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Protoperads I: combinatorics and definitions

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Abstract

This paper is the first of two articles which develop the notion of protoperads. In this one, we construct a new monoidal product on the category of reduced $\mathfrak{S}$-modules. We study the associated monoids, called *protoperads*, which are a type of generalised operad. As operads encode algebraic operations with several inputs and one output, protoperads encode algebraic operations with the same number of inputs and outputs. We describe the underlying combinatorics of protoperads, and show that there exists a notion of free protoperad. We also show that the monoidal product introduced here is related to Vallette's one on the category of $\mathfrak{S}$-bimodules, via the induction functor.

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Introduction

The motivation for this work is to determine what is a double Poisson bracket up to homotopy. Double Poisson structures, defined by Van den Bergh in [21], give Poisson structures in non-commutative algebraic geometry (see [5, 22, 2, 1]). A double Poisson structure is *properadic* in nature; it is encoded by the properad $\mathcal{DP}ois$ (see [11]). This gives a good framework in which to study its up-to-homotopy version (cf. [18, 15, 16] for the definition of properad).

This is the first of two papers in which the author develops the notion of *protoperad*, which is a kind of properad (see [18, 19]). The homotopy theory of these new objects and its applications is treated in the second paper [11]. A properad is an algebraic notion which encodes types of bialgebras, i.e. operations with several inputs and several outputs. Properads are related to other families of algebraic objects:

$$\text{Associative algebras} \subset \text{NS-Operads} \subset \text{Operads} \subset \text{Dioperads} \subset \text{Properads} \subset \text{Props}.$$

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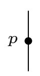
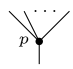
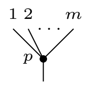
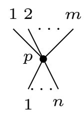
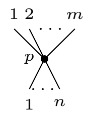
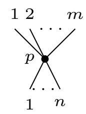
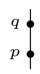
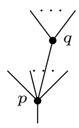
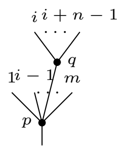
To illustrate these, let V be a k-vector space: associative algebras encode algebraic structures on V with one input and one output $V \to V$; non-symmetric (and symmetric) operads encode algebraic structures with several inputs and one output $V^{\otimes m} \to V$; dioperads, properads and props encodes algebraic structures with several inputs and outputs $V^{\otimes m} \to V^{\otimes n}$. To illustrate the relationship between these, we indicate below in which categories these objects live and their underlying combinatorics.

	Algebra	ns-Operad	Operad	Dioperad	Properad	Prop
Category	Vect_k	$\mathbb{N}\text{-}\mathsf{mod}_k^{\mathrm{red}}$	$\mathfrak{S}\text{-}\mathsf{mod}_k^{\mathrm{red}}$	$\mathfrak{S}\text{-}\mathsf{bimod}_k^{\mathrm{red}}$	$\mathfrak{S}\text{-}\mathsf{bimod}_k^{\mathrm{red}}$	$\mathfrak{S}\text{-}\mathsf{bimod}_k$
Monoid for	$\otimes_k$	$\circ^{\mathrm{ns}}$	$\circ$	$\boxtimes_{c,\varnothing}^{\mathrm{Gan}}$	$\boxtimes_c^{\mathrm{Val}}$	
Generators						
Composition				connected oriented graphs *without genus*	connected oriented graphs with genus	oriented graphs with genus
An example	Chain complexes	Associative algebras	Lie algebras	Lie bialgebras	Involutive Lie bialgebras	Loday infinitesimal bialgebras
A reference	[12, Ch.1]	[12, Ch.5]	[12, Ch.5]	[4]	[19]	[13]

These successive algebraic structures increase in generality. As a consequence, their homotopy theory is also more and more elaborate. If props encode the largest category of algebraic structures, we do not yet have the homotopical tools, such as Koszul duality theory, that hold for properads, operads, etc. Chain complex structures are encoded by the algebra of dual numbers. The non-symmetric operad framework is the minimal one to encode associative algebras, the symmetric operad framework is the minimal one that encodes associative commutative algebras, and so on: dioperads to encode bialgebras without genus in the underlying combinatorics, as Lie bialgebras.

In this paper, we define a new kind of such objects called *protoperads*, which are new intermediaries between algebras and properads:

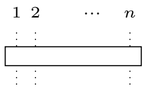
$$\begin{tikzcd} & \text{Operads} \arrow[dr, hook] & \\ \text{Algebras} \arrow[ur, hook] \arrow[dr, hook] & & \text{Properads} \\ & \text{Protoperads} \arrow[ur, hook] & \end{tikzcd} .$$

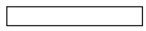
Protoperads are monoids in the same category as operads but for a new and very different monoidal product $\boxtimes$, the connected composition product, that we will define in Section 2. Generators of protoperads have the same number of inputs and outputs and a *diagonal symmetry*:

in fact, the $\mathfrak{S}$-bimodule (which are families of bimodules on permutation groups) of generators is induced by representations of the symmetric groups $\mathfrak{S}_n$, and the diagonal morphism

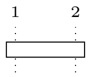
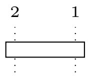
$$\begin{array}{ccc} \mathfrak{S}_n & \longrightarrow & \mathfrak{S}_n \times \mathfrak{S}_n^{op} \\ g & \longmapsto & g \times g^{-1} \end{array} . \tag{1}$$

There are represented by

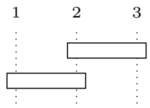
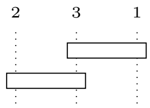
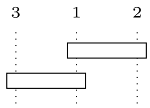


or by


if there is no ambiguity as to the arity i.e the number of inputs and outputs. Just as the combinatorics of operads is given by trees, the combinatorics of protoperads is described by *walls*, defined in Section 1. The main example is the protoperad $\mathcal{DL}ie$, which encodes a part of the double Poisson structure (called the double Lie structure), and which is defined by generators and relations, i.e. $\mathcal{DL}ie = \mathscr{F}(V_{\mathcal{DL}ie})/\langle R_{\mathcal{DJ}}\rangle$, with the following generator with 2 inputs and two outputs:

$V_{\mathcal{DL}ie} =$  $= -$ 

and the double Jacobi relation

$R_{\mathcal{DJ}} =$  $+$  $+$  .

We summarise the most important results of this article in the following theorem.

Theorem (Definition 2.15, Theorem 4.15 and Theorem 5.14)**.** *Let* C *be an abelian symmetric monoidal category. The category* $\mathfrak{S}\text{-}\mathsf{mod}_{\mathsf{C}}^{\mathrm{red}}$ *of reduced* $\mathfrak{S}$*-modules, i.e. the full sub-category of functors* $P : \mathsf{Bij}^{\mathrm{op}} \to \mathsf{C}$ *such that* $P(\varnothing) = 0$*, is monoidal for the connected composition product* $\boxtimes$*. The monoids in this category are called* protoperads*. There exists the free protoperad functor, denoted by* $\mathscr{F}(-)$*, and the monoidal functor*

$$\mathrm{Ind} : \quad \left(\mathfrak{S}\text{-}\mathsf{mod}_{\mathsf{C}}^{\mathrm{red}}, \boxtimes\right) \longrightarrow \left(\mathfrak{S}\text{-}\mathsf{bimod}_{\mathsf{C}}^{\mathrm{red}}, \boxtimes^{\mathrm{Val}}\right) ,$$

which is exact and satisfies $\mathrm{Ind} \circ \mathscr{F} = \mathscr{F}^{\mathrm{Val}} \circ \mathrm{Ind}$*, where* $\mathscr{F}^{\mathrm{Val}}$ *is the free properad functor.*

About the second paper [11] For an object $\mathcal{P}$ in the previous table, e.g. $\mathcal{P}$ an operad, a properad, etc. the notion of $\mathcal{P}$-algebra up to homotopy is encoded by a cofibrant resolution of $\mathcal{P}$. The homotopy theory of such objects is complicated. So it is useful to have the minimal framework to attack the problem of determining of a cofibrant resolution. We have technical tools to construct such a resolution, depending of the framework: the Koszul duality theory, rewriting methods, PBW or Gröbner bases and distributive laws (see [7, 3, 12]). But, apart from the Koszul duality theory developed by Vallette in [18, 19], such tools do not exist for properads. As a direct study of the Koszulness of the properad encoding double Lie structure is difficult, the idea is to use the diagonal symmetry of the generator and the relation of this properad to pass in the simpler world of protoperad and study the associated protoperad $\mathcal{DL}ie$. Using the framework of protoperads is successful. In the second paper [11], the author develops the Bar-Cobar adjunction and the Koszul duality theory of protoperads (see [11, Theorem 2.24]), which is related to Koszul duality theory of properads (see [18, 19]) via the induction functor Ind. We use it to prove the following theorem.

Theorem (Main theorem of [11]). *The properads $\mathcal{DL}ie$ and $\mathcal{DP}ois$ are Koszul.*

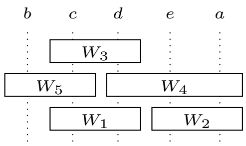
As the double Jacobi relation lives in genus 0, one can choose one other type of such object to encode double Lie structure, like dioperads (see [4] for the definition). But, as remarked by Merkulov–Vallette in [15, Sect. 5.6.], the functor

$$\mathsf{dioperads} \longrightarrow \mathsf{properads}$$

do not preserve the Koszulness. So proving the Koszulness of the dioperad encoding double Lie structure (which is done in [10, Corollaire 7.3.0.5.]) is not enough to prove the same property for the associated properad.

Contents of this article

Section 1 – Bricks and walls We develop the combinatorics for protoperads. This is controlled by *walls*. A wall over a non-empty finite set S is a set of subsets (called *bricks*) of S, equipped with a particular partial order and such that the union of these subsets is S. We represent a wall $W = ((W_1, \dots, W_n), \leqslant)$ diagrammatically as follows:



Here, the set S is $\{a, b, c, d, e\}$ and W has five elements. In this diagram, the dotted lines correspond to elements of S, each white box is a brick of the wall W, i.e. an element of S, and the partial order can be readable on the diagram : we have $W_i \leqslant W_j$ if $W_i \cap W_j \neq \varnothing$ and the brick W_j is above the brick W_i.

This is encoded by the functor $\mathcal{W}^{\mathrm{conn}}$ and certain subfunctors. We define also the notion of *connectedness* for a wall and denote by $\mathcal{W}^{\mathrm{conn}}(S)$, the set of connected walls over S.

Section 2 – Products on $\mathfrak{S}$-modules We review two monoidal products on $\mathfrak{S}\text{-}\mathsf{mod}^{\mathrm{red}}_{\mathsf{C}} := \mathsf{Func}(\mathsf{Bij}^{\mathrm{op}}, \mathsf{C})^{\mathrm{red}}$ which is the full sub-category of functors $P : \mathsf{Bij}^{\mathrm{op}} \to \mathsf{C}$ from the category Bij of finite sets with bijections, to an abelian monoidal category C such that $P(\varnothing) = 0$. These are *the composition product* $\square$, also called the Hadamard product, and *the concatenation product* $\otimes^{\mathrm{conc}}$ (see Section 2.3).

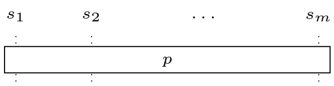
We also define the *connected composition product* on $\mathfrak{S}\text{-}\mathsf{mod}^{\mathrm{red}}_{\mathsf{C}}$ (see Definition 2.15), denoted by $\boxtimes$, which encodes algebraic structures which have the same number of inputs and outputs and a diagonal symmetry. It is the bifunctor

$$- \boxtimes - : \mathfrak{S}\text{-}\mathsf{mod}^{\mathrm{red}}_{\mathsf{C}} \times \mathfrak{S}\text{-}\mathsf{mod}^{\mathrm{red}}_{\mathsf{C}} \longrightarrow \mathfrak{S}\text{-}\mathsf{mod}^{\mathrm{red}}_{\mathsf{C}}$$

defined, for reduced $\mathfrak{S}$-modules P and Q and for a non-empty finite set S, by:

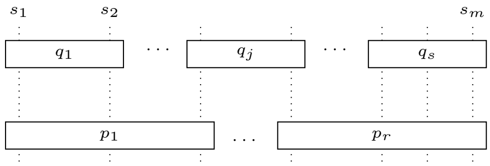
$$P \boxtimes Q(S) := \bigoplus_{(I,J) \in \mathcal{X}^{\mathrm{conn}}(S)} \bigotimes_{\alpha} P(I_\alpha) \otimes \bigotimes_{\beta} Q(J_\beta).$$

For a $\mathfrak{S}$-module P, we represent a element p of $\mathcal{P}(S)$ as a labelled brick



,

where the dotted lines correspond to elements of S. With this graphical representation, an element in the product $P \boxtimes_c Q(S)$ is represented by



where $q_1, \ldots, q_s \in Q$ and $p_1, \ldots, p_r \in P$.

Section 3 – Connected product on $\mathfrak{S}$-bimodules In this section, we recall three monoidal structures on the category of reduced $\mathfrak{S}$-bimodules which are analogous to those on reduced $\mathfrak{S}$-modules: the concatenation product, the composition product, and the connected composition product $\boxtimes^{\mathrm{Val}}$, defined by Vallette in [18, 19]. Here we take a different approach from the original one: we give an equivalent definition of the connected composition product, which is more adapted to *species* and the functorial point of view of $\mathfrak{S}$-bimodules.

Section 4 – Induction functor The new product $\boxtimes$ is the avatar of the product $\boxtimes^{\mathrm{Val}}$ on the category $\mathfrak{S}\text{-}\mathsf{bimod}_{\mathsf{C}}^{\mathrm{red}}$. The most important property of the product $\boxtimes$ is its compatibility with the Vallette product via the induction functor $\mathrm{Ind} : \mathfrak{S}\text{-}\mathsf{mod}_{\mathsf{C}}^{\mathrm{red}} \to \mathfrak{S}\text{-}\mathsf{bimod}_{\mathsf{C}}^{\mathrm{red}}$, defined using Equation (1). We prove the following.

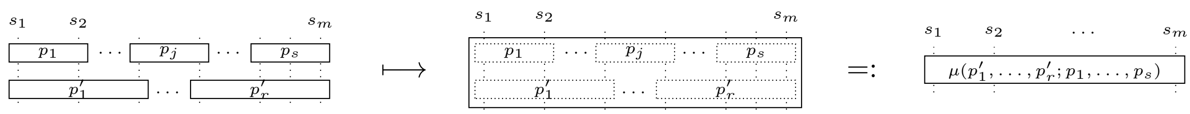
Theorem (Theorem 4.15)**.** *The induction functor*

$$\mathrm{Ind} : \left(\mathfrak{S}\text{-}\mathsf{mod}_{\mathsf{C}}^{\mathrm{red}}, \boxtimes\right) \longrightarrow \left(\mathfrak{S}\text{-}\mathsf{bimod}_{\mathsf{C}}^{\mathrm{red}}, \boxtimes^{\mathrm{Val}}\right)$$

is monoidal. In particular, it sends protoperads to properads,

Section 5 – Protoperads In this section, we define the central object of this paper.

Definition (Protoperad)**.** *Protoperads* are the monoids in the monoidal category $(\mathfrak{S}\text{-}\mathsf{mod}_{\mathsf{C}}^{\mathrm{red}}, \boxtimes_c)$. The product of a protoperad $\mathcal{P}$ is a map $\mu \colon \mathcal{P} \boxtimes_c \mathcal{P} \to \mathcal{P}(S)$ graphically represented



where $p_!, \ldots, p_s, p'_1, \ldots, p'_r \in \mathcal{P}$.

Under the functor of induction, protoperads are identified with properads concentrated in the arities (n, n) with $n \in \mathbb{N}^*$. We give equivalent definition of a protoperad, generalising the definition of operads in terms of partial compositions.

Proposition (Proposition 5.10)**.** *A protoperad $\mathcal{P}$ has an underlying partial composition system. Conversely, a partial composition system on a $\mathfrak{S}$-module P canonically extends to a protoperad structure.*

Using the work of Vallette on free monoids in abelian monoidal categories (see [20]), we show that there exists a free protoperad functor. We also have a combinatorial description of the free protoperad.

Theorem (Theorem 5.14)**.** *Let V be a reduced $\mathfrak{S}$-module. The free protoperad functor is the graded functor $\mathscr{F}^*(-)$ given by the isomorphism of right* $\mathrm{Aut}(S)$*-modules*

$$\mathscr{F}^{\rho}(V)(S) \cong \bigoplus_{\substack{(\{W_\alpha\}_{\alpha\in A},\leqslant)\\ \in\mathcal{W}_{\rho}^{\mathrm{conn}}(S)}} \bigotimes_{\alpha\in A} V(W_\alpha),$$

for S a finite set and ρ a natural number and with $\mathcal{W}_{\rho}^{\mathrm{conn}}(S)$, the set of connected walls with ρ bricks.

Section 6 – Colours on walls In this last section, we define the notion of a coloured wall, and we associate to a wall W over a totally ordered set S, the *colouring complex*, denoted by $\mathrm{C}_{\bullet}^{\mathcal{C}\mathrm{ol}}(W)$. This is motivated by the combinatorial description of the bar construction of the free protoperad (see also [10, 11]). The principal result of this section is the following:

Theorem (see Theorem 6.15)**.** *Let S be a finite totally ordered set, and W a wall over S. If the set* $\mathrm{Succ}(W)$ *(see Section 1.1 for the definition of* Succ*) is non empty, then the colouring complex* $\mathrm{C}_{\bullet}^{\mathcal{C}\mathrm{ol}}(W)$ *is acyclic.*

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Notations

We use the notation $\mathbb{N}^*$ for the set $\mathbb{N} - \{0\}$. We denote by Bij, the category with finite sets as objects and bijections as morphisms and Sets, the category of all sets and all maps. For two integers a and b, we denote by $[\![a,b]\!]$ the set $[a,b]\cap\mathbb{Z}$, and, for n in $\mathbb{N}^*$, $\mathfrak{S}_n$ is the automorphism group of $[\![1,n]\!]$, i.e. $\mathfrak{S}_n \coloneqq \mathrm{Aut}_{\mathsf{Bij}}([\![1,n]\!])$. Let C and D be two categories, we denote by $\mathsf{Func}(\mathsf{C},\mathsf{D})$, the category of the functors from C to D. Let $(\mathsf{D},\odot)$ be a monoidal category: we denote by $\mathcal{A}s(\mathsf{D},\odot)$ the category of monoids without unit (not necessarily unital) in D and $\mathcal{UA}s(\mathsf{D},\odot)$ the category of unital monoids in D. If $(\mathsf{D},\odot)$ is symmetric monoidal, we also denote by $\mathcal{C}om(\mathsf{D},\odot)$ the category of commutative monoids and $\mathcal{UC}om(\mathsf{D},\odot)$ the category of commutative unital monoids in D. A monoidal category $(\mathsf{D},\odot,I)$ is an *abelian monoidal category* if D is also abelian: we do not suppose any compatibility between the monoidal product $\odot$ and the abelian structure $\oplus$.

1. Bricks and walls

We begin by describing the combinatorial framework of this paper. The first section is about posets and, after that, we introduce the *functor of walls*. Walls encode the combinatorics of "diagonal properads", as rooted trees govern the combinatorics of operads (see 1.22 for a pictural explanation of this fact). In this section, we define two important functors:

$$\mathcal{X}^{\mathrm{conn}} : \mathsf{Bij}^{\mathrm{op}} \to \mathsf{Sets}^{\mathrm{op}} \quad \text{and} \quad \mathcal{W}^{\mathrm{conn}} : \mathsf{Bij}^{\mathrm{op}} \to \mathsf{Sets}^{\mathrm{op}} .$$

The first one, $\mathcal{X}^{\mathrm{conn}}$, encodes the combinatorics of the new monoidal structure on the category of $\mathfrak{S}$-modules, *the connected composition product* (see Section 2.4). The second, $\mathcal{W}^{\mathrm{conn}}$ encodes the combinatorics of the free monoid for this monoidal structure (see Theorem 5.14).

Remark 1.1. In this section, we construct (covariant) functors from the opposite category of finite sets to the opposite category of sets, i.e. $\mathcal{F} : \mathsf{Bij}^{\mathrm{op}} \to \mathsf{Sets}^{\mathrm{op}}$ or a abelian symmetric monoidal category C, i.e. $F : \mathsf{Bij}^{\mathrm{op}} \to \mathsf{C}$. We choose to consider the opposite category of Bij to get a right action of the automorphism group $\mathrm{Aut}(S)$ on $\mathcal{F}(S)$. This right action mimics the actions of symmetric groups on the leaves of trees in the operadic case.

1.1 Recollections on posets Let k and l be two elements of a poset $(K, \leqslant)$. We say that k and l are *successors* if $k < l$ and if there does not exist an element t in K such that $k < t < l$. We denote by $\mathrm{Succ}(K)$, the set of pairs of successors of K. A *chain* of a poset K is an increasing sequence of elements of K and the *length* of the chain is the number of elements of the chain: we denote the length of a chain $k_1 < k_2 < \ldots < k_r$ by $\mathrm{len}(k_1 < k_2 < \ldots < k_r)$. The *height* of an element k of a poset $(K, \leqslant)$ is the element $\mathfrak{h}(k)$ of $\mathbb{N} \cup \{\infty\}$ defined by

$$\mathfrak{h}(k) := \max\big\{\mathrm{len}(c) \in \mathbb{N}^* \mid c = (\lambda_1 < \lambda_2 < \ldots < \lambda_{r-1} < k)\big\}.$$

Proposition 1.2. *Let $(K, \leqslant)$ be a poset and (k, l) in $\mathrm{Succ}(K)$. Then the surjection*

$$\pi_k^l : K \twoheadrightarrow K/_{k\sim l}$$

induces a partial order on $K/_{k\sim l}$ defined, for all r and s in K, by

- *$[r] \leqslant [s]$ if $r \leqslant s$ and $r, s \notin \{k, l\}$*
- *$[s] \leqslant [k \sim l]$ (resp. $s \geqslant [k \sim l]$) if $s \leqslant k$ or $s \leqslant l$ (resp. $s \geqslant k$ or $s \geqslant l$).*

Proof. Left to the reader. □

Lemma 1.3. *Let $(R, \leqslant_R)$ and $(S, \leqslant_S)$ be two posets with injections $R \hookrightarrow T \hookleftarrow S$. If, for all a and b in $R \cap S$, $a \leqslant_R b$ if and only if $a \leqslant_S b$, then $R \cup S$ has a canonical partial order which extends the partial orders $\leqslant_R$ and $\leqslant_S$.*

Proof. For any x and y in $R \cup S$, we have $x \leqslant_{R\cup S} y$ if and only if one of the following assumption holds:

- x and y are in R and $x \leqslant_R y$;
- x and y are in S and $x \leqslant_S y$;
- x is in R, y is in S and there exists t in $R \cap S$ such that $x \leqslant_R t \leqslant_S y$.

□

1.2 The functors of walls In the rest of this section, we define some (covariant) functors from the category $\mathsf{Bij}^{\mathrm{op}}$ to the category $\mathsf{Sets}^{\mathrm{op}}$, called *functors of walls*. Let $\mathcal{W}$ be a functor of walls (see below for definitions) and S a finite set with n elements. An element W of $\mathcal{W}(S)$ should represent a morphism $\mathrm{Hom}_{\mathsf{C}}(V^{\otimes n}, V^{\otimes n})$, for V an object of C, with a diagonal action of $\mathfrak{S}_n$ by permutations inputs and outputs at the same time.

Definition 1.4 (Functor of ordered walls $\mathcal{W}^{\mathrm{or}}$)**.** For n in $\mathbb{N}^*$, the covariant functor $\mathcal{W}_n^{\mathrm{or}} : \mathsf{Bij}^{\mathrm{op}} \longrightarrow \mathsf{Bij}^{\mathrm{op}}$ is given, for all finite set S, by

$$\mathcal{W}_n^{\mathrm{or}}(S) := \left\{ W = \big((W_1, \ldots, W_n), \leqslant\big) \;\middle|\; \begin{array}{l} \forall i \in [\![1,n]\!], W_i \subset S,\ W_i \neq \varnothing;\ \cup_i W_i = S\ ; \\ \forall s \in S, \Gamma_s^W := \{W_i | s \in W_i\} \text{ is totally ordered (by } \leqslant) \end{array} \right\}.$$

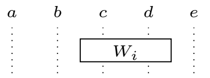
We denote by $(W, \leqslant)$, the elements of $\mathcal{W}_n^{\mathrm{or}}(S)$. The action of an element σ of $\mathrm{Aut}(S)$ on $\big((W_1, \ldots, W_n), \leqslant\big)$ in $\mathcal{W}_n^{\mathrm{or}}$ is induced by the canonical action on S,

$$\big((W_1, \ldots, W_n), \leqslant\big) \cdot \sigma = \big((W_1 \cdot \sigma, \ldots, W_n \cdot \sigma), \leqslant^{\sigma}\big)$$

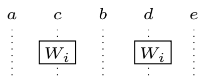
where $\leqslant^{\sigma}$ is induced by the total orders of the sets $\Gamma_s^{W\cdot\sigma} := \{W_i \cdot \sigma | s \in W_i \cdot \sigma\}$. The functor $\mathcal{W}^{\mathrm{or}}$ is defined by

$$\begin{array}{rccc} \mathcal{W}^{\mathrm{or}} : & \mathsf{Bij}^{\mathrm{op}} & \longrightarrow & \mathsf{Sets}^{\mathrm{op}} \\ & S & \longmapsto & \coprod_{n \in \mathbb{N}^*} \mathcal{W}_n^{\mathrm{or}}(S) \end{array}.$$

Remark 1.5. *About the diagrammatic representation of walls.* The terminology introduced in Definition 1.4 comes from the diagrammatic representation of the elements of $\mathcal{W}^{\mathrm{or}}(S)$. Fix an finite set S and an element $W = ((W_1, \ldots, W_n), \leqslant) \in \mathcal{W}^{\mathrm{or}}(S)$: one can represents elements of S by vertical dotted lines and the subset W_i of S by *a brick* :



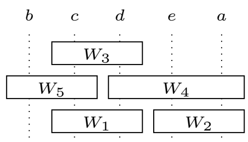
As S does not have a total order, one can choose an other order on S to represent it, as the following



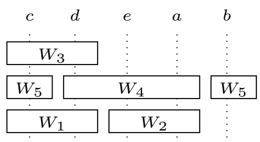
In this case, the representation of the brick W_i is "broken". Just as the combinatorics of operads is controlled by rooted trees (cf. [12, Sect. 5.6]), the combinatorics of protoperads is controlled by a stack of bricks : an element $W = ((W_1, \ldots, W_n), \leqslant)$ of $\mathcal{W}^{\mathrm{or}}(S)$ is called *a wall.* The partial order of $W = ((W_1, \ldots, W_n), \leqslant)$ gives a way to organise the bricks $W_1, \ldots, W_n$ between them. For example, if we consider a set $S = \{a, b, c, d, e\}$ with five elements and a wall W with five bricks $W_1 = \{c, d\} = W_3$, $W_2 = \{a, e\}$, $W_4 = \{a, d, e\}$, and $W_5 = \{b, c\}$ with the partial order

$$W_1 \leqslant W_5 \leqslant W_3,\ W_1 \leqslant W_4 \leqslant W_3,\ W_2 \leqslant W_4,$$

one can represent it by



or

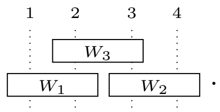


The partial order is readable on the diagram : we have $W_i \leqslant W_j$ if $W_i \cap W_j \neq \varnothing$ and the brick W_j is above the brick W_i in the representation.

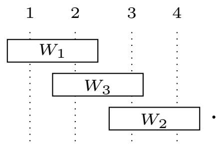
Remark 1.6. The terminology *ordered wall* is chosen because elements of $\mathcal{W}_n^{\mathrm{or}}(S)$ is n-tuples of subsets of S. It *does not* refer to the partial order which corresponds to the combinatorics of the wall.

Example 1.7. We consider $W_1 = \{1,2\}$, $W_2 = \{3,4\}$ and $W_3 = \{2,3\}$, three subsets of $S = [\![1,4]\!]$. We have the following four elements of $\mathcal{W}_3^{\mathrm{or}}(S)$:

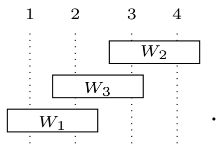
- $\big((W_1, W_2, W_3), <^1\big)$ with $<^1$ given by $W_1 <^1 W_3$ and $W_2 <^1 W_3$, represented by



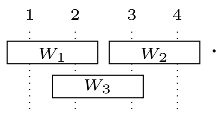
- $\big((W_1, W_2, W_3), <^2\big)$ with $<^2$ given by $W_3 <^2 W_1$ and $W_3 <^2 W_2$, represented by



- $\big((W_1, W_2, W_3), <^3\big)$ with $<^3$ given by $W_1 <^3 W_3$ and $W_3 <^3 W_2$, represented by



- $\big((W_1, W_2, W_3), <^4\big)$ with $<^4$ given by $W_3 <^4 W_1$ and $W_2 <^4 W_3$, represented by



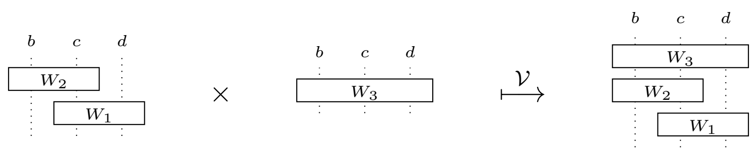
These four elements of $\mathcal{W}_3^{\mathrm{or}}(S)$ are distinct.

Remark 1.8. For all non-empty sets S, we have $\mathcal{W}_0^{\mathrm{or}}(S) = \varnothing$. For all integers $n > 0$, the group $\mathfrak{S}_n$ acts freely on $\mathcal{W}_n^{\mathrm{or}}(S)$ by permuting the position of elements, i.e. for τ in $\mathfrak{S}_n$, we have $\tau \cdot \big((W_1, \dots, W_n), \leqslant\big) = \big((W_{\tau^{-1}(1)}, \dots, W_{\tau^{-1}(n)}), \leqslant\big)$. The partial order is the same because it doesn't depend of the W_i's indexes.

The *vertical composition product* on $\mathcal{W}^{\mathrm{or}}$, is the natural transformation:

$$\mathcal{V} : \big(\mathcal{W}^{\mathrm{or}} \times \mathcal{W}^{\mathrm{or}}\big)(-) \longrightarrow \mathcal{W}^{\mathrm{or}}(-)$$

given, for all finite set S, by $\mathcal{V}_{n,m,S} : \mathcal{W}_n^{\mathrm{or}}(S) \times \mathcal{W}_m^{\mathrm{or}}(S) \longrightarrow \mathcal{W}_{m+n}^{\mathrm{or}}(S)$ which sends the pair $\big((W, \leqslant_W), (L, \leqslant_L)\big)$ on $\big(R = (W_1, \dots, W_n, L_1, \dots L_m), \leqslant_W^L\big)$ where, for all s in S, the total order of the poset Γ_s^R is induced by the ones of Γ_s^W and Γ_s^L and by extension, for all W_i in Γ_s^W and all L_j in Γ_s^L, we have $W_i \leqslant_W^L L_j$. The vertical product is represented as follow:



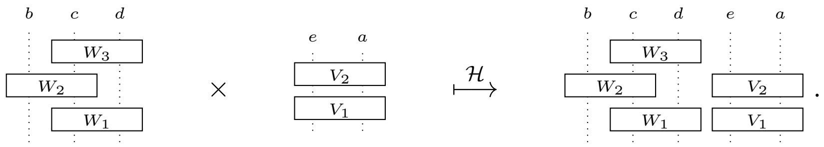
This product is associative, so, for all finite set S, we have the following commutative diagram:

$$\begin{tikzcd}
\mathcal{W}^{\mathrm{or}}(S) \times \mathcal{W}^{\mathrm{or}}(S) \times \mathcal{W}^{\mathrm{or}}(S) \arrow[r, "\mathcal{V}\times\mathrm{id}"] \arrow[d, "\mathrm{id}\times\mathcal{V}"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{W}^{\mathrm{or}}(S) \times \mathcal{W}^{\mathrm{or}}(S) \arrow[d, "\mathcal{V}"] \\
\mathcal{W}^{\mathrm{or}}(S) \times \mathcal{W}^{\mathrm{or}}(S) \arrow[r, "\mathcal{V}"'] & \mathcal{W}^{\mathrm{or}}(S).
\end{tikzcd}$$

The *(horizontal) concatenation product* on $\mathcal{W}^{\mathrm{or}}$, is the natural transformation between bifunctors:

$$\mathcal{H} : \mathcal{W}^{\mathrm{or}}(-_1) \times \mathcal{W}^{\mathrm{or}}(-_2) \longrightarrow \mathcal{W}^{\mathrm{or}}(-_1 \amalg -_2)$$

given, for all finite sets S and T, by $\mathcal{H}_{n,m,S,T} : \mathcal{W}^{\mathrm{or}}_n(S) \times \mathcal{W}^{\mathrm{or}}_m(T) \longrightarrow \mathcal{W}^{\mathrm{or}}_{m+n}(S \amalg T)$ which sends $\big((W, \leqslant_W), (L, \leqslant_L)\big)$ to $\big(R = (W_1, \ldots, W_n, L_1, \ldots L_m), \leqslant_{W,L}\ \big)$ where, for all s in S and t in T, we have the equalities $\Gamma^R_s = \Gamma^W_s$ and $\Gamma^R_t = \Gamma^L_t$. The horizontal product is represented as follow



This product is associative and commutative, so we have the following commutative diagrams:

$$\begin{tikzcd}
\mathcal{W}^{\mathrm{or}}(-_1) \times \mathcal{W}^{\mathrm{or}}(-_2) \times \mathcal{W}^{\mathrm{or}}(-_3) \arrow[r, "\mathcal{H}\times\mathrm{id}"] \arrow[d, "\mathrm{id}\times\mathcal{H}"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{W}^{\mathrm{or}}(-_1) \times \mathcal{W}^{\mathrm{or}}(-_2 \amalg -_3) \arrow[d, "\mathcal{H}"] \\
\mathcal{W}^{\mathrm{or}}(-_1 \amalg -_2) \times \mathcal{W}^{\mathrm{or}}(-_3) \arrow[r, "\mathcal{H}"'] & \mathcal{W}^{\mathrm{or}}(-_1 \amalg -_2 \amalg -_3)
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathcal{W}^{\mathrm{or}}(-_1) \times \mathcal{W}^{\mathrm{or}}(-_2) \arrow[r, "\mathcal{H}"] \arrow[d, "\cong"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{W}^{\mathrm{or}}(-_1 \amalg -_2) \arrow[d, "\cong"] \\
\mathcal{W}^{\mathrm{or}}(-_2) \times \mathcal{W}^{\mathrm{or}}(-_1) \arrow[r, "\mathcal{H}"'] & \mathcal{W}^{\mathrm{or}}(-_2 \amalg -_1)\ .
\end{tikzcd}$$

We also have the following commutative diagram of natural transformations, called *the interchanging law*:

$$\begin{tikzcd}
\left(\mathcal{W}^{\mathrm{or}}\right)^{\times 2}(-_1) \times \left(\mathcal{W}^{\mathrm{or}}\right)^{\times 2}(-_2) \arrow[rr, "\mathrm{id}\times\sigma\times\mathrm{id}"] \arrow[d, "\mathcal{V}\times\mathcal{V}"'] & & \left(\mathcal{W}^{\mathrm{or}}(-_1) \times \mathcal{W}^{\mathrm{or}}(-_2)\right)^{\times 2} \arrow[d, "\mathcal{H}\times\mathcal{H}"] \\
\mathcal{W}^{\mathrm{or}}(-_1) \times \mathcal{W}^{\mathrm{or}}(-_2) \arrow[dr, "\mathcal{H}"] & \circlearrowleft & \left(\mathcal{W}^{\mathrm{or}}(-_1 \amalg -_2)\right)^{\times 2} \arrow[dl, "\mathcal{V}"'] \\
& \mathcal{W}^{\mathrm{or}}(-_1 \amalg -_2) &
\end{tikzcd} .$$

Definition 1.9 (Functor of walls $\mathcal{W}$)**.** We define the functor $\mathcal{W}_n : \mathsf{Bij}^{\mathrm{op}} \to \mathsf{Bij}^{\mathrm{op}}$ by $\mathcal{W}_n := \left(\mathcal{W}^{\mathrm{or}}_n\right)_{\mathfrak{S}_n}$ which is given, for all finite sets S, by

$$\mathcal{W}_n(S) := \left\{ W = \big(\{W_\alpha\}_{\alpha\in A}, \leqslant\ \big) \;\middle|\; \begin{array}{l} |A| = n;\ \forall \alpha \in A,\ W_\alpha \subset S,\ W_\alpha \neq \varnothing;\ \cup_\alpha W_\alpha = S; \\ \forall s \in S,\ \Gamma^W_s := \{W_\alpha | s \in W_\alpha\} \text{ is totally ordered (by } \leqslant) \end{array} \right\} .$$

We have the natural projection $\pi : \mathcal{W}^{\mathrm{or}} \to \mathcal{W}$. The action of an element σ of $\mathrm{Aut}(S)$ on $\big(\{W_\alpha\}_{\alpha\in A}, \leqslant\ \big) \in \mathcal{W}_n$ is induced by the canonical action on S, i.e.

$$\big(\{W_\alpha\}_{\alpha\in A}, \leqslant\ \big) \cdot \sigma = \big(W = \{W_\alpha \cdot \sigma\}_{\alpha\in A}, \leqslant^\sigma\ \big)$$

where $\leqslant^\sigma$ is induced by total orders of $\Gamma^{W\cdot\sigma}_{s\cdot\sigma} := \{W_\alpha \cdot \sigma | s\cdot\sigma \in W_\alpha \cdot \sigma\}$. We also define the functor

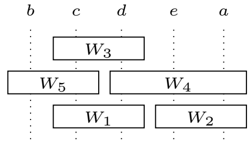
$$\begin{array}{rccc} \mathcal{W} : & \mathsf{Bij}^{\mathrm{op}} & \longrightarrow & \mathsf{Sets}^{\mathrm{op}} \\ & S & \longmapsto & \coprod_{n\in\mathbb{N}^*} \mathcal{W}_n(S) \end{array} .$$

An element W of $\mathcal{W}(S)$ is called a *wall* over S, and an element of a wall W is called a *brick* of W.

Remark 1.10. In Remark 1.5, we consider the set $S = \{a, b, c, d, e\}$ with five elements and the wall W with five bricks $W_1 = \{c, d\} = W_3$, $W_2 = \{a, e\}$, $W_4 = \{a, d, e\}$, and $W_5 = \{b, c\}$ with the partial order

$$W_1 \leqslant W_5 \leqslant W_3,\ W_1 \leqslant W_4 \leqslant W_3,\ W_2 \leqslant W_4,$$

represented by

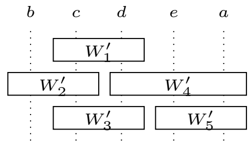


.

One can also consider the wall W' with five bricks $W'_1 = \{c, d\} = W'_3$, $W'_5 = \{a, e\}$, $W'_4 = \{a, d, e\}$, and $W'_2 = \{b, c\}$ with the partial order

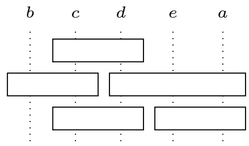
$$W'_3 \leqslant W'_2 \leqslant W'_1,\ W'_3 \leqslant W'_4 \leqslant W'_1,\ W'_5 \leqslant W'_4,$$

represented by

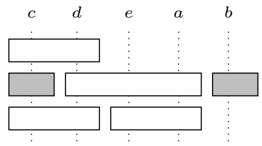


.

The walls W and W' are distinct elements in $\mathcal{W}^{\mathrm{or}}(S)$ which are identified in $\mathcal{W}(S)$ to the element $[W]$ which can be represented by



or



,

the grey colour indicating that this brick is "broken" in this representation.

Proposition 1.11 (Products on $\mathcal{W}$). *The products $\mathcal{V}$ and $\mathcal{H}$ on $\mathcal{W}^{\mathrm{or}}$ pass through the quotient by the actions of the symmetric groups on the indexes of bricks, hence induce natural transformations*

$$\mathcal{V} : \big(\mathcal{W} \times \mathcal{W}\big)(-) \longrightarrow \mathcal{W}(-) \quad \textit{and} \quad \mathcal{H} : \mathcal{W}(-_1) \times \mathcal{W}(-_2) \longrightarrow \mathcal{W}(-_1 \amalg -_2),$$

respectively called the composition product and concatenation product on $\mathcal{W}$, such that we have the following commutative diagrams

$$\begin{tikzcd}
\big(\mathcal{W}^{\mathrm{or}} \times \mathcal{W}^{\mathrm{or}}\big)(-) \arrow[r, "\mathcal{V}"] \arrow[d, "\pi^{\times 2}"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{W}^{\mathrm{or}}(-) \arrow[d, "\pi"] \\
\big(\mathcal{W} \times \mathcal{W}\big)(-) \arrow[r, "\mathcal{V}"'] & \mathcal{W}(-)
\end{tikzcd} ; \quad
\begin{tikzcd}
\mathcal{W}^{\mathrm{or}}(-_1) \times \mathcal{W}^{\mathrm{or}}(-_2) \arrow[r, "\mathcal{H}"] \arrow[d, "\pi^{\times 2}"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{W}^{\mathrm{or}}(-_1 \amalg -_2) \arrow[d, "\pi"] \\
\mathcal{W}(-_1) \times \mathcal{W}(-_2) \arrow[r, "\mathcal{H}"] & \mathcal{W}(-_1 \amalg -_2)
\end{tikzcd} ;$$

$$\begin{tikzcd}
\big(\mathcal{W} \times \mathcal{W}\big)(-_1) \times \big(\mathcal{W} \times \mathcal{W}\big)(-_2) \arrow[rr, "\mathrm{id} \times \sigma \times \mathrm{id}"] \arrow[d, "\mathcal{V} \times \mathcal{V}"'] & & \mathcal{W}(-_1) \times \mathcal{W}(-_2) \times \mathcal{W}(-_1) \times \mathcal{W}(-_2) \arrow[d, "\mathcal{H} \times \mathcal{H}"] \\
\mathcal{W}(-_1) \times \mathcal{W}(-_2) \arrow[dr, "\mathcal{H}"'] & \circlearrowleft & \mathcal{W}(-_1 \amalg -_2) \times \mathcal{W}(-_1 \amalg -_2) \arrow[dl, "\mathcal{V}"] \\
& \mathcal{W}(-_1 \amalg -_2) &
\end{tikzcd} .$$

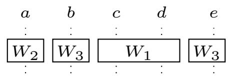
1.3 Two subfunctors of $\mathcal{W}$ We introduce here two important subfunctors of $\mathcal{W}$. For all finite non-empty sets S and all n in $\mathbb{N}^*$, we define the functor of ordered partitions $\mathcal{Y}^{\mathrm{or}}_n : \mathsf{Bij}^{\mathrm{op}} \to \mathsf{Bij}^{\mathrm{op}}$, by

$$\mathcal{Y}^{\mathrm{or}}_n(S) := \{ \ (K_1, \dots, K_n) \ \mid \amalg_{i=1}^n K_i = S; \ \forall i \in [\![1,n]\!], K_i \neq \varnothing \ \}$$

equipped with the natural injection $\mathcal{Y}^{\mathrm{or}}_n \hookrightarrow \mathcal{W}^{\mathrm{or}}_n$. By disjoint union, we also define the functor $\mathcal{Y}^{\mathrm{or}}$ by

$$\begin{array}{rccc} \mathcal{Y}^{\mathrm{or}} : & \mathsf{Bij}^{\mathrm{op}} & \longrightarrow & \mathsf{Sets}^{\mathrm{op}} \\ & S & \longmapsto & \coprod_{n\in\mathbb{N}^*} \mathcal{Y}^{\mathrm{or}}_n(S) \end{array} .$$

Example 1.12. Consider a set $S = \{a, b, c, d, e\}$, the wall $W = (W_1, W_2, W_3)$ with $W_1 = \{c, d\}$, $W_2 = \{a\}$ and $W_3 = \{b, e\}$, represented by



is an element of $\mathcal{Y}^{\mathrm{or}}(S)$.

Via the vertical composition, we have, for all finite sets S and all m and n in $\mathbb{N}^*$, the isomorphism:

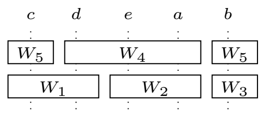
$$\mathcal{Y}^{\mathrm{or}}_m(S) \times \mathcal{Y}^{\mathrm{or}}_n(S) \ \cong$$
$$\left\{ R = \big((K_1, \dots, K_m, L_1, \dots, L_n), \leqslant\big) \ \middle| \ \begin{array}{l} \amalg_i K_i = S = \amalg_j L_j; \\ \forall i \in [\![1,m]\!], K_i \neq \varnothing; \ \forall j \in [\![1,n]\!], L_j \neq \varnothing; \\ \forall s \in S, \ \exists! i \in [\![1,m]\!], \ \exists! j \in [\![1,n]\!] \\ \qquad \text{s.t. } \Gamma^R_s := \{K_i, L_j\} \text{ and } K_i \leqslant L_j \end{array} \right\},$$

which gives us the natural injection $\mathcal{Y}^{\mathrm{or}}_m(S) \times \mathcal{Y}^{\mathrm{or}}_n(S) \hookrightarrow \mathcal{W}^{\mathrm{or}}_{m+n}(S)$. Hence, we define, for all non-empty finite sets S, the functor $\mathcal{X}^{\mathrm{or}}$ of ordered pairs of partitions of finite sets, by

$$\mathcal{X}^{\mathrm{or}}(S) := \coprod_{m,n\in\mathbb{N}^*} \mathcal{Y}^{\mathrm{or}}_n(S) \times \mathcal{Y}^{\mathrm{or}}_m(S).$$

equipped with the natural injection $\mathcal{X}^{\mathrm{or}} \hookrightarrow \mathcal{W}^{\mathrm{or}}$. This functor is important: it encodes the combinatorics of our new monoidal product, up to a property of connectedness (see Section 1.4).

Example 1.13. Let S be a set with five elements. The following wall



represents an element of $\mathcal{X}^{\mathrm{or}}(S)$.

The natural surjection $\pi : \mathcal{W}^{\mathrm{or}} \twoheadrightarrow \mathcal{W}$ gives the following commutative diagrams of natural transformations:

$$\begin{tikzcd} \mathcal{Y}^{\mathrm{or}} \arrow[r, hook] \arrow[d, two heads] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{W}^{\mathrm{or}} \arrow[d, two heads] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{X}^{\mathrm{or}} \arrow[l, hook'] \arrow[d, two heads] \\ \mathcal{Y} \arrow[r, hook] & \mathcal{W} & \mathcal{X} \arrow[l, hook'] \end{tikzcd},$$

where $\mathcal{Y}$ (resp. $\mathcal{X}$) is the quotient of $\mathcal{Y}^{\mathrm{or}}$ (resp. $\mathcal{X}^{\mathrm{or}}$) by the action of the symmetric group on the indexes of bricks. The concatenation product restricts to the subfunctors $\mathcal{X}$ and $\mathcal{Y}$:

$$\begin{tikzcd} \mathcal{Y}(-_1)\times\mathcal{Y}(-_2) \arrow[r, hook] \arrow[d, "\mu^{\mathrm{conc}}"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{W}(-_1)\times\mathcal{W}(-_2) \arrow[d, "\mathcal{H}"] \\ \mathcal{Y}(-_1\amalg -_2) \arrow[r, hook] & \mathcal{W}(-_1\amalg -_2) \end{tikzcd} \quad\text{and}\quad \begin{tikzcd} \mathcal{X}(-_1)\times\mathcal{X}(-_2) \arrow[r, hook] \arrow[d, "\mu^{\mathrm{conc}}"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{W}(-_1)\times\mathcal{W}(-_2) \arrow[d, "\mathcal{H}"] \\ \mathcal{X}(-_1\amalg -_2) \arrow[r, hook] & \mathcal{W}(-_1\amalg -_2) \end{tikzcd}.$$

1.4 Connected walls Now, we introduce the notion of *connectedness* of a wall. Let $(W = \{W_\alpha\}_{\alpha\in A}, \leqslant)$ be a wall in $\mathcal{W}(S)$ (or in $\mathcal{W}^{\mathrm{or}}(S)$). We define on W *the equivalence relation of connectedness* $\overset{conn.}{\sim}$: for two elements a and b of A, we say $W_a \overset{conn.}{\sim} W_b$ if there exist an integer $n \geqslant 2$ and a sequence $W_0, W_1, \ldots, W_{n-1}, W_n$ of elements of W with $W_0 = W_a$ and $W_n = W_b$ such that, for all i in $[\![0, n-1]\!]$,

$$W_i \cap W_{i+1} \neq \varnothing \text{ and } (W_i, W_{i+1}) \in \mathrm{Succ}(W) \text{ or } (W_{i+1}, W_i) \in \mathrm{Succ}(W).$$

Definition 1.14 (Projection $\mathcal{K}$)**.** We define the projection $\mathcal{K}$ as follows: for a finite set S, we have

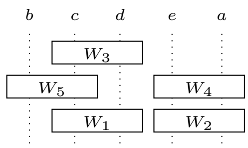
$$\begin{array}{rccc} \mathcal{K}_S : & \mathcal{W}(S) & \longrightarrow & \mathcal{Y}(S) \subset \mathcal{W}(-) \\ & W & \longmapsto & \left\{ \bigcup_{B_\alpha \in \pi^{-1}([B])} B_\alpha \;\middle|\; [B] \in \pi(W) \right\} \end{array},$$

where π is the projection of W to its quotient by $\overset{conn.}{\sim}$.

Example 1.15. Consider the set $S = \{a, b, c, d, e\}$ with five elements and the wall $W \in \mathcal{W}^{\mathrm{or}}(S)$ with five bricks $W_1 = \{c, d\} = W_3,\ W_2 = \{a, e\} = W_4$, and $W_5 = \{b, c\}$ with the partial order

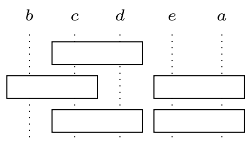
$$W_1 \leqslant W_5 \leqslant W_3,\ W_1 \leqslant W_3,\ W_2 \leqslant W_4,$$

represented by



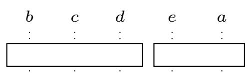
.

Denote the class of W in $\mathcal{W}(S)$ also by W, which is represented by



,

then the wall $\mathcal{K}(W)$ is represented by

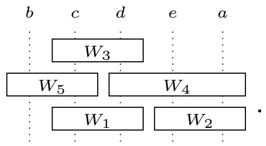


,

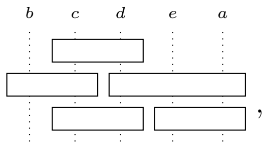
Example 1.16. Consider the set $S = \{a, b, c, d, e\}$ with five elements and the wall W with five bricks $W_1 = \{c, d\} = W_3,\ W_2 = \{a, e\},\ W_4 = \{a, d, e\}$, and $W_5 = \{b, c\}$ with the partial order

$$W_1 \leqslant W_5 \leqslant W_3,\ W_1 \leqslant W_4 \leqslant W_3,\ W_2 \leqslant W_4,$$

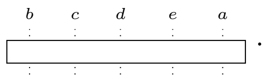
represented by



Denote the class of W in $\mathcal{W}(S)$ also by W, which is represented by



then the wall $\mathcal{K}(W)$ is represented by



We have the natural commutative diagram

$$\begin{tikzcd} \mathcal{X} \arrow[r, hook] \arrow[d, "\mathcal{K}"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{W} \arrow[d, "\mathcal{K}"] \\ \mathcal{Y} \arrow[r, hook] & \mathcal{W}. \end{tikzcd}$$

Lemma 1.17. *The projection $\mathcal{K}$ is associative, i.e. the following diagram of natural transformation commutes:*

$$\begin{tikzcd} \mathcal{Y}^{\times 3} \arrow[r, "\mathcal{Y}\times\mathcal{K}"] \arrow[d, "\mathcal{K}\times\mathcal{Y}"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{Y}^{\times 2} \arrow[d, "\mathcal{K}"] \\ \mathcal{Y}^{\times 2} \arrow[r, "\mathcal{K}"'] & \mathcal{Y} \end{tikzcd} .$$

Definition 1.18 (The functor $\mathcal{W}^{\mathrm{conn}}$)**.** We define the functor

$$\mathcal{W}_n^{\mathrm{conn,or}} : \mathsf{Bij}^{\mathrm{op}} \longrightarrow \mathsf{Bij}^{\mathrm{op}}$$

by, for all non-empty set S, the fiber of $\mathcal{K}_S : \mathcal{W}^{\mathrm{or}}(S) \to \mathcal{Y}^{\mathrm{or}}(S)$ over the wall with one brick $\{S\}$, i.e. the subfunctor of $\mathcal{W}_n^{\mathrm{or}}$ giving by $\mathcal{W}_n^{\mathrm{conn,or}}(S) :=$

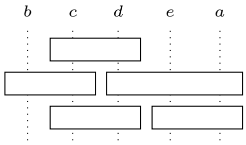
$$\left\{ (W, <_W) \in \mathcal{W}_n^{\mathrm{or}} \;\middle|\; \begin{array}{l} \forall \alpha, \beta \in [\![1, n]\!], \exists W_\alpha =: W_{i_0}, \ldots, W_{i_{m-1}}, W_m := W_\beta \\ \text{s.t. } \forall j \in [\![0, m-1]\!], \quad W_{i_j} \in W,\ W_{i_j} \cap W_{i_{j+1}} \neq \varnothing \\ \text{and } (W_{i_j}, W_{i_{j+1}}) \text{ or } (W_{i_{j+1}}, W_{i_j}) \in \mathrm{Succ}(W) \end{array} \right\}.$$

The natural surjection $\mathcal{W}^{\mathrm{or}} \twoheadrightarrow \mathcal{W}$ gives us the subfunctor

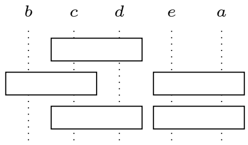
$$\begin{tikzcd} \mathcal{W}^{\mathrm{conn,or}} \arrow[r] \arrow[d, two heads] & \mathcal{W}^{\mathrm{or}} \arrow[d, two heads] \\ \mathcal{W}^{\mathrm{conn}} \arrow[r] & \mathcal{W} \end{tikzcd}$$

called *the functor of connected walls*: an element of $\mathcal{W}^{\mathrm{conn}}(S)$ is called a *connected* wall on S.

Example 1.19. Consider the set $S = \{a, b, c, d, e\}$ and the two elements W and W' in $\mathcal{W}(S)$ respectively represented by



and



.

Then W is connected and W' is not.

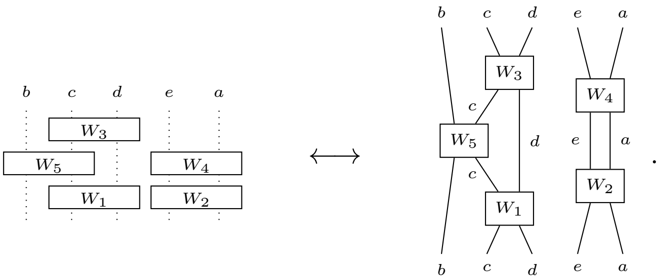
Remark 1.20. By the same arguments as in Section 1.3, we have the natural injection of $\mathcal{X}^{\text{conn}}$ in $\mathcal{W}^{\text{conn}}$.

Proposition 1.21. *Let W be a wall in $\mathcal{W}(S)$. Then, there exist n in $\mathbb{N}$ and $S_1 \amalg \ldots \amalg S_n$ a unique non-ordered partition of S such that*

$$W \in \operatorname{im}\Big(\mathcal{H} : \prod_{i=1}^{n} \mathcal{W}^{\text{conn}}(S_i) \longrightarrow \mathcal{W}(S)\Big).$$

Proof. Let S be a finite set and W be in $\mathcal{W}(S)$, a wall over S. The partition $\mathcal{K}(W)$ in $\mathcal{Y}(S)$ gives the result. $\square$

Remark 1.22. The terminology of bricks and walls is a way of presenting a subclass of directed (from the top to the bottom) graphs without cycles in which each vertex has the same number of inputs and outputs with some labels. For example, consider the same wall as in Example 1.15. It corresponds to the graph given on the right



All this paper could have been written in terms of labelled graphs. However, this is the original combinatorial approach that has allowed the author to find the way to the proof of the Koszulness of the properad of double Poisson algebras (see [11, Theorem 5.11]), especially the results of [11, Section 4].

2. Products on $\mathfrak{S}$-modules

In the rest of this paper, we consider a category C which

- has an initial object denoted by 0 ;
- is additive and has all coproducts $\oplus$;
- is monoidal, with product and unit denoted by $\otimes$ and k respectively, and symmetric for the structural morphism τ ;
- and such that the monoidal structure is distributive relative to the coproduct, i.e. for X, Y and Z three objects in C, we have

$$(X \oplus Y) \otimes Z = (X \otimes Z) \oplus (Y \otimes Z)$$

Example 2.1. The category of $\mathbb{Z}$-graded chain complexes over the field k, denoted by Ch_k, with the classical monoidal product $\otimes_k$ satisfies these axioms.

2.1 Some functorial constructions We give some categorical and functorial constructions.

Definition 2.2 (Category of elements)**.** Let B be a category and $F : \mathsf{B} \to \mathsf{Sets}$ be a functor. We define the *category of elements of F* denoted by B_F as follows:

- the objects of B_F are the pairs (b, x) with b, an object of B and x in $F(b)$;
- a morphism φ in $\mathrm{Hom}_{\mathsf{B}_F}((b_1, x_1), (b_2, x_2))$ is a morphism $\varphi : b_1 \to b_2$ such that $F(\varphi)(x_1) = x_2$.

Remark 2.3. We have the canonical projection functor $\pi : \mathsf{B}_F \to \mathsf{B}$ defined by $\pi(b, x) = b$, which induces by precomposition the functor

$$\pi^* : \mathsf{Func}(\mathsf{B}, \mathsf{C}) \longrightarrow \mathsf{Func}(\mathsf{B}_F, \mathsf{C})$$

which has a left adjoint denoted by π^*

$$\pi^* : \mathsf{Func}(\mathsf{B}_F, \mathsf{C}) \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathsf{Func}(\mathsf{B}, \mathsf{C}) : \pi_* \quad .$$

This left adjoint is given as the left Kan extension operation along F: this extension exists because the category C has all colimits. Consider a functor $F : \mathsf{B} \to \mathsf{Sets}$ and $V : \mathsf{B}_F \to \mathsf{C}$, the functor $\pi^* V$ is given by

$$\begin{array}{rccc} \pi^* V : & \mathsf{B} & \longrightarrow & \mathsf{C} \\ & S & \longmapsto & \displaystyle\bigoplus_{x \in F(S)} V(S, x) \end{array} .$$

Now, we consider a functor $\mathcal{F} : \mathsf{B} \to \mathsf{Sets}$ such that:

- $\mathcal{F}$ is multigraded, i.e. there exists N in $\mathbb{N}$ such that
$$\mathcal{F} = \coprod_{\bar{n} \in (\mathbb{N}^*)^N} \mathcal{F}_{\bar{n}} \ ,$$
where $\amalg$ is the coproduct in the category Sets;
- there exists M in $\mathbb{N}^*$ and a map $\gamma : (\mathbb{N}^*)^N \to (\mathbb{N}^*)^M$ such that, for all multi-indices $\bar{n}$ in $(N^*)^N$, there exists a functor
$$U_{\bar{n}} : \mathsf{B}_{\mathcal{F}_{\bar{n}}} \longrightarrow \mathsf{B}^{\times \gamma(\bar{n})} \ .$$

Example 2.4. The functors $\mathcal{X}^{\mathrm{or}}$, $\mathcal{Y}^{\mathrm{or}}$ and $\mathcal{W}^{\mathrm{or}}$ defined in Section 1 are such functors, from $\mathsf{Bij}^{\mathrm{op}}$ to Sets. For example, the functor $\mathcal{Y}^{\mathrm{or}} = \coprod_{n \in \mathbb{N}^*} \mathcal{Y}^{\mathrm{or}}_n$ is defined, for S in $\mathsf{Bij}^{\mathrm{op}}$ and n in $\mathbb{N}^*$, by

$$\mathcal{Y}^{\mathrm{or}}_n(S) = \{(I_1, \ldots, I_n) \mid I_1 \amalg \ldots \amalg I_n = S\}$$

and

$$\begin{array}{rccc} U_n : & (\mathsf{Bij}^{\mathrm{op}})_{\mathcal{Y}_n} & \longrightarrow & (\mathsf{Bij}^{\mathrm{op}})^{\times n} \\ & \big(S, (I_1, \ldots, I_n)\big) & \longmapsto & (I_1, \ldots, I_n) \end{array}$$

Notation 2.5. For such a multigraded functor $\mathcal{F} : \mathsf{B} \to \mathsf{C}$, a multi-index $\overline{n} = (n_1, \ldots, n_N)$ in $(\mathbb{N}^*)^N$ and a functor $V : \mathsf{B} \to \mathsf{C}$: we can construct the functor

$$\begin{array}{rccc} V^{\mathcal{F}_{\bar{n}}} : & \mathsf{B} & \longrightarrow & \mathsf{C} \\ & S & \longmapsto & \displaystyle\bigoplus_{x \in \mathcal{F}_{\bar{n}}} V^{\times \gamma(\bar{n})} \circ U_{\bar{n}}(S, x) \end{array}$$

In the next sections, we will define several functors following these constructions, the most important one being the free protoperad functor (see Appendix A.2).

2.2 $\mathfrak{S}$-modules Recall that the category Bij is a groupoid which gives us the equivalence of categories $\mathsf{Bij} \cong \mathsf{Bij}^{\mathrm{op}}$ by passage to the inverse. One of the key points of the constructions of this section is that $(\mathsf{Bij}, \amalg)$ is a symmetric monoidal category. We denote by $\mathfrak{S}$ its skeleton i.e. the category where objects are natural numbers, i.e. $\mathrm{Ob}\,\mathfrak{S} = \mathbb{N}$ and where morphisms are given by $\mathrm{Hom}_{\mathfrak{S}}(n,n) = \mathfrak{S}_n$ for $n \neq 0$ and $\mathrm{Hom}(0,0) = \{\mathrm{id}\}$, and which is equivalent to Bij.

Definition 2.6 ($\mathfrak{S}$-module, $\mathfrak{S}$-bimodule)**.** A (right) $\mathfrak{S}$-*module* is an object of $\mathsf{Func}(\mathsf{Bij}^{\mathrm{op}}, \mathsf{C})$, the category of contravariant functors from Bij to C, denoted by $\mathfrak{S}\text{-}\mathsf{mod}_{\mathsf{C}}$. A $\mathfrak{S}$-*bimodule* is an object of the category $\mathsf{Func}(\mathsf{Bij} \times \mathsf{Bij}^{\mathrm{op}}, \mathsf{C})$ which is denoted by $\mathfrak{S}\text{-}\mathsf{bimod}_{\mathsf{C}}$.

Example 2.7. The functor

$$\begin{array}{rccc} k[\mathrm{Aut}(-)]: & \mathsf{Bij}^{\mathrm{op}} & \longrightarrow & \mathsf{C} \\ & S & \longmapsto & \bigoplus\limits_{\mathrm{Aut}(S)} k \end{array}$$

is a $\mathfrak{S}$-module. When $|S| = n$ we also denote $k[\mathrm{Aut}(S)]$ by $k[\mathfrak{S}_n]$.

Remark 2.8. As $\mathfrak{S}$ is the skeletal category of Bij, we can view an $\mathfrak{S}$-module M as a collection $\big(M(n)\big)_{n \in \mathbb{N}^*}$ of objects of C indexed by natural numbers, where the group $\mathfrak{S}_n$ acts (on the left) on $M(n)$, for $n \neq 0$. Similarly, an $\mathfrak{S}$-bimodule P is a collection $\big(P(m,n)\big)_{m,n \in \mathbb{N}}$ of objects of C indexed by pairs of integers where $P(m,n)$ has an action of $\mathfrak{S}_m$ on the left and an action of $\mathfrak{S}_n$ on the right, or equivalently, has an action of the group $\mathfrak{S}_m \times \mathfrak{S}_n^{\mathrm{op}}$ on the left.

Definition 2.9 (Reduced $\mathfrak{S}$-(bi)module)**.** A $\mathfrak{S}$-module (resp. $\mathfrak{S}$-bimodule) P which satisfies $P(\varnothing) = 0$ (resp. $P(\varnothing, S) = 0$ and $P(S, \varnothing) = 0$ for all finite set S) is called *reduced*. We respectively note by $\mathfrak{S}\text{-}\mathsf{mod}_{\mathsf{C}}^{\mathrm{red}}$ and $\mathfrak{S}\text{-}\mathsf{bimod}_{\mathsf{C}}^{\mathrm{red}}$, the full subcategories of $\mathfrak{S}\text{-}\mathsf{mod}_{\mathsf{C}}$ and $\mathfrak{S}\text{-}\mathsf{bimod}_{\mathsf{C}}$ of reduced $\mathfrak{S}$-modules and $\mathfrak{S}$-bimodules.

Remark 2.10. We have the equivalence of categories $\mathfrak{S}\text{-}\mathsf{mod}_{\mathsf{C}} \cong \mathfrak{S}^{\mathrm{op}}\text{-}\mathsf{mod}_{\mathsf{C}}$, induced by taking the inverse of elements in symmetric groups. We use this equivalence without mention.

2.3 Composition and concatenation products on $\mathfrak{S}\text{-}\mathsf{mod}_{\mathsf{C}}$ In this subsection, we recall the classical constructions of composition and concatenation product of $\mathfrak{S}$-modules. The *composition product* (or *vertical product*) is the bifunctor

$$-\square- : \mathfrak{S}\text{-}\mathsf{mod}_{\mathsf{C}}^{\mathrm{red}} \times \mathfrak{S}\text{-}\mathsf{mod}_{\mathsf{C}}^{\mathrm{red}} \longrightarrow \mathfrak{S}\text{-}\mathsf{mod}_{\mathsf{C}}^{\mathrm{red}}$$

defined, for P and Q two reduced $\mathfrak{S}$-modules and S a non-empty finite set, by

$$\big(P \mathbin{\square} Q\big)(S) \coloneqq P(S) \otimes Q(S).$$

This bi-additive bifunctor gives the category $\mathfrak{S}\text{-}\mathsf{mod}_{\mathsf{C}}^{\mathrm{red}}$ a symmetric monoidal structure, with the identity $I_\square$, defined, for all non-empty sets S, by $I_\square(S) \coloneqq k$. In the literature of algebraic operads (cf. [12, Sect. 5.1.12]), the composition product of $\mathfrak{S}$-modules is also called the *Hadamard product*. The *concatenation product* is the bifunctor

$$- \otimes^{\mathrm{conc}} - : \mathfrak{S}\text{-}\mathsf{mod}_{\mathsf{C}}^{\mathrm{red}} \times \mathfrak{S}\text{-}\mathsf{mod}_{\mathsf{C}}^{\mathrm{red}} \longrightarrow \mathfrak{S}\text{-}\mathsf{mod}_{\mathsf{C}}^{\mathrm{red}}$$

defined, for all finite set S and all reduced $\mathfrak{S}$-modules P and Q, by:

$$\big(P \otimes^{\mathrm{conc}} Q\big)(S) \coloneqq \bigoplus_{\{S', S''\} \in \mathcal{Y}_2^{\mathrm{or}}(S)} P(S') \otimes Q(S'').$$

This product has no identity.

Remark 2.11. This product is called the *concatenation product* because it corresponds to a concatenation of operations. It is a particular case of Day's convolution product. For P and Q two reduced $\mathfrak{S}$-modules, we have

$$P \otimes^{\mathrm{conc}} Q(-) \coloneqq \int^{(S',S'')\in \mathrm{Ob}\,(\mathsf{Bij}^{\mathrm{op}})^{\times 2}} k\big[\mathrm{Hom}_{\mathsf{Bij}}(S' \amalg S'', -)\big] \otimes P(S') \otimes Q(S'').$$

Proposition 2.12. *The concatenation product is symmetric, i.e. for all reduced $\mathfrak{S}$-modules P and Q, we have the following isomorphism of $\mathfrak{S}$-modules*

$$P \otimes^{\mathrm{conc}} Q \cong Q \otimes^{\mathrm{conc}} P.$$

Proof. If (S', S'') is an element of $\mathcal{Y}_2^{\mathrm{or}}(S)$, then (S'', S') too. □

Recall that the bifunctor $- \otimes^{\mathrm{conc}} -$ is linear in each of its inputs. We have the functor:

$$\mathbb{T}_{\otimes}(-) : \mathfrak{S}\text{-}\mathsf{mod}_{\mathsf{C}}^{\mathrm{red}} \longrightarrow \mathcal{A}s(\mathfrak{S}\text{-}\mathsf{mod}_{\mathsf{C}}^{\mathrm{red}}, \otimes^{\mathrm{conc}})$$

defined, for all finite sets S and all reduced $\mathfrak{S}$-modules P, by

$$\big(\mathbb{T}_{\otimes}P\big)(S) \coloneqq \bigoplus_{r\in\mathbb{N}^*} \big(\mathbb{T}_{\otimes}^{r}P\big)(S) \text{ with } \big(\mathbb{T}_{\otimes}^{r}P\big)(S) \coloneqq \bigoplus_{I\in\mathcal{Y}_r^{\mathrm{or}}(S)} P(I_1)\otimes\ldots\otimes P(I_r).$$

As $\otimes^{\mathrm{conc}}$ is symmetric, we have also the *commutative free monoid functor*

$$\mathbb{S}(-) : \mathfrak{S}\text{-}\mathsf{mod}_{\mathsf{C}}^{\mathrm{red}} \longrightarrow \mathcal{C}om(\mathfrak{S}\text{-}\mathsf{mod}_{\mathsf{C}}^{\mathrm{red}}, \otimes^{\mathrm{conc}}).$$

defined, for all reduced $\mathfrak{S}$-modules P, by:

$$\mathbb{S}(P) \coloneqq \bigoplus_{b\in\mathbb{N}^*} \mathbb{S}^b(P) \quad \text{with} \quad \mathbb{S}^b(P) \coloneqq \big(\mathbb{T}_{\otimes}^{b}(P)\big)_{\mathfrak{S}_b}$$

where the action of $\mathfrak{S}_b$ is given by the symmetry τ of the product $\otimes^{\mathrm{conc}}$.

Notation 2.13. Let S be a finite set and P be a reduced $\mathfrak{S}$-module, as in [12, Sect. 5.1.14], we use the following notation

$$\bigoplus_{I\in\mathcal{Y}_r(S)} \bigotimes_{\alpha\in A} P(I_\alpha) \coloneqq \Big(\bigoplus_{I\in\mathcal{Y}_r^{\mathrm{or}}(S)} P(I_1)\otimes\ldots\otimes P(I_r)\Big)_{\mathfrak{S}_r}.$$

Let S be a finite set and P and Q be two reduced $\mathfrak{S}$-modules. Since

$$\big(\mathbb{S}^r P\big)(S) \coloneqq \Big(\big(\mathbb{T}^r P\big)(S)\Big)_{\mathfrak{S}_r} \cong \Big(\bigoplus_{I\in\mathcal{Y}_r^{\mathrm{or}}(S)} P(I_1)\otimes\ldots\otimes P(I_r)\Big)_{\mathfrak{S}_r}$$

then, we have

$$\big(\mathbb{S}P\big)(S) \cong \bigoplus_{\{I_\alpha\}_{\alpha\in A}\in\mathcal{Y}(S)} \bigotimes_{\alpha\in A} P(I_\alpha).$$

We also have the following isomorphism:

$$\big(\mathbb{S}P \,\square\, \mathbb{S}Q\big)(S) \cong \bigoplus_{(\{I_\alpha\}_{\alpha\in A},\{J_\beta\}_{\beta\in B})\in\mathcal{X}(S)} \bigotimes_{\alpha\in A} P(I_\alpha)\otimes \bigotimes_{\beta\in B} Q(J_\beta).$$

Remark 2.14. As the bifunctor $- \otimes^{\mathrm{conc}} -$ is biadditive, the functor $\mathbb{S}$ has the exponential property:

$$\mathbb{S}(P\oplus Q) \cong \mathbb{S}(P)\oplus\mathbb{S}(Q)\oplus\mathbb{S}(P)\otimes^{\mathrm{conc}}\mathbb{S}(Q).$$

2.4 Connected composition product of $\mathfrak{S}$-modules In this section, we define the new monoidal structure on the category of reduced $\mathfrak{S}$-modules, which is called the connected composition product. This monoidal structure is the analogous in the category of $\mathfrak{S}$-modules of the product $\boxtimes_c$ defined by Vallette in [18, 19].

Definition 2.15 (Connected composition product of $\mathfrak{S}$-modules)**.** The *connected composition product* of reduced $\mathfrak{S}$-modules is the bifunctor

$$- \boxtimes - : \mathfrak{S}\text{-}\mathsf{mod}^{\mathrm{red}}_{\mathsf{C}} \times \mathfrak{S}\text{-}\mathsf{mod}^{\mathrm{red}}_{\mathsf{C}} \longrightarrow \mathfrak{S}\text{-}\mathsf{mod}^{\mathrm{red}}_{\mathsf{C}}$$

defined, for all reduced $\mathfrak{S}$-modules P and Q and for all non-empty finite set S, by:

$$P \boxtimes Q(S) \coloneqq \bigoplus_{(I,J)\in\mathcal{X}^{\mathrm{conn}}(S)} \bigotimes_{\alpha} P(I_\alpha) \otimes \bigotimes_{\beta} Q(J_\beta).$$

Denote by $I_\boxtimes$, the $\mathfrak{S}$-module defined by

$$I_\boxtimes(S) \coloneqq \begin{cases} k & \text{if } |S| = 1 \\ 0 & \text{otherwise} \end{cases},$$

which is the unit of the product $\boxtimes$.

Remark 2.16. The previous construction is functorial relative to S: as in Section 2.1, we construct the functor

$$S \longmapsto \bigoplus_{(I,J)\in\mathcal{X}^{\mathrm{or,conn}}_{A,B}(S)} P^{\times A} \times Q^{\times B} \circ U_{A,B}(S,(I,J)) = \bigoplus_{(I,J)\in\mathcal{X}^{\mathrm{or,conn}}_{A,B}(S)} \bigotimes_{1\leqslant a\leqslant A} P(I_a) \otimes \bigotimes_{1\leqslant b\leqslant B} Q(J_b)\ ,$$

where the functor $U_{A,B}$ is defined by

$$\begin{array}{rccc} U_{A,B}\colon & (\mathsf{Bij}^{\mathrm{op}})_{\mathcal{X}^{\mathrm{or,conn}}_{A,B}} & \longrightarrow & (\mathsf{Bij}^{\mathrm{op}})^{\times A} \times (\mathsf{Bij}^{\mathrm{op}})^{\times B} \\ & (S,(I,J)) & \longmapsto & (I_1,\ldots,I_A,J_1,\ldots,J_B) \end{array}.$$

The functor $P \boxtimes Q$ is given by taking the invariant under the action of $\mathfrak{S}_A \times \mathfrak{S}_B$ and the sum over (A,B) in $\mathbb{N}^* \times \mathbb{N}^*$.

Lemma 2.17. *The product $\boxtimes$ is associative and, for all reduced $\mathfrak{S}$-modules A and B, the endofunctor*

$$\begin{array}{rccc} \Phi_{A,B}\colon & \mathfrak{S}\text{-}\mathsf{mod}^{\mathrm{red}}_{\mathsf{C}} & \longrightarrow & \mathfrak{S}\text{-}\mathsf{mod}^{\mathrm{red}}_{\mathsf{C}} \\ & X & \longmapsto & A \boxtimes X \boxtimes B \end{array}$$

is split analytic (in the sense of [20]).

Proof. The associativity of the product $\boxtimes$ follow from the associativity of $\mathcal{K}_S$: for P,Q and R three reduced $\mathfrak{S}$-modules, and S a non-empty set, we have the following isomorphism:

$$\begin{aligned} \big((P\boxtimes Q)\boxtimes R\big)(S) &\cong \bigoplus_{(I,J)\in\mathcal{X}^{\mathrm{conn}}(S)} \bigoplus_{\substack{(K,L)\in\mathcal{Y}(S)\times\mathcal{Y}(S)\\ \mathcal{K}_S(K,L)=I}} \bigotimes_{\gamma,\alpha} P(K^\alpha_\gamma) \otimes \bigotimes_{\delta,\alpha} Q(L^\alpha_\delta) \otimes \bigotimes_{\beta} R(J_\beta) \\ &\cong \bigoplus_{\substack{(K,L,J)\in\mathcal{Y}(S)^{\times 3}\\ \mathcal{K}_S(\mathcal{K}_S(K,L),J)=(S)}} \bigotimes_{\gamma} P(K_\gamma) \otimes \bigotimes_{\delta} Q(L_\delta) \otimes \bigotimes_{\beta} R(J_\beta) \\ &\underset{(1.17)}{\cong} \bigoplus_{\substack{(K,L,J)\in\mathcal{Y}(S)^{\times 3}\\ \mathcal{K}_S(K,\mathcal{K}_S(L,J))=(S)}} \bigotimes_{\gamma} P(K_\gamma) \otimes \bigotimes_{\delta} Q(L_\delta) \otimes \bigotimes_{\beta} R(J_\beta) \\ &\cong \bigoplus_{(K,I)\in\mathcal{X}^{\mathrm{conn}}(S)} \bigotimes_{\alpha} P(K_\gamma) \otimes \bigotimes_{\alpha} \big(Q\boxtimes R\big)(I_\alpha) \end{aligned}$$

so we have the associativity of the product $\boxtimes$

$$\big((P \boxtimes Q) \boxtimes R\big)(S) \cong \big(P \boxtimes (Q \boxtimes R)\big)(S)\ .$$

Also, for all reduced $\mathfrak{S}$-modules A and B, the endofunctor $\Phi_{A,B}$ is well defined by

$$\begin{aligned}\Phi_{A,B}(X) &\cong \bigoplus_{\substack{(K,L,J)\in\mathcal{Y}(S)^{\times 3}\\ \mathcal{K}_S(\mathcal{K}_S(K,L),J)=(S)}} \bigotimes_{\gamma} A(K_\gamma) \otimes \bigotimes_{\delta} X(L_\delta) \otimes \bigotimes_{\beta} B(J_\beta)\\ &\cong \bigoplus_{n\in\mathbb{N}} \bigoplus_{L\in\mathcal{Y}_n(S)} \bigoplus_{\substack{(K,J)\in\mathcal{Y}(S)^{\times 2}\\ \mathcal{K}_S(\mathcal{K}_S(K,L),J)=(S)}} \bigotimes_{\gamma} A(K_\gamma) \otimes \bigotimes_{\delta} X(L_\delta) \otimes \bigotimes_{\beta} B(J_\beta) \quad =: \bigoplus_{n\in\mathbb{N}} (\Phi_{A,B})_n(X).\end{aligned}$$

where $(\Phi_{A,B})_n$ is an homogeneous polynomial functor of degree n; so, for all reduced $\mathfrak{S}$-modules A and B, the functor $\Phi_{A,B}$ is a split analytic functor (in the sense of [20, Sect. 4]). $\square$

Proposition 2.18. *The category* $\big(\mathfrak{S}\text{-}\mathsf{mod}^{\mathrm{red}}_{\mathsf{C}}, \boxtimes, I_{\boxtimes}, \tau\big)$ *is an abelian symmetric monoidal category that preserves reflexive coequalizers and sequential colimits.*

Proof. Let P and Q be two reduced $\mathfrak{S}$-modules, we have, for all non-empty finite sets S, the isomorphism of $\mathfrak{S}_{|S|}$-modules

$$P \boxtimes Q(S) \cong \bigoplus_{(I,J)\in\mathcal{X}^{\mathrm{conn}}(S)} \bigotimes_{\beta} Q(J_\beta) \otimes \bigotimes_{\alpha} P(I_\alpha)$$

by symmetry of $\otimes$ of the category C; since (I, J) is in $\mathcal{X}(S)$ if, and only if, (J, I) is in $\mathcal{X}(S)$, we have the isomorphism

$$P \boxtimes Q(S) \cong \bigoplus_{(J,I)\in\mathcal{X}^{\mathrm{conn}}(S)} \bigotimes_{\beta} Q(J_\beta) \otimes \bigotimes_{\alpha} P(I_\alpha) \cong\ Q \boxtimes P(S).$$

We denote this isomorphism $\tau_{P,Q}(S) : P \boxtimes Q(S) \longrightarrow Q \boxtimes P(S)$, which gives us the symmetry of the product. The rest of the proof is similar to [20, Prop 13]. $\square$

We have the following compatibility between these products:

Proposition 2.19. *Let P and Q be two reduced $\mathfrak{S}$-modules. We have the following natural isomorphism of $\mathfrak{S}$-modules:*

$$\mathbb{S}(P \boxtimes Q) \cong \mathbb{S}P \,\square\, \mathbb{S}Q.$$

Proof. Let S be a finite set, we have

$$\begin{aligned}\mathbb{S}\big(P \boxtimes Q\big)(S) &\underset{2.15}{=} \bigoplus_{\Lambda\in\mathcal{Y}(S)} \bigotimes_{\gamma} \bigoplus_{(I^\gamma,J^\gamma)\in\mathcal{X}^{\mathrm{conn}}(\Lambda_\gamma)} \bigotimes_{\alpha} P(I^\gamma_\alpha) \otimes \bigotimes_{\beta} Q(J^\gamma_\beta)\\ &\cong \bigoplus_{\Lambda\in\mathcal{Y}(S)} \bigoplus_{(\widetilde{I},\widetilde{J})\in\mathcal{K}_S^{-1}(\Lambda)} \left(\bigotimes_{a} P(\widetilde{I}_a) \otimes \bigotimes_{b} Q(\widetilde{J}_b)\right)\\ &\cong \bigoplus_{(\widetilde{I},\widetilde{J})\in X(S)} \left(\bigotimes_{\alpha} P(\widetilde{I}_\alpha) \otimes \bigotimes_{\beta} Q(\widetilde{J}_\beta)\right) \ = \ \Big(\mathbb{S}P \,\square\, \mathbb{S}Q\Big)(S)\end{aligned}$$

which conclude the proof. $\square$

3. Connected product on $\mathfrak{S}$-bimodules

Just as in the $\mathfrak{S}$-modules, we give a description of the three monoidal structures on the category of (reduced) $\mathfrak{S}$-bimodules which are the equivalent of the three previous ones defined in $\mathfrak{S}\text{-}\mathsf{mod}_{\mathsf{C}}^{\text{red}}$. We start with a section on the functors that encode the combinatorics of our monoidal products. We express the combinatorics of the different monoidal structures on the $\mathfrak{S}\text{-}\mathsf{bimod}_{\mathsf{C}}$ category, using the same formalism as in the previous section.

3.1 Combinatorics of the connected composition of $\mathfrak{S}$-bimodules

Definition 3.1 (Functors $\mathbb{Y}^{\text{or}}$ and $\mathbb{Y}$)**.** Let S and E be two finite sets. We define

1. the bifunctor $\mathbb{Y}^{\text{or}}(-,-) : \mathsf{Bij}\times\mathsf{Bij}^{\text{op}} \to \mathsf{Sets}$ given on objects by $\mathbb{Y}^{\text{or}}(S,E) = \amalg_{r\in\mathbb{N}^*}\mathbb{Y}^{\text{or}}_r(S,E)$, with $\mathbb{Y}^{\text{or}}_r(S,E) \coloneqq \mathcal{Y}^{\text{or}}_r(S)\times\mathcal{Y}^{\text{or}}_r(E) \cong$
$$\left\{ (I,K) = \big((I_j,K_j)\big)_{j\in[\![1,r]\!]} \;\middle|\; \begin{array}{c} \amalg_{j=1}^r I_j = S;\ \amalg_{j=1}^r K_j = E; \\ \forall j\in[\![1,r]\!], I_j\neq\varnothing, K_j\neq\varnothing \end{array} \right\},$$
where the elements of $\mathbb{Y}^{\text{or}}_r(S,E)$ are ordered sets of pairs of sets.
2. the bifunctor $\mathbb{Y}(-,-) : \mathsf{Bij}\times\mathsf{Bij}^{\text{op}} \to \mathsf{Sets}$ given by $\mathbb{Y}(S,E) = \amalg_{r\in\mathbb{N}^*}\mathbb{Y}_r(S,E)$ with $\mathbb{Y}_r(S,E) =$
$$\left\{ \{I,K\} \coloneqq \{(I_\alpha,K_\alpha)\}_{\alpha\in A} \;\middle|\; \begin{array}{c} A\in \operatorname{Ob}\mathsf{Bij},\ |A| = r \\ \amalg_{\alpha\in A} I_\alpha = S,\ \amalg_{\alpha\in A}K_\alpha = E \\ \forall\alpha\in A, I_\alpha\neq\varnothing,\ K_\alpha\neq\varnothing \end{array} \right\},$$
where the elements of $\mathbb{Y}_r(S,E)$ are non ordered sets of pairs of sets.

Note that $\mathbb{Y}_r(S,E) \ncong \mathcal{Y}(S)\times\mathcal{Y}(E)$. As in the case of $\mathcal{Y}^{\text{or}}(-)$, we have a free action of $\mathfrak{S}_r$ on $\mathbb{Y}^{\text{or}}_r(S,E)$ given, for all (I,K) in $\mathbb{Y}^{\text{ord}}_r(S,E)$ and all permutation σ in $\mathfrak{S}_r$, by
$$\sigma\cdot(I,K) = \big((I_{\sigma^{-1}(j)},K_{\sigma^{-1}(j)})\big)_{j\in[\![1,r]\!]}$$
which induces the surjection $\mathbb{Y}^{\text{or}}_r(S,E) \twoheadrightarrow \mathbb{Y}_r(S,E)$. As in the case of $\mathfrak{S}$-modules, we define a new bifunctor, denoted by $\mathbb{X}^{\text{conn}}$ which encodes connectedness. For r and s in $\mathbb{N}^*$ and for S and E two finite sets, the bifunctor $\mathbb{X}^{n,\text{conn}}_{r,s}(S,E)$ is equal to
$$\left\{ \begin{array}{c} (\{I,K'\},\{K'',J\}) \\ \in \mathbb{Y}_r(S,[\![1,n]\!])\times\mathbb{Y}_s([\![1,n]\!],E) \end{array} \;\middle|\; (K',K'')\in\mathcal{X}^{\text{conn}}([\![1,n]\!]) \right\} \Big/ {\mathfrak{S}_n},$$
where, for all n in $\mathbb{N}^*$, by the functoriality of $\mathcal{X}^{\text{conn}}$, the quotient by $\mathfrak{S}_n$ which identifies
$$\big(\{I,K'\}\cdot\sigma,\{K'',J\}\big) \sim \big(\{I,K'\},\sigma^{-1}\cdot\{K'',J\}\big)$$
is well defined. We also have the functors
$$\mathbb{X}^{n,\text{conn}}(S,E) \coloneqq \coprod_{r,s\in\mathbb{N}^*}\mathbb{X}^{n,\text{conn}}_{r,s}(S,E) \text{ and } \mathbb{X}^{\text{conn}}(S,E) \coloneqq \coprod_{n\in\mathbb{N}^*}\mathbb{X}^{n,\text{conn}}(S,E).$$

3.2 Monoidal products of the category $\mathfrak{S}\text{-}\mathsf{bimod}_{\mathsf{C}}$

3.2.1 Composition product of $\mathfrak{S}$-bimodules $\square$

Definition 3.2 (Composition product of $\mathfrak{S}$-bimodules)**.** The *composition product of* $\mathfrak{S}$*-bimodules* is the bifunctor of

$$-\square- : \mathfrak{S}\text{-}\mathsf{bimod}^{\mathrm{red}}_{\mathsf{C}} \times \mathfrak{S}\text{-}\mathsf{bimod}^{\mathrm{red}}_{\mathsf{C}} \longrightarrow \mathfrak{S}\text{-}\mathsf{bimod}^{\mathrm{red}}_{\mathsf{C}}$$

defined, for all reduced $\mathfrak{S}$-bimodules P and Q, and for all finite sets S and E, by

$$\begin{aligned}(P\square Q)(S,E) &:= \bigoplus_{n\in\mathbb{N}^*}\Big(\bigoplus_{(\{I,K'\},\{K'',J\})\in\mathbb{X}^n_{1,1}(S,E)} P(I,K')\otimes Q(K'',J)\Big)\Big/_{\mathfrak{S}_n}\\ &\cong \bigoplus_{n\in\mathbb{N}^*} P(S,[\![1,n]\!]) \underset{\mathfrak{S}_n}{\otimes} Q([\![1,n]\!],E)\ ;\end{aligned}$$

where the action of $\mathfrak{S}_n$ is induced by the action on $[\![1,n]\!]$.

Remark 3.3. The composition product $\square$ is defined by a coend. In fact, the tensorial product of the category C gives us the external product

$$\begin{array}{ccc}\mathsf{Func}(\mathsf{Bij}\times\mathsf{Bij}^{\mathrm{op}},\mathsf{C})^{\times 2} & \longrightarrow & \mathsf{Func}(\mathsf{Bij}\times\mathsf{Bij}^{\mathrm{op}}\times\mathsf{Bij}\times\mathsf{Bij}^{\mathrm{op}},\mathsf{C})\\ (P,Q) & \longmapsto & \{(S_1,E_1,S_2,E_2)\mapsto P(S_1,E_1)\otimes Q(S_2,E_2)\}\end{array},$$

and, by taking the coend of the functors $P(S_1,-)\otimes Q(-,E_2) : \mathsf{Bij}^{\mathrm{op}}\times\mathsf{Bij}\to\mathsf{Bij}$ for S_1 and E_2 two finite sets, we have

$$\begin{array}{cccc}-\square-: & \mathsf{Func}(\mathsf{Bij}\times\mathsf{Bij}^{\mathrm{op}},\mathsf{C})^{\times 2} & \longrightarrow & \mathsf{Func}(\mathsf{Bij}\times\mathsf{Bij}^{\mathrm{op}},\mathsf{C})\\ & (P,Q) & \longmapsto & \int^{S\in\mathsf{Bij}} P(-,S)\otimes Q(S,-)\end{array}.$$

As the category $\mathfrak{S}$ is a skeleton of the category Bij, we finally have

$$\begin{array}{cccc}-\square-: & \mathsf{Func}(\mathsf{Bij}\times\mathsf{Bij}^{\mathrm{op}},\mathsf{C})^{\times 2} & \longrightarrow & \mathsf{Func}(\mathsf{Bij}\times\mathsf{Bij}^{\mathrm{op}},\mathsf{C})\\ & (P,Q) & \longmapsto & \bigoplus_{n\in\mathbb{N}} P(-,[\![1,n]\!]) \underset{\mathfrak{S}_n}{\otimes} Q([\![1,n]\!],-)\end{array}.$$

Remark 3.4. Let P and Q be two reduced $\mathfrak{S}$-modules. For all m and n in $\mathbb{N}^*$, we have the following isomorphism of $\mathfrak{S}_m\times\mathfrak{S}_n^{\mathrm{op}}$-bimodules:

$$P\square Q(m,n)\cong\bigoplus_{N\in\mathbb{N}^*} P(m,N) \underset{k[\mathfrak{S}_N]}{\otimes} Q(N,n).$$

Proposition 3.5. *The category* $\big(\mathfrak{S}\text{-}\mathsf{bimod}_{\mathsf{C}},\square,I_\square\big)$ *with* $I_\square(S,E)=k[\mathrm{Aut}(S)]$ *for* $S\cong E$ *and* 0 *otherwise, is a monoidal category.*

Proof. Let P, Q and R be three reduced $\mathfrak{S}$-bimodules and S and E be two non-empty finite sets. By definition of $I_\square$, it is clear that $P\square I_\square(S,E)\cong P(S,E)$. By Remark 3.3, we also have the isomorphisms:

$$\big(P\square Q\big)\square R(S,E)\cong\int^{(U,V)\in\mathsf{Bij}^{\times 2}} P(S,V)\otimes Q(V,U)\otimes R(U,E)\cong P\square\big(Q\square R\big)(S,E)$$

$\square$

3.2.2 The concatenation product $\otimes^{\mathrm{conc}}$ As in the case of $\mathfrak{S}$-modules, we define the concatenation product of two reduced $\mathfrak{S}$-bimodules.

Definition 3.6 (Concatenation product of $\mathfrak{S}$-bimodules)**.** The *concatenation product* is the bifunctor

$$- \otimes^{\mathrm{conc}} - : \mathfrak{S}\text{-}\mathsf{bimod}_{\mathsf{C}}^{\mathrm{red}} \times \mathfrak{S}\text{-}\mathsf{bimod}_{\mathsf{C}}^{\mathrm{red}} \longrightarrow \mathfrak{S}\text{-}\mathsf{bimod}_{\mathsf{C}}^{\mathrm{red}},$$

defined, for P and Q two reduced $\mathfrak{S}$-bimodules and for all finite sets S and E, by

$$\big(P \otimes^{\mathrm{conc}} Q\big)(S, E) \coloneqq \bigoplus_{(I,K)\in \mathbb{Y}_2^{\mathrm{ord}}(S,E)} P(I_1, K_1) \otimes Q(I_2, K_2).$$

Remark 3.7. This product is a particular case of a Day convolution product: let P and Q be two reduced $\mathfrak{S}$-bimodules, the bifunctor $P \otimes^{\mathrm{conc}} Q(-_1, -_2)$ is given by

$$\int^{(I_1,I_2,J_1,J_2)} k\left[\mathrm{Hom}_{\mathsf{Bij}^{\mathrm{op}}\times\mathsf{Bij}}\big((I_1 \amalg I_2, J_1 \amalg J_2), (-_1, -_2)\big)\right] \otimes P(I_1, J_1) \otimes Q(I_2, J_2).$$

The two product $\square$ and $\otimes^{\mathrm{conc}}$ satisfy the interchanging law.

Proposition 3.8 (Interchanging law)**.** *Let A, B, C, D, E and F be reduced $\mathfrak{S}$-bimodules. We have the natural injection of $\mathfrak{S}$-bimodules*

$$\Phi_{A,B,C,D} : (A \,\square\, B) \otimes^{\mathrm{conc}} (C \,\square\, D) \hookrightarrow (A \otimes^{\mathrm{conc}} C) \,\square\, (B \otimes^{\mathrm{conc}} D),$$

which is associative, i.e. we have

$$\Phi_{A\otimes C,B\otimes D,E,F}\left((\Phi_{A,B,C,D}) \otimes (E\square F)\right) = \Phi_{A,B,C\otimes E,D\otimes F}\left((A\square B) \otimes \Phi_{C,D,E,F}\right).$$

Proof. Let A, B, C and D be reduced $\mathfrak{S}$-bimodules. We have

$$\begin{aligned}
&(A \,\square\, B)\otimes^{\mathrm{conc}}(C \,\square\, D)(-,-)\\
&\quad= \int^{(I_1,I_2,J_1,J_2)\in\mathsf{Bij}^{\times 4}} k[\mathrm{Hom}((I_1\amalg I_2,J_1\amalg J_2),(-,-))] \otimes \int^{S\in\mathsf{Bij}} A(I_1,S)\otimes B(S,J_1) \otimes \int^{T\in\mathsf{Bij}} C(I_2,T)\otimes D(T,J_2)\\
&\quad\underset{\text{Fubini}}{\cong} \int^{(I_1,I_2,J_1,J_2,S,T)\in\mathsf{Bij}^{\times 6}} k[\mathrm{Hom}((I_1\amalg I_2,J_1\amalg J_2),(-,-))]\otimes A(I_1,S)\otimes C(I_2,T)\otimes B(S,J_1)\otimes D(T,J_2).
\end{aligned}$$

The natural injections

$$\begin{aligned}
k\Big[\mathrm{Hom}\big((I_1\amalg I_2,J_1\amalg J_2),(-,-)\big)\Big] &\hookrightarrow \coprod_S k\Big[\mathrm{Hom}\big((I_1\amalg I_2,J_1\amalg J_2),(-,S)\big)\Big]\otimes k\Big[\mathrm{Hom}\big((I_1\amalg I_2,J_1\amalg J_2),(S,-)\big)\Big]\\
&\hookrightarrow \coprod_{\substack{S\\ U\to S\\ S\to V}} k\Big[\mathrm{Hom}\big((I_1\amalg I_2,U),(-,S)\big)\Big]\otimes k\Big[\mathrm{Hom}\big((V,J_1\amalg J_2),(S,-)\big)\Big]
\end{aligned}$$

and

$$A(I_1,S)\otimes C(I_2,T)\otimes B(S,J_1)\otimes D(T,J_2) \hookrightarrow \coprod_{J_1,J_2,K_1,K_2} A(I_1,J_1)\otimes C(I_2,J_2)\otimes B(K_1,L_1)\otimes D(K_2,L_2)$$

give us the natural injection

$$
\begin{aligned}
&(A \square B) \otimes^{\mathrm{conc}} (C \square D)(-,-) \\
\hookrightarrow\ & \int^{(S,I_1,I_2,J_1,J_2,K_1,K_2,L_1,L_2)} \begin{matrix} k\Big[\mathrm{Hom}\Big((I_1 \amalg I_2, J_1 \amalg J_2),(-,S)\Big)\Big] \\ \otimes k\Big[\mathrm{Hom}\Big((K_1 \amalg K_2, L_1 \amalg L_2),(S,-)\Big)\Big] \\ \otimes A(I_1,J_1)\otimes C(I_2,J_2) \otimes B(K_1,L_1) \otimes D(K_2,L_2) \end{matrix} \\
\underset{\text{Fubini}}{\cong}\ & \int^{S} \int^{I_i,J_i} \begin{matrix} k\Big[\mathrm{Hom}\Big((I_1 \amalg I_2, J_1 \amalg J_2),(-,S)\Big)\Big] \\ \otimes A(I_1,J_1)\otimes C(I_2,J_2) \end{matrix} \otimes \int^{K_i,L_i} \begin{matrix} k\Big[\mathrm{Hom}\Big((K_1 \amalg K_2, L_1 \amalg L_2),(S,-)\Big)\Big] \\ \otimes B(K_1,L_1) \otimes D(K_2,L_2) \end{matrix} \\
=\ & \int^{S} \big(A \otimes^{\mathrm{conc}} C\big)(-,S) \otimes \big(B \otimes^{\mathrm{conc}} D\big)(S,-)\ .
\end{aligned}
$$

The injective natural transformation between bifunctors:

$$
\Phi_{A,B,C,D} : (A \square B) \otimes^{\mathrm{conc}} (C \square D) \hookrightarrow (A \otimes^{\mathrm{conc}} C) \square (B \otimes^{\mathrm{conc}} D)\ ,
$$

is associative, because the following diagram is commutative:

$$
\begin{tikzcd}
\Big((A \square B) \otimes^{\mathrm{conc}} (C \square D)\Big) \otimes^{\mathrm{conc}} (E \square F) \arrow[rr, "\cong"] \arrow[d, "\Phi_{A,B,C,D}\otimes 1"'] & & (A \square B) \otimes^{\mathrm{conc}} \Big((C \square D) \otimes^{\mathrm{conc}} (E \square F)\Big) \arrow[d, "1\otimes\Phi_{C,D,E,F}"] \\
\Big((A \otimes^{\mathrm{conc}} C) \square (B \otimes^{\mathrm{conc}} D)\Big) \otimes^{\mathrm{conc}} (E \square F) \arrow[dr, "\Phi_{A\otimes C, B\otimes D, E, F}"'] & \circlearrowleft & (A \square B) \otimes^{\mathrm{conc}} \Big((C) \otimes^{\mathrm{conc}} E) \square (D \otimes^{\mathrm{conc}} F)\Big) \arrow[dl, "\Phi_{A,B,C\otimes E, D\otimes F}"] \\
& (A \otimes^{\mathrm{conc}} C \otimes^{\mathrm{conc}} E) \square (B \otimes^{\mathrm{conc}} D \otimes^{\mathrm{conc}} F) &
\end{tikzcd}
$$

□

Corollary 3.9. *The categories of monoids* $\big(\mathcal{A}s(\mathfrak{S}\text{-}\mathsf{bimod}_{\mathsf{C}}, \otimes^{\mathrm{conc}}), \square, I_\square\big)$ *and commutative monoids* $\big(\mathcal{C}om(\mathfrak{S}\text{-}\mathsf{bimod}_{\mathsf{C}}, \otimes^{\mathrm{conc}}), \square, I_\square\big)$ *are monoidal. In other words, if* (P, c_P) *and* (Q, c_Q) *are two monoids in the symmetric monoidal category (without unit)* $\big(\mathfrak{S}\text{-}\mathsf{bimod}_{\mathsf{C}}, \otimes^{\mathrm{conc}}\big)$*, then*

1. $(P \square Q, c_{P \square Q})$ *is a monoid in* $\big(\mathfrak{S}\text{-}\mathsf{bimod}_{\mathsf{C}}, \otimes^{\mathrm{conc}}\big)$*;*
2. *if* P *and* Q *are commutative monoids, then* $P \square Q$ *is too.*

Proof. 1. The injection Φ of the Proposition 3.8 gives us the product $c_{P \square Q}$:

$$
\begin{tikzcd}
(P \square Q) \otimes^{\mathrm{conc}} (P \square Q) \arrow[r, "\Phi"] \arrow[dr, dotted, "c_{P \square Q}"'] & (P \otimes^{\mathrm{conc}} P) \square (Q \otimes^{\mathrm{conc}} Q) \arrow[d, "c_P \square c_Q"] \\
& P \square Q
\end{tikzcd}
$$

and the associativity of the product $\square$ gives us the associativity of the product $c_{P \square Q}$.

2. The commutativity of $c_{P \square Q}$ follows from the commutativity of c_P and c_Q.

□

Definition 3.10 (Free monoids)**.** Let P be a reduced $\mathfrak{S}$-bimodule and S and E be two finite sets.

1. The free associative monoid without unit on P is defined by

$$
\mathbb{T}^r P(S,E) := \bigoplus_{(I,K)\in \mathbb{Y}_r^{\mathrm{ord}}(S,E)} P(I_1,K_1) \otimes \ldots \otimes P(I_r,K_r).
$$

2. The free commutative monoid without unit on P is defined by the quotient of the free associative monoid by the action of the symmetric groups; namely
$$\mathbb{S}^r P(S,E) \coloneqq \Big(\mathbb{T}^r P(S,E)\Big)_{\mathfrak{S}_r} \cong \bigoplus_{\{I,K\}\in \mathbb{Y}_r(S,E)} \bigotimes_{\alpha} P(I_\alpha, K_\alpha).$$

3.3 Connected composition product of $\mathfrak{S}$-bimodules

We define the connected composition product of $\mathfrak{S}$-bimodules. The product was defined the first time by Vallette in his PhD thesis [18], for studying the homotopic comportment of algebraic structures with several inputs and outputs. Our definition is not the original one, but we show (cf. Proposition 3.16) that they are equivalent.

Definition 3.11 (Product of connected composition $\boxtimes^{\mathrm{bi}}$)**.** The *product of connected composition* of reduced $\mathfrak{S}$-bimodules is the bifunctor
$$-\boxtimes^{\mathrm{bi}} - : \mathfrak{S}\text{-}\mathsf{bimod}_{\mathsf{C}}^{\mathrm{red}} \times \mathfrak{S}\text{-}\mathsf{bimod}_{\mathsf{C}}^{\mathrm{red}} \longrightarrow \mathfrak{S}\text{-}\mathsf{bimod}_{\mathsf{C}}^{\mathrm{red}}$$
defined, for all reduced $\mathfrak{S}$-bimodules P and Q and all pairs (S,E) of finite sets, by $\big(P\boxtimes^{\mathrm{bi}} Q\big)(S,E) \coloneqq$
$$\bigoplus_{n\in\mathbb{N}^*} \Bigg(\bigoplus_{(\{I,K'\},\{K'',J\})\in \mathbb{X}^{n,\mathrm{conn}}(S,E)} \bigotimes_{\alpha} P(I_\alpha, K'_\alpha) \otimes \bigotimes_{\beta} Q(K''_\beta, J_\beta)\Bigg)\Big/_{\mathfrak{S}_n},$$
where the quotient by $\mathfrak{S}_n$ identifies, for all σ in $\mathfrak{S}_n$, the terms
$$\bigotimes_{\alpha} P(I_\alpha, K'_\alpha) \otimes \bigotimes_{\beta} Q(K''_\beta, J_\beta) \sim \Big(\bigotimes_{\alpha} P(I_\alpha, K'_\alpha)\Big)\cdot\sigma^{-1} \otimes \sigma\cdot\Big(\bigotimes_{\beta} Q(K''_\beta, J_\beta)\Big).$$

Remark 3.12. This construction is functorial, because $P\boxtimes^{\mathrm{bi}} Q$ is a sub-bifunctor of $\mathbb{S}P \,\square\, \mathbb{S}Q$ (see Proposition 3.18).

Notation 3.13. We denote by $/_{\mathfrak{S}}$, the quotient by symmetric groups for
$$\big(P\boxtimes^{\mathrm{bi}} Q\big)(S,E) \coloneqq \Bigg(\bigoplus_{\substack{(\{I,K'\},\{K'',J\})\\ \in \mathbb{X}^{\mathrm{conn}}(S,E)}} \bigotimes_{\alpha} P(I_\alpha, K'_\alpha) \otimes \bigotimes_{\beta} Q(K''_\beta, J_\beta)\Bigg)\Big/_{\mathfrak{S}}$$

The following proposition says that our definition of the connected composition product of $\mathfrak{S}$-bimodules is equivalent to that of Vallette in [18, 19]. First, recall the notion of connected permutation.

Definition 3.14 (Connected permutation – [19, Sect 1.3])**.** Let a,b and N be three integers with a and b in $\mathbb{N}^*$, let $\bar\alpha = (\alpha_1,\ldots,\alpha_a)$ in $(\mathbb{N}^*)^a$ be an a-tuple and $\bar\beta = (\beta_1,\ldots,\beta_b)$ in $(\mathbb{N}^*)^b$ be a b-tuple such that $|\bar\alpha| = N = |\bar\beta|$. A $(\bar\alpha,\bar\beta)$-*connected permutation* σ of $\mathfrak{S}_N$ is a permutation of $\mathfrak{S}_N$ such that the graph of a geometric representation of σ is connected if one gathers the inputs labelled by $\alpha_1+\ldots+\alpha_i+1,\ldots,\alpha_1+\ldots+\alpha_i+\alpha_{i+1}$ for $0\leqslant i\leqslant a-1$, and the outputs labelled by $\beta_1+\ldots+\beta_i+1,\ldots,\beta_1+\ldots+\beta_i+\beta_{i+1}$ for $0\leqslant i\leqslant b-1$. The set of $(\bar\alpha,\bar\beta)$-connected permutations is denoted by $S_c^{\bar\beta,\bar\alpha}$.

Lemma 3.15. *Let r,s and N be in $\mathbb{N}^*$, and let $\bar k$ be an r-tuple in $(\mathbb{N}^*)^r$ and $\bar j$ be a s-tuple in $(\mathbb{N}^*)^s$ such that $\sum_{\alpha=1}^r k_\alpha = N = \sum_{\beta=1}^s j_\beta$. The map*
$$\begin{array}{rccc} \varphi: & \left\{(K,J)\in \mathcal{X}_{r,s}^{\mathrm{conn,ord}}(N) \,\middle|\, \substack{(|K_1|,\ldots,|K_r|)=\bar k,\\ (|J_1|,\ldots,|J_s|)=\bar j} \right\} & \longrightarrow & S_c^{\bar k,\bar j} \\ & (K,J) & \longmapsto & \sigma_K^{-1}\sigma_J \end{array}$$
is surjective.

Proof. By the definition of the connectedness of a pair (K, J) in $\mathcal{X}_{r,s}^{\mathrm{conn,ord}}(N)$, if we consider the graph of a geometric representation of the permutation $\sigma_K^{-1}\sigma_J$, where we gather the inputs and the outputs as in the definition of connected permutation, then there exists a path between every input labelled by i and every output labelled by j. Then, we have $\sigma_K^{-1}\sigma_J$ in $S_c^{\bar{k},\bar{j}}$. Now, let σ be a permutation in $S_c^{\bar{k},\bar{j}}$. We consider

$$[\![1, N]\!]_{\bar{j}} \coloneqq \Big\{[\![1, j_1]\!], [\![j_1 + 1, j_1 + j_2]\!], \ldots, [\![j_1 + \ldots + j_{s-1} + 1, j_1 + \ldots + j_s]\!]\Big\},$$

and we denote by $([\![1, N]\!] \cdot \sigma)_{\bar{k}}$, the following set

$$\Big\{\{\sigma^{-1}(1), \sigma^{-1}(2), \ldots \sigma^{-1}(k_1)\}, \ldots, \{\sigma^{-1}(k_1 + \ldots + k_{r-1} + 1), \ldots, \sigma^{-1}(k_1 + \ldots + k_r)\}\Big\};$$

the pair $(([\![1, N]\!] \cdot \sigma)_{\bar{k}}, [\![1, N]\!]_{\bar{j}})$ is an element of $\mathcal{X}_{r,s}^{\mathrm{conn,ord}}(N)$ by the connectedness of the permutation σ. So the application

$$\begin{array}{rccc} \psi: & S_c^{\bar{k},\bar{j}} & \longrightarrow & \Big\{(K, J) \in \mathcal{X}_{r,s}^{\mathrm{conn,ord}}(N) \,\Big|\, {\scriptstyle (|K_1|,\ldots,|K_r|)=\bar{k}, \atop (|J_1|,\ldots,|J_s|)=\bar{j}} \Big\} \\ & \sigma & \longmapsto & \big(([\![1, N]\!] \cdot \sigma)_{\bar{k}}, [\![1, N]\!]_{\bar{j}}\big) \end{array}$$

is well-defined and gives us a section of φ, so that φ is surjective. □

Proposition 3.16. *For all integers m and n in $\mathbb{N}^*$, we have the isomorphism of $\mathfrak{S}_m \times \mathfrak{S}_n^{\mathrm{op}}$-modules:*

$$\big(P \boxtimes^{\mathrm{bi}} Q\big)([\![1, m]\!], [\![1, n]\!]) \cong \big(P \boxtimes^{\mathrm{Val}} Q\big)(m, n).$$

Proof. Let m and n be two integers in $\mathbb{N}^*$. We have

$$\begin{aligned} &\big(P \boxtimes^{\mathrm{bi}} Q\big)([\![1, m]\!], [\![1, n]\!]) \\ &\cong \Bigg(\bigoplus_{N\in\mathbb{N}} \Bigg(\bigoplus_{\substack{\bar{l}\in(\mathbb{N}^*)^r, |\bar{l}|=m \\ \bar{i}\in(\mathbb{N}^*)^s, |\bar{i}|=n \\ (K',K'')\in \\ X_{r,s}^{\mathrm{conn,ord}}([\![1,N]\!])}} k[\mathfrak{S}_m] \underset{\prod \mathfrak{S}_{l_\alpha}}{\otimes} \bigotimes_{\alpha=1}^{r} P(l_\alpha, K'_\alpha) \otimes \bigotimes_{\beta=1}^{s} Q(K''_\beta, i_\beta) \underset{\prod \mathfrak{S}_{i_\beta}}{\otimes} k[\mathfrak{S}_n] \Bigg)_{\mathfrak{S}_r\times\mathfrak{S}_s} \Bigg) \Big/ \mathfrak{S} \\ &\xrightarrow{\varphi} \bigoplus_{N\in\mathbb{N}} \Bigg(\bigoplus_{\substack{\bar{l},\bar{k}\in(\mathbb{N}^*)^r, |\bar{l}|=m \\ \bar{i},\bar{j}\in(\mathbb{N}^*)^s, |\bar{i}|=n \\ |\bar{k}|=N=|\bar{j}|}} k[\mathfrak{S}_m] \underset{\prod \mathfrak{S}_{l_\alpha}}{\otimes} \Big(\bigotimes_{\alpha=1}^{r} P(l_\alpha, k_\alpha)\Big) \underset{\mathfrak{S}_{\bar{k}}}{\otimes} k[S_c^{\bar{k},\bar{j}}] \underset{\mathfrak{S}_{\bar{j}}}{\otimes} \Big(\bigotimes_{\beta=1}^{s} Q(j_\beta, i_\beta)\Big) \underset{\prod \mathfrak{S}_{i_\beta}}{\otimes} k[\mathfrak{S}_n] \Bigg)_{\mathfrak{S}_r\times\mathfrak{S}_s} \end{aligned}$$

where φ sends, for $(K', K'') \in \mathcal{X}_{r,s}^{\mathrm{conn}}([\![1, N]\!])$, the component

$$\Bigg(k[\mathfrak{S}_m] \underset{\prod \mathfrak{S}_{l_\alpha}}{\otimes} \bigotimes_{\alpha=1}^{r} P(l_\alpha, K'_\alpha) \otimes \bigotimes_{\beta=1}^{s} Q(K''_\beta, i_\beta)\Bigg) \underset{\prod \mathfrak{S}_{i_\beta}}{\otimes} k[\mathfrak{S}_n] \Big/ \mathfrak{S}_N$$

to the following $\mathfrak{S}_m \times \mathfrak{S}_n^{\mathrm{op}}$-module

$$\begin{aligned} k[\mathfrak{S}_m] \underset{\prod \mathfrak{S}_{l_\alpha}}{\otimes} \bigotimes_{\alpha=1}^{r} P\big(l_\alpha, [\![\sum_{j=1}^{\alpha-1} |K'_j| + 1, \sum_{j=1}^{\alpha} |K'_j|]\!]\big) \underset{\mathfrak{S}_{\bar{k}}}{\otimes} \sigma_{K'}^{-1} \underset{\mathfrak{S}_N}{\otimes} \sigma_{K''} \\ \underset{\mathfrak{S}_{\bar{j}}}{\otimes} \bigotimes_{\beta=1}^{s} Q\big([\![\sum_{j=1}^{\beta-1} |K''_j| + 1, \sum_{j=1}^{\beta} |K''_j|]\!], i_\beta\big) \underset{\prod \mathfrak{S}_{i_\beta}}{\otimes} k[\mathfrak{S}_n]\ , \end{aligned}$$

which is isomorphic to

$$k[\mathfrak{S}_m] \underset{\prod \mathfrak{S}_{l_\alpha}}{\otimes} \bigotimes_{\alpha=1}^{r} P\big(l_\alpha, [\![\sum_{j=1}^{\alpha-1} |K'_j| + 1, \sum_{j=1}^{\alpha} |K'_j|]\!]\big) \underset{\mathfrak{S}_{\bar{k}}}{\otimes} \sigma_{K'}^{-1}\sigma_{K''} \underset{\mathfrak{S}_{\bar{j}}}{\otimes}$$
$$\bigotimes_{\beta=1}^{s} Q\big([\![\sum_{j=1}^{\beta-1} |K''_j| + 1, \sum_{j=1}^{\beta} |K''_j|]\!], i_\beta\big) \underset{\prod \mathfrak{S}_{i_\beta}}{\otimes} k[\mathfrak{S}_n]\ .$$

However, by Lemma 3.15, the morphism φ is surjective, and the quotient by the group $\mathfrak{S}_{\bar{k}} \times \mathfrak{S}_{\bar{j}}$ gives us the injectivity. □

Proposition 3.17 ([18, Lem. 49]). *The category* $\big(\mathfrak{S}\text{-}\mathsf{bimod}_{\mathsf{C}}, \boxtimes^{\mathrm{Val}}, I_\boxtimes\big)$ *where the unit* $I_\boxtimes$ *is defined, for all pairs* (S,E) *of finite sets, by:*

$$I_\boxtimes(S,E) := \begin{cases} k & \textit{if } |S| = 1 = |E| \\ 0 & \textit{otherwise} \end{cases} ,$$

is abelian monoidal and preserves coequalizers and sequential colimits.

The $\mathfrak{S}$-bimodule $P \boxtimes^{\mathrm{Val}} Q$ appears as the indecomposables for the product $\otimes^{\mathrm{conc}}$ of $\mathbb{S}P \,\square\, \mathbb{S}Q$.

Proposition 3.18. *Let* P *and* Q *be two reduced* $\mathfrak{S}$*-bimodules. We have the natural isomorphism*

$$\mathbb{S}\big(P \boxtimes^{\mathrm{Val}} Q\big) \cong \mathbb{S}P \,\square\, \mathbb{S}Q.$$

Proof. Let S and E be two finite sets, then

$$\begin{aligned}
\big(\mathbb{S}P \,\square\, \mathbb{S}Q\big)(S,E) &\cong \bigoplus_{n\in\mathbb{N}^*} \Big(\bigoplus_{(\{A,B\},\{C,D\})\in \mathbb{X}^n_{1,1}(m,n)} \mathbb{S}P(A,B) \otimes \mathbb{S}Q(C,D)\Big) \Big/_{\mathfrak{S}_n} \\
&\cong \bigoplus_{n\in\mathbb{N}^*} \Big(\bigoplus_{\substack{(\{A,B\},\{C,D\})\\ \in \mathbb{X}^n_{1,1}(S,E)}} \bigoplus_{\substack{\{I,K'\}\in \mathbb{Y}(A,B)\\ \{K'',J\}\in\mathbb{Y}(C,D)}} \bigotimes_\alpha P(I_\alpha, K'_\alpha) \otimes \bigotimes_\beta Q(K''_\beta, J_\beta)\Big) \Big/_{\mathfrak{S}_n} \\
&\cong \bigoplus_{n\in\mathbb{N}^*} \Big(\bigoplus_{(\{I,K'\},\{K'',J\})\in \mathbb{X}^n(S,E)} \bigotimes_\alpha P(I_\alpha, K'_\alpha) \otimes \bigotimes_\beta Q(K''_\beta, J_\beta)\Big) \Big/_{\mathfrak{S}_n} .
\end{aligned}$$

□

Remark 3.19. The previous proposition is central for the definition of the notion of ∞-*morphism* between $\mathcal{P}$-gebras up to homotopy (see [8, Section 3]).

4. Induction functor

We describe the functor Ind, and its right adjoint, the restriction functor Res, which establishes the link between the two previous sections, since $\mathrm{Ind} : \mathfrak{S}\text{-}\mathsf{mod}_{\mathsf{C}} \to \mathfrak{S}\text{-}\mathsf{bimod}_{\mathsf{C}}$ is strong monoidal for the different products introduced in the sections 2 and 3.

4.1 Adjunction Ind-Res For $\mathcal{C}$ a groupoid, we note $\Delta_{\mathcal{C}}$ the functor

$$\Delta_{\mathcal{C}} \coloneqq (\mathrm{inv}_{\mathcal{C}}, \mathrm{id}_{\mathcal{C}}) : \mathcal{C} \longrightarrow \mathcal{C}^{\mathrm{op}} \times \mathcal{C}$$

where $\mathrm{inv}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}^{\mathrm{op}}$ is the equivalence of categories given by the passage to the inverse. The *restriction functor*, denoted by Res, is given by the following composition:

$$\begin{array}{rccc} \mathrm{Res} : & \mathfrak{S}\text{-}\mathsf{bimod}_{\mathsf{C}} & \longrightarrow & \mathfrak{S}\text{-}\mathsf{mod}_{\mathsf{C}} \\ & P & \longmapsto & P \circ \Delta_{\mathsf{Bij}^{\mathrm{op}}} \end{array} .$$

This functor is exact and has a left adjoint functor called the *induction functor*.

Definition 4.1 (Functor Ind)**.** The *induction functor* is given by:

$$\begin{array}{rccc} \mathrm{Ind} : & \mathfrak{S}\text{-}\mathsf{mod}_{\mathsf{C}} & \longrightarrow & \mathfrak{S}\text{-}\mathsf{bimod}_{\mathsf{C}} \\ & V & \longmapsto & \big(\mathrm{Ind}V\big)(S,E) \coloneqq \displaystyle\bigoplus_{\mathrm{Hom}_{\mathsf{Bij}}(E,S)} V(E) \end{array}$$

where an element f in $\mathrm{Aut}(S)$ acts on the left by

$$f \cdot \Big(\bigoplus_{\varphi \in \mathrm{Hom}_{\mathsf{Bij}}(E,S)} V(E) \Big) = \bigoplus_{f\varphi \in \mathrm{Hom}_{\mathsf{Bij}}(E,S)} V(E)$$

and an element g in $\mathrm{Aut}(E)$ acts on the right by

$$\Big(\bigoplus_{\varphi \in \mathrm{Hom}_{\mathsf{Bij}}(E,S)} V(E) \Big) \cdot g = \bigoplus_{\varphi g \in \mathrm{Hom}_{\mathsf{Bij}}(E,S)} V\big(f^{-1}(E)\big).$$

Remark 4.2. Let V be a reduced $\mathfrak{S}$-module and S and E be two finite sets. If S and E are not isomorphic (i.e. $|S| \neq |E|$) then $\big(\mathrm{Ind}V\big)(S,E) = 0$. Finally, we have:

$$\big(\mathrm{Ind}\ V\big)(S,E) \cong \begin{cases} 0 & \text{if } S \ncong E \\ k[\mathrm{Aut}(S)] \otimes V(S) & \text{otherwise.} \end{cases}$$

Proposition 4.3. *We have the adjunction*

$$\mathrm{Ind} : \mathfrak{S}\text{-}\mathsf{mod}_{\mathsf{C}} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathfrak{S}\text{-}\mathsf{bimod}_{\mathsf{C}} : \mathrm{Res}.$$

Proof. By the classical result [17, Th.13]. □

One of the fundamental properties of Ind is the following.

Proposition 4.4. *The functor* Ind *is exact, preserves quasi-isomorphisms and commutes with colimits.*

Proof. The result is induced by Remark 4.2; more generally, for G a finite group and H a subgroup of G, $k[G]$ is a free $k[H]$-module. As the functor Ind has a right adjoint, it commutes with colimits. □

4.2 Compatibilities with products In this subsection, we show compatibilities of the functor Ind with products defined in previous sections. First, we recall the following proposition, about compatibility between monoidal structures and adjunction.

Proposition 4.5. *Let* $(\mathsf{C}, \otimes, I_{\mathsf{C}})$ *and* $(\mathsf{D}, \odot, I_{\mathsf{D}})$ *be two monoidal categories with the following adjunction:*

$$L \ : \ \mathsf{C} \underset{}{\overset{\perp}{\rightleftarrows}} \mathsf{D} \ : \ R$$

such that the left adjoint is a strong monoidal functor by the following natural equivalence $\mu_L : L(-_1) \odot L(-_2) \xrightarrow{\cong} L(-_1 \otimes -_2)$ *and the natural isomorphism* $e_L : I_{\mathsf{D}} \xrightarrow{\cong} L(I_{\mathsf{C}})$*. Then the right adjoint is a Lax monoidal functor with the natural transformation* μ_R *and the morphism* ϵ_R *given by*

$$\begin{tikzcd}
R(-_1) \otimes R(-_2) \arrow[r, "\eta(R(-_1)\otimes R(-_2))"] \arrow[d, dotted, "\mu_R :="'] & RL\big(R(-_1) \otimes R(-_2)\big) \arrow[d, "R\left(\mu_L^{-1}(R(-_1), R(-_2))\right)"] \\
R(-_1 \otimes -_2) & R\big(LR(-_1) \odot LR(-_2)\big) \arrow[l, "R(\epsilon(-_1)\odot\epsilon(-_2))"]
\end{tikzcd}$$

and $e_R : I_{\mathsf{C}} \xrightarrow{\eta(I_{\mathsf{C}})} RL(I_{\mathsf{C}}) \xrightarrow{R(e_L^{-1})} R(I_{\mathsf{D}})$.

Now, we can study the case of the adjunction given by the functors Ind and Res.

Notation 4.6. We note $\mathfrak{S}\text{-}\mathsf{bimod}_{\mathsf{C}}^{\mathrm{Ind}}$, the essential image of the functor Ind.

Proposition 4.7. *The category* $(\mathfrak{S}\text{-}\mathsf{bimod}_{\mathsf{C}}^{\mathrm{Ind}}, \square)$ *is symmetric monoidal and the induction functor*

$$\mathrm{Ind} : (\mathfrak{S}\text{-}\mathsf{mod}_{\mathsf{C}}, \square) \longrightarrow (\mathfrak{S}\text{-}\mathsf{bimod}_{\mathsf{C}}^{\mathrm{Ind}}, \square)$$

is symmetric monoidal.

Proof. Let P and Q be two $\mathfrak{S}$-modules and, S and E be two finite sets. Using Remark 4.2 and the definition of the product $\square$, the $\mathfrak{S}$-bimodule $\big(\mathrm{Ind}P \,\square\, \mathrm{Ind}Q\big)(S,E)$ is different of zero if $S \cong E$. So, fix $n \in \mathbb{N}$ and consider $S \cong E \cong [\![1,n]\!]$, we have:

$$\begin{aligned}
\big(\mathrm{Ind}P \,\square\, \mathrm{Ind}Q\big)([\![1,n]\!],[\![1,n]\!]) &\cong k[\mathfrak{S}_n] \otimes P([\![1,n]\!]) \otimes_{\mathfrak{S}_n} k[\mathfrak{S}_n] \otimes Q([\![1,n]\!]) \\
&\cong k[\mathfrak{S}_n] \otimes P([\![1,n]\!]) \otimes Q([\![1,n]\!]) \\
&\cong \mathrm{Ind}\big(P \,\square\, Q\big)([\![1,n]\!])
\end{aligned}$$

□

Corollary 4.8. *The functor* Res : $\big(\mathfrak{S}\text{-}\mathsf{bimod}_{\mathsf{C}}^{\mathrm{red}}, \square\big) \to \big(\mathfrak{S}\text{-}\mathsf{mod}_{\mathsf{C}}^{\mathrm{red}}, \square\big)$ *is Lax-monoidal.*

Proof. By adjunction of functors Ind and Res, and Proposition 4.5. □

Remark 4.9. The functor Res is not strongly monoidal with respect to $\square$. For example, if we consider P and Q, the $\mathfrak{S}$-bimodules defined by

$$P(S,E) := \begin{cases} k & \text{if } |S| = 1 \text{ and } |E| = 1 \text{ or } 2 \\ 0 & \text{otherwise} \end{cases}$$

and

$$Q(S,E) := \begin{cases} k & \text{if } |E| = 1 \text{ and } |S| = 1 \text{ or } 2 \\ 0 & \text{otherwise} \end{cases}$$

then $\mathrm{Res}(P \,\square\, Q)(\{*\},\{*\}) = k^2$ and $\big(\mathrm{Res}P \,\square\, \mathrm{Res}Q\big)(\{*\},\{*\}) = k$.

Remark 4.10. We have the monoidal adjunction

$$\mathrm{Ind} : \left(\mathfrak{S}\text{-}\mathsf{mod}_{\mathsf{C}}^{\mathrm{red}}, \square\right) \rightleftarrows \left(\mathfrak{S}\text{-}\mathsf{bimod}_{\mathsf{C}}^{\mathrm{red}}, \square\right) : \mathrm{Res}.$$

The functor Ind is also compatible with the concatenation product.

Proposition 4.11. *The functor of induction*

$$\mathrm{Ind} : (\mathfrak{S}\text{-}\mathsf{mod}_{\mathsf{C}}^{\mathrm{red}}, \otimes^{\mathrm{conc}}) \longrightarrow (\mathfrak{S}\text{-}\mathsf{bimod}_{\mathsf{C}}^{\mathrm{red}}, \otimes^{\mathrm{conc}})$$

is symmetric monoidal.

Proof. Let P and Q be two reduced $\mathfrak{S}$-modules and S and E be two finite sets. We have the isomorphism:

$$\begin{aligned}
\big(\mathrm{Ind}P \otimes^{\mathrm{conc}} \mathrm{Ind}Q\big)(S,E) &\cong \bigoplus_{(I,J)\in\mathbb{Y}_2^{\mathrm{or}}(S,E)} \ \bigoplus_{\mathrm{Hom}_{\mathsf{Bij}}(J_1,I_1)\times\mathrm{Hom}_{\mathsf{Bij}}(J_2,I_2)} P(J_1)\otimes Q(J_2)\\
&\cong \bigoplus_{J\in\mathcal{Y}_2^{\mathrm{or}}(E)} \ \bigoplus_{\mathrm{Hom}_{\mathsf{Bij}}(J_1\amalg J_2,S)} P(J_1)\otimes Q(J_2)\\
&\cong \mathrm{Ind}\big(P\otimes^{\mathrm{conc}} Q\big)(S,E).
\end{aligned}$$

□

Proposition 4.12. *The functor* $\mathrm{Res} : \left(\mathfrak{S}\text{-}\mathsf{bimod}_{\mathsf{C}}^{\mathrm{red}}, \otimes^{\mathrm{conc}}\right) \to \left(\mathfrak{S}\text{-}\mathsf{mod}_{\mathsf{C}}^{\mathrm{red}}, \otimes^{\mathrm{conc}}\right)$ *is Lax-monoidal.*

Proof. Let P and Q be two $\mathfrak{S}$-bimodules. We have the natural injections

$$\begin{aligned}
&\bigoplus_{I\in\mathcal{Y}_2^{\mathrm{or}}(-)} \mathrm{Hom}_{\mathsf{Bij}^{\mathrm{op}}}(I_1\amalg I_2,-)\otimes \mathrm{Res}P(I_1)\otimes\mathrm{Res}Q(I_2)\\
&\qquad\hookrightarrow \bigoplus_{I\in\mathcal{Y}_2^{\mathrm{or}}(-)} k[\mathrm{Hom}_{\mathsf{Bij}\times\mathsf{Bij}^{\mathrm{op}}}\big((I_1\amalg I_2, I_1\amalg I_2),\Delta(-)\big)]\otimes P(I_1,I_1)\otimes Q(I_2,I_2)\\
&\qquad\hookrightarrow \bigoplus_{(I,J)\in\mathbb{Y}_2^{\mathrm{or}}(-)} k[\mathrm{Hom}_{\mathsf{Bij}\times\mathsf{Bij}^{\mathrm{op}}}\big((I_1\amalg I_2, J_1\amalg J_2),\Delta(-)\big)]\otimes P(I_1,J_1)\otimes Q(I_2,J_2)
\end{aligned}$$

which imply the following natural injection

$$\bigoplus_{I\in\mathcal{Y}_2^{\mathrm{or}}(-)} k[\mathrm{Hom}_{\mathsf{Bij}^{\mathrm{op}}}(I_1\amalg I_2,-)]\otimes\mathrm{Res}P(I_1)\otimes\mathrm{Res}Q(I_2)\hookrightarrow \mathrm{Res}\big(P\otimes^{\mathrm{conc}} Q\big).$$

Also, we have the following commutative diagram

$$\begin{tikzcd}
\displaystyle\bigoplus_{I_1\to J_1, I_2\to J_2} k[\mathrm{Hom}_{\mathsf{Bij}^{\mathrm{op}}}(J_1\amalg J_2,-)]\otimes\mathrm{Res}P(I_1)\otimes\mathrm{Res}Q(I_2) \arrow[d, shift left] \arrow[d, shift right] & \\
\displaystyle\bigoplus_{I\in\mathcal{Y}_2^{\mathrm{or}}(-)} k[\mathrm{Hom}_{\mathsf{Bij}^{\mathrm{op}}}(I_1\amalg I_2,-)]\otimes\mathrm{Res}P(I_1)\otimes\mathrm{Res}Q(I_2) \arrow[r, hook] \arrow[d] & \mathrm{Res}(P\otimes^{\mathrm{conc}} Q) \\
\big(\mathrm{Res}P\big)\otimes\big(\mathrm{Res}Q\big) \arrow[ur, dotted, "\exists"'] &
\end{tikzcd}$$

so, by the universal property of the coend, we have the natural morphism

$$\mathrm{Res}P\otimes^{\mathrm{conc}}\mathrm{Res}Q\longrightarrow\mathrm{Res}(P\otimes^{\mathrm{conc}} Q).$$

□

Remark 4.13. The functor Res is not strongly monoidal with respect to $\otimes^{\text{conc}}$. Indeed, if we consider P and Q, the $\mathfrak{S}$-bimodules defined by

$$P(S,E) := \begin{cases} k & \text{if } |E| = 2 \text{ and } |S| = 1; \\ 0 & \text{otherwise} \end{cases}$$

and

$$Q(S,E) := \begin{cases} k & \text{if } |S| = 2 \text{ and } |E| = 1; \\ 0 & \text{otherwise} \end{cases},$$

then $\mathrm{Res}P \otimes^{\text{conc}} \mathrm{Res}Q = 0$, while $\mathrm{Res}(P \otimes^{\text{conc}} Q)(S) = k$ if the cardinal of S is 3.

We have the following compatibility between functors $\mathbb{S}(-)$ and $\mathrm{Ind}(-)$.

Proposition 4.14. *Let P be a reduced $\mathfrak{S}$-module. Then, we have the natural isomorphism of reduced $\mathfrak{S}$-bimodules:*

$$\mathrm{Ind}\big(\mathbb{S}(P)\big) \cong \mathbb{S}\big(\mathrm{Ind}(P)\big).$$

Proof. The functor Ind commutes with the concatenation product $\otimes^{\text{conc}}$ and is compatible with the symmetry, by Proposition 4.11. We conclude by the exactness of the functor Ind. □

One of the most important property of the functor Ind is that it is compatible with connected composition products.

Theorem 4.15. *The functor* $\mathrm{Ind} : \big(\mathfrak{S}\text{-}\mathsf{mod}^{\text{red}}_{\mathsf{C}}, \boxtimes\big) \to \big(\mathfrak{S}\text{-}\mathsf{bimod}^{\text{red}}_{\mathsf{C}}, \boxtimes^{\text{Val}}\big)$ *is monoidal.*

Proof. Let S and E be two finite sets, then $(\mathrm{Ind}\, I_{\boxtimes})(S,E) = k$ if $|S| = 1 = |E|$ and 0 otherwise. Then, the functor Ind respect the unit. Let V and W be two reduced $\mathfrak{S}$-modules and S and E be two finite sets.

$$\big(\mathrm{Ind}V \boxtimes^{\text{Val}} \mathrm{Ind}W\big)(S,E) \cong \bigoplus_{n\in\mathbb{N}^*} \bigoplus_{\substack{(\{I,K'\},\{K'',J\})\\ \in \mathbb{X}^{n,\text{conn}}(S,E)}} \bigotimes_{\alpha} \bigoplus_{\mathrm{Hom}_{\mathsf{Bij}}(K'_\alpha, I_\alpha)} V(K'_\alpha) \underset{\mathfrak{S}_n}{\otimes} \bigotimes_{\beta} \bigoplus_{\mathrm{Hom}_{\mathsf{Bij}}(J_\beta, K''_\beta)} W(J_\beta).$$

Note that the right side is non zero if and only if $I_\alpha \cong K'_\alpha$ for all α and $K''_\beta \cong J_\beta$ for all β. This implies that $(\mathrm{Ind}V \boxtimes \mathrm{Ind}W)(S,E) = 0$ if $|S| \neq |E|$.

$$\begin{aligned}
\big(\mathrm{Ind}V \boxtimes^{\text{Val}} \mathrm{Ind}W\big)(S,E) &\cong \bigoplus_{n\in\mathbb{N}^*} \bigoplus_{\substack{(\{I,K'\},\{K'',J\})\\ \in \mathbb{X}^{n,\text{conn}}(S,E)}} \bigoplus_{\substack{\prod_\alpha \mathrm{Hom}_{\mathsf{Bij}}(K'_\alpha, I_\alpha)\\ \times \prod_\beta \mathrm{Hom}_{\mathsf{Bij}}(J_\beta, K''_\beta)}} \bigotimes_{\alpha} V(K'_\alpha) \underset{\mathfrak{S}_n}{\otimes} \bigotimes_{\beta} W(J_\beta) \\
&\cong \bigoplus_{r,s,n\in\mathbb{N}^*} \bigoplus_{\substack{I\in Y_r(S), J\in Y_s(E)\\ (K',K'')\in \mathcal{X}^{\text{conn}}_{r,s}([\![1,n]\!])\\ I_\alpha \cong K'_\alpha, K''_\beta \cong J_\beta}} \bigoplus_{\mathrm{Hom}_{\mathsf{Bij}}(E,S)} \bigotimes_{\alpha} V(K'_\alpha) \underset{\mathfrak{S}_n}{\otimes} \bigotimes_{\beta} W(J_\beta) \\
&\cong \bigoplus_{\mathrm{Hom}_{\mathsf{Bij}}(E,S)} \bigoplus_{r,s\in\mathbb{N}^*} \bigoplus_{(I,J)\in \mathcal{X}^{\text{conn}}_{r,s}(E)} \bigotimes_{\alpha} V(I_\alpha) \underset{\mathfrak{S}_n}{\otimes} \bigotimes_{\beta} W(J_\beta) \\
&\cong \mathrm{Ind}\,\big(V \boxtimes W\big)(S,E).
\end{aligned}$$

□

Corollary 4.16. *The functor* $\mathrm{Res} : \big(\mathfrak{S}\text{-}\mathsf{bimod}^{\text{red}}_{\mathsf{C}}, \boxtimes^{\text{Val}}\big) \longrightarrow \big(\mathfrak{S}\text{-}\mathsf{mod}^{\text{red}}_{\mathsf{C}}, \boxtimes\big)$ *is Lax-monoidal.*

Proof. By the Proposition 4.5. □

5. Protoperads

In this section, we study monoids in the monoidal category $(\mathfrak{S}\text{-}\mathsf{mod}_{\mathsf{C}}^{\mathrm{red}}, \boxtimes, I_{\boxtimes})$.

Definition 5.1 (Protoperad)**.** A *protoperad* is an unital monoid (P, μ, η) in the monoidal category $(\mathfrak{S}\text{-}\mathsf{mod}_{\mathsf{C}}^{\mathrm{red}}, \boxtimes)$. We note $\mathsf{protoperads}_{\mathsf{C}}$, the category $\mathcal{UA}s(\mathfrak{S}\text{-}\mathsf{mod}_{\mathsf{C}}^{\mathrm{red}}, \boxtimes, I_{\boxtimes})$ of protoperads.

Example 5.2. An associative algebra (A, μ) in the category of chain complex can be viewed as a protoperad concentrated in arity 1.

Example 5.3 (Endomorphism protoperad)**.** Let X be an object of the category C. The reduced $\mathfrak{S}$-module End_X is defined, for each finite set S, by

$$\mathrm{End}_X(S) \coloneqq \mathrm{Hom}_{\mathsf{C}}(X^{\otimes|S|}, X^{\otimes|S|}) \ ,$$

where the action of the group of automorphisms of S is diagonal. The structural map of this protoperad is given by the composition of morphisms.

Proposition 5.4. *The functor* Ind *induces the functor* $\mathrm{Ind}\colon \mathsf{protoperads}_{\mathsf{C}} \longrightarrow \mathsf{properads}_{\mathsf{C}}$.

Proof. By the Theorem 4.15. □

Remark 5.5. There exists the notion of *prop* which is more general than the notion of properad: a prop is an object of the category

$$\mathsf{props}_{\mathsf{C}} \coloneqq \mathcal{UA}s\big(\mathcal{C}om(\mathfrak{S}\text{-}\mathsf{bimod}_{\mathsf{C}}, \otimes^{\mathrm{conc}}), \square, I_{\square}\big),$$

i.e. a $\mathfrak{S}$-bimodule with two products, a horizontal and a vertical ones, which satisfy the *interchanging law* (see [14]). A natural question is the following: what structure puts on a $\mathfrak{S}$-module P such that $\mathrm{Ind}(P)$ is a prop? As the functor Ind is monoidal for $\square$ and $\otimes^{\mathrm{conc}}$, it induces the functor

$$\mathrm{Ind} : \mathsf{protops}_{\mathsf{C}} \coloneqq \mathcal{UA}s\big(\mathcal{C}om(\mathfrak{S}\text{-}\mathsf{mod}, \otimes^{\mathrm{conc}}), \square, I_{\square}\big) \longrightarrow \mathsf{props}_{\mathsf{C}}.$$

We also have the dual notion.

Definition 5.6 (Coprotoperad)**.** A *coprotoperad* is a co-unital comonoid (Q, Δ, ϵ) in the monoidal category $(\mathfrak{S}\text{-}\mathsf{mod}_{\mathsf{C}}^{\mathrm{red}}, \boxtimes, I_{\boxtimes})$: we note $\mathsf{coprotoperads}_k$, the category $\mathrm{co}\mathcal{UA}s(\mathfrak{S}\text{-}\mathsf{mod}_{\mathsf{C}}^{\mathrm{red}}, \boxtimes, I_{\boxtimes})$ of coprotoperads.

Notation 5.7. We note $\mathfrak{S}\text{-}\mathsf{mod}_{\mathsf{C}}^{\mathrm{gr}}$, the category $\mathsf{Func}(\mathsf{Bij}^{\mathrm{op}}, \mathsf{C}^{\mathrm{gr}})$, where C^{gr} is the category of $\mathbb{N}$-graded object of C. This grading is called the *weight*. All the previous constructions extend naturally to graded $\mathfrak{S}$-modules.

Definition 5.8 ((Connected) Weight graded protoperad/coprotoperad)**.** A protoperad (resp. coprotoperad) $\mathcal{P}$ is *weight graded* if $\mathcal{P}$ is a monoid (resp. comonoid) in the category $\mathfrak{S}\text{-}\mathsf{mod}_{\mathsf{C}}^{\mathrm{red,gr}}$ for the product $\boxtimes$. We denote this grading by $\mathcal{P} = \bigoplus_{i\in\mathbb{N}} \mathcal{P}^{[i]}$. We say that a weight graded (co)protoperad $\mathcal{P} = \bigoplus_{i\in\mathbb{N}} \mathcal{P}^{[i]}$ is *connected* if $\mathcal{P}^{[0]} \cong I_{\boxtimes}$.

5.1 Partial compositions One can describe (cf. [12, Sect. 5.3.4]) the operad structure on a $\mathfrak{S}$-module $\mathcal{O}$ just by giving the partial compositions maps $\circ_s : \mathcal{O}(S) \otimes \mathcal{O}(R) \to \mathcal{O}(R \amalg S \backslash \{s\})$. We have a similar property for protoperads

Definition 5.9 (Partial compositions)**.** Let P be a reduced $\mathfrak{S}$-module equipped with a morphism of $\mathfrak{S}$-modules $\epsilon : I_{\boxtimes} \hookrightarrow P$. Let M, N and S be three non-empty finite sets, with two diagrams of injections as follows:

$$\varphi := \big(i : M \hookrightarrow S \hookleftarrow N : j\big) \text{ and } \varphi^{\mathrm{op}} := \big(j : N \hookrightarrow S \hookleftarrow M : i\big)$$

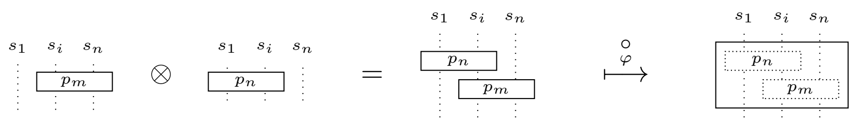
and such that

$$\begin{cases} \mathrm{im}(i) \cup \mathrm{im}(j) & = S \\ \mathrm{im}(i) \cap \mathrm{im}(j) & \neq \varnothing \end{cases} . \tag{2}$$

We say that P has a *partial composition system* if, for all diagrams φ, we have a morphism

$$\underset{\varphi}{\circ} : P(M) \otimes P(N) \longrightarrow P(S) ,$$

graphically represented by



compatible with the action of $\mathrm{Aut}(S)$, i.e. for all $\sigma \in \mathrm{Aut}(S)$ with

$$\sigma \cdot \varphi := \Big(i' : M' := \sigma\big(i(M)\big) \hookrightarrow S \hookleftarrow \sigma\big(j(N)\big) =: N' : j'\Big)$$

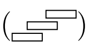
we have the commutative diagram

$$\begin{tikzcd} P(M) \otimes P(N) \arrow[r, "P(\sigma|_M)\otimes P(\sigma|_N)", "\cong"'] \arrow[d, "\underset{\varphi}{\circ}"'] & P(M') \otimes P(N') \arrow[d, "\underset{\sigma\cdot\varphi}{\circ}"] \\ P(S) \arrow[r, "P(\sigma)"'] & P(S) \end{tikzcd} \tag{3}$$

and which satisfies the following compatibility properties, for all commutative diagram of injections

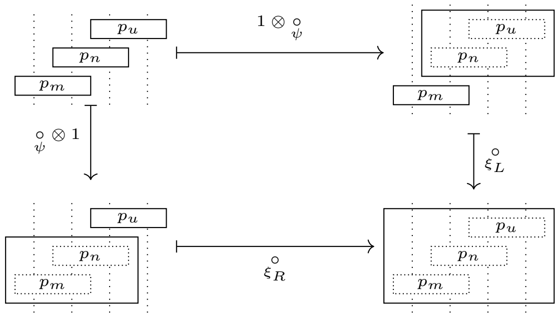
$$\begin{tikzcd} M \arrow[dr] & & N \arrow[dl] \arrow[dr] & & U \arrow[dl] \\ & R \arrow[dr] \arrow[u, phantom, "\varphi"] & & S \arrow[dl] \arrow[u, phantom, "\psi"] & \\ & & T \arrow[u, phantom, "\xi"] & & \end{tikzcd}$$

with $\xi_L := M \to T \leftarrow S$ and $\xi_R := R \to T \leftarrow U$, such that the four pairs of arrows φ, ψ, ξ_L and ξ_R satisfy the condition (2). The partial composition satisfies the three associativity axioms:

Axiom H ()

$$\begin{tikzcd} P(M) \otimes P(N) \otimes P(U) \arrow[r, "1\otimes \underset{\psi}{\circ}"] \arrow[d, "\underset{\varphi}{\circ}\otimes 1"] & P(M) \otimes P(S) \arrow[d, "\underset{\xi_L}{\circ}"] \\ P(R) \otimes P(U) \arrow[r, "\underset{\xi_R}{\circ}"] & P(T) \end{tikzcd}$$

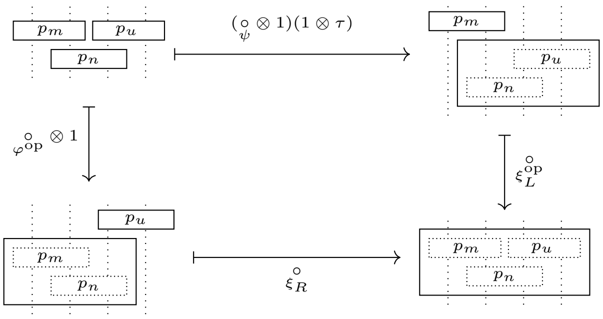
represented by



Axiom V

$$\begin{tikzcd}
P(N)\otimes P(M)\otimes P(U) \arrow[r, "(\underset{\psi}{\circ}\otimes 1)(1\otimes\tau)"] \arrow[d, "\underset{\varphi^{\mathrm{op}}}{\circ}\otimes 1"'] & P(S)\otimes P(M) \arrow[d, "\overset{\circ}{\xi}{}^{\mathrm{op}}_{L}"] \\
P(R)\otimes P(U) \arrow[r, "\overset{\circ}{\xi}_{R}"'] & P(T)
\end{tikzcd}$$

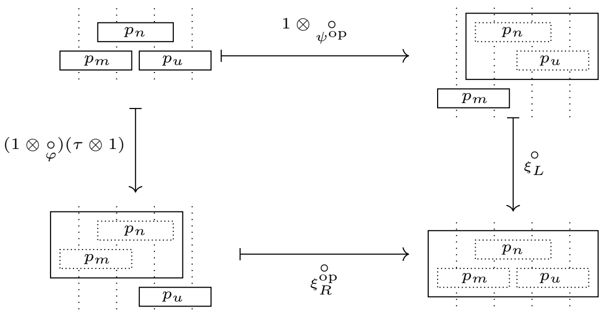
represented by



Axiom Λ

$$\begin{tikzcd}
P(M)\otimes P(U)\otimes P(N) \arrow[r, "1\otimes \underset{\psi^{\mathrm{op}}}{\circ}"] \arrow[d, "(1\otimes\underset{\varphi}{\circ})(\tau\otimes 1)"'] & P(M)\otimes P(S) \arrow[d, "\overset{\circ}{\xi}_{L}"] \\
P(U)\otimes P(R) \arrow[r, "\overset{\circ}{\xi}{}^{\mathrm{op}}_{R}"'] & P(T)
\end{tikzcd}$$

represented by



The partial products also satisfy the following unital property for all diagrams of the form

$$\iota := \big(i : \{*\} \hookrightarrow M \overset{\cong}{\leftarrow} M : \mathrm{id}\big),$$

we have commutative diagrams

$$\begin{tikzcd} I_{\boxtimes}(\{*\}) \otimes P(M) \arrow[r, hook, "\epsilon\otimes\mathrm{id}"] \arrow[dr, bend right, "\cong"'] & P(\{*\}) \otimes P(M) \arrow[d, "\underset{\iota}{\circ}"] \\ & P(M) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} P(M) \otimes I_{\boxtimes}(\{*\}) \arrow[r, hook, "\mathrm{id}\otimes\epsilon"] \arrow[dr, bend right, "\cong"'] & P(M) \otimes P(\{*\}) \arrow[d, "\underset{\iota^{\mathrm{op}}}{\circ}"] \\ & P(M) \end{tikzcd}.$$

Proposition 5.10. *A protoperad $\mathcal{P}$ has canonically a partial compositions system. Conversely, a partial compositions system on a $\mathfrak{S}$-module P canonically extends to a protoperad structure.*

Proof. Let (P, μ) be a monoid in the category $\left(\mathfrak{S}\text{-}\mathsf{mod}_{\mathsf{C}}^{\mathrm{red}}, \boxtimes, I_{\boxtimes}\right)$. By the grading of $\boxtimes$ which is implied by the analycity of $\boxtimes$ (cf. Lemma 2.17), the restriction of the product μ to $(P \boxtimes P)^{(2)_P}$ gives us directly all the partial compositions $\underset{\varphi}{\circ}$ and the associativity and the unit of the product imply all diagrams of the definition hold.

Conversely, let P be a reduced $\mathfrak{S}$-module, with an injection of $\mathfrak{S}$-modules $I_{\boxtimes} \hookrightarrow P$ and a partial composition system. By the associativity of partial compositions $\underset{\varphi}{\circ}$ for P, we define, for all $K \in \mathcal{Y}_m^{\mathrm{or}}(S)$, $L \in \mathcal{Y}_n^{\mathrm{or}}(S)$ with $\mathcal{K}(K, L) = \{S\}$, a morphism

$$\tilde{\mu}_{K,L} : \bigotimes_{i=1}^{m} P(K_i) \otimes \bigotimes_{j=1}^{n} P(L_j) \longrightarrow P(S).$$

The compatibility of the partial compositions with the action of the automorphism group of the target (cf. Equation (3)) implies that the following morphism

$$\sum \tilde{\mu}_{K,L} : \bigoplus_{(\sigma,\tau)\in\mathfrak{S}_m\times\mathfrak{S}_n} \bigotimes_{i=1}^{m} P(K_{\sigma(i)}) \otimes \bigotimes_{j=1}^{n} P(L_{\tau(j)}) \longrightarrow P(S)$$

passes to the quotient

$$\mu_S : \bigoplus_{(K,L)\in\mathcal{X}^{\mathrm{conn}}(S)} \bigotimes_{\alpha\in A} P(K_\alpha) \otimes \bigotimes_{\beta\in B} P(L_\beta) \longrightarrow P(S),$$

which gives us a natural transformation $\mu : P \boxtimes P \longrightarrow P$. This natural transformation makes P a unital associative monoid in $\boxtimes$, because the partial products satisfy the associativity and unital axioms. $\square$

5.2 Free monoid in abelian monoidal categories We briefly recall the construction of the free monoid $\mathscr{F}(-)$ by Vallette in [18, 20] for general abelian monoidal category. Let $(\mathsf{A}, \odot, I_\odot)$ be an abelian monoidal category such that, for all objects A in A, the endofunctors of A R_A and L_A, given by $R_A(M) \coloneqq M \odot A$ and $L_A(M) \coloneqq A \odot M$ for all object $M \in \mathsf{A}$, preserve reflexive coequalizors and sequential colimits (cf. [20]). Fix an object V in the category A. The underlying object $\mathscr{F}(V)$ of the free monoid associated to V, is given by the following sequential colimit:

$$\begin{tikzcd} \widetilde{V}_0 \coloneqq I \arrow[r, "\widetilde{\eta}_V"] \arrow[drr, "j_{V,0}"'] & \widetilde{V}_1 \arrow[r, "\widetilde{\eta}_V"] \arrow[dr, "j_{V,1}"] & \widetilde{V}_2 \arrow[r, "\widetilde{\eta}_V"] \arrow[d, "j_{V,2}"] & \cdots \arrow[r, "\widetilde{\eta}_V"] & \widetilde{V}_n \arrow[r, "\widetilde{\eta}_V"] \arrow[dll, "j_{V,n}"] & \cdots \\ & & \mathscr{F}(V) \coloneqq \operatorname*{colim}_{n\in\mathbb{N}} \widetilde{V}_n, & & & \end{tikzcd}$$

where

- the objects $\widetilde{V}_n$ are defined as a quotient of $(V \oplus I_\odot)^{\odot n}$ where we identify $V \odot I_\odot$ and $I_\odot \odot V$ on each copy of $(V \oplus I_\odot)^{\odot 2}$ in $(V \oplus I_\odot)^{\odot n}$;
- the maps $\widetilde{\eta}_V$ are induced by the maps

$$\eta_{V,i} : (V \oplus I_\odot)^{\odot i} \odot I_\odot \odot (V \oplus I_\odot)^{\odot (n-i)} \xrightarrow{V_i \odot \eta_V \odot V_{n-i}} (V \oplus I_\odot)^{\odot i} \odot (V \oplus I_\odot) \odot (V \oplus I_\odot)^{\odot (n-i)}$$

which are identified in the quotient.

The interested reader can refer to [20] for more details. Using this, we describe the free protoperad $\mathscr{F}(V)$ over a $\mathfrak{S}$-module V. We summarise the results of Appendix A.1 in the following proposition.

Proposition 5.11 (First description of the free protoperad $\mathscr{F}(V)$)**.** *Let V be a reduced $\mathfrak{S}$-module. We have an explicit description of the free protoperad on V. Moreover, the free protoperad is weight-graded:*

$$\mathscr{F}(V) \cong \bigoplus_{\rho \in \mathbb{N}} \mathscr{F}^{(\rho)}(V) \ .$$

with $\mathscr{F}^{(0)}(V) = I_\boxtimes(V)$.

Proof. See Proposition A.6 for the precise statement and the proof. □

The functor Ind commutes to free monoids and sends a protoperad defined by generators $\mathcal{G}$ and relations $\mathcal{R}$ to a properad defined by generators $\mathrm{Ind}(\mathcal{G})$ and relations $\mathrm{Ind}(\mathcal{R})$

Proposition 5.12. *1. The functor* Ind *commutes with the functors $\mathscr{F}$ and $\mathscr{F}^{\mathrm{Val}}$, i.e.*

$$\mathscr{F}^{\mathrm{Val}}\big(\mathrm{Ind}(-)\big) \cong \mathrm{Ind}\big(\mathscr{F}(-)\big),$$

where $\mathscr{F}^{\mathrm{Val}}$ is the functor of free properad (see [18, 19]).

2. Let V be a reduced $\mathfrak{S}$-module and R be a sub-$\mathfrak{S}$-module of the free monoid $\mathscr{F}(V)$. Then, we have the isomorphism:

$$\mathrm{Ind}\Big(\mathscr{F}(V)/\langle R\rangle\Big) \cong \mathscr{F}^{\mathrm{Val}}\big(\mathrm{Ind}(V)\big)\Big/\big\langle \mathrm{Ind}(R)\big\rangle.$$

Proof. 1. The functor Ind commutes with colimits.

2. This follows from Theorem 4.15, Proposition 4.4, [18, Prop. 28] and the construction of the free monoid

□

Remark 5.13 (Description of $\mathscr{F}^{(2)}(V)$)**.** By Proposition A.5 and Proposition A.6, we have an explicit description of the sub-$\mathfrak{S}$-module of weight 2 of the free protoperad:

$$\mathscr{F}^{(2)}(V) \cong \bigoplus_{\substack{K,L \subset S \\ K \cup L = S \\ K \cap L \neq \varnothing}} V(K) \otimes V(L).$$

As we said, the first description of the free protoperad is rather complicated, so we prefer a combinatorial description generalizing the identification of Remark 5.13. This is the purpose of the next subsection.

5.3 Combinatorial version of the free protoperad Fix a reduced $\mathfrak{S}$-module V: by the external product of $\mathfrak{S}$-modules, we define the functor

$$V^{\times n} : \left(\mathsf{Bij}^{\mathrm{op}}\right)^{\times n} \longrightarrow \mathsf{C}.$$

Following Section 2.1, we have the functor

$$V^{\mathcal{W}_n^{\mathrm{or}}}(S) = \bigoplus_{\left((K_1,\ldots,K_n),\leqslant\right)\in\mathcal{W}_n^{\mathrm{or}}(S)} V(K_1)\otimes\ldots\otimes V(K_n)\ .$$

The free action of $\mathfrak{S}_n$ on $V^{\times n}$ acts on $V^{\mathcal{W}_n^{\mathrm{or}}}$, by the symmetry of $(\mathsf{C},\otimes,\tau)$: then, we obtain the functor

$$\left(V^{\mathcal{W}_n^{\mathrm{or}}}\right)_{\mathfrak{S}_n}(S) \cong \bigoplus_{\left(\{K_\alpha\}_{\alpha\in A},\leqslant\right)\in\mathcal{W}_n(S)} \Big(\bigoplus_{(K_1,\ldots,K_n)\in\{K_\alpha\}_{\alpha\in A}} V(K_1)\otimes\ldots\otimes V(K_n)\Big)_{\mathfrak{S}_n}$$

which we denote as follows:

$$V^{\mathcal{W}_n}(S) \coloneqq \bigoplus_{\left(\{K_\alpha\}_{\alpha\in A},\leqslant\right)\in\mathcal{W}_n(S)} \bigotimes_{\alpha\in A} V(K_\alpha).$$

The construction works for the functor of connected walls $\mathcal{W}_n^{conn,ord}$, which gives us the functor

$$V^{\mathcal{W}_n^{\mathrm{conn}}}(S) \coloneqq \bigoplus_{\left(\{K_\alpha\}_{\alpha\in A},\leqslant\right)\in\mathcal{W}_n^{\mathrm{conn}}(S)} \bigotimes_{\alpha\in A} V(K_\alpha).$$

We define the partial composition product. Let φ be a diagram of injections $i : S \hookrightarrow R \hookleftarrow T : j$ with $\mathrm{im}(i)\cap\mathrm{im}(j) \neq \varnothing$ and $\mathrm{im}(i)\cup\mathrm{im}(j) = R$. We have, by Proposition 1.11, the morphism

$$\begin{aligned} V^{\mathcal{W}_m^{\mathrm{or}}}(S)\otimes V^{\mathcal{W}_n^{\mathrm{or}}}(T) &\cong \bigoplus_{\substack{\left((K_1,\ldots,K_m),\leqslant_K\right)\in\mathcal{W}_m^{\mathrm{or}}(S)\\ \left((L_1,\ldots,L_n),\leqslant_L\right)\in\mathcal{W}_n^{\mathrm{or}}(T)}} V(K_1)\otimes\ldots\otimes V(K_m)\otimes V(L_1)\otimes\ldots\otimes V(L_n) \\ &\to \bigoplus_{\substack{\left((i(K_1),\ldots,i(K_m),\right.\\ \left.j(L_1),\ldots,j(L_n)),\leqslant_{i(K)}^{j(L)}\right)\\ \in\mathcal{W}_{m+n}^{\mathrm{or}}(R)}} V(K_1)\otimes\ldots\otimes V(K_m)\otimes V(L_1)\otimes\ldots\otimes V(L_n), \end{aligned}$$

where $\leqslant_{i(K)}^{j(L)}$ is defined as follows: we have on $\cup_a i(K_a)$ (resp. $\cup_b j(L_b)$) the partial order $\leqslant_{i(K)}$ (resp.$\leqslant_{j(L)}$) induced by that of K (resp. L), i.e. $i(K_a) \leqslant_{i(K)}^{j(L)} i(K_b)$ if $K_a \leqslant_K K_b$ (resp. $j(L_a) \leqslant_{i(K)}^{j(L)} j(L_b)$ if $L_a \leqslant_L L_b$) which gives to $(K_1,\ldots,\mathcal{K}_m,L_1,\ldots,L_n)$, the order $\leqslant_{i(K)}^{j(L)}$, by Lemma 1.3. Thus we have the morphism

$$V^{\mathcal{W}_m^{\mathrm{or}}}(S)\otimes V^{\mathcal{W}_n^{\mathrm{or}}}(T) \longrightarrow\ V^{\mathcal{W}_{m+n}}(R)$$

which factorises through $V^{\mathcal{W}_m}(S)\otimes V^{\mathcal{W}_n}(T)$, giving the partial composition product

$$\underset{\varphi}{\circ} : V^{\mathcal{W}_m}(S)\otimes V^{\mathcal{W}_n}(T) \longrightarrow\ V^{\mathcal{W}_{m+n}}(R).$$

As $\mathrm{im}(i)\cap\mathrm{im}(j) \neq \varnothing$, this partial composition product restricts to the connected version: we have the partial composition product

$$\underset{\varphi}{\circ} : V^{\mathcal{W}_m^{\mathrm{conn}}}(S)\otimes V^{\mathcal{W}_n^{\mathrm{conn}}}(T) \longrightarrow\ V^{\mathcal{W}_{m+n}^{\mathrm{conn}}}(R).$$

These partial products make $V^{\mathcal{W}^{\mathrm{conn}}} \coloneqq \coprod_n V^{\mathcal{W}_n^{\mathrm{conn}}}$, a protoperad by Proposition 5.10.

Theorem 5.14 (Description of the free protoperad)**.** *Let V be a reduced $\mathfrak{S}$-module and ρ be an integer in $\mathbb{N}^*$. We have, for all finite sets S, the isomorphism of (right)* $\mathrm{Aut}(S)$*-modules*

$$\mathscr{F}^{\rho}(V)(S) \cong \bigoplus_{\substack{(\{K_\alpha\}_{\alpha\in A},\leqslant)\\ \in \mathcal{W}^{\mathrm{conn}}_{\rho}(S)}} \bigotimes_{\alpha\in A} V(K_\alpha).$$

Proof. See Appendix A.2. □

Remark 5.15 (Wall module endofunctor)**.** This free protoperad construction gives us the *wall module endofunctor*

$$\begin{array}{rccc} \mathscr{F}: & \mathfrak{S}\text{-}\mathsf{mod}^{\mathrm{red}}_{\mathsf{C}} & \longrightarrow & \mathfrak{S}\text{-}\mathsf{mod}^{\mathrm{red}}_{\mathsf{C}} \\ & V & \longmapsto & \bigoplus_{w\in\mathcal{W}^{\mathrm{conn}}} w(V) \end{array}$$

where, for a finite set S and a connected wall $w = (\{W_\alpha\}_{\alpha\in A}, \leqslant) \in \mathcal{W}^{\mathrm{conn}}(S)$, we have

$$w(V)(S) \coloneqq \bigotimes_{\alpha\in A} V(W_\alpha)\ .$$

Remark that, for a reduced $\mathfrak{S}$-module V, the elements of $\mathscr{F}\circ\mathscr{F}(V)$ can be viewed as sums of walls W where each brick B of W is labelled by $w(V)$ with w a wall in $\mathcal{W}^{\mathrm{conn}}(B)$. We endow the wall endofunctor $\mathscr{F}$ with a monad structure where the natural transformation $\mathscr{F}\circ\mathscr{F}\to\mathscr{F}$ amounts to forgetting the partition of walls on the left hand side and where the unit $\mathrm{id}\to\mathscr{F}$ is given by the embeddings $V \hookrightarrow \mathscr{F}(V)$ of walls with one brick.

Proposition 5.16. *The category of algebras over the monad $\mathscr{F}$ is equivalent to the category of protoperads.*

Proof. As $\mathcal{X}^{\mathrm{conn}}$ is a subfunctor of $\mathcal{W}^{\mathrm{conn}}$ (see Remark 1.20), then every algebra over the monad $\mathscr{F}$ is endowed with a structure of protoperad. Conversely, let $\mathcal{P}$ be a protoperad. By Proposition 5.10, $\mathcal{P}$ has a partial compositions system. By induction, it is easy to see that we can reconstruct every connected wall using partial compositions. Then $\mathcal{P}$ is an algebra over the monad $\mathscr{F}$. □

Remark 5.17 (About Feynman category formalism)**.** In [9], the authors develop the notion of *Feynman category*. A Feynman category is a triple $\mathfrak{F} \coloneqq (\mathsf{V}, \mathsf{F}, \iota)$ where V is a groupoid, F is a symmetric monoidal category and $\iota\colon \mathsf{V}\to\mathsf{F}$ is a functor, satisfying three conditions (see [9, Def. 1.1.1.] for a precise statement). Fix C, a symmetric monoidal category (like Ch or Top). The authors gives several examples of Feynman categories $\mathfrak{F}$ such that the associated category of strong symmetric monoidal functors $\mathfrak{F}\text{-}\mathsf{Ops} \coloneqq \mathsf{Func}_{\otimes}(\mathsf{F}, \mathsf{C})$ are algebraic objects which encode algebraic structures.

- The authors define the Feynman category $\mathfrak{O} = (\mathsf{Crl}^{\mathrm{rt}}, \mathsf{Opd}, \iota)$ (see [9, Sect. 2.2.1.]), where $\mathsf{Crl}^{\mathrm{rt}}$ is the groupoid where object are rooted directed corollas, and morphisms are isomorphisms which preserve the directed structure; the objects of Opd are disjoint unions of corollas and morphisms are some morphisms of graphs. This Feynman category encodes *operads*, i.e. the category of strong symmetric monoidal functors $\mathsf{Func}_{\otimes}(\mathsf{Opd}, \mathsf{C})$ is the category of operads.
- The Feynman category $\mathfrak{P}$, also defined with some categories of graphs, encodes *PROPs* (see [9, Sect. 2.2.2.]).

- The Feynman category $\mathfrak{P}^{\mathrm{ctd}}$, which is a Feynman subcategory of $\mathfrak{P}$, encodes *properads* (see [9, Sect. 2.2.2.]).

Using Remark 1.22 and Theorem 5.14, it will be interesting to describe explicitly the Feynman category which encodes protoperads. With such description of this specific Feynman category, some results of this paper could be derived from [9].

5.4 Examples In this section, one gives several examples of protoperads. Using [18, Chap. 1 Sect. 7] and Proposition 2.18, one can define some protoperads by generators and relations.

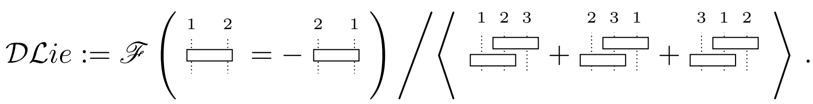
Example 5.18 ($\mathcal{DL}ie$)**.** We have already seen the double Lie protoperad in the introduction. Here, we present a formal definition. The protoperad $\mathcal{DL}ie$ is defined by generators and relations

$$\mathcal{DL}ie := \frac{\mathscr{F}(V)}{\langle R_{\mathcal{DJ}}\rangle}$$

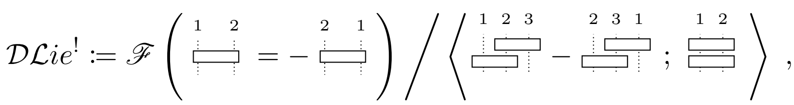
where the $\mathfrak{S}$-modules V and $R_{\mathcal{DJ}}$ are given by

$$V(n) := \begin{cases} \mathrm{sgn}(\mathfrak{S}_2) & \text{if } n = 2; \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad R_{\mathcal{DJ}}(n) := \begin{cases} \mathrm{triv}(\mathbb{Z}/3\mathbb{Z}) \uparrow^{\mathfrak{S}_3} & \text{if } n = 3; \\ 0 & \text{otherwise} \end{cases} .$$

With the presentation in terms of walls, that gives us the following presentation:

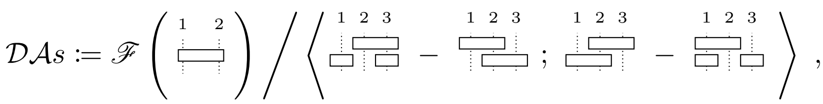


Example 5.19 ($\mathcal{DL}ie^!$)**.** The protoperad $\mathcal{DL}ie^!$, given as the Koszul dual of the protoperad $\mathcal{DL}ie$ (and named $\mathcal{DC}om$ in [11] to mimic the operadic case), is defined by generators and relations



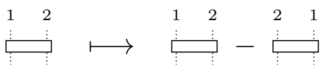
We finish by an example of protoperad which encodes the structure of double associative algebra, introduced by Goncharov and Kolesnikov in [6].

Example 5.20 ($\mathcal{DA}s$)**.** The protoperad $\mathcal{DA}s$, which encodes the double associative algebras, is defined by generators and relations,

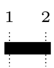
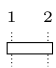


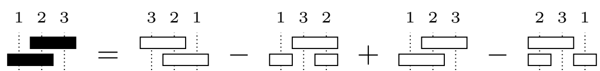
Lemma 5.21. *The map*

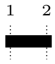
$$\Phi\colon\ \mathcal{DA}s \longrightarrow \mathcal{DL}ie$$



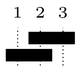
is a morphism of protoperads.

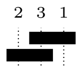
Proof. Denote by  the image of the generator  of $\mathcal{DA}s$ by the map Φ. Then we have

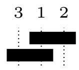


It is easy to see that 

satisfies the double Jacobi relation:



+ 

+ 

= 0 .

□

Remark 5.22. This lemma is a protoperadic incarnation of [6, Lem. 1].

6. Colours on walls

In this section, we associate to a wall W in $\mathcal{W}^{\mathrm{conn}}$, a chain complex, called the *colouring complex*: it is determined by the combinatorics of the wall W. This is a combinatorial introduction to some results of [11]: the colouring complex of a connected wall W over S encodes a part of the differential of the bar construction of a free protoperad (see [11, Sect. 2]).

6.1 Coloured bricks In this subsection, we define the notion of a colouring of a wall: throughout this section, we consider S, a non-empty finite set.

Definition 6.1 (Colouring)**.** Let $(W, \leqslant_W)$ in $\mathcal{W}(S)$ be a wall over S. A (connected) C-*colouring* of W is a surjective morphism of sets $\varphi : W \twoheadrightarrow C$, where C is called *the set of colours*, satisfying the following assertions:

1. the binary relation $\leqslant_\varphi$ induced on C by the partial order of W, defined, for all c_1, c_2 in C, by
$$c_1 \leqslant_\varphi c_2 \text{ if } \exists k_1 \in \varphi^{-1}(c_1), k_2 \in \varphi^{-1}(c_2) \text{ such that } k_1 \leqslant_W k_2;$$
is a partial order;
2. the fibers of φ are connected, i.e. for each colour c in C, the set $\varphi^{-1}(c)$ belongs to $\mathcal{W}^{\mathrm{conn}}(S_c)$ with $S_c \coloneqq \bigcup_{W_\alpha \in \varphi^{-1}(c)} W_\alpha$.

We denote by $\mathrm{Succ}(\varphi)$ or $\mathrm{Succ}(C)$ the set of successive colours. Two colouring $\varphi : W \twoheadrightarrow C$ and $\psi : W \twoheadrightarrow D$ of a wall W are isomorphic if there exists an isomorphism of posets $\Phi : (C, \leqslant_\varphi) \to (D, \leqslant_\psi)$ such that the following diagram commutes:

$$\begin{tikzcd} & W \arrow[dl, bend right, "\varphi"'] \arrow[dr, bend left, "\psi"] & \\ C \arrow[rr, "\cong", "\Phi"'] & & D. \end{tikzcd}$$

We denote by $\mathcal{C}\mathrm{ol}(W)$ the set of isomorphism classes of colourings of W:

$$\mathcal{C}\mathrm{ol}(W) \coloneqq \big\{\varphi : W \to C | \varphi \text{ a colouring}\big\}/_{\cong},$$

which is graded by the number of colours:

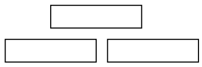
$$\mathcal{C}\mathrm{ol}_\bullet(W) = \coprod_{n \in \mathbb{N}^*} \mathcal{C}\mathrm{ol}_n(W) \;\text{ with }\; \mathcal{C}\mathrm{ol}_n(W) \coloneqq \big\{\varphi \in \mathcal{C}\mathrm{ol}(W) \mid |\varphi(W)| = n\big\}.$$

Remark 6.2. 1. As any colouring $\varphi : W \to C$ is a surjective map, for any colour c in C, the set of C-coloured bricks is non empty, i.e. $\varphi^{-1}(c) \neq \varnothing$. Furthermore, as W is a wall over S, i.e. $W \in \mathcal{W}(S)$, we have $\bigcup_{c \in C} S_c = S$ with
$$S_c \coloneqq \bigcup_{W_\alpha \in \varphi^{-1}(c)} W_\alpha.$$

2. $\mathcal{C}\mathrm{ol}_{|W|}(W)$ is reduced to a unique element.
3. If $n > |W|$, then $\mathcal{C}\mathrm{ol}_n(W) = \varnothing$, as in example 6.3.
4. Let W be a non-connected wall over S. The decomposition into connected parts (cf. Proposition 1.21) of $W = W^1 \amalg \ldots \amalg W^l$ implies that we have the following graded product:

$$\mathcal{C}\mathrm{ol}_\bullet(W) = \prod_{i=1}^{l} \mathcal{C}\mathrm{ol}_\bullet(W^i).$$

Example 6.3 (Top-colouring)**.** For any wall W, the identity morphism $W \to W$ defines the *top-colouring*, in which each brick of W has a different colour. For example, we represent the top-colouring of the following wall



by the coloured diagram:

.

Example 6.4 (Bot-colouring)**.** For any connected wall, the projection to a point $W \to \{*\}$ defines a colouring called the *bot-colouring*, denoted by bot_W, which colours all the bricks of W the same colour; for example, we represent diagrammatically the bot-colouring of the previous wall by:

.

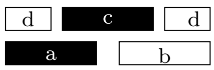
If $W \in \mathcal{W}(S)$ is a non connected wall over S, then, by the proposition 1.21, we have its decomposition in connected component $W^1 \amalg \ldots \amalg W^n$, and the *bot-colouring* of W, also denoted by bot_W, is given by $bot_W = bot_{W_1} \amalg \ldots \amalg bot_{W_n}$; for example:

.

Non-example 6.5. We consider the wall $W = \{W_a, W_b, W_c, W_d\}$ in $\mathcal{W}^{\mathrm{conn}}([\![1,4]\!])$ over $S = [\![1,4]\!]$ with

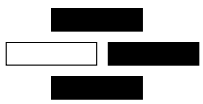
$$W_a = \{1,2\}, W_b = \{3,4\}, W_c = \{2,3\} \text{ and } W_d = \{1,4\}$$

and the partial order given by $W_a < W_c$, $W_a < W_d$, $W_b < W_c$ and $W_b < W_d$. We consider the surjective map $f : W \to \{w, b\}$ which maps W_a and W_c to b and W_b and W_d to w: we diagrammatically represent f by

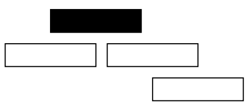


.

However, f does not define a colouring of W, because the binary relation $\leqslant_f$ induced by the order of W is not a partial order. For the same reason, the following coloured diagram:



is not the diagram of a colouring. The diagram



is not the diagram of a colouring, because the white sub-wall is not connected.

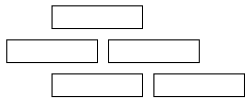
Lemma 6.6. *Let S be a non-empty finite set, W in $\mathcal{W}(S)$, a wall over S and $\varphi : W \twoheadrightarrow C$, a colouring of W with* $\mathrm{Succ}(C) \neq \varnothing$ *(which implies $|C| > 1$). For any pair* $(c_1 < c_2) \in \mathrm{Succ}(C)$ *of successive colours, the composition*

$$\begin{tikzcd} W \arrow[r, two heads, "\varphi"] \arrow[dr, dotted, "\exists!\tilde{\varphi}"'] & C \arrow[d, "\pi_{c_1}^{c_2}"] \\ & C/_{c_1\sim c_2} \end{tikzcd}$$

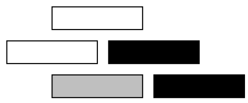
where $\pi_{c_1}^{c_2}$ identified the two colours c_1 and c_2, define a colouring $\tilde{\varphi}$ of W.

Proof. By Proposition 1.2: ☐

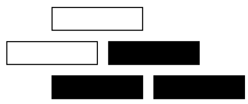
Example 6.7. We consider the wall W with five bricks represented by


,

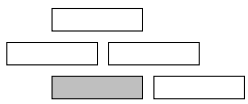
and the colouring $\varphi : W \twoheadrightarrow \{w, b, g\}$ (with w for white, g for grey and b for black), diagrammatically represented by:


,

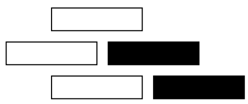
with $g \leqslant b \leqslant w$. As g and b are two successive colours, φ induces a colouring $\tilde{\varphi} : W \twoheadrightarrow C/_{g\sim b}$ with two colours, which is diagrammatically represented by:


.

The colouring φ induces another colouring $\widehat{\varphi} : W \twoheadrightarrow C/_{b\sim w}$ represented by:


.

On the other hand, as G and W are not successive, the map $\bar{\varphi} : W \twoheadrightarrow C/_{g\sim w}$ does not define a colouring of W: the diagram



is not a colouring diagram.

Lemma 6.8. *Let S be a non-empty finite set, W in $\mathcal{W}(S)$ a wall (resp. W in $\mathcal{W}^{\mathrm{conn}}(S)$ a connected wall) over S and $\varphi : W \twoheadrightarrow C$ a colouring of W. We note $\sim_\varphi$, the equivalence relation of W induced by φ, i.e. for k and l, two elements of W, we have $k \sim_\varphi l$ if $\varphi(k) = \varphi(l)$. Then $W/_{\sim_\varphi}$ is a wall (resp. $W/_{\sim_\varphi}$ is a connected wall) over S.*

Proof. By Proposition 1.2 and the definition of a colouring. ☐

6.2 The colouring complex

Definition/Proposition 6.9. Consider $(S, <_S)$, a finite *totally* ordered set: for a wall $(W, <_W)$ in $\mathcal{W}(S)$ over S, we can extend as follows the partial order $<_W$ to a total order $\prec_W$ on W, induced by that of S. For W_a and W_b in W, we have $W_a \prec_W W_b$, if:

- $W_a \cap W_b \neq \varnothing$ and $W_a <_W W_b$ (because $W_a \cap W_b \neq \varnothing$ implies that W_a and W_b are comparable for $<_W$);
- $W_a \cap W_b = \varnothing$ and $\mathfrak{h}(W_a) <_{\mathbb{N}} \mathfrak{h}(W_b)$ with $\mathfrak{h}(W_\alpha)$ the height of the brick W_α in the wall W (cf. Section 1.1);
- $W_a \cap W_b = \varnothing$, $\mathfrak{h}(W_a) = \mathfrak{h}(W_b)$ and $\min(W_a) <_S \min(W_b)$.

Let $\varphi : W \to C$ be a colouring of W, a wall over S. As the order $<_W$ induces a partial order $<_\varphi$ on C, by definition of a colouring, the total order $\prec_W$ induces a total order on C denoted by $\prec_\varphi$:

$$c_1 \prec_\varphi c_2 \text{ if } \exists k_1 \in \varphi^{-1}(c_1), k_2 \in \varphi^{-1}(c_2) \text{ such that } k_1 \prec_W k_2;$$

Lemma 6.10. *For a connected wall W in $\mathcal{W}^{\mathrm{conn}}(S)$ and a colouring φ in $\mathcal{C}\mathrm{ol}(W)$, the set of pairs of successive colours* $\mathrm{Succ}(\varphi)$ *has a total order $\prec_\varphi$ defined as follows: for $c = (c_1, c_2)$ and $d = (d_1, d_2)$, two elements of* $\mathrm{Succ}(\varphi)$, *we have $c \prec_\varphi d$ if*

$$\min\nolimits_{\prec_W}\Big(\varphi^{-1}(c_1) \cup \varphi^{-1}(c_2)\Big) \prec_W \min\nolimits_{\prec_W}\Big(\varphi^{-1}(d_1) \cup \varphi^{-1}(d_2)\Big).$$

By this lemma, we index the projections π_c^d by integers: if $(c < d) \in \mathrm{Succ}(\varphi)$ is the i-th element (for the total order $\prec_\varphi$), we note $\partial_i := \pi_c^d$. Furthermore, we observe that $\partial_i\partial_j = \partial_{j-1}\partial_i$ for all $i <_{\mathbb{N}} j$, as in a semi-simplicial set. However, we will see (cf. Example 6.14) that the set of colouring of a wall W is *not* a semi-simplicial set, but we can still associate a chain complex to the poset of colourings of a wall.

Definition/Proposition 6.11. Let S be a finite totally ordered set. For a connected wall W over S, the $\mathbb{Z}$-linearisation of the graded set $\mathcal{C}\mathrm{ol}_\bullet(W)$ gives a chain complex, called the *colouring complex*, denoted by $C_\bullet^{\mathcal{C}\mathrm{ol}}(W)$, where the differential is given by

$$(W, \varphi) \overset{\partial^{\mathcal{C}\mathrm{ol}}}{\longmapsto} \sum_{(c<d) \in \mathrm{Succ}(\varphi)} (-1)^\Lambda (W, \pi_c^d \circ \varphi)$$

with

$$\begin{aligned}\Lambda :=& \#\big\{x \in \varphi(W) \mid x \prec_\varphi d \text{ and } x \neq c\big\} \\ &+ \#\big\{x \in \varphi(W) \mid c \prec_\varphi x \prec_\varphi d \text{ and } (x < d) \in \mathrm{Succ}(\varphi)\big\}\ .\end{aligned}$$

Remark 6.12. The condition $c \prec_\varphi x \prec_\varphi d$ implies $x \neq c$ and $x \neq d$: the inequalities are strict.

Proof. We need to prove that $\partial^{\mathcal{C}\mathrm{ol}} \circ \partial^{\mathcal{C}\mathrm{ol}} = 0$. We just need to understand what are the signs of terms $\pi_a^b\pi_c^d$ and $\pi_c^d\pi_a^b$ in $\partial^{\mathcal{C}\mathrm{ol}} \circ \partial^{\mathcal{C}\mathrm{ol}}$. We consider two pairs of successive colours $(a < b) \prec_\varphi (c < d)$ for φ, a colouring of W.

- We start with two pairs of successive colours $(a < b) \prec_\varphi (c < d)$ such that $b \neq d$. Then, in the composition $\partial^{\mathcal{C}\mathrm{ol}} \circ \partial^{\mathcal{C}\mathrm{ol}}$, we have the contribution $(-1)^{\Lambda_1}\pi_a^b\pi_c^d + (-1)^{\Lambda_2}\pi_c^d\pi_a^b$, with

$$\begin{aligned}
\Lambda_1 =& \#\big\{x \in \varphi(W) \mid x \prec_\varphi d \text{ and } x \neq c\big\}\\
&+ \#\big\{x \in \varphi(W) \mid c \prec_\varphi x \prec_\varphi d \text{ and } (x < d) \in \mathrm{Succ}(\varphi)\big\}\\
&+ \#\big\{x \in \pi_c^d \circ \varphi(W) \mid x \prec_{\pi_c^d \circ \varphi} b \text{ and } x \neq a\big\}\\
&+ \#\big\{x \in \pi_c^d \circ \varphi(W) \mid a \prec_{\pi_c^d \circ \varphi} x \prec_{\pi_c^d \circ \varphi} b \text{ and } (x < b) \in \mathrm{Succ}(\pi_c^d \circ \varphi)\big\}\\
=& \#\big\{x \in \varphi(W) \mid x \prec_\varphi d \text{ and } x \neq c\big\}\\
&+ \#\big\{x \in \varphi(W) \mid c \prec_\varphi x \prec_\varphi d \text{ and } (x < d) \in \mathrm{Succ}(\varphi)\big\}\\
&+ \#\big\{x \in \varphi(W) \mid x \prec_\varphi b \text{ and } x \neq a\big\}\\
&+ \#\big\{x \in \varphi(W) \mid a \prec_\varphi x \prec_\varphi b \text{ and } (x < b) \in \mathrm{Succ}(\varphi)\big\}
\end{aligned}$$

and

$$\begin{aligned}
\Lambda_2 =& \#\big\{x \in \varphi(W) \mid x \prec_\varphi b \text{ and } x \neq a\big\}\\
&+ \#\big\{x \in \varphi(W) \mid a \prec_\varphi x \prec_\varphi b \text{ and } (x < b) \in \mathrm{Succ}(\varphi)\big\}\\
&+ \#\big\{x \in \pi_a^b \circ \varphi(W) \mid x \prec_{\pi_a^b \circ \varphi} d \text{ and } x \neq c\big\}\\
&+ \#\big\{x \in \pi_a^b \circ \varphi(W) \mid c \prec_{\pi_a^b \circ \varphi} x \prec_{\pi_a^b \circ \varphi} d \text{ and } (x < d) \in \mathrm{Succ}(\pi_a^b \circ \varphi)\big\}\\
=& \#\big\{x \in \varphi(W) \mid x \prec_\varphi b \text{ and } x \neq a\big\}\\
&+ \#\big\{x \in \varphi(W) \mid a \prec_\varphi x \prec_\varphi b \text{ and } (x < b) \in \mathrm{Succ}(\varphi)\big\}\\
&+ \#\big\{x \in \varphi(W) \mid x \prec_\varphi d \text{ and } x \neq c\big\} - 1\\
&+ \#\big\{x \in \varphi(W) \mid c \prec_\varphi x \prec_\varphi d \text{ and } (x < d) \in \mathrm{Succ}(\varphi)\big\}
\end{aligned}$$

so the contribution $(-1)^{\Lambda_1}\pi_a^b\pi_c^d + (-1)^{\Lambda_2}\pi_c^d\pi_a^b$ is null.
- We consider the case which we have $(a < c) \prec_\varphi (b < c)$. The contribution $(-1)^{\Lambda_1}\pi_a^c\pi_b^c + (-1)^{\Lambda_2}\pi_b^c\pi_a^c$ have the signs given by:

$$\begin{aligned}
\Lambda_1 =& \#\big\{x \in \varphi(W) \mid x \prec_\varphi c \text{ and } x \neq b\big\}\\
&+ \#\big\{x \in \varphi(W) \mid b \prec_\varphi x \prec_\varphi c \text{ and } (x < c) \in \mathrm{Succ}(\varphi)\big\}\\
&+ \#\big\{x \in \pi_b^c \circ \varphi(W) \mid x \prec_{\pi_b^c \circ \varphi} c \text{ and } x \neq a\big\}\\
&+ \#\big\{x \in \pi_b^c \circ \varphi(W) \mid a \prec_{\pi_b^c \circ \varphi} x \prec_{\pi_b^c \circ \varphi} c \text{ and } (x < c) \in \mathrm{Succ}(\pi_b^c \circ \varphi)\big\}\\
=& \#\big\{x \in \varphi(W) \mid x \prec_\varphi c \text{ and } x \neq b\big\}\\
&+ \#\big\{x \in \varphi(W) \mid b \prec_\varphi x \prec_\varphi c \text{ and } (x < c) \in \mathrm{Succ}(\varphi)\big\}\\
&+ \#\big\{x \in \varphi(W) \mid x \prec_\varphi c \text{ and } x \neq a\big\} - 1\\
&+ \#\big\{x \in \varphi(W) \mid a \prec_\varphi x \prec_\varphi c \text{ and } (x < c) \in \mathrm{Succ}(\varphi)\big\} - 1
\end{aligned}$$

and

$$\begin{aligned}
\Lambda_2 =& \#\big\{x \in \varphi(W) \mid x \prec_\varphi c \text{ and } x \neq b\big\} - 1\\
&+ \#\big\{x \in \varphi(W) \mid b \prec_\varphi x \prec_\varphi c \text{ and } (x < c) \in \mathrm{Succ}(\varphi)\big\}\\
&+ \#\big\{x \in \varphi(W) \mid x \prec_\varphi c \text{ and } x \neq a\big\}\\
&+ \#\big\{x \in \varphi(W) \mid a \prec_\varphi x \prec_\varphi c \text{ and } (x < c) \in \mathrm{Succ}(\varphi)\big\}
\end{aligned}$$

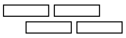
so the contribution of $(-1)^{\Lambda_1}\pi_a^c\pi_b^c + (-1)^{\Lambda_2}\pi_b^c\pi_a^c$ is null.

Then, we have $\partial^{\mathcal{C}\mathrm{ol}} \circ \partial^{\mathcal{C}\mathrm{ol}} = 0$. □

Lemma 6.13. *If $W \in \mathcal{W}(S)$ is a non connected wall over a totally ordered S, and $W^1 \amalg \ldots \amalg W^l$ is the decomposition in connected component of W, we have*

$$C_\bullet^{\mathcal{C}\mathrm{ol}}(W) \cong \bigotimes_{i=1}^{l} C_\bullet^{\mathcal{C}\mathrm{ol}}(W^i).$$

Proof. By Remark 6.2 4, and the fact that the functor of linearisation preserves coproducts. □

Example 6.14. Consider the colouring complex of the wall $W \in \mathcal{W}^{\mathrm{conn}}([\![1,4]\!])$ represented by  .

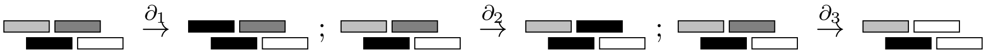
- $\mathcal{C}\mathrm{ol}_4(W)$ contains only the top-colouring of W:

   .

  the poset $\mathrm{Succ}(top)$, where we denote by $\prec_{top}$ its total order, is given by:

  $\mathrm{Succ}(top) = \Big\{$  $\Big\}$

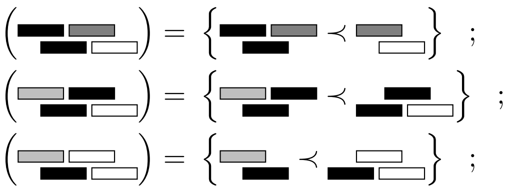
  where any diagram represent a pair of successive bricks. So, we have the following three arrows $\mathcal{C}\mathrm{ol}_4(W) \overset{\partial_i}{\rightrightarrows} \mathcal{C}\mathrm{ol}_3(W)$ represented by:

- $\mathcal{C}\mathrm{ol}_3(W)$ is the set of the following three colourings

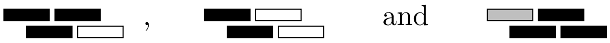
  

  For each colouring in $\mathcal{C}\mathrm{ol}_3(W)$, we have the set of successive colours:

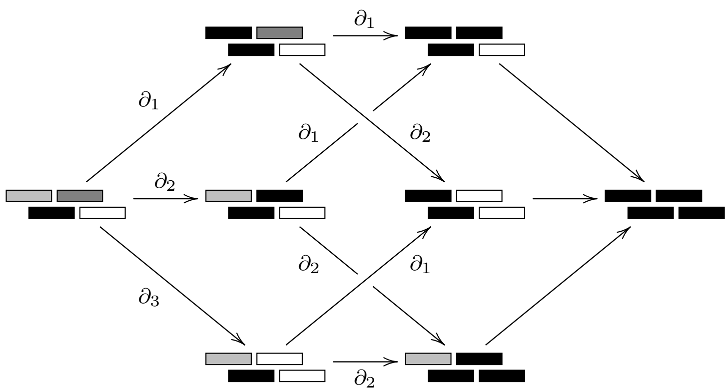
  which are given to us the maps from $\mathcal{C}\mathrm{ol}_3(W)$ to $\mathcal{C}\mathrm{ol}_2(W)$.

- $\mathcal{C}\mathrm{ol}_2(W)$ contains the following colourings:

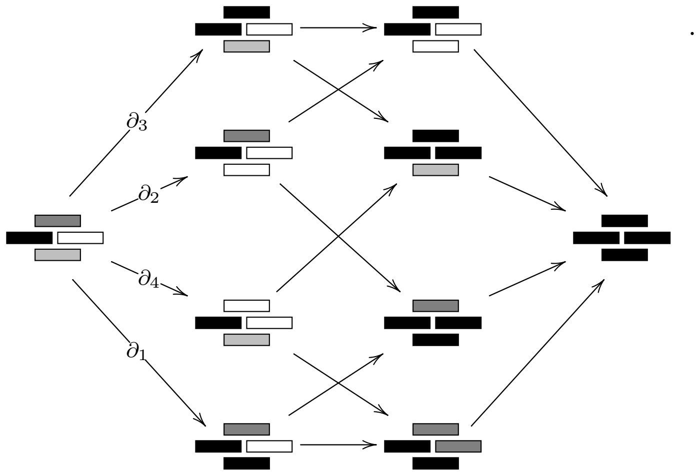
  

  As these colourings just have two colours, the sets of successive colours associated to them are reduced to only one element.

Finally, we have the complete description of the colouring complex of W:

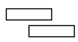
$C_\bullet^{\mathcal{C}\mathrm{ol}}(W) :=$ 

where the differential is given by the sum of ∂_i with the sign rule defined above. We have the following second example:



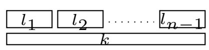
Theorem 6.15. *Let S be a finite totally ordered set and W a wall over S. If the set* $\mathrm{Succ}(W)$ *is not empty, then the colouring complex $C^{\mathcal{C}\mathrm{ol}}_\bullet(W)$ is acyclic.*

Proof. Let S be a totally ordered finite set and W, a wall over S with $\mathrm{Succ}(W) \neq \varnothing$. We prove the proposition by induction on the number of bricks in W. If W has only one brick, then the set $\mathrm{Succ}(W)$ is empty. If $\#W = 2$ then the complex $C^{\mathcal{C}\mathrm{ol}}_\bullet(W)$ is as follows:

$C^{\mathcal{C}\mathrm{ol}}_\bullet(W) = \quad 0 \longrightarrow$  $\xrightarrow{\partial}$  $\longrightarrow 0,$

which is acyclic. We suppose, by induction, that, for all wall W over S such that $2 \leqslant \#W < n$ and $\mathrm{Succ}(W) \neq \varnothing$, the chain complex $C^{\mathcal{C}\mathrm{ol}}_\bullet(W)$ is acyclic. If W is a non-connected wall with n bricks, by Lemma 6.13 and the induction hypothesis, the chain complex $C^{\mathcal{C}\mathrm{ol}}_\bullet(W)$ is acyclic. Now, let W be a connected wall with n bricks.

We start be treating the exceptional case where W has the following shape:



. (4)

We consider the sub-complex $C^{\mathcal{C}\mathrm{ol},k<l_1}_\bullet(W)$ which is isomorphic to $C^{\mathcal{C}\mathrm{ol}}_\bullet(W/_{k\sim l_1})$ which is acyclic by induction. So we consider the following short exact sequence:

$$0 \longrightarrow C^{\mathcal{C}\mathrm{ol},k<l_1}_\bullet(W) \longrightarrow C^{\mathcal{C}\mathrm{ol}}_\bullet(W) \longrightarrow C^{\mathcal{C}\mathrm{ol}}_\bullet(W)/C^{\mathcal{C}\mathrm{ol},k<l_1}_\bullet(W) \longrightarrow 0\ ;$$

The chain complex $C^{\mathcal{C}\mathrm{ol}}_\bullet(W)/C^{\mathcal{C}\mathrm{ol},k<l_1}_\bullet(W)$ is isomorphic to $C^{\mathcal{C}\mathrm{ol}}_\bullet(W\backslash\{l_1\})$, so the term on the right hand side of the exact sequence is also acyclic, so the complex $C^{\mathcal{C}\mathrm{ol}}_\bullet(W)$ too.

Now, we suppose that W does not have the exceptional shape Equation (4). We choose $(k<l) \in \mathrm{Succ}(W)$ with $\mathfrak{h}(k) = 1$ and we consider the subcomplex $C^{\mathcal{C}\mathrm{ol},(k<l)}_\bullet(W) \subset C^{\mathcal{C}\mathrm{ol}}_\bullet(W)$ of colourings φ of W such that $\varphi(k) = \varphi(l)$. We consider $W/_{k\sim l}$, the poset with $\big(W\backslash\{k,l\}\big)\cup\{k\cup l\}$, its underlying set with the partial order induced by the W one: for $j \in W$ such that $j > k$ or $j > l$ (resp. $j < k$ or $j < l$) then we have $j > (k \cup l)$ (resp. $j < (k \cup l)$). By the definition of the differential of the colouring complex, we have the following isomorphism of chain complexes:

$$C^{\mathcal{C}\mathrm{ol},k<l}_\bullet(W) \cong C^{\mathcal{C}\mathrm{ol}}_\bullet\big(W/_{k\sim l}\big).$$

As $|W/_{k\sim l}| = |W|-1$, then $C_\bullet^{\mathcal{C}\mathrm{ol},k<l}(W)$ is acyclic by induction.

We denote by $\mathrm{Succ}_W(k)$, the set $\{l \in W \mid (k<l) \in \mathrm{Succ}(W)\}$ and we consider the chain complex

$$\sum_{l\in \mathrm{Succ}_W(k)} C_\bullet^{\mathcal{C}\mathrm{ol},k<l}(W).$$

By Lemma 6.16 below, this complex is acyclic if, for all non-empty subsets $\widetilde{\mathrm{Succ}}(k) \subset \mathrm{Succ}_W(k)$, the complex

$$\bigcap_{l\in \widetilde{\mathrm{Succ}}(k)} C_\bullet^{\mathcal{C}\mathrm{ol},k<l}(W)$$

is acyclic. Let $\widetilde{\mathrm{Succ}}(k)$ be a non-empty subset of $\mathrm{Succ}_W(k)$: we have the isomorphism of chain complexes

$$\bigcap_{l\in \widetilde{\mathrm{Succ}}(k)} C_\bullet^{\mathcal{C}\mathrm{ol},k<l}(W) \cong C_\bullet^{\mathcal{C}\mathrm{ol},k<l}(W/_{\widetilde{\mathrm{Succ}}(k)}).$$

There are two cases: if $|W/_{\widetilde{\mathrm{Succ}}(k)}| = 1$, which means that $\widetilde{\mathrm{Succ}}(k) = \mathrm{Succ}_W(k)$ and so W has the exceptional shape as in Equation (4), which is excluded by the hypothesis. Otherwise $|W/_{\widetilde{\mathrm{Succ}}(k)}| > 1$: in this case, $C_\bullet^{\mathcal{C}\mathrm{ol},k<l}(W/_{\widetilde{\mathrm{Succ}}(k)})$ is acyclic by induction.

We consider the complex

$$C_\bullet^{\mathcal{C}\mathrm{ol}}(W)\Big/\Big(\sum_{l\in \mathrm{Succ}_W(k)} C_\bullet^{\mathcal{C}\mathrm{ol},k<l}(W)\Big).$$

which is isomorphic to $C_\bullet^{\mathcal{C}\mathrm{ol}}\big(W\backslash\{k\}\big)$. Note that this complex is generally a $\otimes$-product of complexes because $W\backslash\{k\}$ is not necessarily connected. As we have the short exact sequence

$$0 \to \sum_{l\in \mathrm{Succ}_W(k)} C_\bullet^{\mathcal{C}\mathrm{ol},k<l}(W) \to C_\bullet^{\mathcal{C}\mathrm{ol}}(W) \to \frac{C_\bullet^{\mathcal{C}\mathrm{ol}}(W)}{\sum_{l\in \mathrm{Succ}_W(k)} C_\bullet^{\mathcal{C}\mathrm{ol},k<l}(W)} \to 0 \quad ,$$

with the left and the right hand sides acyclic, the complex $C_\bullet^{\mathcal{C}\mathrm{ol}}(W)$ is too. $\square$

Lemma 6.16 (Algebraic Mayer-Vietoris)**.**
1. *Let A and B be acyclic chain complexes. If the complex $A \cap B$ is acyclic, then the complex $A+B$ is too.*
2. *More generally, let m be an integer in $\mathbb{N}^*$ and $\{A_i\}_{i\in[\![1,m]\!]}$ a sequence of m acyclic complexes. If, for all subsets $J \subset [\![1,m]\!]$, the complex $\bigcap_{j\in J} A_j$ is acyclic, then the complex $\sum_{j=1}^m A_j$ is too.*

Proof. 1. We have the square

$$\begin{tikzcd} A\cap B \arrow[r] \arrow[d] & B \arrow[d] \\ A \arrow[r] & A+B \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

which induces the short exact sequence

$$0 \longrightarrow A\cap B \longrightarrow A\oplus B \longrightarrow A+B \longrightarrow 0.$$

We conclude the proof by the associated long exact sequence in homology and the additivity of the functor $\mathrm{H}_\bullet(-)$.

2. We prove the result by induction on m. For $m = 2$, we have (1). Let m be an integer and suppose by induction that for all family $A_1, \dots, A_m$ of m acyclic complexes, the complex $\bigcap_{j\in J} A_j$ is acyclic, then the complex $\sum_{j=1}^m A_j$ is acyclic. Let $A_0, \dots, A_m$ be a family of $m+1$ acyclic complexes such that, for all $J \subset [\![0, m]\!]$, the complex $\bigcap_{j\in J} A_j$ is acyclic: so the complex $\sum_{j=1}^m A_j$ is acyclic, by induction. We have the following commutative square:

$$\begin{tikzcd}
\sum_{j=1}^m A_j \cap A_0 \arrow[r] \arrow[d] & A_0 \arrow[d] \\
\sum_{j=1}^m A_j \arrow[r] & \sum_{j=0}^m A_j \arrow[ul, phantom, "\ulcorner"]
\end{tikzcd}$$

with A_0, $\sum_{j=1}^m A_j$ and $\sum_{j=1}^m A_j \cap A_0$ acyclic. We conclude by (1).

$\square$

Appendix A: Construction of the free protoperad

A.1 A first description of the free protoperad We use the results of [20] to describe the free protoperad $\mathscr{F}(V)$ over a $\mathfrak{S}$-module V.

To a partition K of a finite set S, i.e. K is an element of $\mathcal{Y}(S)$ (see Section 1.3), we associate the non-ordered set $\Gamma(K)$ which label this partition: $S \cong \coprod_{\alpha\in\Gamma(K)} K_\alpha$. Recall that the functor $\mathbb{S}(-)$ is non-unitary and satisfies the exponential property (cf. Remark 2.14): then, for two $\mathfrak{S}$-modules V_1 and V_2, we have the following isomorphism of $\mathfrak{S}$-modules:

$$\mathbb{S}\big(V_1 \oplus V_2\big) \cong \mathbb{S}(V_1) \oplus \mathbb{S}(V_2) \oplus \mathbb{S}(V_1) \otimes^{\mathrm{conc}} \mathbb{S}(V_2).$$

Notation A.1. Let V be a $\mathfrak{S}$-module, we denote by the exponent $(-)_V$, the weight-grading by the number of terms V.

The functor $\mathbb{S}(-)$ is split analytic (cf. [18, 20] for the definition of split analytic functor), so that, for three $\mathfrak{S}$-modules V_1, V_2 and V_3, we have the weight-bigrading:

$$\begin{aligned}
\mathbb{S}(V_1 \oplus V_2) \,\square\, \mathbb{S}(V_3) &\cong \mathbb{S}V_1 \,\square\, \mathbb{S}V_3 \oplus \mathbb{S}V_2 \,\square\, \mathbb{S}V_3 \oplus \big(\mathbb{S}V_1 \otimes^{\mathrm{conc}} \mathbb{S}V_2\big) \,\square\, \mathbb{S}V_3 \\
&\cong \bigoplus_{i,j\in\mathbb{N}^*} \mathbb{S}^j V_1 \,\square\, \mathbb{S}V_3 \oplus \mathbb{S}^j V_2 \,\square\, \mathbb{S}V_3 \oplus \big(\mathbb{S}^i V_1 \otimes^{\mathrm{conc}} \mathbb{S}^j V_2\big) \,\square\, \mathbb{S}V_3 \\
&=: \bigoplus_{i,j\in\mathbb{N}^*} \big(\mathbb{S}(V_1 \oplus V_2) \,\square\, \mathbb{S}(V_3)\big)^{(i)_{V_1},(j)_{V_2}}
\end{aligned}$$

by the bi-additivity of the bifunctors $-\square-$ and $-\otimes^{\mathrm{conc}}-$. This bigrading induces, via the injection $(V_1 \oplus V_2) \boxtimes V_3 \hookrightarrow \mathbb{S}(V_1 \oplus V_2) \,\square\, \mathbb{S}(V_3)$ (cf. Proposition 2.19), the bigrading by weight of V_1 and V_2 on $(V_1 \oplus V_2) \boxtimes V_3$ which is denoted by

$$(V_1 \oplus V_2) \boxtimes V_3 =: \bigoplus_{i,j\in\mathbb{N}^*} \big((V_1 \oplus V_2) \boxtimes V_3\big)^{(i)_{V_1},(j)_{V_2}}.$$

By the symmetry of the product $\boxtimes$, we also have the bigrading

$$V_3 \boxtimes (V_1 \oplus V_2) =: \bigoplus_{i,j\in\mathbb{N}^*} \big(V_3 \boxtimes (V_1 \oplus V_2)\big)^{(i)_{V_1},(j)_{V_2}},$$

and we denote $\big((V_1 \oplus V_2) \boxtimes V_3\big)^{(j)_{V_2}} := \bigoplus_{i\in\mathbb{N}^*} \big((V_1 \oplus V_2) \boxtimes V_3\big)^{(i)_{V_1},(j)_{V_2}}$.

Remark A.2. These gradings are natural: they arise from the split analytic property of the bifunctor $-\boxtimes -$.

Remark A.3. In [15] and [18], for two $\mathfrak{S}$-bimodules $\mathcal{M}$ and $\mathcal{P}$, the weight grading on $\mathcal{M}$ is denoted by

$$\underbrace{\mathcal{M}}_{r}\boxtimes\mathcal{P}.$$

As the functor Ind is monoidal and commutes with the direct sum, we have

$$\mathrm{Ind}\big(((V_1\oplus V_2)\boxtimes V_3)^{(r)_{V_2}}\big) = (\mathrm{Ind}(V_1)\oplus \underbrace{\mathrm{Ind}(V_2))}_{r}\boxtimes \mathrm{Ind}(V_3).$$

As we said in Section 5.2, the construction of the free monoid generated by a $\mathfrak{S}$-module V is based on the formal addition of the unit $I_\boxtimes$ to V. We consider the $\mathfrak{S}$-module $V_+ = V\oplus I_\boxtimes$, so that $V_+(S) = V(S)$ for $|S|\neq 1$ and $V_+(\{*\}) = V(\{*\})\oplus k$. We also need the weight bigrading of the $\mathfrak{S}$-module $V_+\boxtimes W_+$ given by the weight grading on V and the weight grading on W. For all finite sets S, this bigrading allows us to write the product $V_+\boxtimes W_+(S)$ as a direct sum of terms with i copies of V and j copies of W. More precisely, the $\mathfrak{S}$-module $\mathbb{S}(V_+)\,\square\,\mathbb{S}(W_+)$ is bigraded by weights in V and W, and, via the injection

$$V_+\boxtimes W_+ \hookrightarrow \mathbb{S}\big(V_+\boxtimes W_+\big) \underset{(2.19)}{\cong} \mathbb{S}(V_+)\,\square\,\mathbb{S}(W_+),$$

the $\mathfrak{S}$-module $V_+\boxtimes W_+$ naturally inherits a weight bigrading in V and W. To express $V_+\boxtimes W_+(S)$ as a sum of terms indexed by the bigrading, we require the following notation.

Notation A.4. Recall that, for all non-empty set S and for all pairs (I,J) in $\mathcal{Y}_2^{\mathrm{or}}(S)$, by definition, we have that I and J are also non-empty; we note:

$$\mathcal{Y}_2^{\mathrm{or},+}(S) := \mathcal{Y}_2^{\mathrm{or}}(S)\cup\{(S,\varnothing),(\varnothing,S)\}.$$

For a non-ordered partition $K\in\mathcal{Y}_n(S)$ with n terms, we want to distinguish the components of V and these with the unit $I_\boxtimes$ in $\bigotimes_{\alpha\in\Gamma(K)} V_+(K_\alpha)$ (where $\Gamma(K)$ is the non-ordered set labeling the partition K). So, we introduce the following functor:

$$\mathfrak{Q}^V_{(R_1^K,R_2^K)} : \widetilde{\Gamma(K)} \longrightarrow \mathsf{C},$$

where $\widetilde{\Gamma(K)}$ is the discrete category on the set $\Gamma(K)$. For each (R_1^K,R_2^K) in $\mathcal{Y}_2^{\mathrm{or},+}(\Gamma(K))$, the functor $\mathfrak{Q}^V_{(R_1^K,R_2^K)}$ associates, for all K_α with α in $\Gamma(K)$, a chain complex as follows:

$$\mathfrak{Q}^V_{(R_1^K,R_2^K)}(K_\alpha) := \begin{cases} V(K_\alpha) & \text{if } \alpha\in R_1^K\\ I_\boxtimes(K_\alpha) & \text{if } \alpha\in R_2^K\end{cases}.$$

that, for α in R_2^K such that $|K_\alpha|\geqslant 2$, the complex $\mathfrak{Q}^V_{(R_1^K,R_2^K)}(K_\alpha)$ is zero.

So we decompose $V_+\boxtimes W_+$ as follow: for a finite set S, we have

$$\begin{aligned} V_+\boxtimes W_+(S) &= \big((V\oplus I_\boxtimes)\boxtimes(W\oplus I_\boxtimes)\big)(S)\\ &\cong \bigoplus_{\substack{(K,L)\in\mathcal{X}^{\mathrm{conn}}(S)\\ (R_1^K,R_2^K)\in\mathcal{Y}_2^{\mathrm{or},+}(\Gamma(K))\\ (R_1^L,R_2^L)\in\mathcal{Y}_2^{\mathrm{or},+}(\Gamma(L))}} \bigotimes_{\alpha\in\Gamma(K)} \mathfrak{Q}^V_{(R_1^K,R_2^K)}(K_\alpha)\otimes \bigotimes_{\beta\in\Gamma(L)} \mathfrak{Q}^W_{(R_1^L,R_2^L)}(L_\beta).\end{aligned}$$

We collect terms by the number of copies of V and W, then the $\mathrm{Aut}(S)$-module $V_+ \boxtimes W_+(S)$ is isomorphic to

$$\bigoplus_{\substack{(r,s)\in\mathbb{N}^2\\(K,L)\in\mathcal{X}^{\mathrm{conn}}(S)}} \bigoplus_{\substack{(R_1^K,R_2^K)\in\mathcal{Y}_2^{\mathrm{or},+}(\Gamma(K))\\(R_1^L,R_2^L)\in\mathcal{Y}_2^{\mathrm{or},+}(\Gamma(L))\\|R_1^K|=r,|R_1^L|=s}} \bigotimes_{\alpha\in\Gamma(K)} \mathfrak{Q}^V_{(R_1^K,R_2^K)}(K_\alpha)\otimes \bigotimes_{\beta\in\Gamma(L)} \mathfrak{Q}^W_{(R_1^L,R_2^L)}(L_\beta)$$

$$\cong I_\boxtimes(S)\oplus W(S)\oplus V(S)\oplus \bigoplus_{\substack{(r,s)\in(\mathbb{N}^*)^2\\(K,L)\in\mathcal{X}^{\mathrm{conn}}(S)}} \bigoplus_{\substack{((R_1^K,R_2^K),(R_1^L,R_2^L))\in\Xi_{K,L}\\|R_1^K|=r,|R_1^L|=s}} \bigotimes_{\alpha\in R_1^K} V(K_\alpha)\otimes \bigotimes_{\beta\in R_1^L} W(L_\beta)$$

where

$$\Xi_{K,L} \coloneqq \left\{ \left((R_1^K,R_2^K),(R_1^L,R_2^L)\right)\in \begin{matrix}\left(\mathcal{Y}_2^{\mathrm{or}}(\Gamma(K))\cup(\Gamma(K),\varnothing)\right)\\ \times\left(\mathcal{Y}_2^{\mathrm{or}}(\Gamma(L))\cup(\Gamma(L),\varnothing)\right)\end{matrix} \;\middle|\; \begin{matrix}\forall\beta\in R_2^K, |K_\beta|=1,\\ \forall\beta\in R_2^L, |L_\beta|=1\end{matrix} \right\}, \tag{5}$$

which gives us the bigrading. This last isomorphism is given by distinguishing terms arising from the injections $I_\boxtimes \boxtimes W \hookrightarrow V_+ \boxtimes W_c$, $V \boxtimes I_\boxtimes \hookrightarrow V_+ \boxtimes W_c$ and $I_\boxtimes \boxtimes I_\boxtimes \hookrightarrow V_+ \boxtimes W_c$. This gives the bigrading of $V_+ \boxtimes W_+(S) = \bigoplus_{(r,s)\in\mathbb{N}^2}(V_+ \boxtimes W_+)^{(r)_V,(s)_W}$ where, for integers r and s in $\mathbb{N}^*$, the term $(V_+ \boxtimes W_+)^{(r)_V,(s)_W}(S)$ is isomorphic to

$$(V_+ \boxtimes W_+)^{(r)_V,(s)_W}(S) \cong \bigoplus_{(K,L)\in\mathcal{X}^{\mathrm{conn}}(S)} \bigoplus_{\substack{((R_1^K,R_2^K),(R_1^L,R_2^L))\in\Xi_{K,L}\\|R_1^K|=r,|R_1^L|=s}} \bigotimes_{\alpha\in R_1^K} V(K_\alpha)\otimes \bigotimes_{\beta\in R_1^L} W(L_\beta)$$

with

$$\bigoplus_{s\in\mathbb{N}^*}(V_+ \boxtimes W_+)^{(0)_V,(s)_W}(S) \cong (V_+ \boxtimes W_+)^{(0)_V,(1)_W}(S) = W(S)\ ,$$
$$\bigoplus_{r\in\mathbb{N}^*}(V_+ \boxtimes W_+)^{(r)_V,(0)_W}(S) \cong (V_+ \boxtimes W_+)^{(1)_V,(0)_W}(S) = V(S)\ ,$$

and $(V_+ \boxtimes W_+)^{(0)_V,(0)_W}(S) = I_\boxtimes(S)$. We describe $(V_+ \boxtimes W_+)^{(1)_V,(1)_W}$ explicitly.

Proposition A.5 (Case of $(V_+ \boxtimes W_+)^{(1)_V,(1)_W}$)**.** *Let V and W be two reduced $\mathfrak{S}$-modules. For all finite sets S, we have the following isomorphism:*

$$(V_+ \boxtimes W_+)^{(1)_V,(1)_W}(S) \cong \bigoplus_{\substack{K,L\subset S\\K\cup L=S\\K\cap L\neq\varnothing}} V(K)\otimes W(L).$$

Proof. The $\mathrm{Aut}(S)$-module $(V_+ \boxtimes W_+)^{(1)_V,(1)_W}(S)$ is isomorphic to

$$\bigoplus_{(K,L)\in\mathcal{X}^{\mathrm{conn}}(S)} \bigoplus_{\substack{(R_1^K,R_2^K)\in\mathcal{Y}_2^{\mathrm{or},+}(\Gamma(K))\\(R_1^L,R_2^L)\in\mathcal{Y}_2^{\mathrm{or},+}(\Gamma(L))\\|R_1^K|=1,|R_1^L|=1}} \bigotimes_{\alpha\in\Gamma(K)} \mathfrak{Q}^V_{(R_1^K,R_2^K)}(K_\alpha)\otimes \bigotimes_{\beta\in\Gamma(L)} \mathfrak{Q}^W_{(R_1^L,R_2^L)}(L_\beta)$$

$$\cong \bigoplus_{\substack{(K,L)\in\mathcal{X}^{\mathrm{conn}}(S)\\ \exists a\in A,b\in B|\forall\alpha\in\Gamma(K)\setminus\{a\},\beta\in\Gamma(L)\setminus\{b\}\\ K_\alpha\cong\{*\}\cong L_\beta}} V(K_a)\otimes W(L_b)$$

since, if there exists α in $A\backslash\{a\}$ such that $K_\alpha \ncong \{*\}$, then $I_\boxtimes(K_\alpha) = 0$; likewise, for $B\backslash\{b\}$. Finally, we rewrite $(V_+ \boxtimes W_+)^{(1)_V,(1)_W}(S)$ as follows:

$$(V_+ \boxtimes W_+)^{(1)_V,(1)_W}(S) \cong \bigoplus_{\substack{K,L\subset S\\ K\cup L=S\\ K\cap L\neq\varnothing}} V(K)\otimes W(L).$$

□

When we take $V = W$, the bigrading of $V_+ \boxtimes W_+$ induces a weight grading of $V_+ \boxtimes V_+$: for a finite set S, we have $V_+ \boxtimes V_+(S)$, which is isomorphic to

$$\begin{aligned}
&\bigoplus_{\substack{\rho\in\mathbb{N}^*\\ (K,L)\in\mathcal{X}^{\mathrm{conn}}(S)}} \bigoplus_{\substack{(R_1^K,R_2^K)\in\mathcal{Y}_2^{\mathrm{or},+}(\Gamma(K))\\ (R_1^L,R_2^L)\in\mathcal{Y}_2^{\mathrm{or},+}(\Gamma(L))\\ |R_1^K|+|R_1^L|=\rho}} \bigotimes_{\alpha\in\Gamma(K)} \mathfrak{Q}^V_{(R_1^K,R_2^K)}(K_\alpha)\otimes \bigotimes_{\beta\in\Gamma(L)} \mathfrak{Q}^V_{(R_1^L,R_2^L)}(L_\beta)\\
&\cong I_\boxtimes(S)\oplus \underbrace{V(S)\otimes\bigotimes_S I_\boxtimes(*)\oplus\bigotimes_S I_\boxtimes(*)\otimes V(S)}_{=:(V_+\boxtimes V_+)^{(1)_V}(S)}\\
&\quad\oplus\bigoplus_{\rho\in\mathbb{N}\backslash\{0,1\}} \underbrace{\bigoplus_{(K,L)\in\mathcal{X}^{\mathrm{conn}}(S)} \bigoplus_{\substack{((R_1^K,R_2^K),(R_1^L,R_2^L))\in\Xi_{K,L}\\ |R_1^K|+|R_1^L|=\rho}} \bigotimes_{\alpha\in R_1^K} V(K_\alpha)\otimes\bigotimes_{\beta\in R_1^L} V(L_\beta)}_{=:(V_+\boxtimes V_+)^{(\rho)_V}(S)}
\end{aligned}$$

where $\Xi_{K,L}$ is described in (5). More generally, the $\mathfrak{S}$-module $V_n := (V_+)^{\boxtimes n}$ is weight-graded in V; for all finite set S, we have

$$V_n(S) \overset{\mathrm{def}}{\coloneqq} (V_+)^{\boxtimes n}(S) \cong \bigoplus_{\rho\in\mathbb{N}} \bigoplus_{\substack{(J^1,\ldots,J^n)\in\mathcal{Y}^n(S)\\ \mathcal{K}_S^{n-1}(J^1,\ldots,J^n)=S\\ \left((R_1^{J^i},R_2^{J^i})\right)_i\\ \in\prod_{i\in[\![1,n]\!]}\mathcal{Y}_2^{\mathrm{or},+}(\Gamma(J^i))\\ \sum_{i=1}^n |R_1^{J^i}|=\rho}} \bigotimes_{\alpha\in\Gamma(J^1)} \mathfrak{Q}^V_{(R_1^{J^1},R_2^{J^1})}(J^1_\alpha)\otimes\ldots\otimes \bigotimes_{\alpha\in\Gamma(J^n)} \mathfrak{Q}^V_{(R_1^{J^n},R_2^{J^n})}(J^n_\alpha)$$

and, if we note by $V_n^{(\rho)_V}$ the following:

$$\bigoplus_{\substack{l\in[\![1,n]\!]\\ \{r_1,\ldots,r_l\}\subset[\![1,n]\!]\\ r_1<r_2<\ldots<r_l\\ (J^{r_1},\ldots,J^{r_l})\in\mathcal{Y}^s(S)\\ \mathcal{K}_S^{l-1}(J^{r_1},\ldots,J^{r_l})=S}} \bigoplus_{\substack{((R_1^{J^{r_1}},R_2^{J^{r_1}}),\ldots,(R_1^{J^{r_l}},R_2^{J^{r_l}}))\\ \in\Xi_{(J^{r_1},\ldots,J^{r_l})}\\ \sum_{i\in\{r_1,\ldots,r_l\}}|R_1^{J^i}|=\rho}} \bigotimes_{\alpha\in R_1^{J^{r_1}}} V(J^{r_1}_\alpha)\otimes\ldots\otimes\bigotimes_{\alpha\in R_1^{J^{r_l}}} V(J^{r_l}_\alpha)$$

where the set $\Xi_{(J^1,\ldots,J^n)}$ is the following

$$\left\{\begin{array}{c}\left((R_1^{J^1},R_2^{J^1}),\ldots,(R_1^{J^n},R_2^{J^n})\right)\\ \in\prod_{i\in[\![1,n]\!]}\left(\mathcal{Y}_2^{\mathrm{or}}(\Gamma(J^i))\cup(\Gamma(J^i),\varnothing)\right)\end{array}\middle|\forall i\in[\![1,n]\!],\forall\beta\in R_2^{J^i},|J^i_\beta|=1\right\},$$

then we have

$$V_n(S)\cong I_\boxtimes(S)\oplus V_n^{(\rho)_V}.$$

The isomorphisms $\lambda_V : I_\boxtimes \boxtimes V \to$ and $\rho_V : V\boxtimes I_\boxtimes \to V$ preserve the grading because $V\boxtimes I_\boxtimes = (V\boxtimes I_\boxtimes)^{(1)_V}$, and preserve all the constructions which are in the construction of $\widetilde{V}_n$

(cf. Section 5.2), then the grading of V_n carries on $\widetilde{V}_n$. Also, the injection $\widetilde{V}_n \hookrightarrow \widetilde{V}_{n+1}$ preserves the weight-grading, so, finally, the free monoid $\mathscr{F}(V)$ is weight-graded by the number of copies of V.

Proposition A.6 (First description of the free protoperad $\mathscr{F}(V)$). *Let S be a finite set and V be a reduced $\mathfrak{S}$-module. We have the isomorphism:*

$$\mathscr{F}(V)(S) \cong I_\boxtimes(S) \oplus \bigoplus_{n\in\mathbb{N}^*} \bigoplus_{\substack{(J^1,\ldots,J^n)\in\mathcal{Y}^n(S)\\ \mathcal{K}_S^{n-1}(J^1,\ldots,J^n)=S}} \bigoplus_{\widetilde{\Xi}_{(J^1,\ldots,J^n)}} \bigotimes_{\alpha\in R_1^{J^1}} V(J^1_\alpha)\otimes\ldots\otimes \bigotimes_{\alpha\in R_1^{J^n}} V(J^n_\alpha),$$

where

$$\widetilde{\Xi}_{(J^1,\ldots,J^n)} := \left\{ \begin{array}{c} \left((R_1^{J^1},R_2^{J^1}),\ldots,(R_1^{J^n},R_2^{J^n})\right) \\ \in\prod_{i\in[\![1,n]\!]}\left(\mathcal{Y}_2^{\mathrm{or}}(\Gamma(J^i))\cup(\Gamma(J^i),\varnothing)\right) \end{array} \middle| \begin{array}{c} \forall i\in[\![1,n-1]\!],\forall\beta\in R_1^{J^{i+1}}, \\ J_\beta^{i+1}\cap\coprod_{\alpha\in R_1^{J^i}} J^i_\alpha\neq\varnothing \\ \forall i\in[\![1,n]\!],\forall\beta\in R_2^{J^i},|J^i_\beta|=1 \end{array} \right\}. \tag{6}$$

Moreover, the free protoperad is weight-graded:

$$\mathscr{F}(V)\cong\bigoplus_{\rho\in\mathbb{N}}\mathscr{F}^{(\rho)}(V)$$

with $\mathscr{F}^{(0)}(V)=I_\boxtimes(V)$ and, for all integers ρ in $\mathbb{N}^*$, the ρ-weighted part $\mathscr{F}^{(\rho)}(V)(S)$ is isomorphic to:

$$\bigoplus_{n\in\mathbb{N}^*} \bigoplus_{\substack{(J^1,\ldots,J^n)\in\mathcal{Y}^n(S)\\ \mathcal{K}_S^{n-1}(J^1,\ldots,J^n)=S}} \bigoplus_{\substack{\widetilde{\Xi}_{(J^1,\ldots,J^n)}\\ \sum_{i=1}^n|R_1^{J^i}|=\rho}} \bigotimes_{\alpha\in R_1^{J^1}} V(J^1_\alpha)\otimes\ldots\otimes \bigotimes_{\alpha\in R_1^{J^n}} V(J^n_\alpha).$$

Proof. This isomorphism corresponds to the choice of a representative for the quotient $V_n \twoheadrightarrow \widetilde{V}_n$, as we will see below. We define the morphism $\tau_V : V \to V_2$ as the following composition:

$$\underbrace{V \xrightarrow[\lambda_V^{-1}+\rho_V^{-1}]{} I_\boxtimes\boxtimes V\oplus V\boxtimes I_\boxtimes \xrightarrow[\eta\boxtimes i_V - i_V\boxtimes\eta]{} (I_\boxtimes\oplus V)\boxtimes(I_\boxtimes\oplus V)}_{=:\tau} =: V_2$$

with $\eta : I_\boxtimes \hookrightarrow I_\boxtimes\oplus V$, which appears in the definition of $R_{A,B}$ for all reduced $\mathfrak{S}$-modules A and B: the $\mathfrak{S}$-module $R_{A,B}$ is defined as the image of the composition:

$$A\boxtimes(\underline{V}\oplus V_2)\boxtimes B \xhookrightarrow{=:\iota_{A,B}} A\boxtimes(V\oplus V_2)\boxtimes B \xrightarrow{A\boxtimes(\tau+\mathrm{id}_{V_2})\boxtimes B} A\boxtimes V_2\boxtimes B\ .$$

We also define the $\mathfrak{S}$-module $\widetilde{V}_n$ as the cokernel of the morphism

$$\bigoplus_{i=0}^{n-2} R_{V_i,V_{n-i-2}} \longrightarrow V_n.$$

The quotient $\widetilde{V}_n$ corresponds to the identification of the images of morphisms $\left(\mathrm{id}_{V_i}\boxtimes\eta\boxtimes i_V\boxtimes \mathrm{id}_{V_{n-i-2}}\right)\circ\iota_{V_i,V_{n-i-2}}$ and $\left(\mathrm{id}_{V_i}\boxtimes i_V\boxtimes\eta\boxtimes\mathrm{id}_{V_{n-i-2}}\right)\circ\iota_{V_i,V_{n-i-2}}$, for all $i\in[\![1,n]\!]$, in V_n. We choose to identify each class of $\widetilde{V}_n$ with an element of the image of $\Sigma_{i\in[\![0,n-2]\!]}\left(\mathrm{id}_{V_i}\boxtimes\eta\boxtimes i_V\boxtimes\mathrm{id}_{V_{n-i-2}}\right)\circ \iota_{V_i,V_{n-i-2}}$. Then, for all finite sets S, we have

$$\widetilde{V}_n(S)\cong \bigoplus_{h\in[\![1,n]\!]} \bigoplus_{\substack{(J^1,\ldots,J^h)\in\mathcal{Y}^h(S)\\ \mathcal{K}_S^{h-1}(J^1,\ldots,J^h)=S}} \bigoplus_{\widetilde{\Xi}_{(J^1,\ldots,J^h)}} \bigotimes_{\alpha\in R_1^{J^1}} V(J^1_\alpha)\otimes\ldots\otimes \bigotimes_{\alpha\in R_1^{J^h}} V(J^h_\alpha)\otimes\underbrace{\bigotimes_S I(*)\otimes\ldots\otimes\bigotimes_S I(*)}_{n-h \text{ terms}}\ ,$$

where $\widetilde{\Xi}_{(J^1,\dots,J^h)}$ is defined in (6). Then, for all integers $n > 1$, we have

$$\widetilde{V}_n(S) \cong \widetilde{V}_{n-1}(S) \oplus \bigoplus_{\substack{(J^1,\dots,J^n)\in\mathcal{Y}^n(S)\\ \mathcal{K}_S^{n-1}(J^1,\dots,J^n)=S}} \bigoplus_{\widetilde{\Xi}_{(J^1,\dots,J^n)}} \bigotimes_{\alpha\in R_1^{J^1}} V(J^1_\alpha)\otimes\dots\otimes \bigotimes_{\alpha\in R_1^{J^n}} V(J^n_\alpha),$$

which exactly describes the injections $\widetilde{V}_{n-1} \hookrightarrow \widetilde{V}_n$. $\square$

A.2 Proof of Theorem 5.14 By Proposition A.6, for all ρ in $\mathbb{N}^*$, we have the isomorphism

$$\mathscr{F}^{(\rho)}(V)(S) \cong \bigoplus_{n\in\mathbb{N}^*} \bigoplus_{\substack{(J^1,\dots,J^n)\in\mathcal{Y}^n(S)\\ \mathcal{K}_S^{n-1}(J^1,\dots,J^n)=S}} \bigoplus_{\substack{\widetilde{\Xi}_{(J^1,\dots,J^n)}\\ \sum_{i=1}^n |R_1^{J^i}|=\rho}} \bigotimes_{\alpha\in R_1^{J^1}} V(J^1_\alpha)\otimes\dots\otimes \bigotimes_{\alpha\in R_1^{J^n}} V(J^n_\alpha)\ .$$

Let $(J^1,\dots,J^n)$ in $\mathcal{Y}^n(S)$ such that $\mathcal{K}_S^{n-1}(J^1,\dots,J^n) = S$ and $\widetilde{\Xi}_{(J^1,\dots,J^n)} \neq \varnothing$; we associate to $(J^1,\dots,J^n)$ the wall W in $\mathcal{W}_\rho^{\mathrm{conn}}(S)$ with sets $\{J^i_{\alpha^i} \subset S \mid i \in [\![1,n]\!],\ \alpha^i \in R_1^{J^i}\}$ and the partial order induced by the relations $J^i_\alpha < J^j_\beta$ if $J^i_\alpha \cap J^j_\beta \neq \varnothing$, $\alpha \in R_1^{J^i}$, $\beta \in R_1^{J^j}$ and $i < j$. So we have the following morphism of $\mathrm{Aut}(S)$-modules:

$$\Phi : \mathscr{F}^\rho(V)(S) \longrightarrow \bigoplus_{\substack{(\{W_\alpha\}_{\alpha\in A},\leqslant)\\ \in\mathcal{W}_\rho^{\mathrm{conn}}(S)}} \bigotimes_{\alpha\in A} V(W_\alpha).$$

Conversely, to a connected wall $W = \big(\{W_\alpha \mid \alpha \in A\}, \leqslant\big)$ with ρ bricks, i.e. W is in $\mathcal{W}_\rho^{\mathrm{conn}}(S)$, with $\max_{\alpha\in A}\big(\mathfrak{h}(W_\alpha)\big) = n$ (where $\mathfrak{h} : W \to \mathbb{N}\cup\{\infty\}$ is the height in the poset W, see Section 1.1), we associate an element $(J^1,\dots,J^n) \in \mathcal{Y}^n(S)$ such that $\widetilde{\Xi}_{(J^1,\dots,J^n)} \neq \varnothing$ as follows. We construct partitions J^i as the sets $\{W_{\alpha^i} \in W \mid \mathfrak{h}(W_\alpha) = i\}$, extended to a partition by singletons: so we have

$$J^i := \big\{W_{\alpha^i} \in W \mid \mathfrak{h}(W_\alpha) = i\big\} \amalg \Big\{\{s\} \;\Big|\; s \notin \amalg_{\alpha^i} W_{\alpha^i}\Big\} = \big\{J^i_\beta \mid \beta \in B = \Gamma(J^i)\big\}$$

and the decomposition $(R_1^{J^i}, R_2^{J^i}) \in \mathcal{Y}_2^{\mathrm{or}}(\Gamma(J^i)) \cup (\Gamma(J^i), \varnothing)$ is given by the definition of J^i :

$$\beta \in \begin{cases} R_1^{J^i} & \text{if } J^i_\beta \in \big\{W_{\alpha^i} \in W \mid \mathfrak{h}(W_\alpha) = i\big\},\\ R_2^{J^i} & \text{otherwise.}\end{cases}$$

The connectedness of the wall W implies that the element $(J^1,\dots,J^n)$ also satisfies the property of connectedness

$$\mathcal{K}_S^{n-1}(J^1,\dots,J^n) = S.$$

Finally, we have the following morphism of $\mathrm{Aut}(S)$-modules :

$$\Psi : \bigoplus_{\substack{(\{W_\alpha\}_{\alpha\in A},\leqslant)\\ \in\mathcal{W}_\rho^{\mathrm{conn}}(S)}} \bigotimes_{\alpha\in A} V(W_\alpha) \longrightarrow \mathscr{F}^\rho(V)(S)$$

which satisfies $\Phi\circ\Psi = \mathrm{id}$ and $\Psi\circ\Phi = \mathrm{id}$.

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