downarrow & & \downarrow \\ O[\mathsf{t}\,\omega] & \longrightarrow & \partial O[\omega]. \end{array}$$

Lemma 3.14. *Let $n \geq 1$, $\nu \in \mathbb{O}_n$, $[l] \in \nu^|$, and $\psi \in \mathbb{O}_{n-1}$ be such that $\mathsf{e}_{[l]}\,\nu = \mathsf{t}\,\psi$, so that the grafting $\nu \circ_{[l]} \mathsf{Y}_\psi$ is well-defined. Then the following square is a pushout:*

$$\begin{array}{ccc} O[\mathsf{e}_{[l]}\,\nu] & \xrightarrow{\mathsf{t}} & O[\psi] \\ {\scriptstyle \mathsf{e}_{[l]}}\downarrow & & \downarrow{\scriptstyle \mathsf{s}_{[l]}} \\ S[\nu] & \longrightarrow & S[\nu \circ_{[l]} \mathsf{Y}_\psi]. \end{array}$$

[7] Recall that in a topos, the pushout of a monomorphism along any arrow is a monomorphism, and the pushout square is a pullback square. This property is sometimes called "adhesivity", and is a consequence of van Kampenness, or descent, for pushouts of monomorphisms.

Notation 3.15. Let $F : \mathbb{O} \longrightarrow \hom \mathcal{C}$ be a function that maps opetopes to morphisms in some category $\mathcal{C}$, and M the set of maps defined by $\mathsf{M} := \{F(\omega) \mid \omega \in \mathbb{O}\}$. Then for $n \in \mathbb{N}$, we define $\mathsf{M}_{\geq n} := \{F(\omega) \mid \omega \in \mathbb{O}_{\geq n}\}$, and similarly for $\mathsf{M}_{>n}$, $\mathsf{M}_{\leq n}$, $\mathsf{M}_{<n}$, and $\mathsf{M}_{=n}$. For convenience, the latter is abbreviated M_n. If $m \leq n$, we also let $\mathsf{M}_{m,n} = \mathsf{M}_{\geq m} \cap \mathsf{M}_{\leq n}$. By convention, $\mathsf{M}_{\leq n} = \varnothing$ if $n < 0$. For example, $\mathsf{S}_{\geq 2} = \{\mathsf{s}_\omega \mid \omega \in \mathbb{O}_{\geq 2}\}$, and $\mathsf{S}_{n,n+1} = \mathsf{S}_n \cup \mathsf{S}_{n+1}$.

Lemma 3.16. *Let $X \in \mathcal{P}\mathrm{sh}(\mathbb{O})$ be an opetopic set.*

(1) If $\mathsf{S}_{n,n+1} \perp X$, then $\mathsf{B}_{n+1} \perp X$. Equivalently, B_{n+1} is a subset of the $\mathsf{S}_{n,n+1}$-local isomorphisms. Thus every morphism in $\mathsf{B}_{\geq n+1}$ is an $\mathsf{S}_{\geq n}$-local isomorphism.

(2) If $\mathsf{S}_{n,n+1} \perp X$ and $\mathsf{B}_{n+2} \perp X$, then $\mathsf{S}_{n+2} \perp X$. Thus if $\mathsf{S}_{n,n+1} \perp X$ and $\mathsf{B}_{\geq n+2} \perp X$, then $\mathsf{S}_{\geq n} \perp X$.

Proof. (1) Let $\omega \in \mathbb{O}_{n+1}$. Note that the following triangle commutes

$$\begin{array}{ccc} S[\omega] & \xrightarrow{\ i\ } & \partial O[\omega] \\ & \searrow^{\mathsf{s}_\omega} & \downarrow \mathsf{b}_\omega \\ & & O[\omega]. \end{array}$$

Since the class of $\mathsf{S}_{n,n+1}$-local isomorphisms has the 3-for-2 property, in order to show that b_ω is in it, it is enough to show that i is in it. Suppose that $\mathsf{S}_{n,n+1} \perp X$. Take a morphism $f : S[\omega] \longrightarrow X$. The existence of a lift $\partial O[\omega] \longrightarrow X$ follows from the existence of a lift $O[\omega] \longrightarrow X$, since $\mathsf{s}_\omega \perp X$.

For uniqueness, consider two lifts $g, h : \partial O[\omega] \longrightarrow X$ of f. By lemma 3.13, in order to show that they are equal, it suffices to show that they coincide on $O[\mathsf{t}\omega]$. But since they coincide on $S[\omega]$ (as they extend f), they must coincide on the subpresheaf $S[\mathsf{t}\omega] \subseteq S[\omega]$. Since $\mathsf{S}_n \perp X$, g and h coincide on $O[\mathsf{t}\omega]$, and are thus equal.

(2) Let $\omega \in \mathbb{O}_{n+2}$ and $f : S[\omega] \longrightarrow X$. By assumption, the restriction $f|_{S[\mathsf{t}\omega]}$ of f to $S[\mathsf{t}\omega]$ extends to a unique $g : O[\mathsf{t}\omega] \longrightarrow X$. We now show that the following square commutes:

$$\begin{array}{ccc} \partial O[\mathsf{t}\omega] & \longrightarrow & S[\omega] \\ \downarrow & & \downarrow f \\ O[\mathsf{t}\omega] & \xrightarrow{\ g\ } & X. \end{array}$$

By lemma 3.13, it suffices to show that f and g coincide on $S[\mathsf{t}\omega]$ and on $O[\mathsf{t}\mathsf{t}\omega]$. The former is tautological, and the latter follows from the hypothesis that $\mathsf{s}_{\mathsf{t}\mathsf{t}\omega} \perp X$ and that f and g coincide on $S[\mathsf{t}\mathsf{t}\omega] \subseteq S[\mathsf{t}\omega]$. Therefore, the square above commutes, and by lemma 3.13 again, f and g extend to a morphism $h : \partial O[\omega] \longrightarrow X$, which in turn extends to a morphism $i : O[\omega] \longrightarrow X$, since by assumption $\mathsf{B}_{n+2} \perp X$.

For unicity, consider two lifts $i, i' : O[\omega] \longrightarrow X$ of f. By lemma 3.13, they are equal if and only if their restriction $g, g' : O[\mathsf{t}\omega] \longrightarrow X$ are equal. Since $g|_{S[\mathsf{t}\omega]} = f|_{S[\mathsf{t}\omega]} = g'|_{S[\mathsf{t}\omega]}$, and since by assumption $\mathsf{S}_{n+1} \perp X$, we have $g = g'$, and thus $i = i'$. □

Corollary 3.17. *Let X be an opetopic set such that $\mathsf{S}_{n,n+1} \perp X$. Then $\mathsf{S}_{\geq n} \perp X$ if and only if $\mathsf{B}_{\geq n+2} \perp X$.*

Lemma 3.18. *Let $n \in \mathbb{N}$, and $\omega \in \mathbb{O}_{n+2}$. Then the inclusion $S[\mathsf{t}\omega] \hookrightarrow S[\omega]$ is a relative S_{n+1}-cell complex.*[8]

[8] Relative S_{n+1}-cell complexes are transfinite composites of pushouts of maps in S_{n+1}.

Proof. We show that the morphism $S[\mathsf{t}\,\omega] \hookrightarrow S[\omega]$ is a composite of pushouts of elements of S_{n+1}. If ω is degenerate, say $\omega = \mathsf{I}_\phi$ for some $\phi \in \mathbb{O}_n$, then $S[\mathsf{t}\,\omega] = S[\mathsf{Y}_\phi] = O[\phi] = S[\omega]$, so the result trivially holds.

Assume that ω is not degenerate, let $X^{(0)} := S[\mathsf{t}\,\omega]$, and $[p_1] > \cdots > [p_k]$ be the node addresses of ω, sorted in reverse lexicographical order. By induction, assume that $X^{(i-1)}$ is a subpresheaf of $S[\omega]$ containing the $(n+1)$-cells $\mathsf{s}_{[p_1]}\,\omega, \ldots, \mathsf{s}_{[p_{i-1}]}\,\omega \in S[\omega]$. Clearly, this holds when $i = 1$, as $S[\mathsf{t}\,\omega]$ does not contain any $(n+1)$-cell.

Take $[q] \in (\mathsf{s}_{[p_i]}\,\omega)^\bullet$. By induction, and since $[p_i[q]] > [p_i]$, the $(n+1)$-cell $\mathsf{s}_{[p_i[q]]}\,\omega$ is in $X^{(i-1)}$. Further, the n-cell $\mathsf{s}_{[q]}\,\mathsf{s}_{[p_i]}\,\omega$ is present in $X^{(i-1)}$, since by **(Inner)**, $\mathsf{s}_{[q]}\,\mathsf{s}_{[p_i]}\,\omega = \mathsf{t}\,\mathsf{s}_{[p_i[q]]}\,\omega$. Therefore, we have an inclusion $u_i : S[\mathsf{s}_{[p_i]}\,\omega] \longrightarrow X^{(i-1)}$ mapping $\mathsf{s}_{[q]}\,\mathsf{s}_{[p_i]}\,\omega$ to $\mathsf{s}_{[q]}\,\mathsf{s}_{[p_i]}\,\omega$, and let $X^{(i)}$ be the pushout

$$\begin{array}{ccc} S[\mathsf{s}_{[p_i]}\,\omega] & \xrightarrow{u_i} & X^{(i-1)} \\ {\scriptstyle \mathsf{s}_{\mathsf{s}_{[p_i]}\,\omega}}\big\downarrow & & \big\downarrow \\ O[\mathsf{s}_{[p_i]}\,\omega] & \xrightarrow{\quad\ulcorner} & X^{(i)} \end{array}$$

Clearly, $X^{(i)}$ is a subpresheaf of $S[\omega]$ containing the $(n+1)$-cell $\mathsf{s}_{[p_j]}\,\omega$ for $1 \leq j \leq i$, and the induction hypothesis is satisfied.

Finally, $X^{(k)} \subseteq S[\omega]$ contains all the $(n+1)$-cells of $S[\omega]$, whence $X^{(k)} = S[\omega]$. By construction, the chain of inclusions $S[\mathsf{t}\,\omega] = X^{(0)} \hookrightarrow X^{(1)} \hookrightarrow \cdots \hookrightarrow X^{(k)} = S[\omega]$ is a relative S_{n+1}-cell complex. □

Corollary 3.19. *Let $n \in \mathbb{N}$, and $\omega \in \mathbb{O}_{n+2}$. Then the target map $\mathsf{t}\,\omega \longrightarrow \omega$ of ω is an $\mathsf{S}_{n+1,n+2}$-local isomorphism.*

Proof. In the square below

$$\begin{array}{ccc} S[\mathsf{t}\,\omega] & \xrightarrow{\mathsf{s}_{\mathsf{t}\,\omega}} & O[\mathsf{t}\,\omega] \\ {\scriptstyle r}\big\downarrow & & \big\downarrow{\scriptstyle \mathsf{t}} \\ S[\omega] & \xrightarrow{\mathsf{s}_\omega} & O[\omega] \end{array}$$

the map r is an S_{n+1}-local isomorphism by lemma 3.18, and the horizontal maps are in $\mathsf{S}_{n+1,n+2}$. The result follows by 3-for-2. □

Corollary 3.20. *Let $\omega \in \mathbb{O}_n$.*

(1) $\mathsf{t}\,\mathsf{t} = \mathsf{s}_{[]}\,\mathsf{t} : \omega \longrightarrow \mathsf{I}_\omega$ *is in* S_{n+2}.

(2) The morphisms $\mathsf{s}_{[]}, \mathsf{t} : \omega \longrightarrow \mathsf{Y}_\omega$ *are* $\mathsf{S}_{n+1,n+2}$*-local isomorphisms.*

Proof. (1) The map $\mathsf{t}\,\mathsf{t} = \mathsf{s}_{[]}\,\mathsf{t} : \omega \longrightarrow \mathsf{I}_\omega$ is precisely the spine inclusion $\mathsf{s}_{\mathsf{I}_\omega}$ of the degenerate $(n+2)$-opetope I_ω.

(2) The source map $\mathsf{s}_{[]} : \omega \longrightarrow \mathsf{Y}_\omega$ is precisely the spine inclusion $\mathsf{s}_{\mathsf{Y}_\omega}$ of the $(n+1)$-opetope Y_ω. The target map $\mathsf{t} : \omega \longrightarrow \mathsf{Y}_\omega$ is the morphism $\mathsf{t} : \mathsf{t}\,\mathsf{t}\,\mathsf{I}_\omega \longrightarrow \mathsf{t}\,\mathsf{I}_\omega$ and is the vertical arrow in the diagram below.

$$\begin{array}{ccc} \omega = S[\mathsf{I}_\omega] & & \\ {\scriptstyle \mathsf{t}}\big\downarrow & \searrow^{\mathsf{s}_{\mathsf{I}_\omega}} & \\ \mathsf{Y}_\omega = \mathsf{t}\,\mathsf{I}_\omega & \xrightarrow[\mathsf{t}]{} & \mathsf{I}_\omega. \end{array}$$

The horizontal arrow is an $\mathsf{S}_{n+1,n+2}$-local isomorphism by corollary 3.19 and the diagonal arrow is in S_{n+2} by point (1). The result follows by 3-for-2. □

3.5 Extensions

Reminder 3.21. Recall that a functor between small categories $u : \mathcal{A} \longrightarrow \mathcal{B}$ induces a restriction $u^* : \mathcal{P}\mathrm{sh}(\mathcal{B}) \longrightarrow \mathcal{P}\mathrm{sh}(\mathcal{A})$ that admits both adjoints $u_! \dashv u^* \dashv u_*$, given by pointwise left and right Kan extension.

Notation 3.22. Let $m \in \mathbb{N}$ and $n \in \mathbb{N} \cup \{\infty\}$ be such that $m \leq n$, and let $\mathbb{O}_{m,n}$ be the full subcategory of $\mathbb{O}$ spanned by opetopes ω such that $m \leq \dim \omega \leq n$. For instance, $\mathbb{O}_{m,\infty} = \mathbb{O}_{\geq m}$.

Definition 3.23 (Truncation)**.** The inclusion $\iota^{\geq m} : \mathbb{O}_{m,n} \longrightarrow \mathbb{O}_{\geq m}$ induces a restriction functor $(-)_{m,n} : \mathcal{P}\mathrm{sh}(\mathbb{O}_{\geq m}) \longrightarrow \mathcal{P}\mathrm{sh}(\mathbb{O}_{m,n})$, called *truncation*, that has both a left adjoint $\iota_!^{\geq m}$ and a right adjoint $\iota_*^{\geq m}$. Explicitly, for $X \in \mathcal{P}\mathrm{sh}(\mathbb{O}_{m,n})$, the presheaf $\iota_!^{\geq m} X$ is the "extension by 0", i.e. $(\iota_!^{\geq m} X)_{m,n} = X$, and $(\iota_!^{\geq m} X)_\psi = \varnothing$ for all $\psi \in \mathbb{O}_{>n}$. On the other hand, $\iota_*^{\geq m} X$ is the "canonical extension" of X into a presheaf over $\mathbb{O}_{\geq m}$: we have $(\iota_*^{\geq m} X)_{m,n} = X$, and $\mathsf{B}_{>n} \perp \iota_*^{\geq m} X$, which uniquely determines $\iota_*^{\geq m} X$.

Likewise, the inclusion $\iota^{\leq n} : \mathbb{O}_{m,n} \longrightarrow \mathbb{O}_{\leq n}$ induces a restriction functor $\mathcal{P}\mathrm{sh}(\mathbb{O}_{\leq n}) \longrightarrow \mathcal{P}\mathrm{sh}(\mathbb{O}_{m,n})$, also denoted by $(-)_{m,n}$ and again called *truncation*, that has both a left adjoint $\iota_!^{\leq n}$ and a right adjoint $\iota_*^{\leq n}$. Explicitly, for $X \in \mathcal{P}\mathrm{sh}(\mathbb{O}_{m,n})$, the presheaf $\iota_!^{\leq n} X$ is the "canonical extension" of X into a presheaf over $\mathbb{O}_{\leq n}$:

$$\iota_!^{\leq n} X = \operatorname*{colim}_{O[\psi]_{m,n} \to X} O[\psi].$$

On the other hand, $\iota_*^{\leq n} X$ is the "terminal extension" of X in that $(\iota_*^{\leq n} X)_{m,n} = X$, and $(\iota_*^{\leq n} X)_\psi$ is a singleton, for all $\psi \in \mathbb{O}_{<m}$.

For $n < \infty$, we write $(-)_{\leq n}$ for $(-)_{0,n} : \mathcal{P}\mathrm{sh}(\mathbb{O}_{\geq 0}) = \mathcal{P}\mathrm{sh}(\mathbb{O}) \longrightarrow \mathcal{P}\mathrm{sh}(\mathbb{O}_{0,n}) = \mathcal{P}\mathrm{sh}(\mathbb{O}_{\leq n})$, and let $(-)_{<n} = (-)_{\leq n-1}$ if $n \geq 0$. Similarly, we note $(-)_{m,\infty} : \mathcal{P}\mathrm{sh}(\mathbb{O}_{\leq \infty}) = \mathcal{P}\mathrm{sh}(\mathbb{O}) \longrightarrow \mathcal{P}\mathrm{sh}(\mathbb{O}_{m,\infty}) = \mathcal{P}\mathrm{sh}(\mathbb{O}_{\geq m})$ by $(-)_{\geq m}$, and let $(-)_{>m} = (-)_{\geq m+1}$.

Proposition 3.24. *(1) The functors $\iota_!^{\geq m}$, $\iota_*^{\geq m}$, $\iota_!^{\leq n}$, and $\iota_*^{\leq n}$ are fully faithful.*

(2) A presheaf $X \in \mathcal{P}\mathrm{sh}(\mathbb{O}_{\geq m})$ is in the essential image of $\iota_!^{\geq m}$ if and only if $X_{>n} = \varnothing$.

(3) A presheaf $X \in \mathcal{P}\mathrm{sh}(\mathbb{O}_{\geq m})$ is in the essential image of $\iota_*^{\geq m}$ if and only if for all $\omega \in \mathbb{O}_{>n}$ we have $(\mathsf{b}_\omega)_{\geq m} \perp X$.

(4) A presheaf $X \in \mathcal{P}\mathrm{sh}(\mathbb{O}_{\leq n})$ is in the essential image of $\iota_*^{\leq n}$ if and only if for all $\omega \in \mathbb{O}_{<m}$ we have $(\mathsf{o}_\omega)_{\leq n} \perp X$, i.e. X_ω is a singleton.

Proof. The first point follows from the fact that $\iota^{\geq m}$ and $\iota^{\leq n}$ are fully faithful, and [17, exposé I, proposition 5.6]. The rest are straightforward verifications. □

Notation 3.25. To ease notations, we sometimes leave truncations implicit, e.g. point (3) of last proposition can be reworded as: a presheaf $X \in \mathcal{P}\mathrm{sh}(\mathbb{O}_{\geq m})$ is in the essential image of $\iota_*^{\geq m}$ if and only if $\mathsf{B}_{>n} \perp X$.

4. Opetopic algebras

Let $k \leq n \in \mathbb{N}$, and recall that $\mathbb{O}_{n-k,n} \hookrightarrow \mathbb{O}$ is the full subcategory of those opetopes ω such that $n - k \leq \dim \omega \leq n$. A *$k$-coloured, n-dimensional opetopic algebra*, or (k, n)-opetopic algebra, will be an algebraic structure on a presheaf over $\mathbb{O}_{n-k,n}$, whose cells of dimension n are "operations" that can be "composed" in ways encoded by $(n + 1)$-cells[9]. As we will see, the fact that the

[9] Recall that an $(n + 1)$-opetope is precisely a pasting diagram of n-opetopes.

operations and relations of a (k,n)-opetopic algebra are encoded by opetopes of dimension $\geq n$ results in the category $\mathcal{A}\mathrm{lg}_{k,n}$ of (k,n)-opetopic algebras always having a canonical fully faithful *nerve functor* to the category $\mathcal{P}\mathrm{sh}(\mathbb{O})$ of opetopic sets (theorem 4.74).

We begin this chapter by surveying elements of the theory of *parametric right adjoint (p.r.a.) monads*. This will be essential to the definition of the *coloured* $\mathfrak{Z}^n$ monad, which is an extension of $\mathfrak{Z}^n$ (in the sense of definition 3.1) to $\mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n})$. The algebras of this new monad will be the (k,n)-opetopic algebras. Then, we introduce the category $\mathbb{A}$ of *opetopic shapes*, which is the category of free algebras over $\mathbb{O}_{n-k,n}$. We investigate ways to construct algebras from presheaves over $\mathbb{A}$ and $\mathbb{O}$. Specifically, we obtain two adjunctions

$$h : \mathcal{P}\mathrm{sh}(\mathbb{O}) \rightleftarrows \mathcal{A}\mathrm{lg} : M, \qquad \tau : \mathcal{P}\mathrm{sh}(\mathbb{A}) \rightleftarrows \mathcal{A}\mathrm{lg} : N,$$

where the left adjoints are called *algebraic realisations*, and where the right adjoints are their respective *nerve functors*. The theory of p.r.a. monads, which we review in section 4.1, provides remarkable information about the nerves, which we state in theorems 4.38 and 4.74.

4.1 Parametric right adjoint monads In preparation to the main results of this section, we survey elements of the theory of parametric right adjoint (p.r.a.) monads, which will be essential to the definition and description of (k,n)-opetopic algebras. A comprehensive treatment of this theory can be found in [18].

Definition 4.1. If $T : \mathcal{C} \longrightarrow \mathcal{D}$ is a functor, and $\mathcal{C}$ has a terminal object 1, then T factors as

$$\mathcal{C} \xrightarrow{\cong} \mathcal{C}/1 \xrightarrow{T_1} \mathcal{D}/T1 \longrightarrow \mathcal{D}, \tag{4.2}$$

where the second functor is the induced functor between slice categories, and the third is the domain functor. We say that T is a *parametric right adjoint* (abbreviated p.r.a.) if T_1 has a left adjoint E.

Remark 4.3. We shall immediately restrict definition 4.1 to the case where $\mathcal{C} = \mathcal{D} = \mathcal{P}\mathrm{sh}(\mathcal{A})$ for a small category $\mathcal{A}$. Recall that $\mathcal{A}/T1$ is the category of elements of $T1 \in \mathcal{P}\mathrm{sh}(\mathcal{A})$, and using the equivalence $\mathcal{P}\mathrm{sh}(\mathcal{A}/T1) \simeq \mathcal{P}\mathrm{sh}(\mathcal{A})/T1$, the factorisation of (4.2) becomes

$$\mathcal{P}\mathrm{sh}(\mathcal{A}) \xrightarrow{T_1} \mathcal{P}\mathrm{sh}(\mathcal{A}/T1) \longrightarrow \mathcal{P}\mathrm{sh}(\mathcal{A}). \tag{4.4}$$

Let E be the left adjoint of T_1. Then T_1 is the nerve of the restriction $E : \mathcal{A}/T1 \longrightarrow \mathcal{P}\mathrm{sh}(\mathcal{A})$ of E to the representable presheaves, and the usual nerve formula gives

$$(T_1 X)_x = \mathcal{P}\mathrm{sh}(\mathcal{A})(Ex, X),$$

where $X \in \mathcal{P}\mathrm{sh}(\mathcal{A})$ and $x \in \mathcal{A}/T1$. Therefore, for $a \in \mathcal{A}$, we have

$$(TX)_a = \sum_{x \in (T1)_a} \mathcal{P}\mathrm{sh}(\mathcal{A})(Ex, X) \tag{4.5}$$

In fact, it is clear that the data of the object $T1 \in \mathcal{P}\mathrm{sh}(\mathcal{A})$ and of the functor $E : \mathcal{A}/T1 \longrightarrow \mathcal{P}\mathrm{sh}(\mathcal{A})$ completely describe (via equation (4.5)) the functor T up to isomorphism. Let Θ_0 (leaving T implicit) be the full subcategory of $\mathcal{P}\mathrm{sh}(\mathcal{A})$ that is the image of the restriction of the left adjoint $E : \mathcal{A}/T1 \longrightarrow \mathcal{P}\mathrm{sh}(\mathcal{A})$ of T_1. Objects of Θ_0 are called T-*cardinals*.

Definition 4.6. A *p.r.a. monad* is a monad T whose endofunctor is a p.r.a. and whose unit $\mathrm{id} \longrightarrow T$ and multiplication $TT \longrightarrow T$ are cartesian natural transformations.

Remark 4.7. A p.r.a. monad T on a presheaf category is an example of a *monad with arities* [3]. The theory of monads with arities provides a remarkable amount of information about the free-forgetful adjunction $\mathcal{P}\mathrm{sh}(\mathcal{A}) \rightleftarrows \mathcal{A}\mathrm{lg}(T)$ and about the category of algebras $\mathcal{A}\mathrm{lg}(T)$.

Notation 4.8. With a slight (but standard) abuse of notations, let $T : \mathcal{P}\mathrm{sh}(\mathcal{A}) \longrightarrow \mathcal{A}\mathrm{lg}(T)$ be the free T-algebra functor. The (identity-on-objects, fully faithful) factorisation of the composite $\Theta_0 \hookrightarrow \mathcal{P}\mathrm{sh}(\mathcal{A}) \xrightarrow{T} \mathcal{A}\mathrm{lg}(T)$ will be denoted by

$$\Theta_0 \xrightarrow{j_T} \Theta_T \xhookrightarrow{i_T} \mathcal{A}\mathrm{lg}(T). \tag{4.9}$$

In other words, Θ_T is the full subcategory of $\mathcal{A}\mathrm{lg}(T)$ spanned by free algebras over T-cardinals.

Proposition 4.10 ([18, proposition 4.20])**.** *Let $T : \mathcal{P}\mathrm{sh}(\mathcal{A}) \longrightarrow \mathcal{P}\mathrm{sh}(\mathcal{A})$ be a p.r.a. monad, and Θ_0 be as in remark 4.3. Then the Yoneda embedding $\mathsf{y}_{\mathcal{A}}$ factors as*

$$\mathcal{A} \xhookrightarrow{i} \Theta_0 \xhookrightarrow{i_0} \mathcal{P}\mathrm{sh}(\mathcal{A}),$$

where i_0 is the full embedding of T-cardinals into all presheaves. In other words, representable presheaves are T-cardinals.

Lemma 4.11 ([3, lemma 3.6])**.** *Let*

$$\mathcal{A} \xhookrightarrow{i} \mathcal{B} \xhookrightarrow{i_0} \mathcal{P}\mathrm{sh}(\mathcal{A}),$$

be a factorization of the Yoneda embedding with i and i_0 fully faithful, and let

$$\mathsf{J}_{\mathcal{A}} := \{\varepsilon_\theta : i_! i^* b \longrightarrow b \mid b \in \mathcal{B} - \mathrm{im}\, i\},$$

where $\varepsilon : i_! i^* \longrightarrow \mathrm{id}_{\mathcal{P}\mathrm{sh}(\mathcal{B})}$ is the counit of the adjunction $i_! : \mathcal{P}\mathrm{sh}(\mathcal{A}) \rightleftarrows \mathcal{P}\mathrm{sh}(\mathcal{B}) : i^*$. Then i_* is fully faithful, and a presheaf $X \in \mathcal{P}\mathrm{sh}(\mathcal{B})$ is in the essential image of $N_{i_0} \cong i_*$ if and only if $\mathsf{J}_{\mathcal{A}} \perp X$.

Theorem 4.12. *Let T be a p.r.a. monad on $\mathcal{P}\mathrm{sh}(\mathcal{A})$.*

(1) *The functors $i_0 : \Theta_0 \longrightarrow \mathcal{P}\mathrm{sh}(\mathcal{A})$ and $i_T : \Theta_T \longrightarrow \mathcal{A}\mathrm{lg}(T)$ are dense. Equivalently, their nerve functors $N_{i_0} : \mathcal{P}\mathrm{sh}(\mathcal{A}) \longrightarrow \mathcal{P}\mathrm{sh}(\Theta_0)$ and $N_{i_T} : \mathcal{A}\mathrm{lg}(T) \longrightarrow \mathcal{P}\mathrm{sh}(\Theta_T)$ are fully faithful.*

(2) *The following diagram is an exact adjoint square*[10]*.*

$$\begin{array}{ccc} \mathcal{P}\mathrm{sh}(\mathcal{A}) & \underset{U_T}{\overset{F_T}{\rightleftarrows}} & \mathcal{A}\mathrm{lg}(T) \\ {\scriptstyle N_0}\big\downarrow & \perp & \big\downarrow{\scriptstyle N_T} \\ \mathcal{P}\mathrm{sh}(\Theta_0) & \underset{j_T^*}{\overset{j_{T!}}{\rightleftarrows}} & \mathcal{P}\mathrm{sh}(\Theta_T) \end{array}$$

In particular, both squares commute up to natural isomorphism.

(3) *(Segal condition) A presheaf $X \in \mathcal{P}\mathrm{sh}(\Theta_T)$ is in the essential image of N_{i_T} if and only if $j_T^* X$ is in the essential image of N_{i_0}.*

[10] There exists a natural isomorphism $N_0 U_T \cong j_T^* N_T$ whose mate $j_{T!} N_0 \longrightarrow N_T F_T$ is invertible (satisfies the Beck-Chevalley condition).

Proof. Density of i_0 is a direct consequence of lemma 4.11, and density of i_T is [3, theorem 1.10]. Point (2) is [3, proposition 1.9], and the Segal condition is [18, theorem 4.10 (2)]. □

Corollary 4.13. *Let*

$$\mathsf{J}_T := j_{T!}\mathsf{J}_{\mathcal{A}} = \{j_{T!}\varepsilon_\theta : j_{T!}i_!i^*\theta \longrightarrow j_{T!}\theta \mid \theta \in \Theta_0 - \operatorname{im} i\},$$

where ε is the counit of the adjunction $i_! \dashv i^*$. Then a presheaf $X \in \mathcal{P}\mathrm{sh}(\Theta_T)$ is in the essential image of N_{i_T} if and only if $\mathsf{J}_T \perp X$. As a consequence, the left adjoint $\mathcal{P}\mathrm{sh}(\Theta_T) \longrightarrow \mathcal{A}\mathrm{lg}(T)$ of N_{i_T} (i.e. the left Kan extension of i_T along the Yoneda embedding) exhibits an equivalence of categories

$$\mathsf{J}_T^{-1}\mathcal{P}\mathrm{sh}(\Theta_T) \xrightarrow{\sim} \mathcal{A}\mathrm{lg}(T).$$

Proof. The first claim follows from lemma 4.11 and theorem 4.12; the second from section 1.6.3. □

4.2 Coloured $\mathfrak{Z}^n$-algebras

Remark 4.14. Recall the definition of the polynomial monad $\mathfrak{Z}^n$ from definition 3.1. If $X = (X_\psi \mid \psi \in \mathbb{O}_n)$ is in $\mathcal{S}\mathrm{et}/\mathbb{O}_n$, and if $\omega \in \mathbb{O}_n$, then

$$(\mathfrak{Z}^n X)_\omega = \sum_{\substack{\nu \in \mathbb{O}_{n+1} \\ \mathsf{t}\,\nu = \omega}} \prod_{[p] \in \nu^\bullet} X_{\mathsf{s}_{[p]}\,\nu}.$$

Under the equivalence $\mathcal{S}\mathrm{et}/\mathbb{O}_n \simeq \mathcal{P}\mathrm{sh}(\mathbb{O}_n)$, this formula can be rewritten as

$$(\mathfrak{Z}^n X)_\omega = \sum_{\substack{\nu \in \mathbb{O}_{n+1} \\ \mathsf{t}\,\nu = \omega}} \mathcal{P}\mathrm{sh}(\mathbb{O}_n)(S[\nu], X),$$

where $S[\nu]$ is the truncated spine of ν (see definition 3.12).

In this section, we extend the polynomial monad $\mathfrak{Z}^n$ over $\mathcal{S}\mathrm{et}/\mathbb{O}_n = \mathcal{P}\mathrm{sh}(\mathbb{O}_n)$ to a p.r.a. monad over $\mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n})$, where $k \leq n$. This new setup will encompass more known examples than the uncoloured case (see proposition 4.31). For instance, recall that the polynomial monad $\mathfrak{Z}^2$ on $\mathcal{S}\mathrm{et}/\mathbb{O}_2 \cong \mathcal{S}\mathrm{et}/\mathbb{N}$ is exactly the monad of planar operads. The extension of $\mathfrak{Z}^2$ to $\mathcal{P}\mathrm{sh}(\mathbb{O}_{1,2})$ will retrieve *coloured* planar operads as algebras. Similarly, the polynomial monad $\mathfrak{Z}^1$ on $\mathcal{S}\mathrm{et}$ is the free-monoid monad, which we would like to vary to obtain "coloured monoids", i.e. small categories.

The first step of this construction is to define $\mathfrak{Z}^n$ as a p.r.a. functor, i.e. an endofunctor $\mathfrak{Z}^n$ on $\mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n})$ such that in the sequence below, $\mathfrak{Z}^n_1$ is a right adjoint:

$$\mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n}) \xrightarrow{\mathfrak{Z}^n_1} \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n}/\mathfrak{Z}^n 1) \longrightarrow \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n}).$$

Following remark 4.3, it suffices to define its value $\mathfrak{Z}^n 1$ on the terminal presheaf, and to specify a functor $E : \mathbb{O}_{n-k,n}/\mathfrak{Z}^n 1 \longrightarrow \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n})$.

Definition 4.15. Define $\mathfrak{Z}^n 1 \in \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n})$ as

$$(\mathfrak{Z}^n 1)_\psi := \{*\}, \qquad (\mathfrak{Z}^n 1)_\omega := \{\nu \in \mathbb{O}_{n+1} \mid \mathsf{t}\,\nu = \omega\},$$

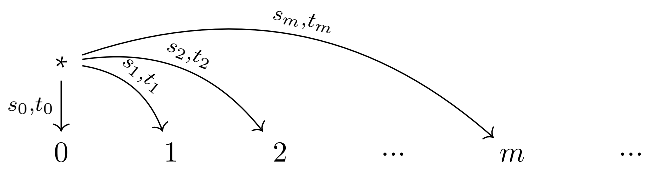
where $\psi \in \mathbb{O}_{n-k,n-1}$ and $\omega \in \mathbb{O}_n$, along with the obvious restriction maps. We now define a functor $E : \mathbb{O}_{n-k,n}/\mathfrak{Z}^n 1 \longrightarrow \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n})$. On objects, for $* \in (\mathfrak{Z}^n 1)_\psi$ and $\nu \in (\mathfrak{Z}^n 1)_\omega$, let[11]

$$E(*) := O[\psi], \qquad E(\nu) := S[\nu]. \tag{4.16}$$

On morphisms, E takes face maps to the canonical inclusions. The functor $\mathfrak{Z}^n_1 : \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n}) \longrightarrow \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n}/\mathfrak{Z}^n 1)$ is defined as the right adjoint to the left Kan extension of E along the Yoneda embedding, i.e. $\mathfrak{Z}^n_1 = N_E$ (see section 1.6.1). We now recover the endofunctor $\mathfrak{Z}^n$ explicitly using equation (4.5): for $\psi \in \mathbb{O}_{n-k,n-1}$ we have $(\mathfrak{Z}^n X)_\psi \cong X_\psi$, and for $\omega \in \mathbb{O}_n$, we recover a formula similar to remark 4.14

$$(\mathfrak{Z}^n X)_\omega \cong \sum_{\substack{\nu \in \mathbb{O}_{n+1} \\ \mathsf{t}\,\nu = \omega}} \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n})(S[\nu], X).$$

Example 4.17. Let us unfold definition 4.15 in the case $n = 1$ and $k = 1$. Here, $\mathcal{P}\mathrm{sh}(\mathbb{O}_{0,1})$ is the category of directed graphs, whose terminal object 1 is the graph with one vertex and a loop. The graph $\mathfrak{Z}^1 1$ also has one vertex, but this time, it has an many loops as there are 2-opetopes, i.e. one loop per element in $\mathbb{N}$. The category of elements $\mathbb{O}_{0,1}/\mathfrak{Z}^1 1$ looks like this:



where $*$ corresponds to the vertex of $\mathfrak{Z}^1 1$, the numbers on the second row correspond to its vertices, and the morphisms are the inclusions of $*$ as the source or target of these vertices. The functor $E : \mathbb{O}_{0,1}/\mathfrak{Z}^1 1 \longrightarrow \mathcal{P}\mathrm{sh}(\mathbb{O}_{0,1})$ maps $*$ to the graph with one vertex and no edges, and maps m to the linear graph with m consecutive edges:

$$E(*) = (\bullet)\,, \qquad E(m) = (\bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \cdots \longrightarrow \bullet)\,.$$

On morphisms, $E(s_n)$ (resp. $E(t_n)$) is the inclusion of $\bullet$ as the first (resp. as the last) vertex of $E(m)$. Then, for $X \in \mathcal{P}\mathrm{sh}(\mathbb{O}_{0,1})$, the graph $\mathfrak{Z}^1 X$ has the same vertices as X, but its edges are paths in X. In other words, $\mathfrak{Z}^1 : \mathcal{P}\mathrm{sh}(\mathbb{O}_{0,1}) \longrightarrow \mathcal{P}\mathrm{sh}(\mathbb{O}_{0,1})$ is the free category monad.

Recall from definition 4.6 that a p.r.a. monad is a monad T whose unit $\mathrm{id} \longrightarrow T$ and multiplication $TT \longrightarrow T$ are cartesian, and such that its underlying functor is a p.r.a. We now endow $\mathfrak{Z}^n$ with the structure of a p.r.a. monad over $\mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n})$. We first specify the unit and multiplication $\eta_1 : 1 \longrightarrow \mathfrak{Z}^n 1$ and $\mu_1 : \mathfrak{Z}^n\mathfrak{Z}^n 1 \longrightarrow \mathfrak{Z}^n 1$ on the terminal object 1, and extend them to cartesian natural transformations (lemma 4.23). Next, we check that the required monad identities hold for 1 (lemma 4.25), which automatically gives us the desired monad structure on $\mathfrak{Z}^n$.

Definition 4.18. Let $\mathbb{O}^{(2)}_{n+2}$ be the set of $(n+2)$-opetopes *of uniform height* 2, i.e. of the form

$$\mathsf{Y}_\alpha \bigcirc_{[[p]]} \mathsf{Y}_{\beta_{[p]}},$$

with $\alpha, \beta_{[p]} \in \mathbb{O}_{n+1}$ and $[p]$ ranging over $\alpha^\bullet$.

[11] Note that in equation (4.16), the presheaves $O[\psi]$ and $S[\nu]$ are considered in $\mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n})$, but the truncations are left implicit.

Proposition 4.19. *If* $X \in \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n})$*, then* $(\mathfrak{Z}^n\mathfrak{Z}^n X)_{<n} = X_{<n}$*, and if* $\omega \in \mathbb{O}_n$*, then*

$$(\mathfrak{Z}^n\mathfrak{Z}^n X)_\omega \cong \sum_{\substack{\xi \in \mathbb{O}^{(2)}_{n+2} \\ \mathsf{t}\mathsf{t}\xi = \omega}} \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n})(S[\mathsf{t}\xi], X).$$

Proof. Take $\omega \in \mathbb{O}_n$, and $x \in (\mathfrak{Z}^n\mathfrak{Z}^n X)_\omega$, say $x : S[\nu] \longrightarrow \mathfrak{Z}^n X$, where $\mathsf{t}\nu = \omega$. For $[p_i] \in \nu^\bullet$, write $x_i := x[p_i] : S[\nu_i] \longrightarrow X$, where $\mathsf{t}\nu_i = \mathsf{s}_{[p_i]}\nu$. Informally, x is a "pasting diagram of pasting diagrams" of X, i.e. a pasting diagram of the x_i's, which are themselves pasting diagrams in X. The goal is to assemble the x_i's in a single pasting diagram $\Phi(x)$. Let

$$\xi := \mathsf{Y}_\nu \bigcirc_{[[p_i]]} \mathsf{Y}_{\nu_i},$$

and note that $\mathsf{t}\mathsf{t}\xi = \mathsf{t}\mathsf{s}_{[]}\xi = \mathsf{t}\nu = \omega$ by **(Glob1)**. We now define a map $\Phi(x) : S[\mathsf{t}\xi] \longrightarrow X$. Note that leaf addresses of ξ are of the form $[[p_i][l]]$, where $[l] \in \nu_i^|$, thus node addresses of $\mathsf{t}\xi$ are of the form $\wp_\xi[[p_i][l]]$. Let

$$\Phi(x)\left(\wp_\xi[[p_i][l]]\right) := x_i\left(\wp_{\nu_i}[l]\right).$$

The construction of $\Phi(x)$ provides a map

$$\Phi : (\mathfrak{Z}^n\mathfrak{Z}^n X)_\omega \longrightarrow \sum_{\substack{\xi \in \mathbb{O}^{(2)}_{n+2} \\ \mathsf{t}\mathsf{t}\xi = \omega}} \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n})(S[\mathsf{t}\xi], X)$$

whose inverse we now construct. Let $\xi \in \mathbb{O}^{(2)}_{n+2}$, say

$$\xi = \mathsf{Y}_\alpha \bigcirc_{[[p]]} \mathsf{Y}_{\beta_{[p]}},$$

be such that $\mathsf{t}\mathsf{t}\xi = \omega$, and take $y : S[\mathsf{t}\xi] \longrightarrow X$. Write $\nu := \mathsf{t}\xi$. As noted in definition 4.18, ξ exhibits a partition of ν into subtrees, and let $\Psi(y) : S[\alpha] \longrightarrow \mathfrak{Z}^n X$ map $[p]$ to the restriction of y to the subtree $\beta_{[p]}$ of ν. It is routine verification to check that Φ and Ψ are mutually inverse. □

Definition 4.20. We now define $\eta_1 : 1 \longrightarrow \mathfrak{Z}^n 1$ and $\mu_1 : \mathfrak{Z}^n\mathfrak{Z}^n 1 \longrightarrow \mathfrak{Z}^n 1$, the monad laws of $\mathfrak{Z}^n$, on the terminal presheaf $1 \in \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n})$. In dimension $< n$, they are the identity. Let $\omega \in \mathbb{O}_n$. Recall from definition 4.15 that $(\mathfrak{Z}^n 1)_\omega = \{\nu \in \mathbb{O}_{n+1} \mid \mathsf{t}\nu = \omega\}$, and by proposition 4.19,

$$(\mathfrak{Z}^n\mathfrak{Z}^n 1)_\omega = \left\{\xi \in \mathbb{O}^{(2)}_{n+2} \mid \mathsf{t}\mathsf{t}\xi = \omega\right\}. \tag{4.21}$$

Now, let $(\eta_1)_\omega$ map the unique element of 1_ω to $\mathsf{Y}_\omega \in (\mathfrak{Z}^n 1)_\omega$, and let $(\mu_1)_\omega$ map $\xi \in (\mathfrak{Z}^n\mathfrak{Z}^n 1)_\omega$ to $\mathsf{t}\xi \in (\mathfrak{Z}^n 1)_\omega$.

Remark 4.22. Let $X \in \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n})$, and consider the terminal map $! : X \longrightarrow 1$. The map $\mathfrak{Z}^n ! : (\mathfrak{Z}^n X)_\omega \longrightarrow (\mathfrak{Z}^n 1)_\omega$ simply maps a pasting diagram $f : S[\nu] \longrightarrow X$ (where $\mathsf{t}\nu = \omega$) to its shape ν.

Lemma 4.23. *Let* $X \in \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n})$*, and consider the terminal map* $! : X \longrightarrow 1$*. To alleviate notations, write* $p := \mathfrak{Z}^n ! : \mathfrak{Z}^n X \longrightarrow \mathfrak{Z}^n 1$*. there exists maps* $\eta_X : X \longrightarrow \mathfrak{Z}^n X$ *and* $\mu_X : \mathfrak{Z}^n\mathfrak{Z}^n X \longrightarrow \mathfrak{Z}^n$ *such that the following squares are cartesian:*

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & \mathfrak{Z}^n X \\ {\scriptstyle !}\downarrow & & \downarrow{\scriptstyle p} \\ 1 & \xrightarrow{\eta_1} & \mathfrak{Z}^n 1, \end{array} \qquad\qquad \begin{array}{ccc} \mathfrak{Z}^n\mathfrak{Z}^n X & \xrightarrow{\mu_X} & \mathfrak{Z}^n X \\ {\scriptstyle \mathfrak{Z}^n p}\downarrow & & \downarrow{\scriptstyle p} \\ \mathfrak{Z}^n\mathfrak{Z}^n 1 & \xrightarrow{\mu_1} & \mathfrak{Z}^n 1. \end{array} \tag{4.24}$$

In particular, the maps η_X and μ_X assemble into cartesian natural transformations $\eta : \mathrm{id} \longrightarrow \mathfrak{Z}^n$ and $\mu : \mathfrak{Z}^n \mathfrak{Z}^n \longrightarrow \mathfrak{Z}^n$.

Proof. All morphisms are identities in dimension $< n$, so it suffices to check that both squares are cartesian in dimension n.

(1) If P is the pullback

$$\begin{array}{ccc} P & \longrightarrow & \mathfrak{Z}^n X \\ {\scriptstyle !}\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ 1 & \xrightarrow{\eta_1} & \mathfrak{Z}^n 1, \end{array}$$

then for $\omega \in \mathbb{O}_n$ we have

$$P_\omega = \{x \in \mathfrak{Z}^n X \mid p(x) = \mathsf{Y}_\omega\} = \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n})(S[\mathsf{Y}_\omega], X) = X_\omega,$$

as $S[\mathsf{Y}_\omega] = O[\omega]$.

(2) Let P be the bullback

$$\begin{array}{ccc} P & \longrightarrow & \mathfrak{Z}^n X \\ \downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ \mathfrak{Z}^n \mathfrak{Z}^n 1 & \xrightarrow{\mu_1} & \mathfrak{Z}^n 1, \end{array}$$

and let $\omega \in \mathbb{O}_n$. By definition, and with equation (4.21), P_ω is the set of all pairs (ξ, x), where $\xi \in \mathbb{O}^{(2)}_{n+2}$ is such that $\mathsf{t}\mathsf{t}\xi = \omega$, $x : S[\nu] \longrightarrow X$ is such that $\mathsf{t}\nu = \omega$, and subject to the constraint that $\mathsf{t}\xi = \nu$. By proposition 4.19, it is clear that $P_\omega \cong (\mathfrak{Z}^n \mathfrak{Z}^n X)_\omega$. $\square$

Lemma 4.25. *The following diagrams commute:*

$$\begin{array}{ccccc} \mathfrak{Z}^n 1 & \xrightarrow{\eta_{\mathfrak{Z}^n 1}} & \mathfrak{Z}^n \mathfrak{Z}^n 1 & \xleftarrow{\mathfrak{Z}^n \eta_1} & \mathfrak{Z}^n 1 \\ & \diagdown\!\!\diagdown & \downarrow{\scriptstyle \mu_1} & \diagup\!\!\diagup & \\ & & \mathfrak{Z}^n 1, & & \end{array} \qquad \begin{array}{ccc} \mathfrak{Z}^n \mathfrak{Z}^n \mathfrak{Z}^n 1 & \xrightarrow{\mathfrak{Z}^n \mu_1} & \mathfrak{Z}^n \mathfrak{Z}^n 1 \\ {\scriptstyle \mu_{\mathfrak{Z}^n 1}}\downarrow & & \downarrow{\scriptstyle \mu_1} \\ \mathfrak{Z}^n \mathfrak{Z}^n 1 & \xrightarrow{\mu_1} & \mathfrak{Z}^n 1. \end{array}$$

Proof. Recall from definition 4.15 that for $X \in \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n})$, $(\mathfrak{Z}^n X)_{<n} = X_{<n}$. Thus all diagrams commute trivially in dimension $< n$.

(1) Let $\omega \in \mathbb{O}_n$ and $\nu \in \mathfrak{Z}^n 1_\omega$, i.e. $\nu \in \mathbb{O}_{n+1}$ such that $\mathsf{t}\nu = \omega$. Then

$$\begin{aligned} \mu_1 \eta_{\mathfrak{Z}^n 1}(\nu) &= \mu_1 \left(\mathsf{Y}_{\mathsf{Y}_{\mathsf{t}\nu}} \underset{[[]]}{\circ} \mathsf{Y}_\nu \right) && \text{see definition 4.20} \\ &= \mathsf{t} \left(\mathsf{Y}_{\mathsf{Y}_{\mathsf{t}\nu}} \underset{[[]]}{\circ} \mathsf{Y}_\nu \right) && \text{see definition 4.20} \\ &= \mathsf{Y}_{\mathsf{t}\nu} \underset{[]}{\square} \nu && \text{by proposition 3.5} \\ &= \nu, \end{aligned}$$

and similarly, if $\{[p_1], \ldots\} = \nu^\bullet$,

$$\begin{aligned} \mu_1(\mathfrak{Z}^n \eta_1)(\nu) &= \mu_1 \left(\mathsf{Y}_\nu \bigcirc_{[[p_i]]} \mathsf{Y}_{\mathsf{Y}_{\mathsf{s}_{[p_i]}\nu}} \right) && \spadesuit \\ &= \mathsf{t} \left(\mathsf{Y}_\nu \bigcirc_{[[p_i]]} \mathsf{Y}_{\mathsf{Y}_{\mathsf{s}_{[p_i]}\nu}} \right) && \spadesuit \\ &= \left(\nu \underset{[p_1]}{\circ} \mathsf{Y}_{\mathsf{s}_{[p_1]}\nu} \right) \underset{[p_2]}{\circ} \mathsf{Y}_{\mathsf{s}_{[p_2]}\nu} \cdots && \text{by proposition 3.5} \\ &= \nu, \end{aligned}$$

where $\spadesuit$ follows from definition 4.20.

(2) Akin to proposition 4.19, one can show that elements of $\mathfrak{Z}^n \mathfrak{Z}^n \mathfrak{Z}^n 1_\omega$ are $(n+2)$-opetopes ξ of uniform height 3 such that $\mathsf{t}\mathsf{t}\xi = \omega$. Let ξ be such an opetope, and write it as

$$\xi = \mathsf{Y}_\alpha \bigcirc_{[[p_i]]} \underbrace{\left(\mathsf{Y}_{\beta_i} \bigcirc_{[[q_{i,j}]]} \mathsf{Y}_{\gamma_{i,j}} \right)}_{A_i :=} = \underbrace{\left(\mathsf{Y}_\alpha \bigcirc_{[[p_i]]} \mathsf{Y}_{\beta_i} \right)}_{B :=} \bigcirc_{[[p_i][q_{i,j}]]} \mathsf{Y}_{\gamma_{i,j}}$$

where $\alpha, \beta_i, \gamma_{i,j} \in \mathbb{O}_n$, $[p_i]$ ranges over $\alpha^\bullet$ and $[q_{i,j}]$ over $\beta_i^\bullet$. Then

$$\begin{aligned} \mu_1(\mathfrak{Z}^n \mu_1)(\xi) &= \mu_1(\mathfrak{Z}^n \mu_1) \left(\mathsf{Y}_\alpha \bigcirc_{[[p_i]]} A_i \right) \\ &= \mu_1 \left(\mathsf{Y}_\alpha \bigcirc_{[[p_i]]} \mathsf{Y}_{\mathsf{t} A_i} \right) \\ &= \mathsf{t} \left(\mathsf{Y}_\alpha \bigcirc_{[[p_i]]} \mathsf{Y}_{\mathsf{t} A_i} \right) \\ &= \mathsf{t} \left(\mathsf{Y}_\alpha \bigcirc_{[[p_i]]} A_i \right) && \text{by proposition 3.5} \\ &= \mathsf{t} \left(B \bigcirc_{[[p_i][q_{i,j}]]} \mathsf{Y}_{\gamma_{i,j}} \right) && \text{by definition} \\ &= \mathsf{t} \left(\mathsf{Y}_{\mathsf{t} B} \bigcirc_{[\wp_B[[p_i][q_{i,j}]]]} \mathsf{Y}_{\gamma_{i,j}} \right) && \text{by proposition 3.5} \\ &= \mu_1 \mu_{\mathfrak{Z}^n 1}(\xi). \end{aligned}$$

□

Proposition 4.26. *The cartesian natural transformations μ and η (whose components are defined in definition 4.20 and lemma 4.23) give $\mathfrak{Z}^n$ a structure of p.r.a. monad on $\mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n})$.*

Proof. This is a direct consequence of lemmas 4.23 and 4.25. □

Remark 4.27. Clearly, when $k = 0$, we recover the usual polynomial monad on $\mathcal{S}\mathrm{et}/\mathbb{O}_n$.

Remark 4.28. Note from definition 4.15 that the $\mathfrak{Z}^n$-cardinals are precisely the representable opetopes in $\mathbb{O}_{n-k,n}$ and the spines $S[\nu]$ for all $\nu \in \mathbb{O}_{n+1}$.

Definition 4.29. A *k-coloured n-dimensional opetopic algebra* is an algebra of $\mathfrak{Z}^n$ in $\mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n})$. We write $\mathcal{A}\mathrm{lg}_{k,n}$ for the Eilenberg–Moore category of $\mathfrak{Z}^n$.

Proposition 4.30. *The category $\mathcal{A}\mathrm{lg}_{k,n}$ is locally finitely presentable.*

Proof. This follows from corollary 4.13 and the fact that every non-representable $\mathfrak{Z}^n$-cardinal $S[\nu]$ is a finite colimit of representables, thus is finitely presentable. □

Proposition 4.31. *Up to equivalence, and for small values of k and n with $k \leq n$, the category $\mathcal{A}\mathrm{lg}_{k,n}$ is given by the following table*[12]*:*

$k \backslash n$	0	1	2	3
0	$\mathcal{S}\mathrm{et}$	$\mathcal{M}\mathrm{on}$	$\mathcal{O}\mathrm{p}$	$\mathcal{C}\mathrm{omb}_{\mathbb{PT}}$
1		$\mathcal{C}\mathrm{at}$	$\mathcal{O}\mathrm{p}_{\mathrm{col}}$	$\mathcal{A}\mathrm{lg}_{1,3}$
2			$\mathcal{A}\mathrm{lg}_{2,2}$	$\mathcal{A}\mathrm{lg}_{2,3}$
3				$\mathcal{A}\mathrm{lg}_{3,3}$

where $\mathcal{M}\mathrm{on}$ is the category of monoids, $\mathcal{O}\mathrm{p}$ of non coloured planar operads, $\mathcal{O}\mathrm{p}_{\mathrm{col}}$ of coloured planar ($\mathcal{S}\mathrm{et}$-)operads, and $\mathcal{C}\mathrm{omb}_{\mathbb{PT}}$ of combinads over the combinatorial pattern of planar trees [15].

Proof (sketch). Let us first treat the case where $k = 0$.

(1) If $n = 0$, then $\mathfrak{Z}^0$ is by definition the identity functor on $\mathcal{S}\mathrm{et}/\mathbb{O}_0 = \mathcal{S}\mathrm{et}$, thus $\mathfrak{Z}^0$-algebras bear no structure, and are simply sets.

(2) The polynomial monad $\mathfrak{Z}^1 = (\mathfrak{Z}^0)^+$ is isomorphic to

$$\{\blacksquare\} \xleftarrow{\mathsf{s}} \mathbb{N}_< \longrightarrow \mathbb{N} \xrightarrow{\mathsf{t}} \{\blacksquare\}$$

where for $m \in \mathbb{N}$, $\mathbb{N}_<(m) := \{0, 1, \ldots, m-1\}$. The result follows by [9, example 1.9].

(3) The functor $\mathfrak{Z}^2 : \mathcal{S}\mathrm{et}/\mathbb{N} \longrightarrow \mathcal{S}\mathrm{et}/\mathbb{N}$ maps a signature $X = (X_m \mid m \in \mathbb{N}) \in \mathcal{S}\mathrm{et}/\mathbb{N}$ to the set of trees whose nodes are adequately decorated by elements of X, i.e. it is the free planar operad monad.

(4) A $\mathfrak{Z}^4$-algebra is a set of "planar trees" (i.e. an element of $\mathcal{S}\mathrm{et}/\mathbb{O}_3$) with an suitable notion of substitution, which is structure encapsulated in the notion of $\mathbb{PT}$-combinad.

Let us now consider higher values of k.

(1) Assume $k = n = 1$. Then $\mathcal{P}\mathrm{sh}(\mathbb{O}_{0,1})$ is the cagegory of graphs, and a $\mathfrak{Z}^1$ maps a graph to its graph of paths. A $\mathfrak{Z}^1$-algebra is just a graph with an adequate notion of composition of paths, i.e. a category.

(2) Similarly, in the case $k = 1$ and $n = 2$, the category $\mathcal{P}\mathrm{sh}(\mathbb{O}_{1,2})$ is the category of signatures whose inputs and output of functions are typed. Extending the reasoning of the case $k = 0$, it is easy to see that a $\mathfrak{Z}^2$-algebra is a coloured planar operad. □

4.3 $\mathfrak{Z}^n$-cardinals and opetopic shapes

In this section, we state and prove the nerve theorem for $\mathfrak{Z}^n$. In particular, we show that the category $\mathcal{A}\mathrm{lg}_{k,n}$ is a localisation of $\mathcal{P}\mathrm{sh}(\mathbb{\Lambda}_{k,n})$, where $\mathbb{\Lambda}_{k,n} := \Theta_{\mathfrak{Z}^n}$ (see definition 4.32). This serves as an intermediate result to obtain a similar nerve theorem over opetopic sets.

[12] Note that if $k > n$, then $\mathcal{A}\mathrm{lg}_{k,n} = \mathcal{A}\mathrm{lg}_{n,n}$.

Definition 4.32. By definitions 4.6 and 4.15, the category Θ_0 of $\mathfrak{Z}^n$-cardinals is the full subcategory of $\mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n})$ spanned by the representables $O[\omega]$, where $\omega \in \mathbb{O}_{n-k,n}$, and the spines $S[\nu]$, where $\nu \in \mathbb{O}_{n+1}$. Analogous to notation 4.8, let $\mathbb{\Lambda}_{k,n}$, the category of *opetopic shapes*, be the full subcategory of $\mathcal{A}\mathrm{lg}_{k,n}$ spanned by $\mathfrak{Z}^n\Theta_0$.

Convention 4.33. Throughout this work, we will frequently fix parameters $k \leq n \in \mathbb{N}$ in an implicit manner, and suppress them in notation whenever it is unambiguous. For example, we write $\mathbb{\Lambda}$ instead of $\mathbb{\Lambda}_{k,n}$, $\mathfrak{Z}$ instead of $\mathfrak{Z}^n$, $\mathcal{A}\mathrm{lg}$ instead of $\mathcal{A}\mathrm{lg}_{k,n}$, etc.

Definition 4.34. Recall from proposition 4.30 that $\mathcal{A}\mathrm{lg}$ is cocomplete. From $\mathbb{\Lambda} \longrightarrow \mathcal{A}\mathrm{lg}$ the inclusion of definition 4.32, we derive an adjunction

$$\tau : \mathcal{P}\mathrm{sh}(\mathbb{\Lambda}) \rightleftarrows \mathcal{A}\mathrm{lg} : N,$$

by left Kan extension along the Yoneda embedding. The left adjoint is called the *algebraic realisation*, and the right adjoint is the *nerve*.

Example 4.35. (1) Take $n = k = 1$. By proposition 4.31, $\mathcal{A}\mathrm{lg}_{1,1} = \mathcal{C}\mathrm{at}$, and $\mathbb{\Lambda}_{1,1}$ is the full subcategory of $\mathcal{C}\mathrm{at}$ spanned by $[m] = \mathfrak{Z}^1 O[\mathbf{m}]$, where $m \in \mathbb{N}$. Therefore, $\mathbb{\Lambda}_{1,1} = \mathbb{\Delta}$. The algebraic realisation $\tau_{1,1} : \mathcal{P}\mathrm{sh}(\mathbb{\Delta}) \longrightarrow \mathcal{C}\mathrm{at}$ is just the realisation of a simplicial set into a category, and its right adjoint $N_{1,1}$ is the classical nerve.

(2) Likewise, $\mathbb{\Lambda}_{1,2}$ is the category of coloured operads generated by trees, thus it is the planar version of Moerdijk and Weiss's category of dendrices Ω. The functor $N_{1,2}$ is the *dendroidal nerve* of [16, section 4], and $\tau_{1,2}$ is its left adjoint. In that paper, they are respectively denoted by N_d and τ_d.

Akin to $\mathcal{P}\mathrm{sh}(\mathbb{O})$, the category $\mathcal{P}\mathrm{sh}(\mathbb{\Lambda})$ enjoys an adequate notion of *spine*. As we shall see, the set S of spine inclusions in $\mathcal{P}\mathrm{sh}(\mathbb{\Lambda})$ will characterize the nerves of algebras in the sense of corollary 4.13.

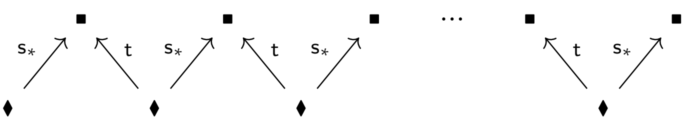
Definition 4.36. For $\nu \in \mathbb{O}_{n+1}$, write $\lambda := \mathfrak{Z}S[\nu]$, and let $S[\lambda]$, the *spine* of the opetopic shape λ, be the colimit

$$S[\lambda] := h_! S[\nu] = \operatorname{colim}\left(\mathbb{O}_{n-k,n}/S[\nu] \longrightarrow \mathbb{O}_{n-k,n} \xrightarrow{\mathfrak{Z}} \mathbb{\Lambda} \xrightarrow{\mathsf{y}} \mathcal{P}\mathrm{sh}(\mathbb{\Lambda})\right).$$

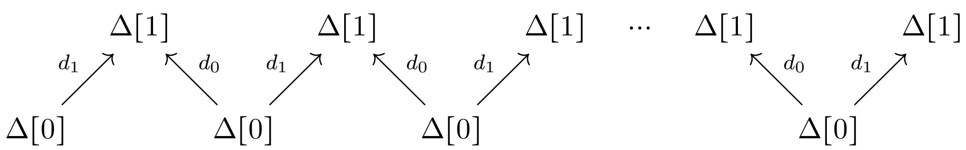
Let $\mathsf{s}_\lambda : S[\lambda] \hookrightarrow \lambda$ be the *spine inclusion of* λ, and let S be the set of spine inclusions in $\mathcal{P}\mathrm{sh}(\mathbb{\Lambda})$:

$$\mathsf{S} := \left\{ \mathsf{s}_\lambda : S[\lambda] \hookrightarrow \lambda \mid \nu \in \mathbb{O}_{n+1} \right\}.$$

Example 4.37. If $k = n = 1$, then $\mathbb{\Lambda}_{1,1} = \mathbb{\Delta}$, and the $(n+1)$-opetopes are the opetopic integers. For $m \in \mathbb{N}$, the diagram $\mathbb{O}_{0,1}/S[\mathbf{m}] \longrightarrow \mathbb{O}_{0,1}$ is



where there are m instances of ▪. By definition, $\mathfrak{Z}\blacklozenge = \Delta[0]$ and $\mathfrak{Z}\blacksquare = \Delta[1]$. Further, $\mathfrak{Z}\mathsf{s}_* = d_1$ and $\mathfrak{Z}\mathsf{t} = d_0$. Thus, if $\lambda := \mathfrak{Z}S[\mathbf{m}]$, then $S[\lambda]$ is the colimit of the following diagram in $\mathcal{P}\mathrm{sh}(\mathbb{\Delta})$:



Therefore, $S[\lambda]$ is the simplicial spine $S[m]$.

Theorem 4.38 (Nerve theorem for $\mathbb{A}$)**.** *(1) The functor $\tau : \mathbb{A} \longrightarrow \mathcal{A}\mathrm{lg}$ is dense, or equivalently, the nerve $N : \mathcal{A}\mathrm{lg} \longrightarrow \mathcal{P}\mathrm{sh}(\mathbb{A})$ is fully faithful.*

(2) A presheaf $X \in \mathcal{P}\mathrm{sh}(\mathbb{A})$ is in the essential image of N if and only if $\mathsf{S} \perp X$. In particular, using point (1), $\mathcal{A}\mathrm{lg}$ is equivalent to the orthogonality class in $\mathcal{P}\mathrm{sh}(\mathbb{A})$ induced by the set S.

(3) (Segal condition) The reflective adjunction $\tau : \mathcal{P}\mathrm{sh}(\mathbb{A}) \rightleftarrows \mathcal{A}\mathrm{lg} : N$ exhibits $\mathcal{A}\mathrm{lg}$ as the localisation of $\mathcal{P}\mathrm{sh}(\mathbb{A})$ at the spine inclusions: $\mathcal{A}\mathrm{lg} \simeq \mathsf{S}^{-1}\mathcal{P}\mathrm{sh}(\mathbb{A})$.

Proof. (1) This is theorem 4.12.

(2) Recall that Θ_0 denotes the category of $\mathfrak{Z}^n$-cardinals (remark 4.3 and definition 4.32). Consider the inclusions

$$\mathbb{O}_{n-k,n} \xrightarrow{i} \Theta_0 \xrightarrow{j} \mathbb{A}$$

as in definition 4.32, and the counit $\varepsilon : i_! i^* \longrightarrow \mathrm{id}_{\mathcal{P}\mathrm{sh}(\Theta_0)}$ of the adjunction $i_! : \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n}) \rightleftarrows \mathcal{P}\mathrm{sh}(\Theta_0) : i^*$. The category $\Theta_0 - \mathrm{im}\, i$ is spanned by the spines $S[\nu]$, for $\nu \in \mathbb{O}_{n+1}$ (see definition 4.32).

Since i maps an opetope $\omega \in \mathbb{O}_{n-k,n}$ to the associated representable $O[\omega] \in \Theta_0$, we have $i^* S[\nu] = \Theta_0(i-, S[\nu]) = S[\nu]$ as presheaves over $\mathbb{O}_{n-k,n}$. Next, by definition of left Kan extensions, the presheaf $i_! i^* S[\nu] = i_! S[\nu]$ is the colimit

$$\operatorname{colim}\left(\mathbb{O}_{n-k,n}/S[\nu] \longrightarrow \mathbb{O}_{n-k,n} \xrightarrow{i} \Theta_0 \xrightarrow{\mathsf{y}} \mathcal{P}\mathrm{sh}(\Theta_0)\right),$$

thus

$$\begin{aligned}
t_! i_! i^* S[\nu] &= t_! \operatorname{colim}\left(\mathbb{O}_{n-k,n}/S[\nu] \longrightarrow \mathbb{O}_{n-k,n} \xrightarrow{i} \Theta_0 \xrightarrow{\mathsf{y}} \mathcal{P}\mathrm{sh}(\Theta_0)\right) \\
&\cong \operatorname{colim}\left(\mathbb{O}_{n-k,n}/S[\nu] \longrightarrow \mathbb{O}_{n-k,n} \xrightarrow{i} \Theta_0 \xrightarrow{\mathsf{y}} \mathcal{P}\mathrm{sh}(\Theta_0) \xrightarrow{t_!} \mathcal{P}\mathrm{sh}(\mathbb{A})\right) \\
&\cong \operatorname{colim}\left(\mathbb{O}_{n-k,n}/S[\nu] \longrightarrow \mathbb{O}_{n-k,n} \xrightarrow{i} \Theta_0 \xrightarrow{j} \mathbb{A} \xrightarrow{\mathsf{y}} \mathcal{P}\mathrm{sh}(\mathbb{A})\right) \\
&= \operatorname{colim}\left(\mathbb{O}_{n-k,n}/S[\nu] \longrightarrow \mathbb{O}_{n-k,n} \xrightarrow{h} \mathbb{A} \xrightarrow{\mathsf{y}} \mathcal{P}\mathrm{sh}(\mathbb{A})\right) && \spadesuit \\
&= S[h\nu] && \diamondsuit
\end{aligned}$$

where $\spadesuit$ is by definition of h on $\mathbb{O}_{n-k,n}$, and $\diamondsuit$ is by definition of $S[h\nu]$. On the other hand, $t_! S[\nu] = \mathfrak{Z}^n S[\nu] = h\nu$, and the counit $\varepsilon_{S[\nu]}$ is simply the spine inclusion $\mathsf{s}_{h\nu} : S[h\nu] \longrightarrow h\nu$. We apply corollary 4.13 to conclude that a presheaf $X \in \mathcal{P}\mathrm{sh}(\mathbb{A})$ is in the essential image of N if and only if $\mathsf{S} \perp X$.

(3) Follows from the previous two points and section 1.6.3. □

Remark 4.39. Then theorem 4.38 generalizes the well-known fact that $\mathcal{C}\mathrm{at}$ (in the case $k = n = 1$) and $\mathcal{O}\mathrm{p}_{\mathrm{col}}$ (in the case $k = 1$ and $n = 2$) have fully faithful nerve functors to $\mathcal{P}\mathrm{sh}(\mathbb{\Delta})$ and $\mathcal{P}\mathrm{sh}(\Omega)$ [16, example 4.2] respectively, exhibiting them as localisations of the respective presheaf categories at a set of *spine inclusions*, sometimes called *Grothendieck–Segal colimits*.

4.4 Algebraic realisation

In this section, we show how to construct opetopic algebras from opetopic sets, by the means of the *algebraic realisation* $h_{k,n} : \mathcal{P}\mathrm{sh}(\mathbb{O}) \longrightarrow \mathcal{A}\mathrm{lg}_{k,n}$, for all $k, n \in \mathbb{N}$ with $k \leq n$. Much in the spirit of the classical realisation functor $\mathcal{P}\mathrm{sh}(\mathbb{\Delta}) \longrightarrow \mathcal{C}\mathrm{at}$, given $X \in \mathcal{P}\mathrm{sh}(\mathbb{O})$, we shall interpret its n-cells as “generators”, and its $(n+1)$-cells as “relations”. The first step to implement this idea is to extend $\mathfrak{Z}^n O[-] : \mathbb{O}_{n-k,n} \longrightarrow \mathbb{A}$ to a functor from $\mathbb{O}_{n-k,n+2}$. Informally, the image of an $(n+1)$-opetope represents an algebra with essentially one relation, and the image of an $(n+2)$-opetope is an algebra, also with essentially a single relation, but which is presented with many smaller composable relations (see example 4.41 for an illustration of this intuition). Thus, realisations of $(n+1)$-opetopes implement the idea of "relation" in opetopic algebras, while realisations of $(n+2)$-opetopes enforce "associativity among relations". Then, in definition 4.43, the realisation $h_{k,n}$ for opetopes is defined as a composite of left adjoints

$$\mathcal{P}\mathrm{sh}(\mathbb{O}) \underset{\perp}{\overset{(-)_{n-k,n+2}}{\rightleftarrows}} \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n+2}) \underset{\perp}{\rightleftarrows} \mathcal{P}\mathrm{sh}(\mathbb{A}_{k,n}) \underset{\perp}{\overset{\tau_{k,n}}{\rightleftarrows}} \mathcal{A}\mathrm{lg}_{k,n}.$$

To declutter notations, we shall use convention 4.33 and omit parameters k and n in most notations, e.g. $\mathbb{A} = \mathbb{A}_{k,n}$, $\mathcal{A}\mathrm{lg} = \mathcal{A}\mathrm{lg}_{k,n}$, $\mathfrak{Z} = \mathfrak{Z}^n$, etc.

Definition 4.40. There is a natural functor $\mathbb{O}_{n-k,n} \longrightarrow \mathbb{A}$, mapping an opetope ω to $\mathfrak{Z}O[\omega]$, see proposition 4.10 and equation (4.9). We now extend it to a functor $h : \mathbb{O}_{n-k,n+2} \longrightarrow \mathbb{A}$. On objects, it is given by

$$\begin{aligned} h : \mathbb{O}_{n-k,n+2} &\longrightarrow \mathbb{A} \\ \omega &\longmapsto \begin{cases} \mathfrak{Z}O[\omega] & \text{if } \dim\omega \leq n, \\ \mathfrak{Z}S[\omega] & \text{if } \dim\omega = n+1, \\ \mathfrak{Z}S[\mathsf{t}\,\omega] & \text{if } \dim\omega = n+2. \end{cases} \end{aligned}$$

We now specify h on morphisms. Since it extends the natural functor $\mathbb{O}_{n-k,n} \longrightarrow \mathbb{A}$, it is enough to consider morphisms in $\mathbb{O}_{n,n+2}$, so take $\nu \in \mathbb{O}_{n+1}$ and $\xi \in \mathbb{O}_{n+2}$.

(1) For $[p] \in \nu^\bullet$, let $h\left(\mathsf{s}_{[p]}\,\nu \xrightarrow{\mathsf{s}_{[p]}} \nu\right) := \mathfrak{Z}\left(O[\mathsf{s}_{[p]}\,\nu] \xrightarrow{\mathsf{s}_{[p]}} S[\nu]\right)$.

(2) In order to define $h\left(\mathsf{t}\,\nu \xrightarrow{\mathsf{t}} \nu\right) = \left(\mathfrak{Z}O[\mathsf{t}\,\nu] \xrightarrow{h\,\mathsf{t}} \mathfrak{Z}S[\nu]\right)$, it is enough to provide a morphism $O[\mathsf{t}\,\nu] \longrightarrow \mathfrak{Z}S[\nu]$, i.e. a cell in $\mathfrak{Z}S[\nu]_{\mathsf{t}\,\nu}$. Let it be $\left(S[\nu] \xrightarrow{\mathrm{id}} S[\nu]\right) \in \mathfrak{Z}S[\nu]_{\mathsf{t}\,\nu}$.

(3) Let $h\left(\mathsf{t}\,\xi \xrightarrow{\mathsf{t}} \xi\right) = \left(\mathfrak{Z}S[\mathsf{t}\,\xi] \xrightarrow{h\,\mathsf{t}} \mathfrak{Z}S[\mathsf{t}\,\xi]\right)$ be the identity map.

(4) Let $[p] \in \xi^\bullet$. In order to define a morphism of $\mathfrak{Z}$-algebras

$$h\left(\mathsf{s}_{[p]}\,\xi \xrightarrow{\mathsf{s}_{[p]}} \xi\right) = \left(\mathfrak{Z}S[\mathsf{s}_{[p]}\,\xi] \xrightarrow{h\,\mathsf{s}_{[p]}} \mathfrak{Z}S[\mathsf{t}\,\xi]\right),$$

it is enough to provide a morphism $h\,\mathsf{s}_{[p]} : S[\mathsf{s}_{[p]}\,\xi] \longrightarrow \mathfrak{Z}S[\mathsf{t}\,\xi]$ in $\mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n})$, which we now construct.

(a) Using equation (2.23), ξ decomposes as

$$\xi = \alpha \underset{[p]}{\circ} \mathsf{Y}_{\mathsf{s}_{[p]}\,\xi} \bigcirc_{[[q_i]]} \beta_i,$$

for some $\alpha, \beta_i \in \mathbb{O}_{n+2}$, and where $[q_i]$ ranges over $(\mathsf{s}_{[p]}\,\xi)^\bullet$. The leaves of any β_i are therefore a subset of the leaves of ξ. More precisely, a leaf address $[l] \in \beta_i^|$ corresponds to the leaf $[p[q_i]l]$ of ξ. This defines an inclusion $f_i : S[\mathsf{t}\,\beta_i] \longrightarrow S[\mathsf{t}\,\xi]$ that maps the node $\wp_{\beta_i}[l] \in (\mathsf{t}\,\beta_i)^\bullet$ to $\wp_\xi[p[q_i]l] \in (\mathsf{t}\,\xi)^\bullet$.

(b) Note that by definition, the map f_i is an element of

$$\mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n})(S[\mathsf{t}\,\beta_i], S[\mathsf{t}\,\xi]) \subseteq \mathfrak{Z}S[\mathsf{t}\,\xi]_{\mathsf{tt}\beta_i},$$

and since $\mathsf{tt}\beta_i = \mathsf{ts}_{[]}\,\beta_i = \mathsf{e}_{[p[q_i]]}\,\xi$ (by **(Glob1)** and **(Inner)**), we have $f_i \in \mathfrak{Z}S[\mathsf{t}\,\xi]_{\mathsf{e}_{[p[q_i]]}\,\xi}$.

(c) Together, the f_i assemble into the required morphism $h\,\mathsf{s}_{[p]} : S[\mathsf{s}_{[p]}\,\xi] \longrightarrow \mathfrak{Z}S[\mathsf{t}\,\xi]$, that maps the node $[q_i] \in (\mathsf{s}_{[p]}\,\xi)^\bullet$ to f_i. So in conclusion, we have

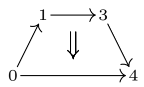
$$\begin{aligned} h\,\mathsf{s}_{[p]} : S[\mathsf{s}_{[p]}\,\xi] &\longrightarrow \mathfrak{Z}S[\mathsf{t}\,\xi] \\ (h\,\mathsf{s}_{[p]})[q_i] : S[\mathsf{t}\,\beta_i] &\longrightarrow S[\mathsf{t}\,\xi] \\ \wp_{\beta_i}[l] &\longmapsto \wp_\xi[p[q_i]l], \end{aligned}$$

for $[q_i] \in (\mathsf{s}_{[p]}\,\xi)^\bullet$ and $[l] \in \beta_i^|$.

This defines h on objects and morphisms, and functoriality is straightforward.

Example 4.41. Consider the case $k = n = 1$, so that $h = h_{1,1}$ is a functor $\mathbb{O}_{0,3} \longrightarrow \mathbb{\Lambda}_{1,1} \cong \mathbb{\Delta}$. In low dimensions, we have $h\blacklozenge = [0]$, $h\blacksquare = [1]$, and $h\mathbf{m} = [m]$ with $m \in \mathbb{N}$, since h is $\mathfrak{Z}$ in this case. For instance,

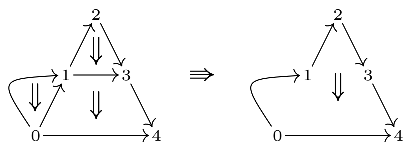
$$h\mathbf{3} = h\left(\quad \right) = [3]$$



is the category with 3 generating morphisms, and the 2-cell of $\mathbf{3}$ just witnesses their composition.

Consider now the following 3-opetope ξ:

$$\xi = \mathsf{Y}_{\mathbf{3}} \underset{[[*]]}{\circ} \mathsf{Y}_{\mathbf{2}} \underset{[[**]]}{\circ} \mathsf{Y}_{\mathbf{1}} = \left(\quad \right)$$



Then $h\xi = \mathfrak{Z}S[\mathsf{t}\,\xi] = \mathfrak{Z}S[\mathbf{4}] = [4]$. This result should be understood as the poset of points of ξ (represented as dots in the pasting diagram above) ordered by the topmost arrows. The 2-dimensional faces of ξ provide several relations among the generating arrows, and the 3-cell is a witness of the composition of those relations.

Take the face map $\mathsf{s}_{[]} : \mathbf{3} \longrightarrow \xi$, corresponding to the trapezoid at the base of the pasting diagram. Then $h\,\mathsf{s}_{[]}$ maps points 0, 1, 2, 3 of $h\mathbf{3} = [3]$ to points 0, 1, 3, 4 of $h\xi$, respectively. In other words, it "skips" point 2, which is exactly what the pasting diagram above depicts: the $[]$-source of ξ does not touch point 2 (the topmost one). Likewise, the map $h\,\mathsf{s}_{[[**]]} : [1] = h\mathbf{1} \longrightarrow h\xi$ maps 0, 1 to 0, 1, respectively.

Consider now the target map $\mathsf{t} : \mathbf{4} \longrightarrow \xi$. Since the target face touches all the points of ξ (this can be checked graphically, but more generally follows from **(Glob2)**), $h\,\mathsf{t}$ should be the identity map on $[4]$, which is precisely what the definition gives.

Remark 4.42. Recall that, as a p.r.a. monad on a presheaf category, $\mathfrak{Z}$ has an associated "generic-free" factorisation system on the category $\mathbb{\Lambda}$ [18, example 4.21]. We note that in definition 4.40, the functor $h : \mathbb{O}_{n-k,n+2} \longrightarrow \mathbb{\Lambda}$ takes a source map $\mathsf{s}_{[]} : \mathsf{s}_{[]}\,\xi \longrightarrow \xi$ (where $\xi \in \mathbb{O}_{n+2}$) to a *generic* morphism $\mathfrak{Z}S[\mathsf{s}_{[]}\,\xi] \longrightarrow \mathfrak{Z}S[\mathsf{t}\,\xi]$. This motivates part (4) of definition 4.40. Namely, h sends a morphism $\mathsf{s}_{[p]} : \mathsf{s}_{[p]}\,\xi \longrightarrow \xi$ to a generic-free composite $\mathfrak{Z}f \circ h\,\mathsf{s}_{[]} : \mathfrak{Z}S[\mathsf{s}_{[p]}\,\xi] \longrightarrow \mathfrak{Z}S[\mathsf{t}\,\nu] \longrightarrow \mathfrak{Z}S[\mathsf{t}\,\xi]$ where $\nu = \mathsf{Y}_{\mathsf{s}_{[p]}\,\xi} \bigcirc_{[[q_i]]} \beta_i$ is the maximal subtree of ξ that "begins" at the node $[p]$, and where $f : S[\mathsf{t}\,\nu] \longrightarrow S[\mathsf{t}\,\xi]$ is the inclusion of the leaves of the subtree ν into the leaves of ξ.

Definition 4.43. With a slight abuse of notation, let $h : \mathcal{P}\mathrm{sh}(\mathbb{O}) \rightleftarrows \mathcal{A}\mathrm{lg} : M$ be the composite adjunction

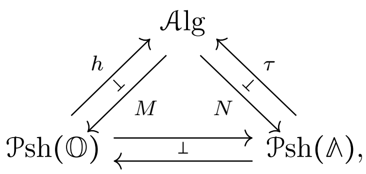
$$\mathcal{P}\mathrm{sh}(\mathbb{O}) \underset{\perp}{\overset{(-)_{n-k,n+2}}{\rightleftarrows}} \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n+2}) \underset{\perp}{\overset{h_!}{\rightleftarrows}} \mathcal{P}\mathrm{sh}(\mathbb{\Lambda}) \underset{\perp}{\overset{\tau}{\rightleftarrows}} \mathcal{A}\mathrm{lg},$$

where $h_!$ is the extension of $h : \mathbb{O}_{n-k,n+2} \longrightarrow \mathbb{\Lambda}$ (definition 4.40) to the presheaf categories.

Remark 4.44. The first adjunction of the composite is just a truncation, and does not carry any information; the part between $\mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n+2})$ and $\mathcal{A}\mathrm{lg}$ is actually what implements the n-cells of a presheaf as operations, and $(n+1)$-cells as relations. The $(n+2)$-cells represent relations among relations (e.g. associativity of composition in categories) and cannot be discarded, i.e. one cannot obtain an adequate realisation adjunction of the form $\mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n+1}) \rightleftarrows \mathcal{A}\mathrm{lg}$. Formally, the nerve theorem 4.74 will not hold if h is defined as the composite

$$\mathcal{P}\mathrm{sh}(\mathbb{O}) \xrightarrow{(-)_{n-k,n+1}} \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n+1}) \xrightarrow{h_!} \mathcal{P}\mathrm{sh}(\mathbb{\Lambda}) \xrightarrow{\tau} \mathcal{A}\mathrm{lg}.$$

Remark 4.45. We now have a commutative triangle of adjunctions:

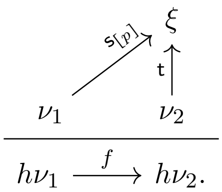


(4.46)

The notation h might seem a bit overloaded, but its meaning is quite simple: it always takes an opetopic set and produces an algebra. If that opetopic set is the representable of an opetope in $\mathbb{O}_{n-k,n+2}$, then it falls within the scope of definition 4.40, and the output algebra is in fact an opetopic shape, i.e. in $\mathbb{\Lambda}$.

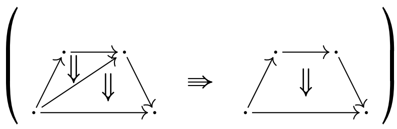
4.5 Diagrammatic morphisms This section is devoted to proving various (rather technical) facts about the functor $h : \mathbb{O}_{n-k,n+2} \longrightarrow \mathbb{\Lambda}$ of definition 4.40, eventually leading to lemma 4.56, stating that most morphisms in $\mathbb{\Lambda}$ admit a good "geometrical description" (see definition 4.47 and example 4.48). This result shall be used when proving the *nerve theorem for* $\mathbb{O}$ (theorem 4.74).

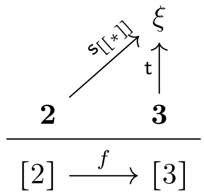
Definition 4.47. Let $\nu_1, \nu_2 \in \mathbb{O}_{n+1}$. A morphism $f : h\nu_1 \longrightarrow h\nu_2$ in $\mathbb{\Lambda}$ is *diagrammatic* if there exists an opetope $\xi \in \mathbb{O}_{n+2}$ and a node address $[p] \in \xi^\bullet$ such that $\mathsf{s}_{[p]}\xi = \nu_1$, $\mathsf{t}\xi = \nu_2$, and $f = (h\,\mathsf{t})^{-1} \cdot (h\,\mathsf{s}_{[p]})$. This situation is summarized by the following diagram, called a *diagram of* f:



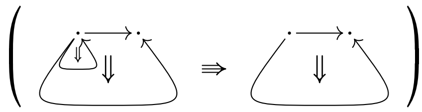
Example 4.48. Consider the case $k = n = 1$ again, and recall from example 4.35 that in this case, $\mathbb{\Lambda} = \mathbb{\Delta}$. Consider the map $f : [2] \longrightarrow [3]$ in $\mathbb{\Delta}$, where $f(0) = 0$, $f(1) = 1$, and $f(2) = 2$. In other words, $f = d^3$ is the 3rd coface map. Taking ξ as on the left, we obtain a diagram of f on

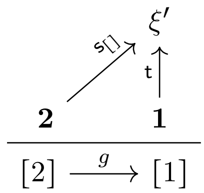
the right:

$\xi = \mathsf{Y}_{\mathbf{2}} \underset{[[*]]}{\circ} \mathsf{Y}_{\mathbf{2}} =$ ,



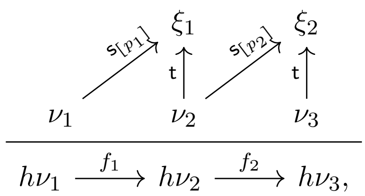
Consider now a non injective map $g : [2] \longrightarrow [1]$ where $g(0) = g(1) = 0$ and $g(2) = 1$. In other words, $g = s^0$ is the 0^{th} codegeneracy map. Taking ξ' as on the left, we obtain a diagram of g on the right:

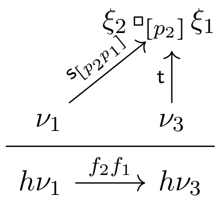
$\xi' = \mathsf{Y}_{\mathbf{2}} \underset{[[*]]}{\circ} \mathsf{Y}_{\mathbf{0}} =$ ,



On the one hand, lemma 4.49 below states that diagrammatic morphisms are stable under composition, and on the other hand, those two examples seem to indicate that all simplicial cofaces and codegeneracies are diagrammatic. One might thus expect all morphisms of $\mathbb{\Delta}$ to be in the essential image of $h_{1,1} : \mathbb{O}_{0,3} \longrightarrow \mathbb{\Delta}$. This is indeed true, and a more general statement is proved in proposition 4.60.

Lemma 4.49. *If f_1 and f_2 are diagrammatic as on the left, the diagram on the right is well-defined, and is a diagram of $f_2 f_1$.*





Proof. It is a simple but lengthy matter of unfolding the definition of h. First, note that

$$\begin{aligned} \mathsf{t}(\xi_2 \underset{[p_2]}{\square} \xi_1) &= \mathsf{t}\,\mathsf{t}(\mathsf{Y}_{\xi_2} \underset{[[p_2]]}{\circ} \mathsf{Y}_{\xi_1}) && \text{by proposition 3.5} \\ &= \mathsf{t}\,\mathsf{s}_{[]}(\mathsf{Y}_{\xi_2} \underset{[[p_2]]}{\circ} \mathsf{Y}_{\xi_1}) && \text{by } \textbf{(Glob2)} \\ &= \mathsf{t}\,\xi_2 = \nu_3. \end{aligned}$$

Using equation (2.23), we decompose ξ_1 as

$$\xi_1 = \alpha_1 \underset{[p_1]}{\circ} \mathsf{Y}_{\nu_1} \bigcirc_{[[q_i]]} \beta_i, \tag{4.50}$$

where $[q_i]$ ranges over $\nu_1^\bullet$. If $\beta_i^| = \{[l_{i,j}] \mid j\}$, then $\xi_1^| = \{[p_1[q_i]l_{i,j}] \mid i,j\}$, and so we have $\nu_2^\bullet = (\mathsf{t}\,\xi_1)^\bullet = \left\{\wp_{\xi_1}[p_1[q_i]l_{i,j}] \mid i,j\right\}$. Using equation (2.23) again, we decompose ξ_2 as

$$\xi_2 = \alpha_2 \underset{[p_2]}{\circ} \mathsf{Y}_{\nu_2} \bigcirc_{[\wp_{\xi_1}[p_1[q_i]l_{i,j}]]} \gamma_{i,j} \tag{4.51}$$

and write

$$
\begin{aligned}
\xi_2 \underset{[p_2]}{\square} \xi_1 &= \left(\alpha_2 \underset{[p_2]}{\circ} \mathsf{Y}_{\nu_2} \bigcirc_{[\wp_{\xi_1}[p_1[q_i]l_{i,j}]]} \gamma_{i,j} \right) \underset{[p_2]}{\square} \xi_1 && \text{see (4.50)} \\
&= \alpha_2 \underset{[p_2]}{\circ} \xi_1 \bigcirc_{[p_1[q_i]l_{i,j}]} \gamma_{i,j} && \text{see definition 2.22} \\
&= \alpha_2 \underset{[p_2]}{\circ} \left(\alpha_1 \underset{[p_1]}{\circ} \mathsf{Y}_{\nu_1} \bigcirc_{[[q_i]]} \beta_i \right) \bigcirc_{[[q_i]l_{i,j}]} \gamma_{i,j} && \text{see (4.51)} \\
&= \left(\alpha_2 \underset{[p_2]}{\circ} \alpha_1 \right) \underset{[p_2 p_1]}{\circ} \mathsf{Y}_{\nu_1} \bigcirc_{[[q_i]]} \underbrace{\left(\beta_i \bigcirc_{[l_{i,j}]} \gamma_{i,j} \right)}_{\delta_i} && \text{rearranging terms.}
\end{aligned}
$$

Applying the definition of h we have, for $[q_i] \in \nu_1^\bullet$, $[l_{i,j}] \in \beta_i^|$, and $[r] \in \gamma_{i,j}^|$,

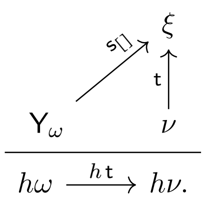
$$
\begin{aligned}
h\,\mathsf{s}_{[p_2 p_1]} : S[\nu_1] &\longrightarrow \mathfrak{Z} S[\nu_3] \\
(h\,\mathsf{s}_{[p_2 p_1]})[q_i] : S[\mathsf{t}\,\delta_i] &\longrightarrow S[\nu_3] \\
\wp_{\delta_i}[l_{i,j} r] &\longmapsto \wp_\zeta[p_2 p_1 [q_i] l_{i,j} r]; \qquad (4.52) \\
h\,\mathsf{s}_{[p_1]} : S[\nu_1] &\longrightarrow \mathfrak{Z} S[\nu_2] \\
(h\,\mathsf{s}_{[p_1]})[q_i] : S[\mathsf{t}\,\beta_i] &\longrightarrow S[\nu_2] \\
\wp_{\beta_i}[l_{i,j}] &\longmapsto \wp_{\xi_1}[p_1[q_i]l_{i,j}]; \qquad (4.53) \\
h\,\mathsf{s}_{[p_2]} : S[\nu_2] &\longrightarrow \mathfrak{Z} S[\nu_3] \\
(h\,\mathsf{s}_{[p_2]})(\wp_{\xi_1}[p_1[q_i]l_{i,j}]) : S[\mathsf{t}\,\gamma_{i,j}] &\longrightarrow S[\nu_3] \\
\wp_{\gamma_{i,j}}[r] &\longmapsto \wp_{\xi_2}[p_2 \ \wp_{\xi_1}[p_1[q_i]l_{i,j}] \ r]. \qquad (4.54)
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&(h\,\mathsf{s}_{[p_2 p_1]})([q_i])(\wp_{\delta_i}[l_{i,j} r]) \\
&= \wp_\zeta[p_2 p_1 [q_i] l_{i,j} r] && \text{by (4.52)} \\
&= \wp_{\xi_2}[p_2 \ \wp_{\xi_1}[p_1[q_i]l_{i,j}] \ r] && \spadesuit \\
&= (h\,\mathsf{s}_{[p_2]})(\wp_{\xi_1}[p_1[q_i]l_{i,j}])(\wp_{\gamma_{i,j}}[r]) && \text{by (4.54)} \\
&= (h\,\mathsf{s}_{[p_2]}) \left((h\,\mathsf{s}_{[p_1]})([q_i])(\wp_{\beta_i}[l_{i,j}]) \right) (\wp_{\gamma_{i,j}}[r]) && \text{by (4.53)} \\
&= \left(h\,\mathsf{s}_{[p_2]} \cdot h\,\mathsf{s}_{[p_1]} \right) ([q_i])(\wp_{\delta_i}[l_{i,j} r]), && \diamondsuit
\end{aligned}
$$

where equality $\spadesuit$ comes from the monad structure on $\mathfrak{Z}$, and $\diamondsuit$ from the definition of the composition in $\mathbb{A}$ when considered as the Kleisli category of $\mathfrak{Z}$. □

Lemma 4.55. *(1) Let $\nu \in \mathbb{O}_{n+1}$, $\omega := \mathsf{t}\,\nu$, and $\xi := \mathsf{Y}_{\mathsf{Y}_\omega} \circ_{[[]]} \mathsf{Y}_\nu$. Note that $\nu = \mathsf{t}\,\xi$. The following is a diagram of $h\,\mathsf{t} : h\omega \longrightarrow h\nu$:*



(2) *Let $\beta, \nu \in \mathbb{O}_{n+1} = \operatorname{tr} \mathfrak{Z}^{n-1}$, and $i : S[\beta] \longrightarrow S[\nu]$ a morphism of presheaves. Then i corresponds to an inclusion $\beta \hookrightarrow \nu$ of $\mathfrak{Z}^{n-1}$ trees, mapping node at address $[q]$ to $[pq]$, where $[p] := i([]) \in \nu^\bullet$ is the address of the image of the root node. Write $\nu = \bar{\beta} \square_{[p]} \beta$, for an adequate $\bar{\beta} \in \mathbb{O}_{n+1}$, and let $\xi := \mathsf{Y}_{\bar{\beta}} \circ_{[[p]]} \mathsf{Y}_\beta$. Note that $\nu = \mathsf{t}\,\xi$ by proposition 3.5. The following is a diagram of hi:*

$$\frac{\begin{array}{ccc} & & \xi \\ & \overset{\mathsf{s}_{[p]}}{\nearrow} & \uparrow \mathsf{t} \\ \beta & & \nu \end{array}}{h\beta \xrightarrow{hi} h\nu.}$$

Proof. Tedious but straightforward matter of unfolding definition 4.40. □

Lemma 4.56 (Diagrammatic lemma)**.** *Let $\nu, \nu' \in \mathbb{O}_{n+1}$ with ν non degenerate, and $f : h\nu \longrightarrow h\nu'$ be a morphism in $\mathbb{A}$. Then f is diagrammatic.*

Proof. Let us first sketch the proof. The idea is to proceed by induction on ν. The case $\nu = \mathsf{Y}_\psi$ for some $\psi \in \mathbb{O}_n$ is fairly simple. In the inductive case, we essentially show that f exhibits an inclusion $\nu \hookrightarrow \nu'$ of $\mathfrak{Z}^{n-1}$-trees by constructing an $(n+1)$-opetope $\bar{\nu}$ such that $\nu' = \bar{\nu} \square_{[q]} \nu$. Thus by lemma 4.55, the following is a diagram of hf:

$$\frac{\begin{array}{ccc} & & \xi \\ & \overset{\mathsf{s}_{[[q_1]]}}{\nearrow} & \uparrow \mathsf{t} \\ \nu & & \nu' \end{array}}{h\nu \xrightarrow{f} h\nu',}$$

where $\xi := \mathsf{Y}_{\bar{\nu}} \circ_{[[q_1]]} \mathsf{Y}_\nu$.

Let us now dive into the details. As advertised, the proof proceeds by induction on ν, which by assumption is not degenerate.

(1) Assume $\nu = \mathsf{Y}_\psi$ for some $\psi \in \mathbb{O}_n$. Then

$$\mathbb{A}(h\mathsf{Y}_\psi, h\nu') = \mathbb{A}(\mathfrak{Z}S[\mathsf{Y}_\psi], \mathfrak{Z}S[\nu']) \cong (\mathfrak{Z}S[\nu])_\psi.$$

Thus f corresponds to a unique morphism $\tilde{f} : S[\nu''] \longrightarrow S[\nu']$, for some $\nu'' \in \mathbb{O}_{n+1}$ such that $\mathsf{t}\,\nu'' = \psi$, and f is the composite

$$h\mathsf{Y}_\psi = h\psi \xrightarrow{h\,\mathsf{t}} h\nu'' \xrightarrow{\mathfrak{Z}\tilde{f}} h\nu'.$$

Those two arrows are diagrammatic by lemma 4.55, and by lemma 4.49, so is f.

(2) By induction, write $\nu = \nu_1 \circ_{[l]} \mathsf{Y}_{\psi_2}$ for some $\nu_1 \in \mathbb{O}_{n+1}$, $[l] \in \nu_1^|$, and $\psi_2 \in \mathbb{O}_n$. Write $\psi_1 := \mathsf{t}\,\nu_1$, and $\nu_2 := \mathsf{Y}_{\psi_2}$. Then f restricts as f_i, $i = 1, 2$, given by the composite $h\nu_i \longrightarrow h\nu \xrightarrow{f} h\nu'$. Let $[l']$ be the edge address of ν' (or equivalently, the $(n-1)$-cell of $S[\nu'] \subseteq h\nu'$) such that $\mathsf{e}_{[l']}\,\nu' = f(\mathsf{e}_{[l]}\,\nu)$. Then ν' decomposes as $\nu' = \beta_1 \circ_{[l']} \beta_2$, for some $\beta_1, \beta_2 \in \mathbb{O}_{n+1}$ (in particular, β_1 and β_2 are sub $\mathfrak{Z}^{n-1}$-trees of ν'), and f_1 and f_2 factor as

$$\begin{array}{ccc} h\nu_i & \xrightarrow{\bar{f}_i} & h\beta_i \\ & {\scriptstyle f_i}\searrow & \downarrow {\scriptstyle b_i} \\ & & h\nu', \end{array}$$

where b_i correspond to the subtree inclusion $\beta_i \longrightarrow \nu'$. By induction, $\bar{f}_i$ is diagrammatic, say with the following diagram:

$$\frac{\begin{array}{ccc} & & \xi_i \\ & \nearrow^{\mathsf{s}_{[p_i]}} & \uparrow \mathsf{t} \\ \nu_i & & \beta_i \end{array}}{h\nu_i \xrightarrow{\bar{f}_i} h\beta_i,}$$

and thus β_i can be written as $\beta_i = \bar{\nu}_i \,\square_{[q_i]}\, \nu_i$, for some $\bar{\nu}_i \in \mathbb{O}_{n+1}$ and $[q_i] \in \bar{\nu}_i^{\bullet}$. In the case $i = 2$, note that $\beta_2 = \bar{\nu}_2 \,\square_{[q_2]}\, \nu_2 = \bar{\nu}_2 \,\square_{[q_2]}\, \mathsf{Y}_{\psi_2} = \bar{\nu}_2$.
On the one hand we have

$$\begin{aligned}
\mathsf{e}_{[l']}\,\nu' &= f(\mathsf{e}_{[l]}\,\nu) && \text{by definition of } [l'] \\
&= f_1(\mathsf{e}_{[l]}\,\nu_1) && \text{since } \nu = \nu_1 \underset{[l]}{\circ} \mathsf{Y}_{\psi_2} \\
&= b_1\bar{f}_1(\mathsf{e}_{[l]}\,\nu_1) && \text{since } f_1 = b_1\bar{f}_1 \\
&= b_1(\mathsf{e}_{[q_1 l]}\,\beta_1) && \text{since } \beta_1 = \bar{\nu}_1 \underset{[q_1]}{\square} \nu_1 \\
&= \mathsf{e}_{[q_1 l]}\,\nu,
\end{aligned}$$

showing $[l'] = [q_1 l]$, and thus that $\bar{\nu}_1$ is of the form

$$\bar{\nu}_1 = \mu_1 \underset{[q_1]}{\circ} \mathsf{Y}_{\psi_1} \bigodot_{[[r_{1,j}]]} \mu_{1,j}, \tag{4.57}$$

where $[r_{1,j}]$ ranges over $\psi_1^{\bullet} - \{\wp_{\nu_1}[l]\}$, and $\mu_1, \mu_{1,j} \in \mathbb{O}_{n+1}$. On the other hand,

$$\begin{aligned}
\mathsf{e}_{[l']}\,\nu' &= f(\mathsf{e}_{[l]}\,\nu) && \text{by definition of } [l'] \\
&= f_2(\mathsf{e}_{[]}\,\nu_2) && \text{since } \nu = \nu_1 \underset{[l]}{\circ} \mathsf{Y}_{\psi_2} \\
&= b_2\bar{f}_2(\mathsf{e}_{[]}\,\nu_2) && \text{since } f_1 = b_2\bar{f}_2 \\
&= b_2(\mathsf{e}_{[q_2]}\,\beta_2) && \text{since } \beta_2 = \bar{\nu}_2 \underset{[q_2]}{\square} \nu_2 \\
&= \mathsf{e}_{[l']}\,\nu',
\end{aligned}$$

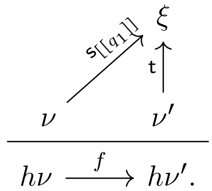
showing $[q_2] = []$, and so $\mathsf{s}_{[]}\,\beta_2 = \mathsf{s}_{[]}\,\bar{\nu}_2 = \psi_2$, and we can write β_2 as

$$\beta_2 = \mathsf{Y}_{\psi_2} \bigodot_{[[r_{2,j}]]} \mu_{2,j}, \tag{4.58}$$

where $[r_{2,j}]$ ranges over $\psi_2^{\bullet}$, and $\mu_{2,j} \in \mathbb{O}_{n+1}$. Finally, we have

$$\begin{aligned}
\nu' &= \beta_1 \underset{[l']}{\circ} \beta_2 = \left(\bar{\nu}_1 \underset{[q_1]}{\square} \nu_1\right) \underset{[l']}{\circ} \beta_2 \\
&= \left(\mu_1 \underset{[q_1]}{\circ} \nu_1 \bigodot_{\wp_{\nu_1}^{-1}[r_{1,j}]} \mu_{1,j}\right) \underset{[l']}{\circ} \left(\mathsf{Y}_{\psi_2} \bigodot_{[[r_{2,j}]]} \mu_{2,j}\right) && \text{by (4.57) and (4.58)} \\
&= \left(\left(\mu_1 \underset{[q_1]}{\circ} \underbrace{\nu_1 \underset{[l]}{\circ} \mathsf{Y}_{\psi_2}}_{=\nu}\right) \bigodot_{[q_1]\cdot\wp_{\nu_1}^{-1}[r_{1,j}]} \mu_{1,j}\right) \bigodot_{[l'[r_{2,j}]]} \mu_{2,j} && \text{rearranging terms} \\
&= \bar{\nu} \underset{[q_1]}{\square} \nu,
\end{aligned}$$

for some $\bar{\nu}' \in \mathbb{O}_{n+1}$. Finally, by lemma 4.55, we have a diagram of hf, where $\xi := \mathsf{Y}_{\bar{\nu}} \circ_{[[q_1]]} \mathsf{Y}_{\nu}$:



∎

Lemma 4.59. *(1) If $\omega \in \mathbb{O}_{n-1}$, then h maps $\mathsf{tt} : \omega \longrightarrow \mathsf{I}_\omega$ to an identity.*

(2) If $\omega \in \mathbb{O}_n$, then h maps $\mathsf{s}_{[]} : \omega \longrightarrow \mathsf{Y}_\omega$ to an identity.

(3) If $\omega \in \mathbb{O}_{n+2}$, then h maps $\mathsf{t} : \mathsf{t}\omega \longrightarrow \omega$ to an identity.

Proof. By inspection of definition 4.40. ∎

The following proposition shows that h is essentially surjective on morphisms. While pleasant, this fact is not put to use in the present work, so the reader may skip it at first.

Proposition 4.60. *The functor $h : \mathbb{O}_{n-k,n+2} \longrightarrow \mathbb{A}$ is essentially surjective on morphisms.*

Proof. Let $\omega, \omega' \in \mathbb{O}_{n-k,n+2}$.

(1) If $\dim \omega, \dim \omega' < n-1$, then by definition 4.40, $h\omega = \omega$ and $h\omega' = \omega'$ as presheaves over $\mathbb{O}_{n-k,n+2}$, and thus

$$\mathbb{A}(h\omega, h\omega') = \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n+2})(\omega, \omega') = \mathbb{O}(\omega, \omega').$$

(2) Assume that $\dim \omega < n-1$ and $\dim \omega' \geq n-1$. We first show that $O[\omega']_{<n-1} = (h\omega')_{<n-1}$ by inspection of definition 4.40. If $\dim \omega' \leq n$, then the claim trivially holds. If $\dim \omega' = n+1$, then $h\omega' = \mathfrak{Z}S[\omega']$, and

$$\begin{aligned} (h\omega')_{<n-1} &= (\mathfrak{Z}S[\omega'])_{<n-1} \\ &= S[\omega']_{<n-1} && \text{see definition 4.15} \\ &= O[\omega']_{<n-1}. \end{aligned}$$

The case where $\dim \omega' = n+2$ is proved similarly. Thus, $O[\omega']_{<n-1} = (h\omega')_{<n-1}$, and in particular, $O[\omega']_\omega = (h\omega')_\omega$. Finally,

$$\begin{aligned} \mathbb{A}(h\omega, h\omega') &\cong \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n+2})(\omega, h\omega') \\ &= \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n+2})(\omega, \omega') && \spadesuit \\ &= \mathbb{O}(\omega, \omega'), \end{aligned}$$

where ♠ results from the observation above.

(3) If $\dim \omega \geq n-1$ and $\dim \omega' < n-1$, then $\mathbb{A}(h\omega, h\omega') = \varnothing$.

(4) Lastly, assume $\dim \omega, \dim \omega' \geq n-1$. By lemma 4.59, we may assume that $\dim \omega = \dim \omega' = n+1$. If ω is non degenerate, then by lemma 4.56, every morphism in $\mathbb{A}(h\omega, h\omega')$ is diagrammatic, thus in the essential image of h. Assume that ω is degenerate, say $\omega = \mathsf{I}_\phi$ for some $\phi \in \mathbb{O}_{n-1}$. Akin to point (2), by inspection of definition 4.40, one can prove that $O[\omega']_\phi = (h\omega')_\phi$. Finally,

$$\begin{aligned} \mathbb{A}(h\omega, h\omega') &\cong \mathbb{A}(h\phi, h\omega') && \text{by corollary 3.20} \\ &\cong \mathbb{O}(\phi, \omega') && \spadesuit, \end{aligned}$$

where ♠ results from the observation above. ∎

Remark 4.61. It is worthwhile to note that $h : \mathbb{O}_{n-k,n+2} \longrightarrow \mathbb{\Lambda}$ is *not* full. Take for example $n = k = 1$, so that h is a functor $\mathbb{O}_{0,3} \longrightarrow \mathbb{\Delta}$. Let $a, b \in \mathbb{N}$, $a \neq b$, and consider the corresponding opetopic integers $\mathbf{a}, \mathbf{b} \in \mathbb{O}_2$. Since they are different but have the same dimension, $\mathbb{O}(\mathbf{a}, \mathbf{b}) = \varnothing$, but of course, $\mathbb{\Delta}(h\mathbf{a}, h\mathbf{b}) = \mathbb{\Delta}([a], [b])$ is not empty. The diagrammatic lemma says that if $a \neq 0$, then a morphism in $\mathbb{\Delta}([a], [b])$ can be recovered as the image of a face map of $\mathbf{a}$ in some 3-opetope whose target is $\mathbf{b}$.

4.6 Nerve theorem Recall from corollary 4.13 that we have a reflective adjunction

$$\tau : \mathcal{P}\mathrm{sh}(\mathbb{\Lambda}) \rightleftarrows \mathcal{A}\mathrm{lg} : N$$

that exhibits $\mathcal{A}\mathrm{lg}$ as the localisation of $\mathcal{P}\mathrm{sh}(\mathbb{\Lambda})$ at the set S of spine inclusions. This result is part of what we call the *nerve theorem for* $\mathbb{\Lambda}$. In this section, we prove a similar result in $\mathcal{P}\mathrm{sh}(\mathbb{O})$. The strategy is to study the adjunction $h_! : \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n+2}) \rightleftarrows \mathcal{P}\mathrm{sh}(\mathbb{\Lambda}) : h^*$, and to show that it preserves the orthogonality classes of spine inclusions. It follows that it restricts and corestricts as an adjunction $\mathsf{S}^{-1}_{n+1,n+2}\mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n+2}) \rightleftarrows \mathsf{S}^{-1}\mathcal{P}\mathrm{sh}(\mathbb{\Lambda}) \simeq \mathcal{A}\mathrm{lg}$, and it remains to prove that it is an equivalence. More formally, we make use of the following observation:

Lemma 4.62. *Let $F : \mathcal{A} \rightleftarrows \mathcal{B} : U$ be an adjunction, and write $\eta : \mathrm{id} \longrightarrow UF$ for the unit and $\varepsilon : FU \longrightarrow \mathrm{id}$ for the counit. If $\mathcal{A}'$ (resp. $\mathcal{B}'$) is a full subcategory of $\mathcal{A}$ (resp. $\mathcal{B}$) such that*

(1) $F\mathcal{A}' \subseteq \mathcal{B}'$ and $U\mathcal{B}' \subseteq \mathcal{A}'$,

(2) for all $a \in \mathcal{A}'$, the unit $\eta_a : a \longrightarrow UFa$ is an isomorphism, and dually, for all $b \in \mathcal{B}'$, ε_b is an isomorphism,

then the adjunction restricts and corestricts to an adjoint equivalence $F : \mathcal{A}' \rightleftarrows \mathcal{B}' : U$. In particular, if $\mathcal{A}'$ (resp. $\mathcal{B}'$) is an orthogonality class induced by a class of morphism K (resp. K'), then condition (1) above translates as follows:

(1') for all $a \in \mathcal{A}$, if $\mathsf{K} \perp a$, then $\mathsf{K}' \perp Fa$, and dually, for all $b \in \mathcal{B}$, if $\mathsf{K}' \perp b$, then $\mathsf{K} \perp Ub$.

Proposition 4.63. *The functor $h_! : \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n+2}) \longrightarrow \mathcal{P}\mathrm{sh}(\mathbb{\Lambda})$ (see definition 4.43) takes the set $\mathsf{S}_{n+1} \subseteq \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n+2})^{[1]}$ (of definition 3.12) to $\mathsf{S} \subseteq \mathcal{P}\mathrm{sh}(\mathbb{\Lambda})^{[1]}$ (of definition 4.36), and takes morphisms in S_{n+2} to S-local isomorphisms.*

Proof. (1) Let $\nu \in \mathbb{O}_{n+1}$, and recall from definition 4.36 that $\mathbb{O}_{n-k,n}/S[\nu]$ is the category of elements of $S[\nu]$. We have

$$\begin{aligned} h_! S[\nu] &= h_! \operatorname*{colim}_{\psi \in \mathbb{O}_{n-k,n}/S[\nu]} O[\psi] \\ &\cong \operatorname*{colim}_{\psi \in \mathbb{O}_{n-k,n}/S[\nu]} h_! O[\psi] \\ &= \operatorname*{colim}_{\psi \in \mathbb{O}_{n-k,n}/S[\nu]} \mathsf{y}_{\mathbb{\Lambda}}(h\psi) \\ &= S[h\nu] && \text{see definition 4.36.} \end{aligned}$$

(2) For $\xi \in \mathbb{O}_{n+2}$, the inclusion $S[\mathsf{t}\,\xi] \longrightarrow S[\xi]$ is a relative S_{n+1}-cell complex by lemma 3.18. Since $h_!$ preserves colimits, and since $h_! \mathsf{S}_{n+1} = \mathsf{S}$, we have that $h_!(S[\mathsf{t}\,\xi] \longrightarrow S[\xi])$ is a relative S-cell complex, and thus an S-local isomorphism. In the square below

$$\begin{array}{ccc} h_! S[\mathsf{t}\,\xi] & \longrightarrow & h_! S[\xi] \\ {\scriptstyle h_! \mathsf{s}_{\mathsf{t}\,\xi}} \downarrow & & \downarrow {\scriptstyle h_! \mathsf{s}_{\xi}} \\ h_! O[\mathsf{t}\,\xi] & \xrightarrow{h_! \, \mathsf{t}} & h_! O[\xi] \end{array}$$

the top arrow is an S-local isomorphism, the right arrow is in S by the previous point, and the bottom arrow is an isomorphism by definition. By 3-for-2, we conclude that $h_! \mathsf{s}_\xi$ is an S-local isomorphism. □

Lemma 4.64. *Let $X \in \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n+2})$ be such that $\mathsf{S}_{n+1,n+2} \perp X$, and take $\omega \in \mathbb{O}_{n-k,n+2}$. The following are spans of isomorphisms:*

(1) for $\psi \in \mathbb{O}_{n-1}$,

$$\mathbb{A}(h\omega, h\psi) \times X_\psi \xleftarrow{\mathrm{id}\times\mathsf{tt}} \mathbb{A}(h\omega, h\psi) \times X_{\mathsf{I}_\psi} \xrightarrow{\mathbb{A}(h\omega, h\,\mathsf{tt})\times\mathrm{id}} \mathbb{A}(h\omega, h\mathsf{I}_\psi) \times X_{\mathsf{I}_\psi};$$

(2) for $\psi \in \mathbb{O}_n$,

$$\mathbb{A}(h\omega, h\psi) \times X_\psi \xleftarrow{\mathrm{id}\times\mathsf{s}_{[]}} \mathbb{A}(h\omega, h\psi) \times X_{\mathsf{Y}_\psi} \xrightarrow{\mathbb{A}(h\omega, h\,\mathsf{s}_{[]})\times\mathrm{id}} \mathbb{A}(h\omega, h\mathsf{Y}_\psi) \times X_{\mathsf{Y}_\psi};$$

(3) for $\psi \in \mathbb{O}_{n+2}$,

$$\mathbb{A}(h\omega, h\,\mathsf{t}\,\psi) \times X_{\mathsf{t}\,\psi} \xleftarrow{\mathrm{id}\times\mathsf{t}} \mathbb{A}(h\omega, h\,\mathsf{t}\,\psi) \times X_\psi \xrightarrow{\mathbb{A}(h\omega, h\,\mathsf{t})\times\mathrm{id}} \mathbb{A}(h\omega, h\psi) \times X_\psi.$$

Proof. Follows from lemma 4.59. □

Lemma 4.65. *Let $\omega \in \mathbb{O}_{n-k,n+2}$. If $\psi \in \mathbb{O}_{n-k,n-1}$, then $\mathbb{A}(h\omega, h\psi) \cong \mathbb{O}_{n-k,n+2}(\omega, \psi)$.*

Proof. Easy verification. □

Proposition 4.66. *Let $X \in \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n+2})$. If $\mathsf{S}_{n+1,n+2} \perp X$, then the unit $\eta_X : X \longrightarrow h^* h_! X$ is an isomorphism.*

Proof. It suffices to show that for each $\omega \in \mathbb{O}_{n-k,n+2}$, the following map is a bijection:

$$X_\omega \xrightarrow{\eta_X} h^* h_! X_\omega = \int^{\psi \in \mathbb{O}_{n-k,n+2}} \mathbb{A}(h\omega, h\psi) \times X_\psi.$$

If $\omega \in \mathbb{O}_{n-k,n-1}$, then $h\omega = O[\omega]$, and $\mathbb{A}(h\omega, h-) \cong \mathbb{O}_{n-k,n+2}(\omega, -)$. Thus,

$$\begin{aligned} h^* h_! X_\omega &= \int^{\psi \in \mathbb{O}_{n-k,n+2}} \mathbb{A}(h\omega, h\psi) \times X_\psi && \text{by definition} \\ &\cong \int^{\psi \in \mathbb{O}_{n-k,n+2}} \mathbb{O}_{n-k,n+2}(\omega, \psi) \times X_\psi && \text{since } \dim\omega \le n-1 \\ &\cong X_\omega && \text{by the density formula.} \end{aligned}$$

Assume how that $\dim\omega \ge n$. We construct an inverse of η_X via a cowedge $\mathbb{A}(h\omega, h-) \times X_- \overset{..}{\longrightarrow} X_\omega$.

(1) Assume $\omega \in \mathbb{O}_n$. By lemma 4.64, it suffices to consider the case $\psi \in \mathbb{O}_{n+1}$. To unclutter notations, write $\mathcal{P} := \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n+2})$. We have the sequence of morphisms

$$\begin{aligned} \mathbb{A}(h\omega, h\psi) \times X_\psi &\xrightarrow{\cong} \Big(\sum_{\substack{\nu \in \mathbb{O}_{n+1} \\ \mathsf{t}\,\nu = \omega}} \mathcal{P}(S[\nu], S[\psi]) \Big) \times \mathcal{P}(S[\psi], X) && \spadesuit \\ &\xrightarrow{\text{comp.}} \sum_{\substack{\nu \in \mathbb{O}_{n+1} \\ \mathsf{t}\,\nu = \omega}} \mathcal{P}(S[\nu], X) \\ &\xrightarrow{\cong} \sum_{\substack{\nu \in \mathbb{O}_{n+1} \\ \mathsf{t}\,\nu = \omega}} X_\nu && \spadesuit \\ &\xrightarrow{\mathsf{t}} X_\omega, \end{aligned}$$

where $\spadesuit$ follow from the assumption that $\mathsf{S}_{n+1} \perp X$. It is straightforward to verify that this defines a cowedge whose induced map is the required inverse.

(2) Assume $\omega \in \mathbb{O}_{n+1}$. If ω is degenerate, say $\omega = \mathsf{I}_\phi$ for some $\phi \in \mathbb{O}_{n-1}$, then $\mathbb{A}(h\omega, h-) \cong \mathbb{A}(h\phi, h-)$ and we are in a case we have treated before. So let ω be non-degenerate. By lemmas 4.64 and 4.65, we may suppose $\psi \in \mathbb{O}_{n,n+1}$. Recall that for every $f \in \mathbb{A}(h\omega, h\psi)$, the diagrammatic lemma 4.56 computes a $\xi \in \mathbb{O}_{n+2}$ and $[p] \in \xi^\bullet$ such that $\mathsf{s}_{[p]}\xi = \omega$, $\mathsf{t}\xi = \psi$ and $h\,\mathsf{s}_{[p]} \cong f$. By corollary 3.19, the target fully faithful $\mathsf{t} : \psi \longrightarrow \xi$ is an $\mathsf{S}_{n+1,n+2}$-local isomorphism, and by assumption, $\mathsf{S}_{n+1,n+2} \perp X$. Therefore, we have an isomorphism $\mathsf{t} : X_\xi \longrightarrow X_\psi$, which gives rise to a map

$$\begin{aligned} \mathbb{A}(h\omega, h\psi) \times X_\psi &\longrightarrow X_\omega \\ (f, x) &\longmapsto \mathsf{s}_{[p]}\,\mathsf{t}^{-1}\,x. \end{aligned}$$

It is straightforward to verify that this assignment defines a cowedge, whose associated map is the required inverse.

(3) Assume $\omega \in \mathbb{O}_{n+2}$. Then by definition of h, $\mathbb{A}(h\omega, h-) \cong \mathbb{A}(h\,\mathsf{t}\,\omega, h-)$, and this is the case we have just treated. □

Corollary 4.67. *Let $X \in \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n+2})$. If $\mathsf{S}_{n+1,n+2} \perp X$, then $\mathsf{S} \perp h_! X$.*

Proof. Recall from proposition 4.63 that $\mathsf{S} = h_! \mathsf{S}_{n+1}$. Let $\nu \in \mathbb{O}_{n+1}$. To unclutter notations, write $\mathcal{P} := \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n+2})$. We have

$$\begin{aligned} \mathcal{P}\mathrm{sh}(\mathbb{A})(h_!\nu, h_!X) &\cong \mathcal{P}(\nu, h^* h_! X) && \text{since } h_! \dashv h^* \\ &\cong \mathcal{P}(\nu, X) && \text{by proposition 4.66} \\ &\cong \mathcal{P}(S[\nu], X) && \text{since } \mathsf{s}_\nu \perp X \\ &\cong \mathcal{P}(S[\nu], h^* h_! X) && \text{by proposition 4.66,} \\ &\cong \mathcal{P}\mathrm{sh}(\mathbb{A})(h_! S[\nu], h_! X) && \text{since } h_! \dashv h^* \end{aligned}$$

and by construction, this isomorphism is the precomposition by $h_! \mathsf{s}_\nu$. Therefore, $h_! \mathsf{s}_\nu \perp X$. □

Notation 4.68. Let $\mathcal{C}$ be a small category, $X : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{S}\mathrm{et}$, and $Y : \mathcal{C} \longrightarrow \mathcal{S}\mathrm{et}$. The coend $\int^c Xc \times Yc$ admits the following simple description as a quotient in $\mathcal{S}\mathrm{et}$:

$$\int^{c \in \mathcal{C}} Xc \times Yc = \frac{\sum_{c \in \mathcal{C}} Xc \times Yc}{\sim}$$

where for $f : c \longrightarrow d$, $x \in Xd$, $y \in Yc$, we have an identification

$$(x, Yf(y)) \sim (Xf(x), y).$$

The class of a pair $(u, v) \in Xc \times Yc$ will be denoted by $u \otimes v$. Abusing notations a little bit, the equivalence relation $\sim$ above then translates to the very familiar identity $x \otimes f(y) = f(x) \otimes y$.

This second proposition will provide the other half of the equivalence between $\mathcal{A}\mathrm{lg}$ and the localisation $\mathsf{S}^{-1}_{n+1,n+2}\mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n+2})$.

Proposition 4.69. *Let $Y \in \mathcal{P}\mathrm{sh}(\mathbb{A})$. If $\mathsf{S} \perp Y$, then the counit map $\varepsilon_Y : h_! h^* Y \longrightarrow Y$ is an isomorphism.*

Proof. We have to prove that for each $\lambda \in \mathbb{A}$, the map

$$h_! Y_\lambda = \int^{\psi \in \mathbb{O}_{n-k,n+2}} \mathbb{A}(\lambda, h\psi) \times Y_{h\psi} \xrightarrow{(\varepsilon_Y)_\lambda} Y_\lambda \tag{4.70}$$

is a bijection. Consider the map

$$s : Y_\lambda \longrightarrow \int^{\psi \in \mathbb{O}_{n-k,n+2}} \mathbb{\Lambda}(\lambda, h\psi) \times Y_{h\psi}$$

mapping $y \in Y_\lambda$ to $\mathrm{id}_\lambda \otimes y$ (see notation 4.68). It is well-defined, as h is surjective on objects, and it is easy to verify that $s(y)$ it is independent of the choice of an antecedent $h\nu = \lambda$. Note that s is a section of $(\varepsilon_Y)_\lambda$, and we proceed to prove that s is surjective. In other words, we show that that every element $f \otimes y$, with $f \in \mathbb{\Lambda}(\lambda, h\psi)$ for some $\psi \in \mathbb{O}_{n-k,n+2}$ and $y \in Y_\lambda$, is equal to an element of the form $\mathrm{id}_\lambda \otimes y'$, for some $y' \in Y_\lambda$.

(1) Assume $\lambda = h\phi$ for some $\phi \in \mathbb{O}_{n-k,n-1}$. Then $\mathbb{\Lambda}(\lambda, h\psi) = \mathbb{O}_{n-k,n+2}(\phi, \psi)$, and $f \otimes y = \mathrm{id}_\phi \otimes f(y)$ has the required form.

(2) Assume $\lambda = h\nu = hS[\nu]$ for some $\nu \in \mathbb{O}_{n+1}$. If ν is degenerate, say $\nu = \mathsf{I}_\phi$, then by lemma 4.59, $h\nu = h\phi$, so we fall in the previous case. Thus, we may assume that ν is not degenerate. Further, by lemmas 4.64 and 4.65, we may consider only the case where $\psi \in \mathbb{O}_{n+1}$. By lemma 4.56, f admits a diagram, say

$$\frac{\begin{array}{ccc} & & \xi \\ & \nearrow^{\mathsf{s}_{[p]}} & \uparrow{\scriptstyle \mathsf{t}} \\ \nu & & \psi \end{array}}{h\nu \xrightarrow{\ f\ } h\psi,}$$

i.e. $f \cong h\,\mathsf{s}_{[p]}$. We then have $f \otimes y = \mathrm{id}_{\mathsf{Y}_\omega} \otimes (h\,\mathsf{s}_{[p]})(y)$.

(3) Assume $\lambda = h\omega$ for some $\omega \in \mathbb{O}_n$. By lemma 4.59, $h\omega = h\mathsf{Y}_\omega$, and we fall in the previous case. □

Definition 4.71. Let the adjunction induced by the localisation of $\mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n+2})$ at the set of spine inclusions $\mathsf{S}_{n+1,n+2}$ be denoted by

$$u : \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n+2}) \rightleftarrows \mathsf{S}^{-1}_{n+1,n+2}\mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n+2}) : N_u$$

On the other hand, recall from theorem 4.38 that we have an adjunction $\tau \dashv N$ that exhibits $\mathcal{A}\mathrm{lg}$ as the localisation $\mathsf{S}^{-1}\mathcal{P}\mathrm{sh}(\mathbb{\Lambda})$. We are now well-equipped to prove that $\mathcal{A}\mathrm{lg}$ is equivalent to the localized category $\mathsf{S}^{-1}_{n+1,n+2}\mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n+2})$.

Lemma 4.72. *The adjunction* $h_! : \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n+2}) \rightleftarrows \mathcal{P}\mathrm{sh}(\mathbb{\Lambda}) : h^*$ *restricts to an adjoint equivalence* $\tilde{h}_! \dashv \tilde{h}^*$*, as shown below.*

$$\begin{array}{ccc} \mathsf{S}^{-1}_{n+1,n+2}\mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n+2}) & \underset{\tilde{h}^*}{\overset{\tilde{h}_!}{\rightleftarrows}} & \mathsf{S}^{-1}\mathcal{P}\mathrm{sh}(\mathbb{\Lambda}) \simeq \mathcal{A}\mathrm{lg} \\ {\scriptstyle N_u}\downarrow & & \downarrow{\scriptstyle N} \\ \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n+2}) & \underset{h^*}{\overset{h_!}{\rightleftarrows}} & \mathcal{P}\mathrm{sh}(\mathbb{\Lambda}). \end{array}$$

Proof. We check the conditions of lemma 4.62.

(1) By proposition 4.63, for all $Y \in \mathcal{A}\mathrm{lg} \simeq \mathsf{S}^{-1}\mathcal{P}\mathrm{sh}(\mathbb{\Lambda})$, we have that $h_!\mathsf{S}_{n+1,n+2} \perp N_u Y$, or equivalently, that $\mathsf{S}_{n+1,n+2} \perp h^* N_u Y$. Thus $h^* N_u$ factors through the localisation $\mathsf{S}^{-1}_{n+1,n+2}\mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n+2})$. Next, by corollary 4.67, $h_! N_u$ factors through $\mathcal{A}\mathrm{lg}$.

(2) By proposition 4.66, if $X \in \mathsf{S}^{-1}_{n+1,n+2}\mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n+2})$, then the unit map $\eta_X : X \longrightarrow h^* h_! X$ is an isomorphism, and dually, by proposition 4.69, if $Y \in \mathsf{S}^{-1}\mathcal{P}\mathrm{sh}(\mathbb{A})$, then the counit map ε_Y is an isomorphism. □

Definition 4.73. Recall the definition of O and S from definitions 3.10 and 3.12, and let $\mathsf{A} = \mathsf{A}_{k,n} := \mathsf{O}_{<n-k} \cup \mathsf{S}_{\geq n+1}$.

Theorem 4.74 (Nerve theorem for $\mathbb{O}$)**.** *The reflective adjunction* $h : \mathcal{P}\mathrm{sh}(\mathbb{O}) \rightleftarrows \mathcal{A}\mathrm{lg} : M$ *exhibits* $\mathcal{A}\mathrm{lg}$ *as the localisation* $\mathsf{A}^{-1}\mathcal{P}\mathrm{sh}(\mathbb{O})$*, or equivalently, as the orthogonality class induced by* A *in* $\mathcal{P}\mathrm{sh}(\mathbb{O})$.

Proof. Recall from definition 4.43 that h is the composite

$$\mathcal{P}\mathrm{sh}(\mathbb{O}) \xrightarrow{(-)_{n-k,n+2}} \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n+2}) \xrightarrow{h_!} \mathcal{P}\mathrm{sh}(\mathbb{A}) \xrightarrow{\tau} \mathcal{A}\mathrm{lg},$$

and by lemma 4.72, it is isomorphic to the composite

$$\mathcal{P}\mathrm{sh}(\mathbb{O}) \xrightarrow{(-)_{n-k,n+2}} \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n+2}) \xrightarrow{u} \mathsf{S}^{-1}_{n+1,n+2}\mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n+2}) \xrightarrow{\simeq} \mathcal{A}\mathrm{lg}.$$

The truncation $(-)_{n-k,n+2}$ is the localisation at $\mathsf{O}_{<n-k} \cup \mathsf{B}_{>n+2}$. By section 1.6.3, u is the localisation at $\mathsf{S}_{n+1,n+2}$. Therefore, h is the localisation at $\mathsf{O}_{<n-k} \cup \mathsf{S}_{n+1,n+2} \cup \mathsf{B}_{>n+2}$, which by lemma 3.16 is the localisation at A. By section 1.6.3, $\mathcal{A}\mathrm{lg}$ is equivalent to the orthogonality class induced by A. □

5. The algebraic trompe-l'œil

As we saw in definition 4.29, for all $k, n \in \mathbb{N}$ with $k \leq n$, we have a notion of *k-coloured n-opetopic algebra*. For such an algebra $B \in \mathcal{A}\mathrm{lg}_{k,n}$, operations are n-cells (so that their shape are n-opetopes), and colours are cells of dimension $n-k$ to $n-1$, thus the "colour space" is stratified over k dimensions. Notable examples include

$$\mathcal{C}\mathrm{at} \simeq \mathcal{A}\mathrm{lg}_{1,1}, \qquad \mathcal{O}\mathrm{p}_{\mathrm{col}} \simeq \mathcal{A}\mathrm{lg}_{1,2}.$$

(see proposition 4.31). But are all $\mathcal{A}\mathrm{lg}_{k,n}$ fundamentally different?

In this section, we answer this question negatively: in a sense that we make precise, the most "algebraically rich" notion of opetopic algebra is given in the case $k = 1$ and $n = 3$. Although opetopes can be arbitrarily complex, the algebraic data they carry can be expressed by 3-opetopes, a.k.a. trees. We call this phenomenon *algebraic trompe-l'œil*, a French expression that literally translates as "fools-the-eye". And indeed, the eye is fooled in two ways: by colour (proposition 5.4) and shape (proposition 5.15). In the former, we argue that the colour space of an algebra $B \in \mathcal{A}\mathrm{lg}_{k,n}$, expressing how operations may or may not be composed, only needs 1 dimension, and thus that cells of dimension less than $n-1$ do not bring new algebraic data, only geometrical one. For the latter, recall from definition 3.3 that opetopes are trees of opetopes. In particular, 3-opetopes are just plain trees, and $\mathbb{O}_3$ already contains all the possible underlying tree shapes of all opetopes. Consequently, operations of B, which are its n-cells, may be considered as 3-cells in a very similar 3-algebra $B^\dagger$. Finally, we combine those two results in theorem 5.16, which states that an algebra $B \in \mathcal{A}\mathrm{lg}_{k,n}$ is exactly a presheaf $B \in \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n})$ with a 1-coloured 3-algebra structure on ${B_{n-1,n}}^\dagger$ (see definition 3.23).

5.1 Colour For $B \in \mathcal{A}\mathrm{lg}_{k,n}$, recall that the colours of B are its cells of dimension $n-k$ to $n-1$. They express which operations (n-cells) of B may or may not be composed. However, since that criterion only depends on $(n-1)$-cells, constraints expressed by lower dimensional cells should be redundant. We confirm this in proposition 5.4, in that the algebra structure on B is completely determined by a 1-coloured n-algebra structure on the truncation $B_{n-1,n}$.

Lemma 5.1. *Let $k, n \geq 1$, and $\nu \in \mathbb{O}_{n+1}$. Then*

$$S[\nu]_{n-k,n} \cong \iota_!(S[\nu]_{n-1,n}),$$

where $\iota_!$ is the left adjoint to the truncation $\mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n}) \longrightarrow \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-1,n})$.

Proof. It follows from the fact that $S[\nu]$ is completely determined by the incidence relation of the n- and $(n-1)$-faces of ν (see lemma 3.14). □

Proposition 5.2. *For $X \in \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n})$ we have $\mathfrak{Z}^n(X_{n-1,n}) \cong (\mathfrak{Z}^n X)_{n-1,n}$. Consequently, the truncation functor $(-)_{n-1,n} : \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n}) \longrightarrow \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-1,n})$ lifts as*

$$\begin{array}{ccc} \mathcal{A}\mathrm{lg}_{k,n} & \xrightarrow{(-)_{n-1,n}} & \mathcal{A}\mathrm{lg}_{1,n} \\ \downarrow & & \downarrow \\ \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n}) & \xrightarrow{(-)_{n-1,n}} & \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-1,n}). \end{array} \tag{5.3}$$

Proof. To unclutter notations, write $\mathcal{P} := \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-1,n})$. First, $\mathfrak{Z}^n(X_{n-1,n})_{n-1} = X_{n-1} = (\mathfrak{Z}^n X)_{n-1}$. Then, for $\omega \in \mathbb{O}_n$, we have

$$\begin{aligned} \mathfrak{Z}^n(X_{n-1,n})_\omega &= \sum_{\substack{\nu \in \mathbb{O}_{n+1} \\ \mathsf{t}\,\nu = \omega}} \mathcal{P}(S[\nu]_{n-1,n}, X_{n-1,n}) && \text{see definition 4.15} \\ &\cong \sum_{\substack{\nu \in \mathbb{O}_{n+1} \\ \mathsf{t}\,\nu = \omega}} \mathcal{P}(\iota_! S[\nu], X) && \text{since } \iota_! \dashv (-)_{n-1,n} \\ &\cong \sum_{\substack{\nu \in \mathbb{O}_{n+1} \\ \mathsf{t}\,\nu = \omega}} \mathcal{P}(S[\nu]_{n-k,n}, X) && \text{by lemma 5.1} \\ &= \mathfrak{Z}^n X_\omega. \end{aligned}$$

□

Proposition 5.4. *The square (5.3) is a pullback. That is, a $\mathfrak{Z}^n$-algebra structure on $X \in \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n})$ is completely determined by a $\mathfrak{Z}^n$-algebra structure on $X_{n-1,n}$.*

Proof. Let $X \in \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n})$. By proposition 5.2, a $\mathfrak{Z}^n$-algebra structure on X restricts to one on $X_{n-1,n}$. Since the truncation functor $(-)_{n-1,n} : \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n}) \longrightarrow \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-1,n})$ is faithful, its lift $\mathcal{A}\mathrm{lg}_{k,n} \longrightarrow \mathcal{A}\mathrm{lg}_{1,n}$ is injective on objects. In particular, different algebra structures on X truncate to different algebra structures on $X_{n-1,n}$. Conversely, since $(\mathfrak{Z}^n X)_{<n} = X_{<n}$, a $\mathfrak{Z}^n$-algebra structure on $X_{n-1,n}$ extends to one on X. Therefore, the truncation functor establishes a bijective correspondence between the algebra structures on X and on $X_{n-1,n}$. □

5.2 Shape We start by defining the *flattening operator* $(-)^\dagger : \mathbb{O}_{n-1,n} \longrightarrow \mathbb{O}_{2,3}$, for $n \geq 1$, mapping an n-opetope ω to a 3-opetope $\omega^\dagger$ having the same underlying polynomial tree, i.e. $\langle \omega^\dagger \rangle \cong \langle \omega \rangle$ (see notation 2.8).

Definition 5.5. (1) If $n = 1$, then $(-)^\dagger$ simply maps $\mathbb{O}_{0,1} = \left(\blacklozenge \xrightarrow{\mathsf{s}_*, \mathsf{t}} \blacksquare \right)$ to the diagram $\left(\mathbf{0} \xrightarrow{\mathsf{s}_{[]}, \mathsf{t}} \mathsf{Y}_{\mathbf{0}} \right)$.

(2) Assume now that $n \geq 2$. Recall from definition 3.3 that a 3-opetope is a $\mathfrak{Z}^1$-tree, where $\mathfrak{Z}^1$ is given by

$$\{\blacksquare\} \xleftarrow{\mathsf{s}} E_2 \xrightarrow{p} \mathbb{O}_2 \xrightarrow{\mathsf{t}} \{\blacksquare\},$$

where $\mathbb{O}_2 = \{\mathbf{m} \mid m \in \mathbb{N}\}$, and where $E_2(\mathbf{m}) = \mathbf{m}^\bullet = \left\{ [*^i] \mid 0 \leq i < m \right\}$. Let $f : \mathfrak{Z}^{n-2} \longrightarrow \mathfrak{Z}^1$ be the morphism of polynomial functors given by

$$\begin{array}{ccccccc}
\mathbb{O}_{n-2} & \xleftarrow{\mathsf{s}} & E_{n-2} & \xrightarrow{p} & \mathbb{O}_{n-1} & \xrightarrow{\mathsf{t}} & \mathbb{O}_{n-2} \\
\downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_2} & \lrcorner & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} \\
\{\blacksquare\} & \xleftarrow{\mathsf{s}} & E_2 & \xrightarrow{p} & \mathbb{O}_2 & \xrightarrow{\mathsf{t}} & \{\blacksquare\},
\end{array}$$

where $f_1(\psi) = \mathbf{m}$, $m = \# \psi^\bullet$, and where f_2 is fiberwise increasing with respect to the lexicographical order $\preceq$ on addresses. This morphism induces a functor $f_* : \mathbb{O}_n = \operatorname{tr} \mathfrak{Z}^{n-2} \longrightarrow \operatorname{tr} \mathfrak{Z}^1 = \mathbb{O}_3$ (see definition 2.7) mapping an n-opetope to its underlying tree, seen as a 3-opetope. Explicitly,

$$f_*(\mathsf{I}_\phi) = \mathsf{I}_\blacksquare, \qquad f_* \left(\mathsf{Y}_\psi \bigcirc_{[[p_i]]} \omega_i \right) = \mathbf{m} \bigcirc_{[[*^i]]} f_*(\omega_i),$$

where $\phi \in \mathbb{O}_{n-2}$, $\psi \in \mathbb{O}_{n-1}$, $\psi^\bullet = \{[p_0] < [p_1] < \cdots\}$, and $\omega_0, \ldots, \omega_{m-1} \in \mathbb{O}_n$. For $\omega \in \mathbb{O}_n$, since ω and $\omega^\dagger$ have the same underlying tree, they have the same number of source faces, i.e. $\# \omega^\bullet = \# (\omega^\dagger)^\bullet$, and we write $a_\omega : \omega^\bullet \longrightarrow (\omega^\dagger)^\bullet$ for the unique increasing map with respect to the lexicographical order. Intuitively, a_ω maps a node of the underlying tree $\langle \omega \rangle$ of ω to that same node in $\langle \omega^\dagger \rangle$. However, since the source faces of ω and $\omega^\dagger$ are not the same, a_ω is not strictly speaking an identity, but rather a conversion of a "walking instruction in the tree ω" (which is what an address is) to one in $\omega^\dagger$. Explicitly, a node address $[[q_1] \cdots [q_k]] \in \omega^\bullet$ (with $[q_{i+1}] \in \mathsf{s}_{[[q_1] \cdots [q_i]]} \omega$) is mapped to $[f_{2, \mathsf{s}_{[]} \omega}[q_1] \cdots f_{2, \mathsf{s}_{[[q_1] \cdots [q_{k-1}]]} \omega}[q_k]]$.

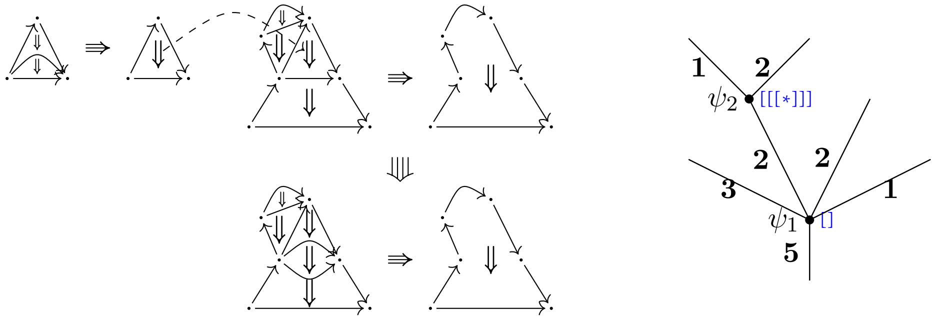
(3) Define now the *flattening operator* $(-)^\dagger : \mathbb{O}_{n-1,n} \longrightarrow \mathbb{O}_{2,3}$ as follows: for $\psi \in \mathbb{O}_{n-1}$ and $\omega \in \mathbb{O}_n$,

(a) $\psi^\dagger := f_1(\psi)$ and $\omega^\dagger := f_*(\omega)$ as above;

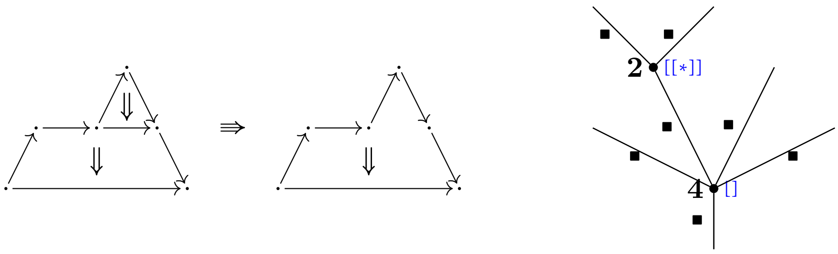
(b) clearly, $(\mathsf{t}\,\omega)^\dagger = \mathbf{m} = \mathsf{t}\,\omega^\dagger$, where $m := \# (\mathsf{t}\,\omega)^\bullet$, so let $\left(\mathsf{t}\,\omega \xrightarrow{\mathsf{t}} \omega \right)^\dagger$ simply be $\left((\mathsf{t}\,\omega)^\dagger \xrightarrow{\mathsf{t}} \omega^\dagger \right)$;

(c) likewise, for $[p] \in \omega^\bullet$, we have $\left(\mathsf{s}_{[p]}\,\omega \right)^\dagger = \mathsf{s}_{a_\omega [p]}\,\omega^\dagger$, and let $\left(\mathsf{s}_{[p]}\,\omega \xrightarrow{\mathsf{s}_{[p]}} \omega \right)^\dagger$ simply be $\left((\mathsf{s}_{[p]}\,\omega)^\dagger \xrightarrow{\mathsf{s}_{a_\omega [p]}} \omega^\dagger \right)$.

Example 5.6. Consider the 4-opetope ω, represented graphically and in tree form below:



where ψ_1 and ψ_2 are the 3-opetopes on the top right and top left hand corner respectively. Then its flattening $\omega^\dagger$ is as follows:



Although the graphical representations of ω and $\omega^\dagger$ look nothing alike, their underlying (undecorated) trees are identical.

Remark 5.7. Clearly, $(-)^\dagger : \mathbb{O}_{n-1,n} \longrightarrow \mathbb{O}_{2,3}$ is faithful, and if $n \leq 3$, then $(-)^\dagger$ is also injective on objects. Note that this is no longer the case if $n \geq 4$, as distinct n-opetopes may have the same underlying tree. For example, the underlying tree of the any degenerate is just a single edge, and for all $n \geq 4$, there exists infinitely many degenerate n-opetopes.

Definition 5.8. With a slight abuse of notations, let

$$(-)^\dagger : \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-1,n}) \longrightarrow \mathcal{P}\mathrm{sh}(\mathbb{O}_{2,3}),$$

the *flattening operation*, be the left Kan extension of $\mathbb{O}_{n-1,n} \xrightarrow{(-)^\dagger} \mathbb{O}_{2,3} \longrightarrow \mathcal{P}\mathrm{sh}(\mathbb{O}_{2,3})$ along the Yoneda embedding.

Lemma 5.9. *Explicitly, for $X \in \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-1,n})$, we have*

$$X^\dagger_{\mathbf{m}} \cong \sum_{\substack{\psi \in \mathbb{O}_{n-1} \\ \# \psi^\bullet = m}} X_\psi, \qquad\qquad X^\dagger_\gamma \cong \sum_{\substack{\omega \in \mathbb{O}_n \\ \omega^\dagger = \gamma}} X_\omega,$$

with $m \in \mathbb{N}$ and $\gamma \in \mathbb{O}_3$.

Proof. (1) Assume that $X = O[\omega]$ for some $\omega \in \mathbb{O}_{n-1}$, and let $d := \# \omega^\bullet$. If $m \in \mathbb{N}$, then by definition

$$O[\omega]^\dagger_{\mathbf{m}} = O[\mathbf{d}]_{\mathbf{m}} = \begin{cases} \{\mathrm{id}_{\mathbf{d}}\} & \text{if } d = m \\ \varnothing & \text{otherwise.} \end{cases}$$

On the other hand, if $\psi \in \mathbb{O}_{n-1}$,

$$O[\omega]_\psi = \mathbb{O}(\psi, \omega) = \begin{cases} \{\mathrm{id}_\psi\} & \text{if } \omega = \psi \\ \varnothing & \text{otherwise.} \end{cases}$$

Thus,

$$\sum_{\substack{\psi \in \mathbb{O}_{n-1} \\ \# \psi^\bullet = m}} O[\omega]_\psi \cong \begin{cases} \{\mathrm{id}_{\mathbf{d}}\} & \text{if } d = m \\ \varnothing & \text{otherwise} \end{cases} = O[\omega]^\dagger_{\mathbf{m}}.$$

On the other hand, if $\gamma \in \mathbb{O}_3$, then

$$O[\omega]^\dagger_\gamma = O[\mathbf{d}]_\gamma = \varnothing = \sum_{\substack{\omega' \in \mathbb{O}_n \\ \omega'^\dagger = \gamma}} X_{\omega'}.$$

(2) With the same reasoning, one can prove the lemma in the case $X = O[\omega]$ for some $\omega \in \mathbb{O}_n$.

(3) Let us now consider the general case. If $m \in \mathbb{N}$, then

$$\begin{aligned} X^\dagger_{\mathbf{m}} &= \int^{\omega \in \mathbb{O}_{n-1,n}} X_\omega \times O[\omega^\dagger]_{\mathbf{m}} \\ &\cong \int^{\omega \in \mathbb{O}_{n-1,n}} X_\omega \times \sum_{\substack{\psi \in \mathbb{O}_{n-1} \\ \# \psi^\bullet = m}} O[\omega]_\psi && \text{by the previous points} \\ &\cong \sum_{\substack{\psi \in \mathbb{O}_{n-1} \\ \# \psi^\bullet = m}} \int^{\omega \in \mathbb{O}_{n-1,n}} X_\omega \times O[\omega]_\psi \\ &\cong \sum_{\substack{\psi \in \mathbb{O}_{n-1} \\ \# \psi^\bullet = m}} X_\psi \end{aligned}$$

We prove the second isomorphism of the lemma in a similar manner. □

Remark 5.10. Take $n \geq 1$ and $X \in \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-1,n})$. By lemma 5.9, X and $X^\dagger$ essentially have the same cells and the same incidence relations among them. Formally, there is a canonical isomorphism $\mathbb{O}_{n-1,n}/X \longrightarrow \mathbb{O}_{2,3}/X^\dagger$ between the categories of elements of X and $X^\dagger$, which maps source (resp. target) maps to source (resp. target) maps. Further, if $f : X \longrightarrow Y$ is a morphism in $\mathcal{P}\mathrm{sh}(\mathbb{O}_{n-1,n})$, then we have a commutative square

$$\begin{array}{ccc} \mathbb{O}_{n-1,n}/X & \xrightarrow{f} & \mathbb{O}_{n-1,n}/Y \\ \cong\downarrow & & \downarrow\cong \\ \mathbb{O}_{2,3}/X^\dagger & \xrightarrow{f^\dagger} & \mathbb{O}_{2,3}/Y^\dagger \end{array}$$

In particular, $(-)^\dagger : \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-1,n}) \longrightarrow \mathcal{P}\mathrm{sh}(\mathbb{O}_{2,3})$ is faithful.

Lemma 5.11. *Let $n \geq 1$, and consider the flattening operator $(-)^\dagger : \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-1,n}) \longrightarrow \mathcal{P}\mathrm{sh}(\mathbb{O}_{2,3})$.*

(1) For $\nu \in \mathbb{O}_{n+1}$, there exists a unique 4-opetope $\nu' \in \mathbb{O}_4$ such that $S[\nu]_{n-1,n}{}^\dagger \cong S[\nu']_{2,3}$.

(2) Let $X \in \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-1,n})$, $\nu \in \mathbb{O}_4$, and $f : S[\nu] \longrightarrow X^\dagger$. Then there exists a unique $\nu' \in \mathbb{O}_{n+1}$ and $g : S[\nu']_{2,3} \longrightarrow X$ such that $S[\nu']_{n-1,n}{}^\dagger = S[\nu]_{2,3}$, and $g^\dagger = f$.

Proof. (1) If $\nu = \mathsf{I}_\phi$ for $\phi \in \mathbb{O}_{n-1}$, let $\nu' = \mathsf{I}_{\phi^\dagger}$. If $\nu = \mathsf{Y}_\omega \bigcirc_{[[p_i]]} \nu_i$, let

$$\nu' := \mathsf{Y}_{\omega^\dagger} \bigcirc_{[a_\omega[p_i]]} \nu'_i,$$

where the ν'_i are given by induction. The graftings are well defined since

$$\mathsf{t}\,\mathsf{s}_{[]}\,\nu'_i = \mathsf{t}\left(\mathsf{s}_{[]}\,\nu_i\right)^\dagger = \left(\mathsf{t}\,\mathsf{s}_{[]}\,\nu_i\right)^\dagger = \left(\mathsf{s}_{[p_i]}\,\omega\right)^\dagger = \mathsf{s}_{a_\omega[p_i]}\,\omega^\dagger.$$

The isomorphism $S[\nu]_{n-1,n}{}^\dagger \cong S[\nu']_{2,3}$ can easily be shown by induction on the structure of ν and using lemma 3.14.

(2) For $\nu^\bullet = \{[p_1], \ldots, [p_m]\}$, f maps $[p_i]$ to a cell $x_i \in X^\dagger = X_{n-1}$, and let $\psi_i \in \mathbb{O}_{n-1}$ be the shape of x_i as a cell of X. If $[p_i] = [p_j[q]]$ for some j and $[q]$, then $\mathsf{s}_{[q]} x_j = \mathsf{t} x_i$ in $X^\dagger$, so $\mathsf{s}_{a^{-1}_{\psi_j}[q]} x_j = \mathsf{t} x_i$ in X, and in particular, $\mathsf{s}_{a^{-1}_{\psi_j}[q]} \psi_j = \mathsf{t} \psi_i$. Consequently, the ψ_is may be grafted together into a $(n+1)$-opetope ν' such that $\nu'^\dagger = \nu$, and $\mathsf{s}_{a^{-1}_{\nu'}[p_i]} = \psi_i$. Define $g : S[\nu']_{2,3} \longrightarrow X$ mapping $\mathsf{s}_{a^{-1}_{\nu'}[p_i]} \nu'$ to x_i, and observe that $g^\dagger = f$. □

Proposition 5.12. *For $X \in \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-1,n})$ we have $\mathfrak{Z}^3(X^\dagger) \cong (\mathfrak{Z}^n X)^\dagger$. Consequently, the functor $(-)^\dagger$ lifts as*

$$\begin{array}{ccc} \mathcal{A}\mathrm{lg}_{1,n} & \xrightarrow{(-)^\dagger} & \mathcal{A}\mathrm{lg}_{1,3} \\ \downarrow & & \downarrow \\ \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-1,n}) & \xrightarrow{(-)^\dagger} & \mathcal{P}\mathrm{sh}(\mathbb{O}_{2,3}). \end{array} \tag{5.13}$$

Proof. First, $\mathfrak{Z}^3(X^\dagger)_2 = X_2^\dagger \cong X_{n-1} = (\mathfrak{Z}^n X)_{n-1} = (\mathfrak{Z}^n X)_2^\dagger$. Then,

$$\begin{aligned} \mathfrak{Z}^3(X^\dagger)_3 &= \sum_{\nu \in \mathbb{O}_4} \mathcal{P}\mathrm{sh}(\mathbb{O}_{2,3})(S[\nu], X^\dagger) \\ &\cong \sum_{\nu \in \mathbb{O}_{n+1}} \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-1,n})(S[\nu], X) && \text{by lemma 5.11} \\ &= (\mathfrak{Z}^n X)_n \\ &= (\mathfrak{Z}^n X)_3^\dagger. \end{aligned}$$

□

Lemma 5.14. *Let $X \in \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-1,n})$ and $m : \mathfrak{Z}^n X \longrightarrow X$. Then m is a $\mathfrak{Z}^n$-algebra structure on X if and only if $m^\dagger : \mathfrak{Z}^3 X^\dagger \longrightarrow X^\dagger$ is a $\mathfrak{Z}^3$-algebra structure on $X^\dagger$.*

Proof. Clearly, $(-)^\dagger$ maps the multiplication $\mu^n : \mathfrak{Z}^n\mathfrak{Z}^n \longrightarrow \mathfrak{Z}^n$ to μ^3, and the unit $\eta^n : \mathrm{id} \longrightarrow \mathfrak{Z}^n$ to η^3. By remark 5.10, $(-)^\dagger$ is faithful, and the square on the left commutes if and only if the square on the right commutes

$$\begin{array}{ccc} \mathfrak{Z}^n\mathfrak{Z}^n X & \xrightarrow{\mathfrak{Z}^n m} & \mathfrak{Z}^n X \\ {\scriptstyle \mu^n}\downarrow & & \downarrow{\scriptstyle m} \\ \mathfrak{Z}^n X & \xrightarrow{m} & X, \end{array} \qquad \begin{array}{ccc} \mathfrak{Z}^3\mathfrak{Z}^3 X^\dagger & \xrightarrow{\mathfrak{Z}^3 m^\dagger} & \mathfrak{Z}^3 X^\dagger \\ {\scriptstyle \mu^3}\downarrow & & \downarrow{\scriptstyle m^\dagger} \\ \mathfrak{Z}^3 X^\dagger & \xrightarrow{m^\dagger} & X^\dagger, \end{array}$$

and likewise for the diagram involving η^n and η^3. □

Proposition 5.15. *The square (5.13) is a pullback. In other words, a $\mathfrak{Z}^n$-algebra structure on a presheaf $X \in \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-1,n})$ is completely determined by a $\mathfrak{Z}^3$-algebra structure on $X^\dagger$.*

Proof. Let $X \in \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-1,n})$. By proposition 5.12, a $\mathfrak{Z}^n$-algebra structure on X induces a $\mathfrak{Z}^3$-algebra structure on $X^\dagger$. By remark 5.10, $(-)^\dagger : \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-1,n}) \longrightarrow \mathcal{P}\mathrm{sh}(\mathbb{O}_{2,3})$ is faithful, and thus its lift $\mathcal{A}\mathrm{lg}_{1,n} \longrightarrow \mathcal{A}\mathrm{lg}_{1,3}$ is injective on object. In particular, different algebra structures on X result in different algebra structures on $X^\dagger$.

Conversely, let $m : \mathfrak{Z}^3 X^\dagger \longrightarrow X^\dagger$ be a $\mathfrak{Z}^3$-algebra structure on $X^\dagger$, and define $m' : \mathfrak{Z}^n X \longrightarrow X$ as the identity in dimension $n-1$, and mapping $f : S[\nu] \longrightarrow X$ to $m(f^\dagger) \in X_2^\dagger \cong X_{n-1}$. Recall that $f^\dagger$ is a map of the form $S[\nu'] \longrightarrow X^\dagger$, for some ν' such that $\mathsf{t}\nu' = (\mathsf{t}\nu)^\dagger$, and thus m' is a map of opetopic sets. By lemma 5.14, it is a $\mathfrak{Z}^n$-algebra structure on X.

Finally, the flattening operation establishes a bijective correspondence between the $\mathfrak{Z}^n$-algebra structures on X and the $\mathfrak{Z}^3$-algebra structures on $X^\dagger$. □

Theorem 5.16 (Algebraic trompe-l'œil)**.** *The following square is a pullback:*

$$\begin{array}{ccc} \mathcal{A}\mathrm{lg}_{k,n} & \xrightarrow{(-)_{n-1,n}{}^\dagger} & \mathcal{A}\mathrm{lg}_{1,3} \\ \downarrow & & \downarrow \\ \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n}) & \xrightarrow{(-)_{n-1,n}{}^\dagger} & \mathcal{P}\mathrm{sh}(\mathbb{O}_{2,3}). \end{array} \tag{5.17}$$

In other words, a $\mathfrak{Z}^n$-algebra structure on $X \in \mathcal{P}\mathrm{sh}(\mathbb{O}_{n-k,n})$ is completely determined by a $\mathfrak{Z}^3$-algebra structure on $X_{n-1,n}{}^\dagger$.

Proof. This is a direct consequence of propositions 5.4 and 5.15, and the pasting lemma for pullbacks. □

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