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Orientals as free algebras

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Abstract

The aim of this paper is to give an alternative construction of Street's cosimplicial object of orientals, based on an idea of Burroni that orientals are free algebras for some algebraic structure on strict ω-categories. More precisely, following Burroni, we define the notion of an expansion on an ω-category and we show that the forgetful functor from strict ω-categories endowed with an expansion to strict ω-categories is monadic. By iterating this monad starting from the empty ω-category, we get a cosimplicial object in strict ω-categories. Our main contribution is to show that this cosimplicial object is the cosimplicial objects of orientals. To do so, we prove, using Steiner's theory of augmented directed chain complexes, a general result for comparing polygraphs having same generators and same linearized sources and targets.

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Introduction

The n-th oriental $\mathcal{O}_n$, introduced by Street in [18], is a (strict, globular) n-category shaped on the standard n-simplex. More precisely, $\mathcal{O}_n$ is an n-category freely generated by a polygraph (or computad) whose generating k-cells correspond to the k-faces of the standard n-simplex. Orientals organize themselves into a cosimplicial object into $\mathbf{Cat}_\omega$, the category of (strict, globular) ω-categories and (strict) ω-functors, that is, into a functor

$$\mathcal{O} : \Delta \to \mathbf{Cat}_\omega,$$

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where Δ denotes the simplex category. This cosimplicial object induces a functor

$$N : \mathbf{Cat}_\omega \to \widehat{\Delta},$$

called *Street's nerve*, taking each ω-category C to the simplicial set

$$NC : \Delta_n \mapsto \mathbf{Cat}_\omega(\mathcal{O}_n, C).$$

The original motivation of Street was to define a cohomology with coefficients in an ω-category.

The combinatorics involved in [18] is notoriously hard. This led Street to extract in [19] (see also [20]) the essential properties making it work. This was formalized in his notion of a *parity complex*. Using [19], the n-th oriental becomes the n-category associated to a simple structure, the parity complex given by the faces of the standard n-simplex, all the difficulty being now hidden in the general machinery of parity complexes. In the same paper, he defined a join construction for parity complexes, leading to an inductive construction of the orientals, that is, a construction of $\mathcal{O}_{n+1}$ from $\mathcal{O}_n$.

Alternative definitions of the orientals were given by various people. Burroni proposed during a presentation [9] an inductive definition with explicit formulas but he didn't compare his definition to Street's one. A short summary of this work was published a few years later [7]. A similar approach was taken independently by Buckley and Garner in [5] who did compare their definition to Street's one. Another definition was given by Steiner using his theory of augmented directed complexes [16] (see [17] for a comparison of the two definitions). Finally, the first author and Maltsiniotis defined in [4] a join construction for strict ω-categories and showed that the cosimplicial object of orientals is induced by the unique monoid structure for this join supported by the terminal ω-category.

One of the drawbacks of the inductive definitions of the orientals is that they don't give for free a cosimplicial object, except if one can show that the iterated construction is equipped with the structure of a monad. It is claimed in [9] and [7] that it is indeed the case but this is far from obvious from the defining formulas. Burroni gave a beautiful solution to this difficulty in a draft [8] that was meant to be the extended version of [7]: the iterated construction is the monad corresponding to some explicit algebraic structure on ω-categories that he called "ω-initial".

The purpose of this paper is two-fold. First, we give a formal, complete account of Burroni's ideas on orientals — up to now only circulated as short papers, preprints and presented in talks. Second, we show that Burroni's definition is equivalent to Street's one. For this purpose, we prove a general result for comparing polygraphs with same generators and same linearized sources and targets using Steiner's theory of augmented directed complexes.

Let us explain in more details Burroni's construction. If C is an ω-category, an *expansion* on C (called an "ω-initial structure" in [8], and a contraction in [3], where the notion was introduced independently) consists of a 0-cell o of C, called the *origin*, and a directed homotopy, that is, an oplax natural transformation, from the constant ω-functor $o : C \to C$ to the identity ω-functor $\mathrm{id}_C : C \to C$, satisfying some degeneracy conditions. When C is a category (seen as an ω-category with only trivial cells from dimension 2 on), the possible origins for an expansion on C are precisely the initial objects of C. In general, the origin of an expansion should be thought of as an ω-initial object (hence Burroni's terminology). By abstract nonsense, the forgetful functor $U : \mathbf{Cat}_{\omega,e} \to \mathbf{Cat}_\omega$, where $\mathbf{Cat}_{\omega,e}$ stands for the category of ω-categories endowed with an expansion, admits a left adjoint and we thus get a monad $T : \mathbf{Cat}_\omega \to \mathbf{Cat}_\omega$. This monad induces a cosimplicial object $\mathcal{O} : \Delta \to \mathbf{Cat}_\omega$ defined on objects by $\mathcal{O}_n = T^{n+1}(\varnothing)$, where $\varnothing$ is the empty ω-category. This is the definition of orientals given in [8].

The paper is organized as follows: Section 1 recalls the basic definitions about ω-categories and polygraphs, and sets related notations. We particularly stress the role of the endofunctor of cylinders in $\mathbf{Cat}_\omega$ first introduced in [14]. Section 2 contains the main definition of the paper, namely the notion of *ω-category with expansion*, and introduces the associated adjunction between $\mathbf{Cat}_{\omega,e}$ and $\mathbf{Cat}_\omega$, leading to an abstract, very compact definition of the orientals. We then give an explicit description of the resulting monad when applied to an ω-category freely generated by a polygraph. In particular, we get that our orientals are freely generated by polygraphs. In Section 3 we give a refined description of the combinatorics of these objects by means of a convenient notation, the *oriental calculus*. Section 4 finally establishes that our orientals coincide with those originally defined by Street. To do so, we prove, using Steiner's theory of augmented directed complexes, a general result for comparing polygraphs having same generators and same linearized sources and targets.

1. Basic notions on higher dimensional categories

This section briefly recalls the basic definitions concerning (strict, globular) ω-categories and fixes the notations to be used throughout this work.

1.1 Globular sets and higher dimensional categories

1.1.1. We write $\mathbf{Glob}_\omega$ for the *category of globular sets*:

- A *globular set* C is an infinite sequence of sets $C_0, C_1, C_2, \ldots$ together with infinitely many *source* maps ∂^- and *target* maps ∂^+

$$\begin{tikzcd} C_0 & C_1 \arrow[l, shift right, "\partial^-"'] \arrow[l, shift left, "\partial^+"] & C_2 \arrow[l, shift right, "\partial^-"'] \arrow[l, shift left, "\partial^+"] & \cdots \arrow[l, shift right, "\partial^-"'] \arrow[l, shift left, "\partial^+"] \end{tikzcd}$$

  satisfying the *globular conditions*:

$$\partial^-\partial^- = \partial^-\partial^+, \qquad \partial^+\partial^- = \partial^+\partial^+.$$

- A *globular morphism* $f : C \to D$ is an infinite sequence of maps

$$f_0 : C_0 \to D_0, f_1 : C_1 \to D_1, f_2 : C_2 \to D_2, \ldots$$

  commuting to the above maps, that is, making the diagram

$$\begin{tikzcd} C_0 \arrow[d, "f_0"'] & C_1 \arrow[l, shift right, "\partial^-"'] \arrow[l, shift left, "\partial^+"] \arrow[d, "f_1"'] & C_2 \arrow[l, shift right, "\partial^-"'] \arrow[l, shift left, "\partial^+"] \arrow[d, "f_2"'] & \cdots \arrow[l, shift right, "\partial^-"'] \arrow[l, shift left, "\partial^+"] \\ D_0 & D_1 \arrow[l, shift right, "\partial^-"'] \arrow[l, shift left, "\partial^+"] & D_2 \arrow[l, shift right, "\partial^-"'] \arrow[l, shift left, "\partial^+"] & \cdots \arrow[l, shift right, "\partial^-"'] \arrow[l, shift left, "\partial^+"] \end{tikzcd}$$

  commute, in the sense that $f_n\partial^\varepsilon = \partial^\varepsilon f_{n+1}$ for $\varepsilon = \pm$ and $n \geq 0$. Whenever x belongs to C_n, we shall simply write $f(x)$ for $f_n(x)$.

1.1.2. An element x of C_n is called a *cell of dimension* n, or simply an *n-cell, in* C.

- If $n > 0$, its *source* $x^- = \partial^- x$ and its *target* $x^+ = \partial^+ x$ are $(n-1)$-cells in C.
- For $i \leq n$, its *i-source* (resp. its *i-target*) is the i-cell $\partial_i^\varepsilon x = \partial^\varepsilon \cdots \partial^\varepsilon x$, where ε stands for $-$ (resp. for $+$) and ∂^ε is applied $n-i$ times to x. We shall also write x_i^ε for $\partial_i^\varepsilon x$. In particular, we get $x_n^\varepsilon = x$.

If $p < n$, to make the i-source and i-target of the n-cell x explicit for $p \leq i < n$, we write

$$x : x_{n-1}^- \to x_{n-1}^+ : \cdots : x_p^- \to x_p^+.$$

We say that two n-cells x and y of C are *parallel* if, either $n = 0$, or $n > 0$ and x and y have same source and same target.

1.1.3 Examples. We get $x : x_0^- \to x_0^+$ if x is a 1-cell, and $x : x_1^- \to x_1^+ : x_0^- \to x_0^+$ if x is a 2-cell.

$$\begin{tikzcd} x_0^- \cdot \arrow[r, "x"] & \cdot\, x_0^+ \end{tikzcd} \qquad \begin{tikzcd} x_0^- \cdot \arrow[r, bend left, "x_1^-"] \arrow[r, bend right, "x_1^+"'] \arrow[r, phantom, "\Downarrow x"] & \cdot\, x_0^+ \end{tikzcd}$$

1.1.4. By restriction to finite sequences $C_0, \ldots, C_n$, we get the *category* $\mathbf{Glob}_n$ *of n-globular sets.* In particular, $\mathbf{Glob}_0$ is $\mathbf{Set}$, the category of sets, and $\mathbf{Glob}_1$ is the category of (directed) graphs. Note that there is an obvious *truncation functor* from $\mathbf{Glob}_\omega$ to $\mathbf{Glob}_n$, mapping C to the n-globular set $C^{(n)}$ obtained by removing all cells of dimension $> n$.

1.1.5. We write $\mathbf{Cat}_\omega$ for the *category of ω-categories*:

- An *ω-category* is a globular set C, together with *compositions* and *units* satisfying the laws of *associativity*, *unit*, *interchange* and *functoriality of units.*
- An *ω-functor* is a globular morphism $f : C \to D$ preserving compositions and units.

1.1.6. For $n > p$ and for any n-cells x, y such that $x_p^+ = y_p^-$ in an ω-category C, we get an n-cell $z = y *_p x$, called *p-composition of y and x*, with the following iterated sources and targets:

$$z_i^\varepsilon = x_i^\varepsilon = y_i^\varepsilon \text{ for } i < p, \qquad z_p^- = x_p^-, \qquad z_p^+ = y_p^+, \qquad z_i^\varepsilon = y_i^\varepsilon *_p x_i^\varepsilon \text{ for } p < i < n.$$

We shall omit parentheses by giving priority to the lowest dimensional composition, namely:

$$z *_p y *_q x = \begin{cases} (z *_p y) *_q x & \text{if } p \leq q, \\ z *_p (y *_q x) & \text{if } p \geq q. \end{cases}$$

By associativity, both conventions are indeed compatible in case $p = q$.

1.1.7. For any p-cell u in an ω-category C, we get a $(p{+}1)$-cell $1_u : u \to u$, called *p-unit on u.* By iterating this operator i times for $i > 0$, we get the following *p-unit of dimension $p + i$ on u*:

$$1_u^i : 1_u^{i-1} \to 1_u^{i-1} : \cdots : 1_u \to 1_u : u \to u.$$

By the law of functoriality of units, any p-cell u can be identified with the n-cell 1_u^{n-p} for $n > p$. In fact, we shall not identify them, but we shall use the following abbreviations for $n > p > q$, for any n-cell x and any p-cell u such that $u_q^+ = x_q^-$ (resp. $x_q^+ = u_q^-$):

$$x *_q u = x *_q 1_u^{n-p}, \qquad u *_q x = 1_u^{n-p} *_q x.$$

1.1.8 Example. For any 2-cell x and any 1-cell u such that $u_0^+ = x_0^-$, we get

$$x *_0 u = x *_0 1_u : x_1^- *_0 u \to x_1^+ *_0 u : u_0^- \to x_0^+.$$

$$\begin{tikzcd} u_0^- \arrow[r, "u"] & u_0^+ = x_0^- \arrow[r, bend left, "x_1^-"] \arrow[r, bend right, "x_1^+"'] \arrow[r, phantom, "\Downarrow x"] & x_0^+ \end{tikzcd}$$

1.1.9. By using n-globular sets instead of globular sets, we get the *category* $\mathbf{Cat}_n$ *of n-categories.* In particular, $\mathbf{Cat}_0$ is **Set** and $\mathbf{Cat}_1$ is **Cat**, the category of small categories.

The truncation functor $C \mapsto C^{(n)}$ from globular sets to n-globular sets extends to a *truncation functor* from $\mathbf{Cat}_\omega$ to $\mathbf{Cat}_n$, which we will denote in the same way. This functor admits a left adjoint mapping any n-category C to the ω-category obtained by adding an n-unit 1_u^i of dimension $n+i$ for $i > 0$ and for each n-cell u in C. This *canonical embedding* yields an equivalence between $\mathbf{Cat}_n$ and the full subcategory of $\mathbf{Cat}_\omega$ whose objects only have unit cells beyond dimension n. In other words, any n-category, and in particular any set, can be seen as an ω-category.

1.1.10. The category $\mathbf{Cat}_\omega$ is complete and cocomplete. In particular, we get the following two ω-categories:

- the *initial ω-category* is the empty set $\varnothing$, which has no cell,
- the *terminal ω-category* is the singleton set $1 = \{o\}$, which has a single 0-cell o, and a single n-cell 1_o^n for each $n > 0$.

1.1.11. The category $\mathbf{Cat}_\omega$ is the limit of the following diagram of categories, where arrows are truncation functors:

$$\mathbf{Cat}_0 \longleftarrow \mathbf{Cat}_1 \longleftarrow \mathbf{Cat}_2 \longleftarrow \cdots$$

Moreover, the category $\mathbf{Cat}_{n+1}$ is *enriched over* $\mathbf{Cat}_n$, and likewise, $\mathbf{Cat}_\omega$ is enriched over itself. For any ω-category C and for any 0-cells u, v in C, we get indeed another ω-category $C(u, v)$:

- An n-cell in $C(u, v)$ is an $(n+1)$-cell x in C such that $x_0^- = u$ and $x_0^+ = v$.
- The p-composition and p-units in $C(u, v)$ are the $(p+1)$-composition and $(p+1)$-units in C.

1.2 Polygraphs

The forgetful functor from $\mathbf{Cat}_\omega$ to $\mathbf{Glob}_\omega$ has a left adjoint, yielding a notion of *ω-category freely generated by a globular set.* Here, we describe a more general notion of *ω-category freely generated by a polygraph*, or *computad*, introduced independently in [18, Section 4] and [6].

1.2.1. Consider the following commutative diagram of categories, where the horizontal arrows are forgetful functors, and the vertical ones are truncation functors:

$$\begin{tikzcd} \mathbf{Cat}_{n+1} \arrow[r] \arrow[d] & \mathbf{Glob}_{n+1} \arrow[d] \\ \mathbf{Cat}_n \arrow[r] & \mathbf{Glob}_n \end{tikzcd}$$

We get a functor $U_n : \mathbf{Cat}_{n+1} \to \mathbf{Cat}_n^+$, where $\mathbf{Cat}_n^+$ is defined by the following pullback square:

$$\begin{tikzcd} \mathbf{Cat}_n^+ \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{Glob}_{n+1} \arrow[d] \\ \mathbf{Cat}_n \arrow[r] & \mathbf{Glob}_n \end{tikzcd}$$

It happens that this functor has a left adjoint $L_n : \mathbf{Cat}_n^+ \to \mathbf{Cat}_{n+1}$. See [6] or [15].

1.2.2. More concretely, an object of $\mathbf{Cat}_n^+$ is a pair (C, S_{n+1}), where C is an n-category and S_{n+1} is a set of $(n+1)$-*generators*, together with maps $\partial^-, \partial^+ : S_{n+1} \to C_n$ satisfying the globular conditions in case $n > 0$. The functor L_n maps this object to the $(n+1)$-category

$$C_0 \underset{\partial^+}{\overset{\partial^-}{\leftleftarrows}} \cdots \underset{\partial^+}{\overset{\partial^-}{\leftleftarrows}} C_n \underset{\partial^+}{\overset{\partial^-}{\leftleftarrows}} S_{n+1}^*,$$

where S_{n+1}^* consists of formal compositions of $(n+1)$-generators and n-units, quotiented by the laws of associativity, unit, interchange and functoriality of units.

By construction, the map $S_{n+1} \hookrightarrow S_{n+1}^*$, which can be shown to be an injection, commutes to the source and target maps, and the above $(n+1)$-category $C' = L_n(C, S_{n+1})$ has the following universal property:

1.2.3 Lemma. *Consider some $(n+1)$-category D and some n-functor $f : C \to D^{(n)}$. Then any map $e_{n+1} : S_{n+1} \to D_{n+1}$ such that $\partial^\varepsilon e_{n+1} = f_n \partial^\varepsilon$ for $\varepsilon = \pm$ extends to a unique map $f_{n+1} : S_{n+1}^* \to D_{n+1}$ such that $f_0, \ldots, f_{n+1}$ form an $(n+1)$-functor from C' to D.*

$$\begin{tikzcd} C_0 \arrow[d, "f_0"'] & \cdots \arrow[l, shift left, "\partial^-"'] \arrow[l, shift right, "\partial^+"] & C_n \arrow[l, shift left, "\partial^-"'] \arrow[l, shift right, "\partial^+"] \arrow[d, "f_n"'] & S_{n+1}^* \arrow[l, shift left, "\partial^-"'] \arrow[l, shift right, "\partial^+"] \arrow[d, dotted, "f_{n+1}"'] & S_{n+1} \arrow[l, hook'] \arrow[dl, "e_{n+1}"] \\ D_0 & \cdots \arrow[l, shift left, "\partial^-"'] \arrow[l, shift right, "\partial^+"] & D_n \arrow[l, shift left, "\partial^-"'] \arrow[l, shift right, "\partial^+"] & D_{n+1} \arrow[l, shift left, "\partial^-"'] \arrow[l, shift right, "\partial^+"] & \end{tikzcd}$$

1.2.4. By induction on n, we define the *category* $\mathbf{Pol}_n$ *of n-polygraphs*, together with a functor $F_n : \mathbf{Pol}_n \to \mathbf{Cat}_n$ mapping any n-polygraph S to the *free n-category S^* generated by S*:

- The category $\mathbf{Pol}_0$ is $\mathbf{Cat}_0$, that is $\mathbf{Set}$, and $F_0 : \mathbf{Set} \to \mathbf{Set}$ is the identity functor.
- Suppose that the category $\mathbf{Pol}_n$ and the functor $F_n : \mathbf{Pol}_n \to \mathbf{Cat}_n$ have been defined. Then the category $\mathbf{Pol}_{n+1}$ is given by the pullback square

$$\begin{tikzcd} \mathbf{Pol}_{n+1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{Cat}_n^+ \arrow[d] \\ \mathbf{Pol}_n \arrow[r, "F_n"'] & \mathbf{Cat}_n \end{tikzcd}$$

and $F_{n+1} : \mathbf{Pol}_{n+1} \to \mathbf{Cat}_{n+1}$ is the composition of $L_n : \mathbf{Cat}_n^+ \to \mathbf{Cat}_{n+1}$ by the top arrow.

In particular, $\mathbf{Pol}_1$ is $\mathbf{Cat}_0^+$, that is $\mathbf{Glob}_1$, and $F_1(S) = S^*$ is the free category generated by S.

1.2.5 Definition (polygraphs)**.**

The *category* $\mathbf{Pol}_\omega$ *of polygraphs* is the limit of the following diagram, where each arrow is the truncation functor given by the previous pullback square:

$$\mathbf{Pol}_0 \longleftarrow \mathbf{Pol}_1 \longleftarrow \mathbf{Pol}_2 \longleftarrow \cdots$$

The functors F_n induce a functor $F_\omega : \mathbf{Pol}_\omega \to \mathbf{Cat}_\omega$ mapping S to the *free ω-category S^* generated by S.*

1.2.6. More concretely:

- A polygraph S is given by an infinite diagram of the form

$$\begin{tikzcd}
S_0 \arrow[d, hook] & S_1 \arrow[d, hook] \arrow[dl, shift right, "\partial^+"'] \arrow[dl, shift left, "\partial^-"] & S_2 \arrow[d, hook] \arrow[dl, shift right, "\partial^+"'] \arrow[dl, shift left, "\partial^-"] & \cdots \arrow[dl, shift right, "\partial^+"'] \arrow[dl, shift left, "\partial^-"] \\
S_0^* & S_1^* \arrow[l, shift right, "\partial^+"'] \arrow[l, shift left, "\partial^-"] & S_2^* \arrow[l, shift right, "\partial^+"'] \arrow[l, shift left, "\partial^-"] & \cdots \arrow[l, shift right, "\partial^+"'] \arrow[l, shift left, "\partial^-"]
\end{tikzcd}$$

  where S_i is a set of *i-generators*, and the bottom row displays S^*, starting from $S_0^* = S_0$.
- A morphism $f : S \to T$ is given by an infinite sequence of maps

$$f_0 : S_0 \to T_0,\ f_1 : S_1 \to T_1,\ f_2 : S_2 \to T_2, \ldots$$

  compatible with sources and targets so that they induce maps

$$f_0^* : S_0^* \to T_0^*, f_1^* : S_1^* \to T_1^*, f_2^* : S_2^* \to T_2^*, \ldots$$

  defining an ω-functor $f^* : S^* \to T^*$. This means that $F_\omega(f) = f^*$ is *rigid*: it preserves generators.

In case $S_i = \varnothing$ for $i > n$, the polygraph S is in fact an n-polygraph, and S^* is an n-category. This *canonical embedding* of $\mathbf{Pol}_n$ into $\mathbf{Pol}_\omega$ is the left adjoint of the obvious truncation functor.

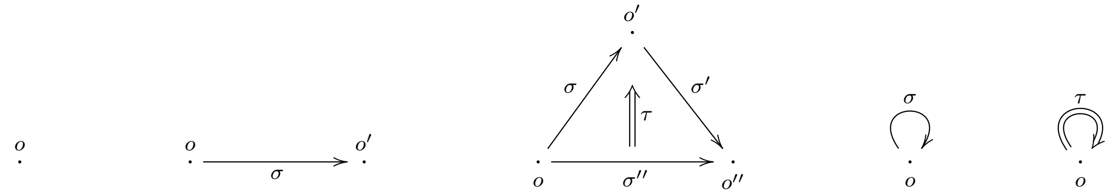
1.2.7 Examples.

- If S consists of a single 0-generator o, then S^* is the singleton set $1 = \{o\}$.
- If S consists of two 0-generators o, o' and a single 1-generator $\sigma : o \to o'$, then S^* consists of the generators and the 0-units $1_o : o \to o$, $1_{o'} : o' \to o'$.
- If S consists of three 0-generators o, o', o'', three 1-generators $\sigma : o \to o'$, $\sigma' : o' \to o''$, $\sigma'' : o \to o''$, and a single 2-generator $\tau : \sigma'' \to \sigma' *_0 \sigma$, then S^* consists of the generators, the 0-composition $\sigma' *_0 \sigma : o \to o''$, the 0-units $1_o : o \to o, 1_{o'} : o' \to o'$, $1_{o''} : o'' \to o''$, the 1-units $1_\sigma : \sigma \to \sigma, 1_{\sigma'} : \sigma' \to \sigma'$, $1_{\sigma''} : \sigma'' \to \sigma''$, $1_{\sigma' *_0 \sigma} : \sigma' *_0 \sigma \to \sigma' *_0 \sigma$, and the 0-units $1^2_o : 1_o \to 1_o$, $1^2_{o'} : 1_{o'} \to 1_{o'}$, $1^2_{o''} : 1_{o''} \to 1_{o''}$ of dimension 2.
- If S consists of a single 0-generator o and a single 1-generator $\sigma : o \to o$, then S^* consists of the 0-generator, and infinitely many 1-cells, which are of the form

$$\sigma^0 = 1_o : o \to o, \qquad \sigma^i = \sigma *_0 \cdots *_0 \sigma : o \to o \text{ for } i > 0.$$

- If S consists of a single 0-generator o and a single 2-generator $\tau : 1_o \to 1_o$, then S^* consists of the 0-generator, the 0-unit $1_o : o \to o$, and infinitely many 2-cells, which are of the form

$$\tau^0 = 1^2_o : 1_o \to 1_o : o \to o, \qquad \tau^i = \tau *_0 \cdots *_0 \tau = \tau *_1 \cdots *_1 \tau : 1_o \to 1_o : o \to o \text{ for } i > 0.$$



1.2.8 Remarks. The first three examples are (isomorphic to) the first three *orientals* $\mathcal{O}_0, \mathcal{O}_1, \mathcal{O}_2$. The last two ones are (isomorphic to) the additive monoid $\mathbb{N}$, respectively seen as a 1-category with a single 0-cell and as a 2-category with a single 0-cell and a single 1-cell.

1.3 Cylinders and oplax transformations Here, we recall the construction of the *endofunctor of small cylinders*, and the resulting notion of *oplax transformation* between two ω-functors, called *homotopy* in [14] and [10].

For that purpose, we first define the set $\Gamma_n(C)$ of *n-cylinders* in an ω-category C, together with two maps $\partial^-, \partial^+ : \Gamma_n(C) \to \Gamma_{n-1}(C)$ in case $n > 0$. We can then define a structure of ω-category on the corresponding globular set, which gives the expected endofunctor.

1.3.1 Definition (cylinders)**.**
If x, y are n-cells in C, the notion of *n-cylinder* $\alpha : x \curvearrowright y$ is given inductively:

- If $n = 0$, then $\alpha : x \curvearrowright y$ consists of a single 1-cell $\alpha_0 : x \to y$ in C.
- If $n > 0$, then $\alpha : x \curvearrowright y$ consists of two 1-cells $\alpha_0^- : x_0^- \to y_0^-$ and $\alpha_0^+ : x_0^+ \to y_0^+$ in C, together with an $(n-1)$-cylinder $\sharp\alpha : \alpha_0^+ *_0 x \curvearrowright y *_0 \alpha_0^-$ in $C(x_0^-, y_0^+)$.

In case $n > 0$, we also get two $(n-1)$-cylinders $\partial^-\alpha : \partial^- x \curvearrowright \partial^- y$ and $\partial^+\alpha : \partial^+ x \curvearrowright \partial^+ y$ which are given inductively:

- If $n = 1$, then $\partial^\varepsilon\alpha : x_0^\varepsilon \curvearrowright y_0^\varepsilon$ is given by the 1-cell $\alpha_0^\varepsilon : x_0^\varepsilon \to y_0^\varepsilon$.
- If $n > 1$, then $\partial^\varepsilon\alpha : x_{n-1}^\varepsilon \curvearrowright y_{n-1}^\varepsilon$ is given by the 1-cells $\alpha_0^- : x_0^- \to y_0^-$ and $\alpha_0^+ : x_0^+ \to y_0^+$, together with the $(n-2)$-cylinder $\sharp\partial^\varepsilon\alpha = \partial^\varepsilon\sharp\alpha : \alpha_0^+ *_0 x_{n-1}^\varepsilon \curvearrowright y_{n-1}^\varepsilon *_0 \alpha_0^-$ in $C(x_0^-, y_0^+)$.

If $\alpha : x \curvearrowright y$ is such an n-cylinder, we write $\overline{\alpha}$ and $\underline{\alpha}$ for the n-cells x and y respectively.

1.3.2. More concretely, an n-cylinder $\alpha : x \curvearrowright y$ in C is given by a finite sequence of cells

$$\alpha_0^-, \alpha_0^+, \dots, \alpha_{n-1}^-, \alpha_{n-1}^+, \alpha_n \text{ in } C,$$

where the *auxiliary cells* α_i^ε have dimension $i+1$, the *principal cell* $|\alpha| = \alpha_n$ has dimension $n+1$, and their sources and targets are given as follows:

$$\alpha_i^\varepsilon : \alpha_{i-1}^+ *_{i-1} \cdots *_1 \alpha_0^+ *_0 x_i^\varepsilon \to y_i^\varepsilon *_0 \alpha_0^- *_1 \cdots *_{i-1} \alpha_{i-1}^- \text{ for } i < n,$$

$$|\alpha| = \alpha_n : \alpha_{n-1}^+ *_{n-1} \cdots *_1 \alpha_0^+ *_0 x \to y *_0 \alpha_0^- *_1 \cdots *_{n-1} \alpha_{n-1}^-.$$

For any $i < n$, the i-cylinder $\partial_i^\varepsilon\alpha : x_i^\varepsilon \curvearrowright y_i^\varepsilon$ is given by the following sequence of cells:

$$\alpha_0^-, \alpha_0^+, \dots, \alpha_{i-1}^-, \alpha_{i-1}^+, \alpha_i^\varepsilon.$$

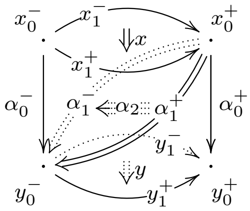
1.3.3 Remark. Beware of a slight discrepancy in our notations:

- If x is a cell, then x_i^ε stands for the i-cell $\partial_i^\varepsilon x$.
- If α is a cylinder, then α_i^ε does not stand for the i-cylinder $\partial_i^\varepsilon\alpha$, but for its *principal cell* $|\partial_i^\varepsilon\alpha|$.

1.3.4 Examples.

- If x, y are 0-cells, a 0-cylinder $\alpha : x \curvearrowright y$ is given by the 1-cell $|\alpha| = \alpha_0 : x \to y$.
- If x, y are 1-cells, a 1-cylinder $\alpha : x \curvearrowright y$ is given by the 1-cells $\alpha_0^\varepsilon : x_0^\varepsilon \to y_0^\varepsilon$ and the 2-cell $|\alpha| = \alpha_1 : \alpha_0^+ *_0 x \to y *_0 \alpha_0^-$.
- If x, y are 2-cells, a 2-cylinder $\alpha : x \curvearrowright y$ is given by the 1-cells $\alpha_0^\varepsilon : x_0^\varepsilon \to y_0^\varepsilon$, the 2-cells $\alpha_1^\varepsilon : \alpha_0^+ *_0 x_1^\varepsilon \to y_1^\varepsilon *_0 \alpha_0^-$, and the 3-cell $|\alpha| = \alpha_2 : \alpha_1^+ *_1 \alpha_0^+ *_0 x \to y *_0 \alpha_0^- *_1 \alpha_1^-$.

$$\begin{tikzcd} x \arrow[d, "\alpha_0"] \\ y \end{tikzcd} \qquad \begin{tikzcd} x_0^- \arrow[r, "x"] \arrow[d, "\alpha_0^-"'] & x_0^+ \arrow[d, "\alpha_0^+"] \arrow[dl, Rightarrow, "\alpha_1"'] \\ y_0^- \arrow[r, "y"'] & y_0^+ \end{tikzcd}$$



1.3.5. If we write $\Gamma_n(C)$ for the set of n-cylinders in C, we get the following globular set $\Gamma(C)$:

$$\Gamma_0(C) \overset{\partial^-}{\underset{\partial^+}{\leftleftarrows}} \Gamma_1(C) \overset{\partial^-}{\underset{\partial^+}{\leftleftarrows}} \Gamma_2(C) \overset{\partial^-}{\underset{\partial^+}{\leftleftarrows}} \cdots$$

The globular set $\Gamma(C)$ supports a structure of ω-category we describe below (see also [10, 11]).

1.3.6 Definition (ω-category of small cylinders)**.**
The *ω-category of small cylinders* in C is $\Gamma(C)$ endowed with the following operations:

- If $n > p$, if x, y, z, t are n-cells such that $x_p^+ = z_p^-$ and $y_p^+ = t_p^-$, and if $\alpha : x \curvearrowright y$, $\beta : z \curvearrowright t$ are n-cylinders such that $\partial_p^+ \alpha = \partial_p^- \beta$, the *$p$-composition* $\gamma = \beta *_p \alpha : z *_p x \curvearrowright t *_p y$ is the n-cylinder given by the following cells:

$$\gamma_i^\varepsilon = \alpha_i^\varepsilon = \beta_i^\varepsilon \text{ for } i < p, \qquad \gamma_p^- = \alpha_p^-, \qquad \gamma_p^+ = \beta_p^+,$$

$$\gamma_{p+1}^\varepsilon = (t_{p+1}^\varepsilon *_0 \alpha_0^- *_1 \cdots *_{p-1} \alpha_{p-1}^- *_p \alpha_{p+1}^\varepsilon) *_{p+1} (\beta_{p+1}^\varepsilon *_p \beta_{p-1}^+ *_{p-1} \cdots *_1 \beta_0^+ *_0 x_{p+1}^\varepsilon) \text{ if } p+1 < n,$$

$$\gamma_i^\varepsilon = (t_{p+1}^+ *_0 \alpha_0^- *_1 \cdots *_{p-1} \alpha_{p-1}^- *_p \alpha_i^\varepsilon) *_{p+1} (\beta_i^\varepsilon *_p \beta_{p-1}^+ *_{p-1} \cdots *_1 \beta_0^+ *_0 x_{p+1}^-) \text{ for } p+1 < i < n,$$

$$|\gamma| = \gamma_n = (t_{p+1}^+ *_0 \alpha_0^- *_1 \cdots *_{p-1} \alpha_{p-1}^- *_p \alpha_n) *_{p+1} (\beta_n *_p \beta_{p-1}^+ *_{p-1} \cdots *_1 \beta_0^+ *_0 x_{p+1}^-).$$

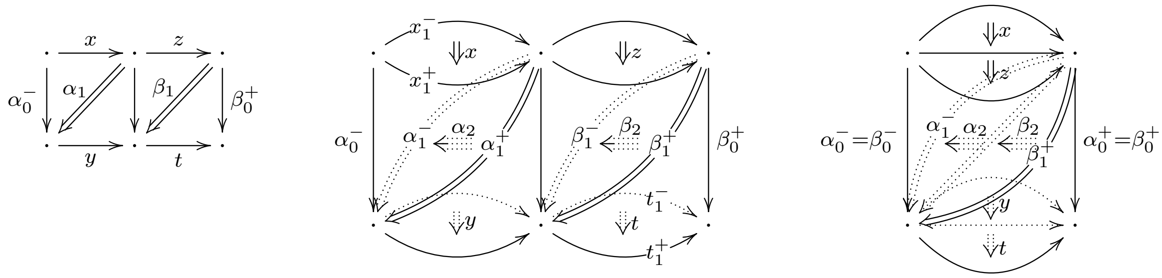
- If $\alpha : x \curvearrowright y$ is a p-cylinder, the *p-unit* $1_\alpha : 1_x \curvearrowright 1_y$ is given by the following cells:

$$\alpha_0^-, \alpha_0^+, \ldots, \alpha_{p-1}^-, \alpha_{p-1}^+, \alpha_p, \alpha_p, 1_{\alpha_p}.$$

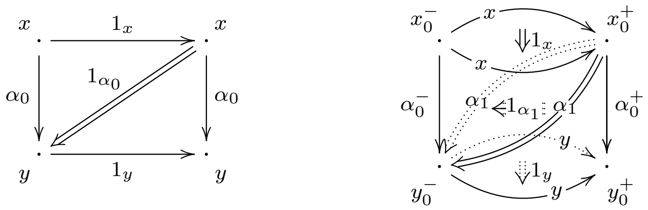
We refer to [14, Appendix A] for a proof that the axioms of (strict) ω-categories hold.

1.3.7 Examples.

- If x, y, z, t are 1-cells such that $x_0^+ = z_0^-$ and $y_0^+ = t_0^-$, and if $\alpha : x \curvearrowright y, \beta : z \curvearrowright t$ are 1-cylinders such that $\alpha_0^+ = \beta_0^-$, the 0-composition $\gamma = \beta *_0 \alpha : z *_0 x \curvearrowright t *_0 y$ is given by the 1-cells $\gamma_0^- = \alpha_0^-$ and $\gamma_0^+ = \beta_0^+$, and the 2-cell $|\gamma| = \gamma_1 = (t *_0 \alpha_1) *_1 (\beta_1 *_0 x)$.
- If x, y, z, t are 2-cells such that $x_0^+ = z_0^-$ and $y_0^+ = t_0^-$, and if $\alpha : x \curvearrowright y, \beta : z \curvearrowright t$ are 2-cylinders such that $\alpha_0^+ = \beta_0^-$, the 0-composition $\gamma = \beta *_0 \alpha : z *_0 x \curvearrowright t *_0 y$ is given by the 1-cells $\gamma_0^- = \alpha_0^-$ and $\gamma_0^+ = \beta_0^+$, the 2-cells $\gamma_1^\varepsilon = (t_1^\varepsilon *_0 \alpha_1^\varepsilon) *_1 (\beta_1^\varepsilon *_0 x_1^\varepsilon)$, and the 3-cell $|\gamma| = \gamma_2 = (t_1^+ *_0 \alpha_2) *_1 (\beta_2 *_0 x_1^-)$.
- If x, y, z, t are 2-cells such that $x_1^+ = z_1^-$ and $y_1^+ = t_1^-$, and if $\alpha : x \curvearrowright y, \beta : z \curvearrowright t$ are 2-cylinders such that $\alpha_0^\varepsilon = \beta_0^\varepsilon$ and $\alpha_1^+ = \beta_1^-$, the 1-composition $\gamma = \beta *_1 \alpha : z *_1 x \curvearrowright t *_1 y$ is given by the 1-cells $\gamma_0^\varepsilon = \alpha_0^\varepsilon = \beta_0^\varepsilon$, the 2-cells $\gamma_1^- = \alpha_1^-$ and $\gamma_1^+ = \beta_1^+$, and the 3-cell $|\gamma| = \gamma_2 = (t *_0 \alpha_0^- *_1 \alpha_2) *_2 (\beta_2 *_1 \beta_0^+ *_0 x)$.



- If $\alpha : x \curvearrowright y$ is a 0-cylinder, the 0-unit $1_\alpha : 1_x \curvearrowright 1_y$ is given by the cells $\alpha_0, \alpha_0, 1_{\alpha_0}$.
- If $\alpha : x \curvearrowright y$ is a 1-cylinder, the 1-unit $1_\alpha : 1_x \curvearrowright 1_y$ is given by the cells $\alpha_0^-, \alpha_0^+, \alpha_1, \alpha_1, 1_{\alpha_1}$.



1.3.8. Since this construction is clearly functorial, we get an endofunctor $\Gamma : \mathbf{Cat}_\omega \to \mathbf{Cat}_\omega$, and by the above formulas, the maps $\alpha \mapsto \overline{\alpha}$ and $\alpha \mapsto \underline{\alpha}$ define ω-functors $\overline{\pi}_C, \underline{\pi}_C : \Gamma(C) \to C$. Hence, we also get two natural transformations $\overline{\pi}, \underline{\pi} : \Gamma \to \mathrm{id}_{\mathbf{Cat}_\omega}$.

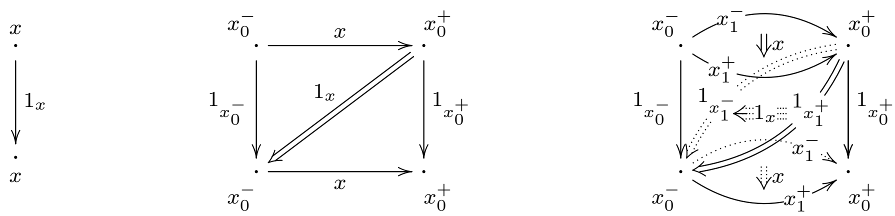
1.3.9. Any n-cell x yields a *trivial n-cylinder* $\tau(x) : x \curvearrowright x$ given by the following cells:

$$1_{x_0^-}, 1_{x_0^+}, \ldots, 1_{x_{n-1}^-}, 1_{x_{n-1}^+}, 1_{x_n}.$$

It is in fact a unit for another composition of cylinders, which is called *concatenation* in [10].

1.3.10 Examples.

- If x is a 0-cell, the 0-cylinder $\tau(x) : x \curvearrowright x$ is given by the cell 1_x.
- If x is a 1-cell, the 1-cylinder $\tau(x) : x \curvearrowright x$ is given by the cells $1_{x_0^-}, 1_{x_0^+}, 1_x$.
- If x is a 2-cell, the 2-cylinder $\tau(x) : x \curvearrowright x$ is given by the cells $1_{x_0^-}, 1_{x_0^+}, 1_{x_1^-}, 1_{x_1^+}, 1_x$.



1.3.11 Definition (oplax transformations)**.**
If $f, g : C \to D$ are two ω-functors, an *oplax transformation θ from f to g* is an ω-functor $\theta : C \to \Gamma(D)$ making the following diagram commutative:

$$\begin{tikzcd} & D \\ C \arrow[ur, "f"] \arrow[r, "\theta"] \arrow[dr, "g"'] & \Gamma(D) \arrow[u, "\overline{\pi}_D"'] \arrow[d, "\underline{\pi}_D"] \\ & D \end{tikzcd}$$

1.3.12. More concretely, if θ is an oplax transformation from f to g, then we get an n-cylinder $\theta(x) : f(x) \curvearrowright g(x)$ in D for each n-cell x in C. Its principal cell $\theta_x = |\theta(x)|$ is an $(n{+}1)$-cell in D, with the following source and target:

$$\theta_x : \theta_{x^+_{n-1}} *_{n-1} \cdots *_1 \theta_{x^+_0} *_0 f(x) \to g(x) *_0 \theta_{x^-_0} *_1 \cdots *_{n-1} \theta_{x^-_{n-1}}.$$

By ω-functoriality of θ and by construction of the ω-category $\Gamma(D)$, the following two equalities hold for $n > p$, for any n-cells x, y such that $x^+_p = y^-_p$, and for any cell u:

$$\theta_{y *_p x} = \Big(g(y^+_{p+1}) *_0 \theta_{x^-_0} *_1 \cdots *_{p-1} \theta_{x^-_{p-1}} *_p \theta_x\Big) *_{p+1} \Big(\theta_y *_p \theta_{y^+_{p-1}} *_{p-1} \cdots *_1 \theta_{y^+_0} *_0 f(x^-_{p+1})\Big),$$
$$\theta_{1_u} = 1_{\theta_u}.$$

Conversely, if for each n-cell x in C, θ_x is an $(n{+}1)$-cell in D with the above source and target, and if the above two axioms hold, then we get a unique oplax transformation θ from f to g, which is defined as follows for each n-cell x in C:

$$\theta(x)^\varepsilon_i = \theta_{x^\varepsilon_i} \text{ for } i < n, \qquad |\theta(x)| = \theta(x)_n = \theta_x.$$

In other words, θ can be reconstructed from the θ_x. See [4, Section B.2] for more details.

1.3.13 Example. For any ω-category C, the ω-functor $\tau : C \to \Gamma(C)$ mapping any cell x in C to the trivial cylinder $\tau(x) : x \curvearrowright x$ defines an oplax transformation from id_C to itself, which is given by the following $(n{+}1)$-cell for each n-cell x in C:

$$\tau_x = 1_x : x \to x.$$

Hence, we get a natural transformation $\tau : \mathrm{id}_{\mathbf{Cat}_\omega} \to \Gamma$, which is a common section of $\overline{\pi}$ and $\underline{\pi}$.

2. Orientals from the expansion monad

This section addresses the main goal of this work, namely a construction of the *cosimplicial object of orientals* $\mathcal{O} : \Delta \to \mathbf{Cat}_\omega$, which is obtained by iterating a monad $T : \mathbf{Cat}_\omega \to \mathbf{Cat}_\omega$. In particular, we get the following definition of orientals:

$$\mathcal{O}_0 = T(\varnothing), \mathcal{O}_1 = T^2(\varnothing), \mathcal{O}_2 = T^3(\varnothing), \ldots$$

This monad comes from the forgetful functor from the category of ω-categories with expansion to the category of ω-categories. This notion of *expansion* is central in this paper:

- It was first introduced under the name of ω-*initial structure* by Burroni in the unpublished paper [8].
- It was then introduced independently under the name of *contraction* by the first author and Maltsiniotis in [3], in their study of homotopical properties of orientals.

2.1 Cones

2.1.1 Definition (ω-category of small cones)**.**
If o is a 0-cell in an ω-category C, which amounts to an ω-functor $o : \mathbf{1} \to C$ where $\mathbf{1}$ is the terminal ω-category, the *ω-category $\Lambda(C, o)$ of small cones of origin o in C* is given by the

following pullback square:

$$\begin{tikzcd} \Lambda(C,o) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma(C) \arrow[d, "\overline{\pi}_C"] \\ \mathbf{1} \arrow[r, "o"'] & C \end{tikzcd}$$

2.1.2 Remark. As the bottom arrow is a monomorphism, so is the top one, and in fact, $\Lambda(C,o)$ is a full subcategory of $\Gamma(C)$.

2.1.3. More concretely, an n-cell in $\Lambda(C,o)$ amounts to an n-cylinder $\alpha : 1_o^n \curvearrowright x$ in C, which is written $\alpha : o \curvearrowright x$ and called *n-cone of origin o*. The n-cell $\underline{\alpha} = x$ is called the *basis* of the cone α.

The formulas of paragraph 1.3.2 for sources of (auxiliary and principal) cells of cylinders are simpler in the case of cones. Indeed, an n-cone $\alpha : o \curvearrowright x$ is given by a finite sequence of cells

$$\alpha_0^-, \alpha_0^+, \ldots, \alpha_{n-1}^-, \alpha_{n-1}^+, \alpha_n$$

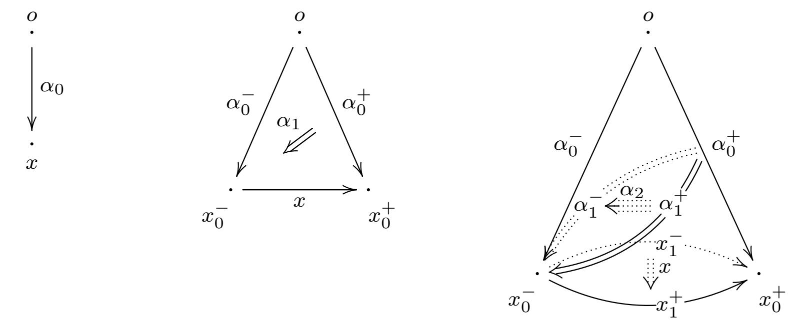
with the following sources and targets:

$$\alpha_0^\varepsilon : o \to x_0^\varepsilon, \qquad \alpha_i^\varepsilon : \alpha_{i-1}^+ \to x_i^\varepsilon *_0 \alpha_0^- *_1 \cdots *_{i-1} \alpha_{i-1}^- \text{ for } 0 < i < n,$$
$$|\alpha| = \alpha_0 : o \to x \text{ if } n = 0, \qquad |\alpha| = \alpha_n : \alpha_{n-1}^+ \to x *_0 \alpha_0^- *_1 \cdots *_{n-1} \alpha_{n-1}^- \text{ if } n > 0.$$

Note that the formulas for targets are unchanged, except that y is replaced by x.

2.1.4 Examples.

- If x is a 0-cell, a 0-cone $\alpha : o \curvearrowright x$ is given by the 1-cell $|\alpha| = \alpha_0 : o \to x$.
- If x is a 1-cell, a 1-cone $\alpha : o \curvearrowright x$ is given by the 1-cells $\alpha_0^\varepsilon : o \to x_0^\varepsilon$ and the 2-cell $|\alpha| = \alpha_1 : \alpha_0^+ \to x *_0 \alpha_0^-$.
- If x is a 2-cell, a 2-cone $\alpha : o \curvearrowright x$ is given by the 1-cells $\alpha_0^\varepsilon : o \to x_0^\varepsilon$, the 2-cells $\alpha_1^\varepsilon : \alpha_0^+ \to x_1^\varepsilon *_0 \alpha_0^-$, and the 3-cell $|\alpha| = \alpha_2 : \alpha_1^+ \to x *_0 \alpha_0^- *_1 \alpha_1^-$.



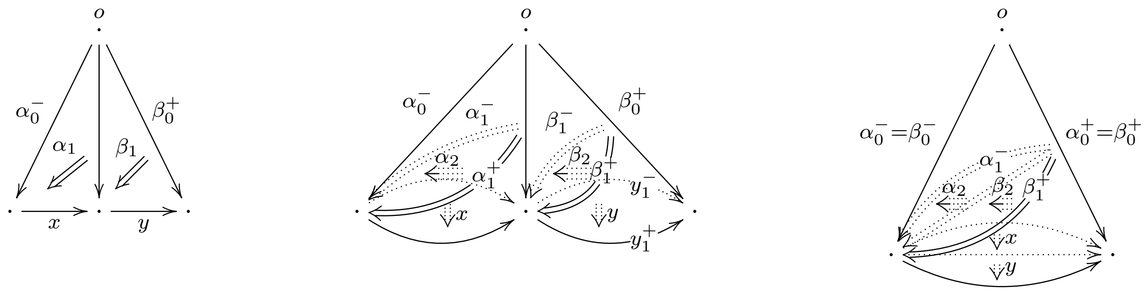
2.1.5. The formulas of definition 1.3.6 for p-composition are simpler in the case of cones. Indeed, if $n > p$, if x, y are n-cells such that $x_p^+ = y_p^-$, and if $\alpha : o \curvearrowright x, \beta : o \curvearrowright y$ are n-cones such that $\partial_p^+ \alpha = \partial_p^- \beta$, the p-composition $\gamma = \beta *_p \alpha : o \curvearrowright y *_p x$ is the n-cone given by the following cells:

$$\gamma_i^\varepsilon = \alpha_i^\varepsilon = \beta_i^\varepsilon \text{ for } i < p, \qquad \gamma_p^- = \alpha_p^-, \qquad \gamma_p^+ = \beta_p^+,$$
$$\gamma_{p+1}^\varepsilon = y_{p+1}^\varepsilon *_0 \alpha_0^- *_1 \cdots *_{p-1} \alpha_{p-1}^- *_p \alpha_i^\varepsilon *_{p+1} \beta_{p+1}^\varepsilon \text{ if } p+1 < n,$$
$$\gamma_i^\varepsilon = y_{p+1}^+ *_0 \alpha_0^- *_1 \cdots *_{p-1} \alpha_{p-1}^- *_p \alpha_i^\varepsilon *_{p+1} \beta_i^\varepsilon \text{ for } p+1 < i < n,$$
$$|\gamma| = \gamma_n = y_{p+1}^+ *_0 \alpha_0^- *_1 \cdots *_{p-1} \alpha_{p-1}^- *_p \alpha_n *_{p+1} \beta_n.$$

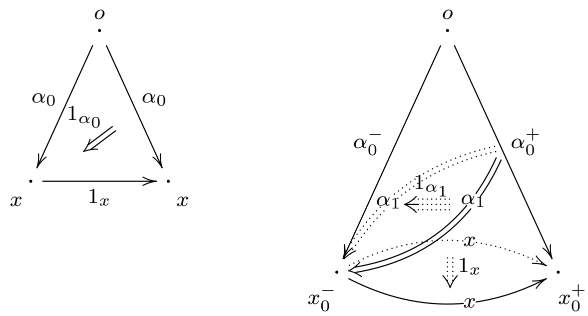
On the other hand, the formulas for p-units are unchanged.

2.1.6 Examples.

- If x, y are 1-cells such that $x_0^+ = y_0^-$, and if $\alpha : o \curvearrowright x, \beta : o \curvearrowright y$ are 1-cones such that $\alpha_0^+ = \beta_0^-$, the 0-composition $\gamma = \beta *_0 \alpha : o \curvearrowright y *_0 x$ is given by the 1-cells $\gamma_0^- = \alpha_0^-$ and $\gamma_0^+ = \beta_0^+$, and the 2-cell $\gamma_1 = y *_0 \alpha_1 *_1 \beta_1$.
- If x, y are 2-cells such that $x_0^+ = y_0^-$, and if $\alpha : o \curvearrowright x, \beta : o \curvearrowright y$ are 2-cones such that $\alpha_0^+ = \beta_0^-$, the 0-composition $\gamma = \beta *_0 \alpha : o \curvearrowright y *_0 x$ is given by the 1-cells $\gamma_0^- = \alpha_0^-$ and $\gamma_0^+ = \beta_0^+$, the 2-cells $\gamma_1^\varepsilon = y_1^\varepsilon *_0 \alpha_1^\varepsilon *_1 \beta_1^\varepsilon$, and the 3-cell $\gamma_2 = y_1^+ *_0 \alpha_2 *_1 \beta_2$.
- If x, y are 2-cells such that $x_1^+ = y_1^-$, and if $\alpha : o \curvearrowright x, \beta : o \curvearrowright y$ are 2-cones such that $\alpha_0^\varepsilon = \beta_0^\varepsilon$ and $\alpha_1^+ = \beta_1^-$, the 1-composition $\gamma = \beta *_1 \alpha : o \curvearrowright y *_1 x$ is given by the 1-cells $\gamma_0^\varepsilon = \alpha_0^\varepsilon = \beta_0^\varepsilon$, the 2-cells $\gamma_1^- = \alpha_1^-$ and $\gamma_1^+ = \beta_1^+$, and the 3-cell $\gamma_2 = y *_0 \alpha_0^- *_1 \alpha_2 *_2 \beta_2$.



- If $\alpha : o \curvearrowright x$ is a 0-cone, the 0-unit $1_\alpha : o \curvearrowright 1_x$ is given by the cells $\alpha_0, \alpha_0, 1_{\alpha_0}$.
- If $\alpha : o \curvearrowright x$ is a 1-cone, the 1-unit $1_\alpha : o \curvearrowright 1_x$ is given by the cells $\alpha_0^-, \alpha_0^+, \alpha_1, \alpha_1, 1_{\alpha_1}$.



2.1.7 Remark. The only trivial n-cylinder defining an n-cone of origin o is $\tau(1_o^n) : 1_o^n \curvearrowright 1_o^n$, but there is a weaker notion, which is used to define the notion of expansion:

2.1.8 Definition (degenerate cones)**.**
An n-cone $\alpha : o \curvearrowright x$ is called *degenerate* if its principal cell $|\alpha| = \alpha_n$ is a unit, as well as its *negative auxiliary cells* $\alpha_0^-, \ldots, \alpha_{n-1}^-$.

2.1.9 Remark. If $\alpha : o \curvearrowright x$ is a degenerate n-cone, then $x_0^- = o$, as α_0^- is an identity.

2.1.10 Lemma. *Let x be an n-cell such that $x_0^- = o$. There is a unique degenerate cone $\alpha : o \curvearrowright x$ of base x. This cone is determined by the following cells:*

$$\begin{aligned}
\alpha_i^- &= 1_{x_i^-} \quad \textit{for } 0 \le i \le n-1, \\
\alpha_i^+ &= \begin{cases} x_{i+1}^- & \textit{for } 0 \le i \le n-2, \\ x & \textit{for } i = n-1, \end{cases} \\
\alpha_n &= 1_x.
\end{aligned}$$

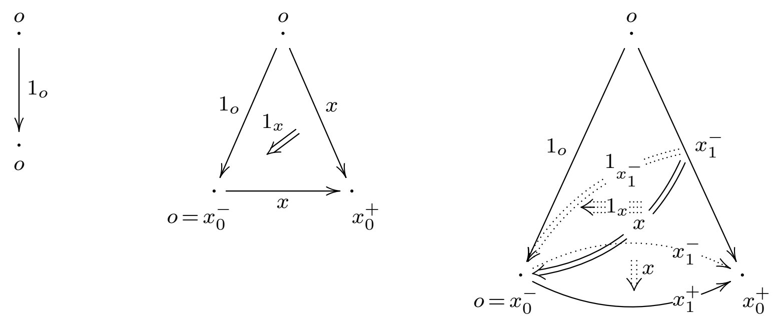
Proof. By induction on n.

- For $n = 0$, there is a unique degenerate cone $\alpha : o \curvearrowright x$ determined by $x = o$ and $\alpha_0 = 1_o$.
- Suppose that the statement holds up to dimension n and let us show that the it holds in dimension $n+1$. Thus, let x be an $(n{+}1)$-cell such that $x_0^- = o$. For the uniqueness part, suppose that $\alpha : o \curvearrowright x$ is a degenerate $(n{+}1)$-cone and consider the n-cone $\beta = \partial^- \alpha : o \curvearrowright x_n^-$. By 1.3.2, $\beta_i^- = \alpha_i^-$ for $0 \leq i \leq n-1$ and $\beta_n = \alpha_n^-$, so that all negative auxiliary cells of β are identities, as well as its principal cell β_n. By induction hypothesis, β is the unique degenerate n-cone of base x_n^-, and we know by 1.3.2 that $\alpha_i^\varepsilon = \beta_i^\varepsilon$ for $0 \leq i \leq n-1$ and $\varepsilon = \pm$. Now by 2.1.3 the principal cell of α is

$$\alpha_{n+1} : \alpha_n^+ \to x *_0 \alpha_0^- *_1 \cdots *_n \alpha_n^-$$

and because α is degenerate, the right-hand side of the previous formula is just x, whereas α_{n+1} is an identity so that $\alpha_n^+ = x$ and $\alpha_{n+1} = 1_x$. This proves uniqueness. Finally, the above formulas for α_i^ε, $0 \leq i \leq n$, and α_{n+1} satisfy the relations of 2.1.3 and define a degenerate $(n{+}1)$-cone. $\square$

2.1.11 Examples. For $n = 0, 1, 2$, we get the following degenerate cones:



2.1.12. We write $\mathbf{Cat}_{\omega,*}$ for the *category of pointed ω-categories*:

- An object is a pair (C, o) where C is an ω-category, and o is a 0-cell in C, called *origin*.
- A morphism $f : (C, o) \to (D, o)$ is an ω-functor $f : C \to D$ preserving the origin.

2.1.13. For any such morphism, we get two ω-functors $f : C \to D$ and $\Gamma(f) : \Gamma(C) \to \Gamma(D)$, which induce an ω-functor $\Lambda(f) : \Lambda(C, o) \to \Lambda(D, o)$ by the pullback square of definition 2.1.1. Hence, we get a functor $\Lambda : \mathbf{Cat}_{\omega,*} \to \mathbf{Cat}_\omega$.

The natural transformation $\underline{\pi} : \Gamma \to \mathrm{id}_{\mathbf{Cat}_\omega}$ induces a natural transformation $\pi : \Lambda \to \Pi$, where $\Pi : \mathbf{Cat}_{\omega,*} \to \mathbf{Cat}_\omega$ stands for the forgetful functor. This means that, for any pointed ω-category (C, o), we get an ω-functor $\pi_{(C,o)} : \Lambda(C, o) \to C$, which maps any cone to its basis. In practice, we shall simply write π_C for this ω-functor.

2.1.14. The ω-category of small cones can be used to describe particular oplax transformations. Indeed, by the pullback square of definition 2.1.1, if C, D are ω-categories and o is a 0-cell in D, an oplax transformation from the constant ω-functor $o : C \to D$ to another ω-functor $f : C \to D$ amounts to an ω-functor $\theta : C \to \Lambda(D, o)$ making the following triangle commutative:

$$\begin{tikzcd} C \arrow[r, "\theta"] \arrow[dr, "f"'] & \Lambda(D, o) \arrow[d, "\pi_D"] \\ & D \end{tikzcd}$$

In particular, an oplax transformation from the constant ω-functor $o : C \to C$ to the identity ω-functor $\mathrm{id}_C : C \to C$ amounts to a section $\theta : C \to \Lambda(C, o)$ of $\pi_C : \Lambda(C, o) \to C$.

2.1.15. By the formulas of paragraph 1.3.12, such an oplax transformation amounts to the data of an $(n{+}1)$-cell θ_x for each n-cell x in C, with the following source and target:

$$\theta_x : o \to x \text{ for } n = 0, \qquad \theta_x : \theta_{x^+_{n-1}} \to x *_0 \theta_{x^-_0} *_1 \cdots *_{n-1} \theta_{x^-_{n-1}} \text{ for } n > 0,$$

such that the following two axioms hold for $n > p$, for any n-cells x, y such that $x^+_p = y^-_p$, and for any cell u:

$$\theta_{y *_p x} = y^+_{p+1} *_0 \theta_{x^-_0} *_1 \cdots *_{p-1} \theta_{x^-_{p-1}} *_p \theta_x *_{p+1} \theta_y, \qquad \theta_{1_u} = 1_{\theta_u}.$$

2.2 Expansion monad

2.2.1 Definition (expansions)**.**
If C is an ω-category, an *expansion* on C consists of:

- a 0-cell o in C, called *origin*,
- a section $\xi : C \to \Lambda(C, o)$ of the ω-functor $\pi_C : \Lambda(C, o) \to C$, called *expanding homotopy*,

such that the cone $\xi(x)$ is degenerate whenever x is of the form o or $|\xi(u)|$ for some u.

2.2.2. By paragraph 2.1.14, ξ is an oplax transformation from the constant ω-functor $o : C \to C$ to the identity ω-functor $1_C : C \to C$. More concretely, it amounts by paragraph 2.1.15 to the data of an $(n{+}1)$-cell ξ_x for each n-cell x in C, with the following source and target:

$$\xi_x : o \to x \text{ for } n = 0, \qquad \xi_x : \xi_{x^+_{n-1}} \to x *_0 \xi_{x^-_0} *_1 \cdots *_{n-1} \xi_{x^-_{n-1}} \text{ for } n > 0,$$

such that the following four axioms hold for $n > p$, for any n-cells x, y such that $x^+_p = y^-_p$, and for any cell u:

$$\xi_{y *_p x} = y^+_{p+1} *_0 \xi_{x^-_0} *_1 \cdots *_{p-1} \xi_{x^-_{p-1}} *_p \xi_x *_{p+1} \xi_y, \qquad \xi_{1_u} = 1_{\xi_u}, \qquad \xi_{\xi_u} = 1_{\xi_u}, \qquad \xi_o = 1_o.$$

We will sometimes call the first two axioms the *functoriality conditions* and the last two axioms the *degeneracy conditions*.

2.2.3. We write $\mathbf{Cat}_{\omega,e}$ for the *category of ω-categories with expansion*:

- An object is a triple (C, o, ξ), where C is an ω-category, and o, ξ define an expansion on C.
- A morphism $f : (C, o, \xi) \to (D, o, \xi)$ is an ω-functor $f : C \to D$ preserving the structure, which means that $f(o) = o$ and the following square commutes:

$$\begin{tikzcd} C \arrow[r, "\xi"] \arrow[d, "f"'] & \Lambda(C, o) \arrow[d, "\Lambda(f)"] \\ D \arrow[r, "\xi"'] & \Lambda(D, o) \end{tikzcd}$$

2.2.4 Proposition. *The obvious forgetful functor $U : \mathbf{Cat}_{\omega,e} \to \mathbf{Cat}_\omega$ admits a left adjoint.*

Proof. This follows from the fact that our structures are "equational" in the sense of the theory of sketches. See for instance [1] for an introduction to this theory. See also Remark 2.4.4 for a more concrete proof of the existence of this left adjoint.

Following the usual definition of ω-categories, $\mathbf{Cat}_\omega$ is indeed sketchable by a limit sketch Σ:

- Objects are the following symbols:

$$C_n \text{ for } n \geq 0, \qquad C_n \times_{C_p} C_n \text{ and } C_n \times_{C_p} C_n \times_{C_p} C_n \text{ for } n > p \geq 0,$$
$$(C_n \times_{C_p} C_n) \times_{C_q} (C_n \times_{C_p} C_n) \text{ for } n > p > q \geq 0.$$

- Generators (for morphisms) are given by sources and targets, compositions and units.
- Relations are given by laws of associativity, unit, interchange and functoriality of units.
- Distinguished cones are suggested by the notation of objects.

Similarly, the equational definition of paragraph 2.2.2, produces a limit sketch Σ_e whose models are ω-categories with expansion. More precisely, Σ_e is obtained by adding to Σ the object 1 as well as suitable generators, relations and distinguished cones.

Now, the canonical inclusion of Σ into Σ_e defines a morphism of sketches $\iota : \Sigma \hookrightarrow \Sigma_e$, and the induced functor $\mathrm{Mod}(\iota) : \mathrm{Mod}(\Sigma_e) \to \mathrm{Mod}(\Sigma)$ is $U : \mathbf{Cat}_{\omega,e} \to \mathbf{Cat}_\omega$. By [12, Lemma p. 6], this implies that U admits a left adjoint. □

2.2.5. We write $F : \mathbf{Cat}_\omega \to \mathbf{Cat}_{\omega,e}$ for this left adjoint, $T = UF : \mathbf{Cat}_\omega \to \mathbf{Cat}_\omega$ for the induced *expansion monad*, $\mu : T^2 \to T$ for its multiplication, and $\eta : \mathrm{id}_{\mathbf{Cat}_\omega} \to T$ for its unit.

It happens that the algebras of this monad are precisely the ω-categories with expansion:

2.2.6 Proposition. *The forgetful functor $U : \mathbf{Cat}_{\omega,e} \to \mathbf{Cat}_\omega$ is monadic.*

Proof. We use again the sketches Σ, Σ_e and the morphism $\iota : \Sigma \hookrightarrow \Sigma_e$ introduced in the proof of Proposition 2.2.4. This morphism has the following properties:

- The base of any distinguished cone of Σ_e factors though ι.
- Every object of Σ_e not reached by Σ (namely only 1) is the tip of a distinguished cone.

It thus fulfills the hypothesis of [12, Corollary 1] and it follows that U is monadic. □

2.3 Cosimplicial object of orientals

2.3.1. We write Δ for the *simplex category*:

- Its objects are the ordered sets $\Delta_n = \{0 < 1 < \cdots < n\}$ for $n \geq 0$.
- Its morphisms are the order-preserving maps.

The morphisms of Δ are generated by

$$\delta_i^n : \Delta_{n-1} \to \Delta_n, \text{ for } n > 0 \text{ and } 0 \leq i \leq n, \qquad \sigma_i^n : \Delta_{n+1} \to \Delta_n, \text{ for } n \geq 0 \text{ and } 0 \leq i \leq n,$$

where δ_i^n is the unique order-preserving injection such that the preimage of $\{i\}$ is empty, and σ_i^n is the unique order-preserving surjection such that the preimage of $\{i\}$ has two elements.

2.3.2. Similarly, we write Δ_+ for the *augmented simplex category*:

- Its objects are those of Δ, plus an additional one: $\Delta_{-1} = \varnothing$.
- Its morphisms are again the order-preserving maps.

By definition, Δ is a full subcategory of Δ_+. Moreover, the morphisms of Δ_+ are generated by the generating morphisms of Δ, plus an additional one: $\delta_0^0 : \Delta_{-1} \to \Delta_0$.

2.3.3. Recall that Δ_+ is the universal monoidal category endowed with a monoid object. More precisely, the disjoint union $\Delta_m \amalg \Delta_n = \Delta_{m+1+n}$ defines a strict monoidal structure on Δ_+, with unit $\varnothing = \Delta_{-1}$, and Δ_0 is endowed with a unique structure of monoid for this monoidal structure:

$$\sigma_0^0 : \Delta_0 \amalg \Delta_0 = \Delta_1 \to \Delta_0, \qquad \delta_0^0 : \varnothing = \Delta_{-1} \to \Delta_0.$$

The universal property of Δ_+ can then be expressed as follows (see for instance [13, Chapter VII, Section 5]):

2.3.4 Lemma. *For any monoid object M in a strict monoidal category $\mathcal{C}$, there exists a unique strict monoidal functor $\Phi : \Delta_+ \to \mathcal{C}$ sending the monoid Δ_0 to the monoid M.*

2.3.5. In particular, a monad on a category $\mathcal{C}$ amounts to a monoid object in the strict monoidal category $\mathrm{End}(\mathcal{C})$ of endofunctors on $\mathcal{C}$. Hence, for any such monad T, we get a canonical functor $c_T : \Delta_+ \to \mathrm{End}(\mathcal{C})$, which is given as follows on objects and on generators:

$$c_T(\Delta_n) = T^{n+1}, \qquad c_T(\delta_i^n) = T^{n-i}\eta T^i : T^n \to T^{n+1}, \qquad c_T(\sigma_i^n) = T^{n-i}\mu T^i : T^{n+2} \to T^{n+1}.$$

2.3.6 Definition (orientals)**.**
The *augmented cosimplicial object of orientals* $\mathcal{O}_+ : \Delta_+ \to \mathbf{Cat}_\omega$ is the composition

$$\Delta_+ \xrightarrow{c_T} \mathrm{End}(\mathbf{Cat}_\omega) \xrightarrow{\mathrm{ev}_\varnothing} \mathbf{Cat}_\omega$$

where c_T is given in the previous paragraph and $\mathrm{ev}_\varnothing$ is the evaluation functor at $\varnothing$.

By restricting $\mathcal{O}_+$ to Δ, we get the *cosimplicial object of orientals* $\mathcal{O} : \Delta \to \mathbf{Cat}_\omega$, and for $n \geq 0$, the *n-th oriental* is $\mathcal{O}_n = \mathcal{O}(\Delta_n)$.

2.3.7. Explicitly, we have:

$$\mathcal{O}_n = T^{n+1}(\varnothing), \qquad \mathcal{O}(\delta_i^n) = T^{n-i}\eta T^i(\varnothing) : \mathcal{O}_{n-1} \to \mathcal{O}_n, \quad \mathcal{O}(\sigma_i^n) = T^{n-i}\mu T^i(\varnothing) : \mathcal{O}_{n+1} \to \mathcal{O}_n.$$

In the remainder of the paper, we will describe explicitly this "abstract" cosimplicial object of orientals and show that it corresponds to the classical one defined by Street in [18].

2.4 Free expansion on a polygraph We know from Proposition 2.2.4 that the forgetful functor $U : \mathbf{Cat}_{\omega,e} \to \mathbf{Cat}_\omega$ admits a left adjoint $F : \mathbf{Cat}_\omega \to \mathbf{Cat}_{\omega,e}$ taking an ω-category C to the *free expansion* $FC = (TC, o, \xi)$ on C. We now present a concrete description of FC in the particular case where the ω-category C is freely generated by a polygraph.

2.4.1. Let S be a polygraph and $C = S^*$ the free ω-category generated by S. We shall define an ω-category $C^\triangleleft$, freely generated by a polygraph, together with

- an inclusion morphism $\eta : C \to C^\triangleleft$,
- a distinguished 0-cell o of $C^\triangleleft$,
- a morphism $\xi : C^\triangleleft \to \Lambda(C^\triangleleft, o)$,

in such a way that, eventually, $(C^\triangleleft, o, \xi)$ becomes an ω-category with expansion, and in fact coincides with FC.

We now define $C^\triangleleft$, η and ξ by simultaneous induction on the dimension. In each dimension n, $C^\triangleleft$ will be defined by freely adjoining a set R_n of new n-generators to those of C, together with source and target maps from R_n to $C^\triangleleft_{n-1}$. The morphism η will be induced by the natural inclusion of generators $S_n \to S_n \amalg R_n$. Throughout the construction, we abuse notations by identifying any n-cell $x \in C_n$ with $\eta x \in C_n^{\triangleleft}$.

- For $n = 0$, R_0 is a singleton and $\eta : C_0 \to C_0^{\triangleleft}$ is the natural inclusion $S_0 \to S_0 \amalg R_0$. The unique element of R_0 is denoted by o and eventually becomes the distinguished 0-cell of $C^{\triangleleft}$.
- For $n = 1$, to each $a \in S_0$ corresponds a generator $r_a \in R_1$ such that $r_a : o \to a$. Now to any generator $a : u \to v$ in S_1 correspond source and target cells u and v in $C_0 \subset C_0^{\triangleleft}$. Therefore the 1-cells in $C_1^{\triangleleft}$ are defined as freely generated by $S_1 \amalg R_1$, and $\eta : C_1 \to C_1^{\triangleleft}$ is induced by the natural inclusion $S_1 \to S_1 \amalg R_1$.
  Moreover the 0-cells of $\Lambda(C^{\triangleleft}, o)$ are now defined as the 0-cones of origin o in $C^{\triangleleft}$, that is, the 1-cells u of $C^{\triangleleft}$ with $u_0^- = o$. Finally $\xi : C_0^{\triangleleft} \to \Lambda(C^{\triangleleft}, o)_0$ is defined by
  - $\xi(a) = r_a$ for each $a \in S_0$,
  - $\xi(o) = 1_o$.

  In particular, the first degeneracy condition holds.
- Let $n \geq 1$ and suppose we have defined $C^{\triangleleft}$ together with a morphism $\eta : C \to C^{\triangleleft}$, up to dimension n, as well as an expanding homotopy $\xi : C^{\triangleleft} \to \Lambda(C^{\triangleleft}, o)$ up to dimension $n-1$. So we get the following diagram:

$$\begin{tikzcd}
C_{n-1} \arrow[d, "\eta"'] & C_n \arrow[l, shift left] \arrow[l, shift right] \arrow[d, "\eta"] \\
C_{n-1}^{\triangleleft} \arrow[d, "\xi"'] & C_n^{\triangleleft} \arrow[l, shift left] \arrow[l, shift right] \\
\Lambda(C^{\triangleleft}, o)_{n-1} \arrow[ur, "|-|"'] &
\end{tikzcd}$$

  In addition, we suppose that $C_n^{\triangleleft}$ is freely generated by the set $S_n \amalg R_n$ where

$$R_n = \{\xi_a \mid a \in S_{n-1}\}.$$

  We must now extend $C^{\triangleleft}$ together with $\eta : C \to C^{\triangleleft}$ up to dimension $n+1$, and the expanding homotopy ξ up to dimension n. The $(n{+}1)$-generators of $C^{\triangleleft}$ are twofold:
  - each $a : u \to v$ in S_{n+1} becomes an $(n{+}1)$-generator of $C^{\triangleleft}$ with source and target u and v in $C_n \subset C_n^{\triangleleft}$;
  - to each $a : u \to v$ in S_n corresponds a new generator r_a in R_{n+1}. By induction hypothesis, $x = \xi(u)$ and $y = \xi(v)$ are two parallel $(n{-}1)$-cones of origin o in $C^{\triangleleft}$ such that $\pi x = u$ and $\pi y = v$. Therefore, by 2.1.3, we may define the source and target of r_a by

$$r_a^- = \xi_{a_{n-1}^+} = \xi_v,$$
$$r_a^+ = a *_0 \xi_{a_0^-} *_1 \cdots *_{n-2} \xi_{a_{n-2}^-} *_{n-1} \xi_{a_{n-1}^-} = a *_0 \xi_{u_0^-} *_1 \cdots *_{n-2} \xi_{u_{n-2}^-} *_{n-1} \xi_u.$$

  Thus $C_{n+1}^{\triangleleft}$ is defined as the set of freely generated $(n{+}1)$-cells over $S_{n+1} \amalg R_{n+1}$, and the natural inclusion $S_{n+1} \to S_{n+1} \amalg R_{n+1}$ induces $\eta : C_{n+1} \to C_{n+1}^{\triangleleft}$. Now, having defined $C^{\triangleleft}$ up to dimension $n+1$, the n-cones in $\Lambda(C^{\triangleleft}, o)$ are determined and the remaining task is to define $\xi : C_n^{\triangleleft} \to \Lambda(C^{\triangleleft}, o)_n$. By Lemma 1.2.3, it is sufficient to define ξ on the generators of $C_n^{\lhd}$, that is, on the elements of $S_n \amalg R_n$, provided the commutation conditions for source and target are satisfied. There are two cases to consider:

- if $a : u \to v$ is in S_n, we have defined $r_a \in R_{n+1}$ in such a way that r_a is the principal cell of an n-cone $z : x \to y$ where $x = \xi(u)$, $y = \xi(v)$ and $\pi z = a$. Therefore $\partial^\varepsilon z = \xi(\partial^\varepsilon a)$ for $\varepsilon = \pm$ so that we may define $\xi(a) = z$;
- if $a \in R_n$, by induction hypothesis, a is of the form $r_b = \xi_b$ for some $b \in S_{n-1}$. Therefore $a_0^- = o$ and by 2.1.10, we may define $\xi(a)$ as the unique degenerate cone of base a. If $n = 1$, $\xi(\partial^- a) = \xi(o) = 1_o = \partial^- \xi(a)$ and $\xi(\partial^+ a) = \xi(b) = \partial^+ \xi(a)$, which entails compatibility with source and target. If $n \geq 2$, then

$$\begin{aligned}\partial^- a &= r_b^- = \xi_{b_{n-2}^+},\\ \partial^+ a &= r_b^+ = b *_0 \xi_{b_0^-} *_1 \cdots *_{n-2} \xi_{b_{n-2}^-}.\end{aligned}$$

  By induction, applying ξ to the above equations, and using the degeneracy conditions up to dimension n gives

$$\begin{aligned}\xi(\partial^- a) &= \partial^- \xi(a),\\ \xi(\partial^+ a) &= \xi(b) *_0 \xi(\xi_{b_0^-}) *_1 \cdots *_{n-2} \xi(\xi_{b_{n-2}^-}),\end{aligned}$$

  and it remains to show that $\partial^+ \xi(a) = \xi(\partial^+ a)$. First, both cones have the same base $\partial^+ a$, and by induction, for $\varepsilon = \pm$,

$$\partial^\varepsilon \xi(\partial^+ a) = \xi(\partial^\varepsilon \partial^+ a) = \xi(\partial^\varepsilon \partial^- a) = \partial^\varepsilon \xi(\partial^- a) = \partial^\varepsilon \partial^- \xi(a) = \partial^\varepsilon \partial^+ \xi(a),$$

  so that both cones have same source and target. Finally, the principal cell of $\partial^+ \xi(a)$ is just a by Lemma 2.1.10, whereas the principal cell of $\xi(\partial^+ a)$ is the one of

$$\xi(b) *_0 \xi(\xi_{b_0^-}) *_1 \cdots *_{n-2} \xi(\xi_{b_{n-2}^-})$$

  by the above formula. As all terms $|\xi(\xi_{b_i^-})|$ are identities, one gets $|\xi(\partial^+ a)| = |\xi(b)| = a$ and we are done.

By construction, R_{n+1} consists in generators of the form ξ_a where $a \in S_n$, and for all $a \in S_{n+1} \amalg R_{n+1}$, $\pi \xi a = a$. Now, for each $u \in C_{n-1}^{\lhd}$, $\xi_{\xi u} = 1_{\xi_u}$: this holds by construction for generators and extends to any $(n-1)$-cell by the functoriality conditions of 2.2.2. Therefore ξ satisfies the expanding homotopy conditions up to dimension n, which completes the induction step.

2.4.2. To sum up, given C a free ω-category on a polygraph S, we have defined an ω-category $C^{\lhd}$, endowed with a distinguished 0-cell $o : 1 \to C^{\lhd}$ and an expanding homotopy $\xi : C^{\lhd} \to \Lambda(C^{\lhd}, o)$. By construction, $C^{\lhd}$ is itself free on a polygraph whose set of n-generators is $S_n \amalg R_n$, where

$$R_0 = \{o\} \qquad \text{and} \qquad R_n = \{r_a \mid a \in S_{n-1}\} \quad \text{for } n \geq 1.$$

The sources and the targets of the generators in S_n are inherited by those of S and, if a is in S_{n-1}, for $n \geq 1$, the source and the target of r_a are

$$r_a^- = \xi_{a_{n-1}^+} \qquad \text{and} \qquad r_a^+ = a *_0 \xi_{a_0^-} *_1 \cdots *_{n-1} \xi_{a_{n-1}^-}$$

and the expanding homotopy is defined by

$$\xi_a = r_a, \qquad \xi_o = 1_o \qquad \text{and} \qquad \xi_{r_a} = 1_{r_a}.$$

We will denote this polygraph by $S^{\lhd}$, so that $C^{\lhd} = S^{\lhd *}$.

2.4.3 Proposition. *For any polygraph S, the left adjoint $F : \mathbf{Cat}_\omega \to \mathbf{Cat}_{\omega,e}$ to the functor $U : \mathbf{Cat}_{\omega,e} \to \mathbf{Cat}_\omega$ takes $C = S^*$ to $(C^{\lhd}, o, \xi)$, and therefore the expansion monad T takes $C = S^*$ to $C^{\lhd} = S^{\lhd *}$.*

Proof. Let (D, o, ξ) be an ω-category with expansion, and $f : S^* \to D$ a morphism in $\mathbf{Cat}_\omega$. We have to show that there is a unique morphism

$$f^* : (C^{\lhd}, o, \xi) \to (D, o, \xi)$$

in $\mathbf{Cat}_{\omega,e}$ such that the following diagram commutes:

$$\begin{tikzcd} S^* \arrow[d, "\eta"'] \arrow[dr, "f"] & \\ C^{\lhd} \arrow[r, "f^*"'] & D \end{tikzcd}$$

Note that, by abuse of notation, we identify here f^* with Uf^*, as Uf^* entirely determines f^*, as soon as it commutes with the origins and the expanding homotopies. The construction is by induction on the dimension.

- In dimension $n = 0$, $S_0^{\lhd} = S_0 \amalg \{o\}$, and we define $f^*a = fa$ if $a \in S_0$ and $f^*a = o$ if $a = o$. This is clearly the only possible choice.
- In dimension $n = 1$, $S_1^{\lhd} = S_1 \amalg R_1$. If $a \in S_1$, we set $f^*a = fa$, the commutation with source and target being straightforward. If $a \in R_1$, $a = r_b = \xi_b$ for some $b \in S_0$, and we define $f^*a = \xi_{fb}$, which again ensures commutation with source and target, and defines f^* up to dimension 1, in the unique possible way. Moreover, for each $u \in C_0^{\lhd}$, $\xi(f^*u) = f^*(\xi u)$. In fact, either $u = o$, in which case $\xi_{f^*o} = \xi_o = 1_o = f^*(1_o) = f^*(\xi_o)$, or $u \in S_0$, in which case $\xi_{f^*u} = f^*(r_u) = f^*(\xi_u)$ by definition.
- Let $n \geq 1$ and suppose that, up to dimension n, we have defined a morphism $f^* : C^{\lhd} \to D$ such that $f^* \circ \eta = f$ and f^* commutes to the expanding homotopies up to dimension $n{-}1$. We extend f^* to dimension $n + 1$ by defining first f^*a when $a \in S_{n+1}^{\lhd}$. As $S_{n+1}^{\lhd} = S_{n+1} \amalg R_{n+1}$, there are two cases to consider:
  - if $a \in S_{n+1}$, we take $f^*a = fa$, and the commutation with source and target is straightforward. The condition $f^* \circ \eta = f$ implies that this choice is unique;
  - if $a \in R_{n+1}$, a is of the form r_b with $b \in S_n$. By induction hypothesis, we already have an n-cell $f^*b \in D_n$. Now D is endowed with an expanding homotopy ξ so that we get an n-cone $x = \xi(f^*b)$, whose principal cell $u = |x|$ is an $(n{+}1)$-cell of D. Thus, we may define $f^*a = u$. The construction of r_b given in 2.4.1 ensures the commutation with source and target. Moreover, because f^* must commute to the expanding homotopies, the above choice for f^*r_b is unique.

  By Lemma 1.2.3, the above values determine a unique extension of the morphism f^* in dimension $n + 1$. It remains to check that the morphism f^* so defined actually commutes to expanding homotopies up to dimension n, that is, $\xi(f^*u) = f^*\xi(u)$ for all $u \in C_n^{\lhd}$.

By functoriality, it suffices to check this commutation on generators. Thus, if $a \in S_n$, $\xi_a = r_a \in R_{n+1}$ and $f^*\xi_a = \xi_{f^*a}$ by definition. If $a \in R_n$, $a = r_b = \xi_b$ for some $b \in S_{n-1}$, and the degeneracy conditions together with the induction hypothesis yield

$$\xi_{f^*a} = \xi_{f^*\xi_b} = \xi_{\xi_{f^*b}} = 1_{\xi_{f^*b}} = 1_{f^*\xi_b} = f^*1_{\xi_b} = f^*\xi_{\xi_b} = f^*\xi_a. \qquad \square$$

2.4.4 Remark. Proposition 2.2.4, whose proof is based on an abstract argument using the theory of sketches, states that the forgetful functor $U : \mathbf{Cat}_{\omega,e} \to \mathbf{Cat}_\omega$ admits a left adjoint. Proposition 2.4.3 gives an alternate proof. Indeed, it shows that this forgetful functor admits a left adjoint relative to the subcategory of ω-categories freely generated by a polygraph. As this subcategory contains a small dense subcategory (for instance, the category of globular pasting schemes indexing operations of ω-categories), this implies that the forgetful functor admits a left adjoint (provided that we know that the category $\mathbf{Cat}_{\omega,e}$ is cocomplete).

2.4.5 Corollary. *For each $n \geq 0$, the n-th oriental $\mathcal{O}_n$ is a free ω-category on a polygraph.*

Proof. By induction on n. As $\mathcal{O}_0$ is the terminal ω-category, it is freely generated by the polygraph having a single 0-generator and no generator of higher dimensions. Let $n \geq 0$ and suppose $\mathcal{O}_n$ is free on a polygraph. By definition, $\mathcal{O}_{n+1} = T(\mathcal{O}_n) = \mathcal{O}_n^{\triangleleft}$, which is again free on a polygraph by Proposition 2.4.3. $\square$

2.4.6 Proposition. *Let $C = S^*$, where S is a polygraph, and let $f : C \to D$ be an ω-functor. Then the action of the ω-functor $Tf : TC \to TD$ on generators is given by*

$$(Tf)(a) = \eta f(a) \quad \text{for } a \text{ in } S_n \text{ with } n \geq 0,$$

$$(Tf)(o) = o \qquad \text{and} \qquad (Tf)(r_a) = \xi_{\eta f(a)} \quad \text{for } a \in S_{n-1} \text{ with } n \geq 1,$$

where o and ξ respectively denote the origin and the expanding homotopy in TC and TD.

Proof. Let $g = \eta f : S^* \to TD$. By naturality of η, the following diagram commutes:

$$\begin{tikzcd} S^* \arrow[r, "f"] \arrow[d, "\eta"'] \arrow[dr, "g"] & D \arrow[d, "\eta"] \\ TC \arrow[r, "Tf"'] & TD \end{tikzcd}$$

Therefore, by Proposition 2.4.3, Tf is the unique ω-morphism commuting to origins and expanding homotopies making the bottom-left triangle commute, and the equations follow from the description of $S^{\triangleleft}$. $\square$

By Proposition 2.4.3, if C is freely generated by a polygraph, then the unit of the expansion monad is given by the morphism $\eta : C \to C^{\triangleleft}$ of paragraph 2.4.1. We end the section by a description of the multiplication of the monad:

2.4.7 Proposition. *Let C be an ω-category freely generated by a polygraph. Then the multiplication of the expansion monad $\mu : T^2C \to TC$ is the ω-functor defined on generators by*

$$\mu(\eta x) = x, \qquad \mu(o) = o \qquad \text{and} \qquad \mu(\xi_{\eta x}) = \xi_x,$$

where x is a generator of $TC = C^{\triangleleft}$.

Proof. The first equation holds for any monad. As for the other ones, they follow from the fact that $\mu = U\varepsilon F$ is induced by a morphism of ω-categories with expansion. $\square$

3. Oriental calculus

3.1 Syntax for expansion We consider an ω-category with expansion (C, o, ξ).

3.1.1. For any n-cell x in C, we write $\langle x\rangle$ for the $(n{+}1)$-cell ξ_x defined in paragraph 2.2.2, which we call *chevron* of x.

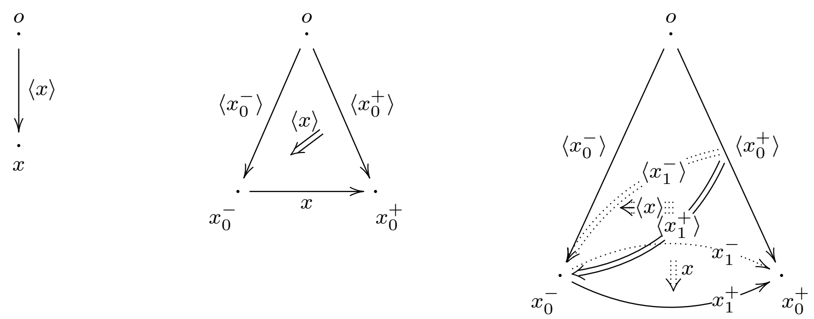
- It has the following source and target:

$$\langle x\rangle : o \to x \text{ if } n = 0, \qquad \langle x\rangle : \langle x^+_{n-1}\rangle \to x *_0 \langle x^-_0\rangle *_1 \cdots *_{n-1} \langle x^-_{n-1}\rangle \text{ if } n > 0.$$

- It is the principal cell of the n-cone $\xi(x) : o \curvearrowright x$ given by the following cells:

$$\langle x^-_0\rangle, \langle x^+_0\rangle, \ldots, \langle x^-_{n-1}\rangle, \langle x^+_{n-1}\rangle, \langle x\rangle.$$

3.1.2 Examples. Starting from $\langle x\rangle : o \to x$ for $n = 0$, we get $\langle x\rangle : \langle x^+_0\rangle \to x *_0 \langle x^-_0\rangle : o \to x^+_0$ for $n = 1$, and $\langle x\rangle : \langle x^+_1\rangle \to x *_0 \langle x^-_0\rangle *_1 \langle x^-_1\rangle : \langle x^+_0\rangle \to x^+_1 *_0 \langle x^-_0\rangle : o \to x^+_0$ for $n = 2$.



3.1.3. The four axioms of paragraph 2.2.2 can be rewritten as follows for any $n > p$, for any n-cells x, y such that $x^+_p = y^-_p$, and for any cell u:

$$\langle y *_p x\rangle = y^+_{p+1} *_0 \langle x^-_0\rangle *_1 \cdots *_{p-1} \langle x^-_{p-1}\rangle *_p \langle x\rangle *_{p+1} \langle y\rangle, \quad \langle 1_u\rangle = 1_{\langle u\rangle}, \quad \langle\langle u\rangle\rangle = 1_{\langle u\rangle}, \quad \langle o\rangle = 1_o.$$

3.1.4 Remarks.

- In case $n = p + 1$, we get $y^+_{p+1} = y$, so that our first axiom can be rewritten as follows:

$$\langle y *_p x\rangle = y *_0 \langle x^-_0\rangle *_1 \cdots *_{p-1} \langle x^-_{p-1}\rangle *_p \langle x\rangle *_{p+1} \langle y\rangle.$$

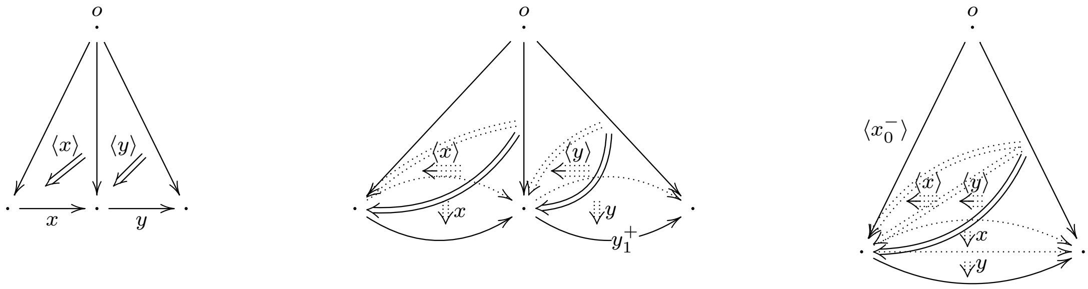
- The last axiom has a single occurrence:

$$\begin{tikzcd} o \arrow[d, "\langle o\rangle = 1_o"] \\ o \end{tikzcd}$$

  If we write $o = \langle *\rangle$ where $*$ is an extra cell of dimension -1, this axiom becomes a particular case of the previous one : $\langle\langle *\rangle\rangle = 1_{\langle * \rangle}$. We shall not introduce such a cell but we shall use a similar idea in our *simplicial notation* for generators of orientals.

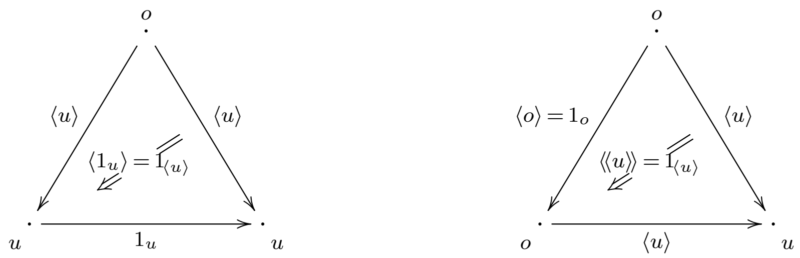
3.1.5 Examples.

- If x, y are 1-cells such that $x_0^+ = y_0^-$, we get $\langle y *_0 x\rangle = y *_0 \langle x\rangle *_1 \langle y\rangle$.
- If x, y are 2-cells such that $x_0^+ = y_0^-$, we get $\langle y *_0 x\rangle = y_1^+ *_0 \langle x\rangle *_1 \langle y\rangle$.
- If x, y are 2-cells such that $x_1^+ = y_1^-$, we get $\langle y *_1 x\rangle = y *_0 \langle x_0^-\rangle *_1 \langle x\rangle *_2 \langle y\rangle$.



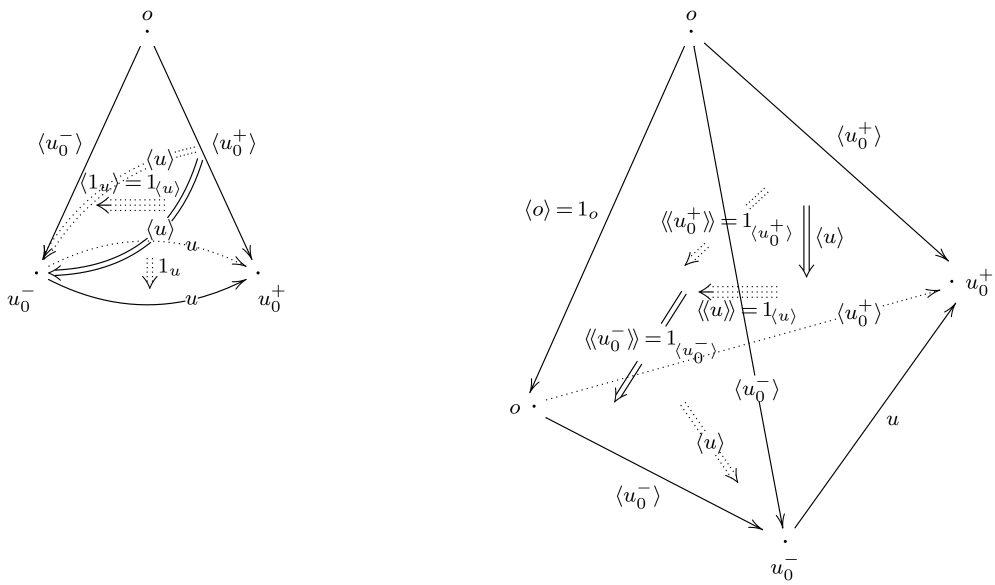
- If u is a 0-cell, we have $1_u : u \to u$ and $\langle u\rangle : o \to u$. So we get

$$\langle 1_u\rangle = 1_{\langle u\rangle} : \langle u\rangle \to 1_u *_0 \langle u\rangle = \langle u\rangle \text{ and } \langle\!\langle u\rangle\!\rangle = 1_{\langle u\rangle} : \langle u\rangle \to \langle u\rangle *_0 \langle o\rangle = \langle u\rangle *_0 1_o = \langle u\rangle.$$



- If u is a 1-cell, we have $1_u : u \to u : u_0^- \to u_0^+$ and $\langle u\rangle : \langle u_0^+\rangle \to u *_0 \langle u_0^-\rangle : o \to u_0^+$. So we get $\langle 1_u\rangle = 1_{\langle u\rangle} : \langle u\rangle \to 1_u *_0 \langle u_0^-\rangle *_1 \langle u\rangle = \langle u\rangle$ and

$$\langle\!\langle u\rangle\!\rangle = 1_{\langle u\rangle} : \langle\!\langle u\rangle_1^+\rangle = \langle u *_0 \langle u_0^-\rangle\!\rangle = u *_0 \langle\!\langle u_0^-\rangle\!\rangle *_1 \langle u\rangle = u *_0 1_{\langle u_0^-\rangle} *_1 \langle u\rangle = \langle u\rangle \to$$
$$\langle u\rangle *_0 \langle\!\langle u\rangle_0^-\rangle *_1 \langle\!\langle u\rangle_1^-\rangle = \langle u\rangle *_0 \langle o\rangle *_1 \langle\!\langle u_0^+\rangle\!\rangle = \langle u\rangle *_0 1_o *_1 1_{\langle u_0^+\rangle} = \langle u\rangle.$$



3.2 Syntax for orientals In paragraph 2.4.1, the unit $\eta : C \to C^{\triangleleft}$ of the expansion monad is considered as an inclusion, but since orientals are obtained by iterating this monad, our *simplicial notation* for orientals uses an explicit *shift*.

3.2.1. Let us introduce the following notations:

- If x is an m-cell in $\mathcal{O}_n$, we write $\lceil x \rceil$, called *shift* of x, for the m-cell $\eta(x)$ in $\mathcal{O}_{n+1}$.
- We write $\langle 0 \rangle$ for the origin o, which is a 0-cell in $\mathcal{O}_n$ for any n.
- More generally, if $0 \le i \le n$, we write $\langle i \rangle$ for the 0-cell $\eta^i(o)$ in $\mathcal{O}_n$.
- If x is an m-cell in $\mathcal{O}_n$, we write $\langle 0, x \rangle$ for the chevron $\langle x \rangle$, which is an $(m+1)$-cell in $\mathcal{O}_n$.
- More generally, if $0 \le i \le n$ and x' is an m-cell in $\mathcal{O}_n$ of the form $\eta^i(x)$ for some x in $\mathcal{O}_{n-i}$, we write $\langle i, x' \rangle$ for the $(m+1)$-cell $\eta^i \langle x \rangle$ in $\mathcal{O}_n$.

3.2.2 Remarks.

- Shift is a notation for the embedding $\eta : \mathcal{O}_n \hookrightarrow \mathcal{O}_n^{\triangleleft} = \mathcal{O}_{n+1}$ induced by the map $i \mapsto i+1$ from $\Delta_n = \{0, \ldots, n\}$ to $\Delta_{n+1} = \{0, \ldots, n+1\}$, which must not be confused with the *canonical inclusion* $\mathcal{O}_n \subset \mathcal{O}_{n+1}$ induced by the inclusion $\Delta_n \subset \Delta_{n+1}$.
- In practice, $\lceil x \rceil$ is obtained by incrementing all integers occurring in x, or more precisely, by applying the following rules:

$$\lceil \langle i \rangle \rceil = \langle i{+}1 \rangle, \qquad \lceil \langle i, x \rangle \rceil = \langle i{+}1, \lceil x \rceil \rangle, \qquad \lceil y *_p x \rceil = \lceil y \rceil *_p \lceil x \rceil, \qquad \lceil 1_u \rceil = 1_{\lceil u \rceil}.$$

3.2.3. By paragraph 2.4.2, the oriental $\mathcal{O}_n$ has a 0-generator $\langle 0 \rangle$, and any m-generator s of $\mathcal{O}_n$ yields two generators of $\mathcal{O}_{n+1}$:

- a *shifted* m-generator $s' = \lceil s \rceil$ standing for $\eta(s)$, with shifted source and target if $m > 0$,
- an *expanded* $(m{+}1)$-generator $\langle 0, s' \rangle$, which is just a new notation for the chevron $\langle s' \rangle$, whose source and target are given by the same formulas as in paragraph 3.1.1.

3.2.4. By induction on n, we get that any 0-generator of $\mathcal{O}_n$ is of the form $\langle i \rangle$ with $0 \le i \le n$, and more generally, any m-generator of $\mathcal{O}_n$ is of the form

$$\langle i_0, \langle i_1, \ldots, \langle i_m \rangle \cdots \rangle\rangle \text{ with } 0 \le i_0 < i_1 < \cdots < i_m \le n.$$

In other words, the set of m-generators of $\mathcal{O}_n$ is in canonical bijection with the set of injective order-preserving maps from Δ_m to Δ_n.

We write $\langle i_0, i_1, \ldots, i_m \rangle$ for the above m-generator, which is in fact defined by induction on m:

$$\langle i_0, i_1, \ldots, i_m \rangle = \langle i_0, \langle i_1, \ldots, i_m \rangle\rangle.$$

In particular, we get a single generator $|\mathcal{O}_n| = \langle 0, 1, \ldots, n \rangle$ of maximal dimension n, which is called the *principal generator* of $\mathcal{O}_n$.

3.2.5. The above notation extends to the case of a *nondecreasing* sequence $i_0 \le i_1 \le \cdots \le i_m$. This is easily seen by induction on m, since we get $\langle i_0, i_1, \ldots, i_m \rangle = \eta^{i_0} \langle 0, \langle i_1 - i_0, \ldots, i_m - i_0 \rangle\rangle$, but this defines a generator only if the sequence is (strictly) increasing. Otherwise, we get a unit. For instance, if $0 = i_0 = i_1 \le i_2 \cdots \le i_m$, then we get the following unit:

$$\langle 0, 0, i_2, \ldots, i_m \rangle = \langle 0, \langle 0, \langle i_2, \ldots, i_m \rangle\rangle\rangle = 1_{\langle 0, i_2, \ldots, i_m \rangle}.$$

The second equality follows indeed from the degeneracy axiom $\langle\langle u \rangle\rangle = 1_{\langle u \rangle}$ where $u = \langle i_2, \ldots, i_m \rangle$ and $\langle u \rangle$ is written $\langle 0, u \rangle$.

3.2.6 Examples.

- $\mathcal{O}_0 = \varnothing^{\triangleleft}$ has a single 0-generator $\langle 0\rangle$.
- $\mathcal{O}_1 = \mathcal{O}_0^{\triangleleft}$ has the generator of $\mathcal{O}_0$ and the following ones:
  - the 0-generator $\lceil\langle 0\rangle\rceil = \langle 1\rangle$,
  - the 1-generator $\langle 0,\langle 1\rangle\rangle = \langle 0,1\rangle : \langle 0\rangle \to \langle 1\rangle$.
- $\mathcal{O}_2 = \mathcal{O}_1^{\triangleleft}$ has the generators of $\mathcal{O}_1$ and the following ones:
  - the 0-generator $\lceil\langle 1\rangle\rceil = \langle 2\rangle$,
  - the 1-generator $\langle 0,\langle 2\rangle\rangle = \langle 0,2\rangle : \langle 0\rangle \to \langle 2\rangle$,
  - the 1-generator $\lceil\langle 0,1\rangle\rceil = \langle 1,2\rangle : \langle 1\rangle \to \langle 2\rangle$,
  - the 2-generator $\langle 0,\langle 1,2\rangle\rangle = \langle 0,1,2\rangle : \langle 0,2\rangle \to \langle 1,2\rangle *_0 \langle 0,1\rangle$.

  For $s = \langle 1,2\rangle : \langle 1\rangle \to \langle 2\rangle$, we get indeed

  $$\langle 0,s\rangle : \langle 0,s_0^+\rangle = \langle 0,\langle 2\rangle\rangle = \langle 0,2\rangle \to s *_0 \langle 0,s_0^-\rangle = \langle 1,2\rangle *_0 \langle 0,\langle 1\rangle\rangle = \langle 1,2\rangle *_0 \langle 0,1\rangle.$$

- $\mathcal{O}_3 = \mathcal{O}_2^{\triangleleft}$ has the generators of $\mathcal{O}_2$ and the following ones:
  - the 0-generator $\lceil\langle 2\rangle\rceil = \langle 3\rangle$,
  - the 1-generator $\langle 0,3\rangle : \langle 0\rangle \to \langle 3\rangle$,
  - the 1-generator $\lceil\langle 0,2\rangle\rceil = \langle 1,3\rangle : \langle 1\rangle \to \langle 3\rangle$,
  - the 2-generator $\langle 0,1,3\rangle : \langle 0,3\rangle \to \langle 1,3\rangle *_0 \langle 0,1\rangle$,
  - the 1-generator $\lceil\langle 1,2\rangle\rceil = \langle 2,3\rangle : \langle 2\rangle \to \langle 3\rangle$,
  - the 2-generator $\langle 0,2,3\rangle : \langle 0,3\rangle \to \langle 2,3\rangle *_0 \langle 0,2\rangle$,
  - the 2-generator $\lceil\langle 0,1,2\rangle\rceil = \langle 1,2,3\rangle : \langle 1,3\rangle \to \langle 2,3\rangle *_0 \langle 1,2\rangle$,
  - the 3-generator $\langle 0,1,2,3\rangle : \langle 2,3\rangle *_0 \langle 0,1,2\rangle *_1 \langle 0,2,3\rangle \to \langle 1,2,3\rangle *_0 \langle 0,1\rangle *_1 \langle 0,1,3\rangle$.

  For $s = \langle 1,2,3\rangle : \langle 1,3\rangle \to \langle 2,3\rangle *_0 \langle 1,2\rangle : \langle 1\rangle \to \langle 3\rangle$, we get that the source of $\langle 0,s\rangle$ is

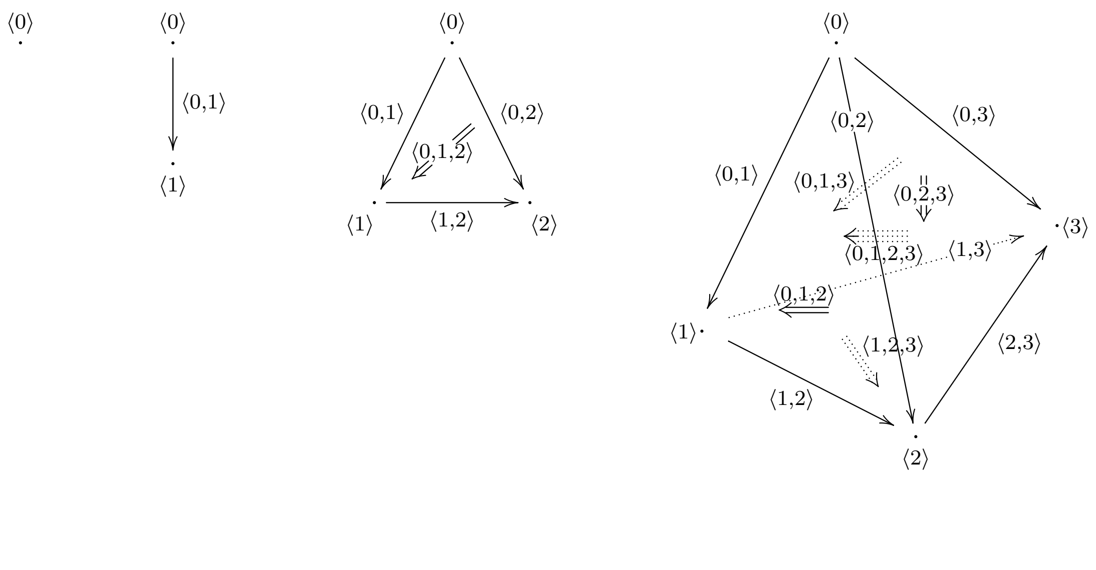
  $$\langle 0,s_1^+\rangle = \langle 0,\langle 2,3\rangle *_0 \langle 1,2\rangle\rangle = \langle 2,3\rangle *_0 \langle 0,\langle 1,2\rangle\rangle *_1 \langle 0,\langle 2,3\rangle\rangle = \langle 2,3\rangle *_0 \langle 0,1,2\rangle *_1 \langle 0,2,3\rangle$$

  and its target is

  $$s *_0 \langle 0,s_0^-\rangle *_1 \langle 0,s_1^-\rangle = \langle 1,2,3\rangle *_0 \langle 0,\langle 1\rangle\rangle *_1 \langle 0,\langle 1,3\rangle\rangle = \langle 1,2,3\rangle *_0 \langle 0,1\rangle *_1 \langle 0,1,3\rangle,$$

  so that we have

  $$\langle 0,s\rangle : \langle 2,3\rangle *_0 \langle 0,1,2\rangle *_1 \langle 0,2,3\rangle \to \langle 1,2,3\rangle *_0 \langle 0,1\rangle *_1 \langle 0,1,3\rangle.$$



3.2.7 Remarks.

- We have a canonical inclusion $\mathcal{O}_n \subset \mathcal{O}_{n+1}$, so that once the generators of $\mathcal{O}_n$ are known, it suffices to give the *last generation* (of generators) of $\mathcal{O}_{n+1}$. The latter is obtained by applying $s \mapsto s' = \lceil s \rceil$ and then $s' \mapsto \langle 0, s' \rangle$ to the last generation (of generators) of $\mathcal{O}_n$. In particular, it does not contain the origin $\langle 0 \rangle$.
- Once we have computed the source u and the target v of the principal generator $|\mathcal{O}_m|$, we get the source and target of any other m-generator $\langle i_0, \ldots, i_m \rangle$ of $\mathcal{O}_n$ for $n > m$ by applying the substitution $0 \mapsto i_0$, $\ldots$, $m \mapsto i_m$ to u and to v. In other words, we apply the functor $\mathcal{O} : \Delta \to \mathbf{Cat}_\omega$, which is described in the next subsection, to this substitution, seen as an injective order-preserving map from Δ_m to Δ_n.

3.3 Expansion monad on orientals

3.3.1. By construction, the expansion monad restricts to orientals:

- It maps the oriental $\mathcal{O}_n$ to the oriental $\mathcal{O}_n^{\lhd} = \mathcal{O}_{n+1}$, and the ω-functor $f : \mathcal{O}_n \to \mathcal{O}_{n'}$ to the ω-functor $f^{\lhd} : \mathcal{O}_n^{\lhd} = \mathcal{O}_{n+1} \to \mathcal{O}_{n'}^{\lhd} = \mathcal{O}_{n'+1}$ defined as follows by proposition 2.4.6:

$$f^{\lhd}\langle 0 \rangle = \langle 0 \rangle, \qquad \begin{cases} f^{\lhd}\lceil s \rceil = \lceil f\, s \rceil, \\ f^{\lhd}\langle 0, \lceil s \rceil \rangle = \langle 0, \lceil f\, s \rceil \rangle \end{cases} \text{for any } m\text{-generator } s \text{ of } \mathcal{O}_n.$$

- Its *unit* is the ω-functor $\eta : \mathcal{O}_n \hookrightarrow \mathcal{O}_n^{\lhd} = \mathcal{O}_{n+1}$ defined as follows:

$$\eta(s) = \lceil s \rceil \text{ for any } m\text{-generator } s \text{ of } \mathcal{O}_n.$$

- Its *multiplication* is the ω-functor $\mu : \mathcal{O}_n^{\lhd\lhd} = \mathcal{O}_{n+2} \to \mathcal{O}_n^{\lhd} = \mathcal{O}_{n+1}$ defined as follows by proposition 2.4.7:

$$\mu\langle 0 \rangle = \langle 0 \rangle, \qquad \begin{cases} \mu\lceil s \rceil = s, \\ \mu\langle 0, \lceil s \rceil \rangle = \langle 0, s \rangle \end{cases} \text{for any } m\text{-generator } s \text{ of } \mathcal{O}_{n+1}.$$

More explicitly, we get the following formulas for η and μ:

$$\eta\langle i_0, \ldots, i_m \rangle = \langle i_0{+}1, \ldots, i_m{+}1 \rangle \text{ for } 0 \le i_0 < \cdots < i_m \le n,$$

$$\begin{cases} \mu\langle i_0{+}1, \ldots, i_m{+}1 \rangle = \langle i_0, \ldots, i_m \rangle, \\ \mu\langle 0, i_0{+}1, \ldots, i_m{+}1 \rangle = \langle 0, i_0, \ldots, i_m \rangle \end{cases} \text{for } 0 \le i_0 < \cdots < i_m \le n+1.$$

3.3.2 Remarks.

- Unlike η, the ω-functor μ is not rigid, since we get the following degenerate case for $i_0 = 0$:

$$\mu\langle 0, 1, i_1{+}1, \ldots, i_m{+}1 \rangle = \langle 0, 0, i_1, \ldots, i_m \rangle = 1_{\langle 0, i_1, \ldots, i_m \rangle}.$$

- Our formulas for μ and η are *generic*, since they do not depend on the dimension n of $\mathcal{O}_n$. For that reason, objects are omitted in our notation for those natural transformations.

3.3.3 Example. The ω-functor $\mu : \mathcal{O}_3 \to \mathcal{O}_2$ is defined as follows:

$$\mu\langle 0 \rangle = \mu\langle 1 \rangle = \langle 0 \rangle, \quad \mu\langle 2 \rangle = \langle 1 \rangle, \quad \mu\langle 3 \rangle = \langle 2 \rangle,$$

$$\mu\langle 0,1\rangle = 1_{\langle 0\rangle}, \quad \mu\langle 0,2\rangle = \mu\langle 1,2\rangle = \langle 0,1\rangle, \quad \mu\langle 0,3\rangle = \mu\langle 1,3\rangle = \langle 0,2\rangle, \quad \mu\langle 2,3\rangle = \langle 1,2\rangle,$$
$$\mu\langle 0,1,2\rangle = 1_{\langle 0,1\rangle}, \quad \mu\langle 0,1,3\rangle = 1_{\langle 0,2\rangle}, \quad \mu\langle 0,2,3\rangle = \mu\langle 1,2,3\rangle = \langle 0,1,2\rangle,$$
$$\mu\langle 0,1,2,3\rangle = 1_{\langle 0,1,2\rangle}.$$

3.3.4 Proposition. *The cosimplicial object of orientals $\mathcal{O} : \Delta \to \mathbf{Cat}_\omega$ maps the object Δ_n to the oriental $\mathcal{O}_n$, and the order-preserving map $\phi : \Delta_n \to \Delta_{n'}$ to the ω-functor $f : \mathcal{O}_n \to \mathcal{O}_{n'}$ defined as follows:*

$$f\langle i_0, \ldots, i_m\rangle = \langle \phi(i_0), \ldots, \phi(i_m)\rangle \text{ for } 0 \le i_0 < \cdots < i_m \le n.$$

Proof. By the description of the functor $\mathcal{O} : \Delta \to \mathbf{Cat}_\omega$ given in paragraph 2.3.7, it suffices to apply the formulas of paragraph 3.3.1. □

4. Comparison with Street's orientals

4.1 Steiner's theory All the definitions and results presented in this subsection are extracted from [16].

4.1.1 Definition (Augmented directed complexes)**.**
An *augmented directed complex* K consists of an augmented chain complex of abelian groups in non-negative degrees

$$\cdots \xrightarrow{d} K_n \xrightarrow{d} \cdots \xrightarrow{d} K_1 \xrightarrow{d} K_0 \xrightarrow{e} \mathbb{Z}$$

(meaning that we have $dd = 0$ and $ed = 0$) endowed with a submonoid K_n^* of K_n for every $n \ge 0$.

If K and L are two such augmented directed complexes, a *morphism* from K to L is a morphism f of augmented chain complexes

$$\begin{tikzcd}
\cdots \arrow[r, "d"] & K_n \arrow[r, "d"] \arrow[d, "f"'] & \cdots \arrow[r, "d"] & K_1 \arrow[r, "d"] \arrow[d, "f"'] & K_0 \arrow[r, "e"] \arrow[d, "f"'] & \mathbb{Z} \arrow[d, equal] \\
\cdots \arrow[r, "d"] & L_n \arrow[r, "d"] & \cdots \arrow[r, "d"] & L_1 \arrow[r, "d"] & L_0 \arrow[r, "e"] & \mathbb{Z}
\end{tikzcd}$$

such that $f(K_n^*) \subset L_n^*$ for every $n \ge 0$.

We will denote by **ADC** the category of augmented directed complexes.

4.1.2. We define a functor $\lambda : \mathbf{Cat}_\omega \to \mathbf{ADC}$ in the following way. Let C be an ω-category. For $n \ge 0$, the abelian group $\lambda(C)_n$ is defined to be the quotient of the free abelian group on the set of n-cells of C by the subgroup generated by the elements of the form $x *_j y - x - y$, where x, y is a pair of n-cells, $0 \le j < n$ and $x *_j y$ is defined. We will denote by $[x]$ the image of an n-cell x of C in $\lambda(C)_n$. If $n > 0$, the differential $d : \lambda(C)_n \to \lambda(C)_{n-1}$ is defined on the generators by $d[x] = [\partial^+ x] - [\partial^- x]$. If $n = 0$, the abelian group $\lambda(C)_0$ is free on the set of objects of C and the augmentation $e : \lambda(C)_0 \to \mathbb{Z}$ is the sum of the coefficients. Finally, for $n \ge 0$, the monoid K_n^* is the submonoid of K_n generated by the generators $[x]$.

If $f : C \to D$ is an ω-functor, then $\lambda(f)$ is defined on generators by $\lambda(f)[x] = [f(x)]$.

One can check that these constructions are well-defined and indeed define a functor λ.

4.1.3. We now define a functor $\nu : \mathbf{ADC} \to \mathbf{Cat}_\omega$.

Let K be an augmented directed complex. For $n \geq 0$, an n-cell of $\nu(K)$ is a table

$$\begin{pmatrix} x_0^- & \cdots & x_n^- \\ x_0^+ & \cdots & x_n^+ \end{pmatrix}$$

where

- x_i^ε belongs to K_i^* for $\varepsilon = \pm$,
- $x_n^- = x_n^+$,
- $d(x_i^\varepsilon) = x_{i-1}^+ - x_{i-1}^-$ for $0 < i \leq n$ et $\varepsilon = \pm$,
- $e(x_0^\varepsilon) = 1$ for $\varepsilon = \pm$.

When $n > 0$, the source and target of such a table are given by the tables

$$\begin{pmatrix} x_0^- & \cdots & x_{n-2}^- & x_{n-1}^- \\ x_0^+ & \cdots & x_{n-2}^+ & x_{n-1}^- \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} x_0^- & \cdots & x_{n-2}^- & x_{n-1}^+ \\ x_0^+ & \cdots & x_{n-2}^+ & x_{n-1}^+ \end{pmatrix}.$$

For $n \geq 0$, the identity of such a table is the table

$$\begin{pmatrix} x_0^- & \cdots & x_n^- & 0 \\ x_0^+ & \cdots & x_n^+ & 0 \end{pmatrix}.$$

Finally, if

$$x = \begin{pmatrix} x_0^- & \cdots & x_n^- \\ x_0^+ & \cdots & x_n^+ \end{pmatrix} \qquad \text{and} \qquad y = \begin{pmatrix} y_0^- & \cdots & y_n^- \\ y_0^+ & \cdots & y_n^+ \end{pmatrix}$$

are two n-cells such that $\partial_j^- y = \partial_j^+ x$ for a j such that $0 \leq j < n$, then the cell $y *_j x$ is the table

$$\begin{pmatrix} x_0^- & \cdots & x_j^- & x_j^- + y_j^- & \cdots & x_n^- + y_n^- \\ y_0^+ & \cdots & y_j^+ & x_j^+ + y_j^+ & \cdots & x_n^+ + y_n^+ \end{pmatrix}.$$

One can check that these cells and operations define an ω-category $\nu(K)$.

If $f : K \to L$ is a morphism of augmented directed complexes, then, for $n \geq 0$, the action of $\nu(f) : \nu(K) \to \nu(L)$ on n-cells is defined by

$$\begin{pmatrix} x_0^- & \cdots & x_n^- \\ x_0^+ & \cdots & x_n^+ \end{pmatrix} \mapsto \begin{pmatrix} f(x_0^-) & \cdots & f(x_n^-) \\ f(x_0^+) & \cdots & f(x_n^+) \end{pmatrix}.$$

One can check that these constructions define a functor $\nu : \mathbf{ADC} \to \mathbf{Cat}_\omega$.

4.1.4 Proposition (Steiner). *The functors*

$$\lambda : \mathbf{Cat}_\omega \to \mathbf{ADC} \qquad\qquad \nu : \mathbf{ADC} \to \mathbf{Cat}_\omega$$

form a pair of adjoint functors.

Proof. This is [16, Theorem 2.11]. □

4.1.5. A *basis* of an augmented directed complex K is a subgraded set $\coprod_{n\geq 0} B_n$ of $\coprod_{n\geq 0} K_n^*$ such that B_n is a basis of the $\mathbb{Z}$-module K_n that generates the monoid K_n^*. We will say that an augmented directed complex is *free* if it admits a basis.

4.1.6 Remark. If such a basis exists, then we have, for $n \geq 0$,

$$K_n \simeq \mathbb{Z}^{(B_n)} \quad \text{and} \quad K_n^* \simeq \mathbb{N}^{(B_n)}.$$

It follows from the second isomorphism that B_n is uniquely determined by K_n^*. In other words, a free augmented directed complex admits a unique basis.

4.1.7. Let K be a free augmented directed complex with basis B. Let z be in K_n for some $n \geq 0$. This element can be written in a unique way

$$z = \sum_{b \in B_n} z_b b,$$

where B_n is the basis of K_n. The *support* of z is the subset of B_n consisting of those b such that $z_b \neq 0$. We define z^+ and z^- to be the unique elements of K_n^* with disjoint supports such that

$$z = z^+ - z^-.$$

For $n \geq 1$ and x in K_n, we set

$$d^-(x) = d(x)^- \quad \text{and} \quad d^+(x) = d(x)^+,$$

and, for $0 \leq i < n$, we set

$$d_i^-(x) = (d^-)^{n-i}(x) \quad \text{and} \quad d_i^+(x) = (d^+)^{n-i}(x).$$

Note that $d_i^-(x)$ and $d_i^+(x)$ are elements of K_i^*.

4.1.8. Let K be a free augmented directed complex with basis B. To any x in K_n^* for some $n \geq 0$, we associate a table

$$\begin{pmatrix} d_0^-(x) & \cdots & d_{n-1}^-(x) & x \\ d_0^+(x) & \cdots & d_{n-1}^+(x) & x \end{pmatrix}.$$

This table satisfies all the conditions to be an n-cell of $\nu(K)$ except maybe that $e(d_0^-(x)) = 1$ and $e(d_0^+(x)) = 1$, where $e : K_0 \to \mathbb{Z}$ is the augmentation of K.

The complex K is said to be *unital* if, for every $n \geq 0$ and every x in B_n, we have $e(d_0^-(x)) = 1$ and $e(d_0^+(x)) = 1$. In this case, the table associated to x is indeed an n-cell of $\nu(K)$ that is called the *atom* of x.

4.1.9. A free augmented directed complex K with basis B is said to be *strongly loop-free* if there exists a partial order $\leq$ on the set $B = \coprod_{n \geq 0} B_n$ such that, for every x in B_m and y in B_n, if

$$\begin{cases} & m \geq 1 \text{ and } y \text{ belongs to the support of } d^+(x) \\ \text{or} & \\ & n \geq 1 \text{ and } x \text{ belongs to the support of } d^-(y) \end{cases}$$

then we have $x \leq y$.

4.1.10. A *strong Steiner complex* is a free augmented directed complex that is both unital and strongly loop-free.

Theorem 4.1.11 (Steiner). *The functor ν : $\mathbf{ADC} \to \mathbf{Cat}_\omega$ is fully faithful when restricted to strong Steiner complexes.*

Proof. This follows from [16, Theorem 5.6 and Proposition 3.7]. □

Theorem 4.1.12 (Steiner)**.** *For any strong Steiner complex K, the ω-category $\nu(K)$ is freely generated, in the sense of polygraphs, by its atoms.*

Proof. This follows from [16, Theorem 6.1 and Proposition 3.7]. □

4.1.13 Remark. Steiner actually proved the two previous theorems for a more general class of complexes, where the strong loop-freeness condition is replaced by a weaker one (see [16, Definition 3.5]).

4.2 A uniqueness result

4.2.1 Proposition. *If S is a polygraph, then $\lambda(S^*)$ is free and its basis consists of the $[x]$, where x varies among the generators of S.*

Proof. Let $n \geq 0$. For every x in S_n, we will denote by e_x the corresponding element of the canonical basis of $\mathbb{Z}^{(S_n)}$. Consider the morphism $\gamma : \mathbb{Z}^{(S_n)} \to \lambda(S^*)_n$ defined by sending e_x to $[x]$, for every x in S_n. We claim that this morphism is an isomorphism. Indeed, by [14, paragraph 3.3], there exists a map $S^*_n \to \mathbb{Z}^{(S_n)}$ sending x in S_n to $[x]$ in $\mathbb{Z}^{(S_n)}$ and compositions in S^* to sums. In particular, we get a morphism $\lambda(S^*)_n \to \mathbb{Z}^{(S_n)}$ sending $[x]$ in $\lambda(S^*)_n$, for x in S_n, to e_x in $\mathbb{Z}^{(S_n)}$. This morphism provides an inverse to γ. □

4.2.2. Let S be a polygraph.

We say that a generator x in S_n is *atomic* if, for every i such that $0 \leq i < n$, the supports of $[x^-_i]$ and $[x^+_i]$ are disjoint. The polygraph S is said to be *atomic* if all its generators are atomic.

The polygraph S is *strongly loop-free* if there exists a partial order $\leq$ on the generators of S such that, for every $m \geq 0$ and $n \geq 0$, every x in S_m and y in S_n, if

$$\begin{cases} & m \geq 1 \text{ and } y \text{ belongs to the support of } [x^+] \\ \text{or} & \\ & n \geq 1 \text{ and } x \text{ belongs to the support of } [y^-] \end{cases}$$

then we have $x \leq y$.

Finally, S is a *strong Steiner polygraph* if it is both atomic and strongly loop-free.

Here is a reformulation of a result of Steiner based on [2]:

Theorem 4.2.3. *The adjoint pair*

$$\lambda : \mathbf{Cat}_\omega \to \mathbf{ADC} \qquad \nu : \mathbf{ADC} \to \mathbf{Cat}_\omega$$

induces an equivalence of categories between the full subcategory of $\mathbf{Cat}_\omega$ *consisting of ω-categories freely generated by a strong Steiner polygraph and the full subcategory of* $\mathbf{ADC}$ *consisting of strong Steiner complexes.*

Proof. This follows from [2, Theorem 2.30], based on [16, Theorem 5.11]. □

4.2.4 Proposition. *Let S and T be two polygraphs and let f be a dimension-preserving bijection between the generators of S and the generators of T. Suppose that, for every $n \geq 1$ and every x in S_n, we have*

$$f[x^-] = [f(x)^-] \quad \textit{and} \quad f[x^+] = [f(x)^+].$$

Suppose moreover that S is a strong Steiner polygraph and that T is atomic. Then T is a strong Steiner polygraph and the map f induces an isomorphism between S^* and T^*.

Proof. If x is in S_n, we have

$$fd[x] = f([x^+] - [x^-]) = f[x^+] - f[x^-] = [f(x)^+] - [f(x)^-] = df[x]$$

and, by Proposition 4.2.1, the map f defines an isomorphism from $\lambda(S^*)$ to $\lambda(T^*)$. Using the previous theorem, to conclude the proof, it thus suffices to show that T is strongly loop-free. But being strongly loop-free only depends on the generators and on the operations $z \mapsto [z^-]$ and $z \mapsto [z^+]$, where z is a generator. Since the bijection f is compatible with these, we get the result. □

4.3 Uniqueness of orientals We shall now give a "linear characterization" of $\mathcal{O}_n$, aiming at proving it is isomorphic to Street's oriental. We saw in paragraph 3.2.4 that the m-generators of $\mathcal{O}_n$ correspond to the injections $\Delta_m \hookrightarrow \Delta_n$. We now describe the linear source and target of such a generator:

4.3.1 Proposition. *Fix $n \geq -1$. For every $m \geq 1$ and every m-generator x of $\mathcal{O}_n$ considered as an injection $x : \Delta_m \hookrightarrow \Delta_n$, we have*

$$[x^-] = \sum_{\substack{0 \leq i \leq m \\ i \text{ odd}}} [x\delta_i^m] \quad \textit{and} \quad [x^+] = \sum_{\substack{0 \leq i \leq m \\ i \text{ even}}} [x\delta_i^m].$$

Proof. We prove the result by induction on n. The assertion is clear if $n = -1$ or $n = 0$. Suppose $n > 0$. Let $m \geq 1$ and let $x = \langle i_0, \ldots, i_m \rangle$ be a generator of $\mathcal{O}_n$ (see paragraph 3.2.4).

1. Suppose first that $i_0 \neq 0$. This means that $x = \eta(y)$, with $y = \langle i_0 - 1, \ldots, i_m - 1 \rangle$ an m-generator of $\mathcal{O}_{n-1}$, where $\eta : \mathcal{O}_{n-1} \hookrightarrow \mathcal{O}_n$ is the ω-functor coming from the fact that $\mathcal{O}_n$ is the free expansion on $\mathcal{O}_{n-1}$. By induction, we have

$$[y^-] = \sum_{\substack{0 \leq i \leq m \\ i \text{ odd}}} [y\delta_i^m] \quad \text{and} \quad [y^+] = \sum_{\substack{0 \leq i \leq m \\ i \text{ even}}} [y\delta_i^m],$$

so that

$$\begin{aligned} [x^-] &= [\eta(y)^-] = [\eta(y^-)] = \lambda(\eta)[y^-] \\ &= \lambda(\eta)\Big(\sum_{\substack{0 \leq i \leq m \\ i \text{ odd}}} [y\delta_i^m] \Big) = \sum_{\substack{0 \leq i \leq m \\ i \text{ odd}}} [\eta(y\delta_i^m)] \\ &= \sum_{\substack{0 \leq i \leq m \\ i \text{ odd}}} [\eta(y)\delta_i^m] = \sum_{\substack{0 \leq i \leq m \\ i \text{ odd}}} [x\delta_i^m], \end{aligned}$$

whence the desired formula, and similarly for $[x^+]$.

2. Suppose now that $i_0 = 0$. This means

$$x = \xi_{\eta(y)},$$

with $y = \langle i_1 - 1, \ldots, i_m - 1 \rangle$ an $(m-1)$-generator of $\mathcal{O}_{n-1}$, where ξ is the expansion of $\mathcal{O}_n$.

In particular,

$$\eta(y) = \langle i_1, \dots, i_m \rangle = x\delta_0^m.$$

(a) If $m = 1$, so that $x = \langle 0, i \rangle$, then $\eta(y) = \langle i \rangle$ and

$$\xi_{\eta(y)} : \langle 0 \rangle \to \langle i \rangle.$$

Thus

$$[x^-] = [\langle 0 \rangle] = [x\delta_1^m] \quad \text{and} \quad [x^+] = [\langle 1 \rangle] = [x\delta_0^m],$$

whence the result.

(b) If $m > 1$, then

$$\xi_{\eta(y)} : \xi_{\eta(y)^+_{m-1}} \to \eta(y) *_0 \xi_{\eta(y)^-_0} *_1 \cdots *_{m-1} \xi_{\eta(y)^-_{m-1}},$$

so that

$$[x^-] = [\xi_{\eta(y)^+_{m-1}}] = [\xi_{\eta(y)^+}] = [\xi_{\eta(y^+)}]$$

and

$$[x^+] = [\eta(y)] + [\xi_{\eta(y)^-_{m-1}}] = [\eta(y)] + [\xi_{\eta(y^-)}],$$

since $[z] = 0$ if z is an identity. To be able to use this, we will need the fact that the oplax transformation ξ induces a $\mathbb{Z}$-linear map (and actually even a chain homotopy)

$$\begin{aligned} \lambda(\xi) : \lambda(\mathcal{O}_n)_k &\to \lambda(\mathcal{O}_n)_{k+1} \\ [z] &\mapsto [\xi_z] \end{aligned}$$

for every $k \geq 0$ (see the proof of Theorem 6.1 of [14]). Now by induction, we have

$$[y^-] = \sum_{\substack{0 \leq i \leq m-1 \\ i \text{ odd}}} [y\delta_i^{m-1}] \quad \text{and} \quad [y^+] = \sum_{\substack{0 \leq i \leq m-1 \\ i \text{ even}}} [y\delta_i^{m-1}],$$

so that

$$\begin{aligned} [x^-] = [\xi_{\eta(y^+)}] = \lambda(\xi)\lambda(\eta)[y^+] &= \lambda(\xi)\lambda(\eta)\Big(\sum_{\substack{0 \leq i \leq m-1 \\ i \text{ even}}} [y\delta_i^{m-1}] \Big) \\ &= \lambda(\xi)\Big(\sum_{\substack{0 \leq i \leq m-1 \\ i \text{ even}}} [\eta(y\delta_i^{m-1})] \Big) = \lambda(\xi)\Big(\sum_{\substack{0 \leq i \leq m-1 \\ i \text{ even}}} [\eta(y)\delta_i^{m-1}] \Big) \\ &= \sum_{\substack{0 \leq i \leq m-1 \\ i \text{ even}}} [\xi_{\eta(y)\delta_i^{m-1}}] = \sum_{\substack{0 \leq i \leq m-1 \\ i \text{ even}}} [\xi_{\eta(y)}\delta_{i+1}^m] \\ &\qquad \text{(the equality } \xi_{\eta(y)\delta_i^{m-1}} = \xi_{\eta(y)}\delta_{i+1}^m \text{ being more transparent} \\ &\qquad \text{under the form } \langle 0, \langle i_1, \dots, i_m \rangle \delta_i^{m-1} \rangle = \langle 0, i_1, \dots, i_m \rangle \delta_{i+1}^m) \\ &= \sum_{\substack{0 \leq i \leq m-1 \\ i \text{ even}}} [x\delta_{i+1}^m] = \sum_{\substack{1 \leq j \leq m \\ j \text{ odd}}} [x\delta_j^m] = \sum_{\substack{0 \leq j \leq m \\ j \text{ odd}}} [x\delta_j^m] \end{aligned}$$

as wanted. Similarly, one gets that

$$[\xi_{\eta(y^-)}] = \sum_{\substack{1\leq j\leq m \\ j \text{ even}}} [x\delta^m_j]$$

and hence that

$$[x^+] = [\eta(y)] + [\xi_{\eta(y^-)}] = [x\delta^m_0] + \sum_{\substack{1\leq j\leq m \\ j \text{ even}}} [x\delta^m_j] = \sum_{\substack{0\leq j\leq m \\ j \text{ even}}} [x\delta^m_j],$$

thereby ending the proof. □

4.3.2 Proposition. *For every $n \geq -1$, the polygraph defining $\mathcal{O}_n$ is atomic.*

Proof. We will prove more generally that if S is an atomic polygraph, then so is the polygraph $S^{\triangleleft}$ of paragraph 2.4.2. The result will then follow by induction as the polygraph defining $\mathcal{O}_n$ is obtained by iterating this construction from the empty polygraph, which is atomic.

Let thus S be an atomic polygraph. Consider a generator x of $S^{\triangleleft}$ of dimension $n \geq 1$.

- If $x = \eta(y)$ for y a generator of S, where $\eta : S \to S^{\triangleleft}$ is the canonical morphism, then, as y is atomic by hypothesis, so is x, as η is injective on cells.
- Otherwise, $x = r_y$ for y a generator of S, with the notation of paragraph 2.4.2. If $n = 1$, then

$$(r_y)^- = o \qquad \text{and} \qquad (r_y)^+ = y,$$

where o is the origin of $S^{\triangleleft}$. If $n > 1$, using the formulas

$$(r_y)^- = r_{y^+_{n-2}} \qquad \text{and} \qquad (r_y)^+ = y *_0 r_{y^-_0} *_1 \cdots *_{n-2} r_{y^-_{n-2}},$$

where y was identified with $\eta(y)$, we get by induction that, for i such that $0 < i < n$,

$$(r_y)^-_i = r_{y^+_{i-1}} \qquad \text{and} \qquad (r_y)^+_i = y^+_i *_0 r_{y^-_0} *_1 \cdots *_{i-1} r_{y^-_{i-1}},$$

and that

$$(r_y)^-_0 = o \qquad \text{and} \qquad (r_y)^+_0 = y^+_0.$$

The supports of $(r_y)^-_i$ and $(r_y)^+_i$, for $0 \leq i < n$, are thus disjoint and x is atomic, whence the result. □

4.3.3 Proposition. *Fix $n \geq -1$ and let S be an atomic polygraph such that*

1. *for every $m \geq 0$, we have $S_m = \{x : \Delta_m \hookrightarrow \Delta_n \mid x \text{ injective and order-preserving}\}$,*
2. *for every $m \geq 1$ and every $x : \Delta_m \hookrightarrow \Delta_n$ in S_m, we have*

$$[x^-] = \sum_{\substack{0\leq i\leq m \\ i \text{ odd}}} [x\delta^m_i] \quad \textit{and} \quad [x^+] = \sum_{\substack{0\leq i\leq m \\ i \text{ even}}} [x\delta^m_i].$$

Then S^* is canonically isomorphic to $\mathcal{O}_n$.

Proof. By paragraph 3.2.4 and Proposition 4.3.1, these two properties are satisfied by the polygraph defining $\mathcal{O}_n$, which is atomic by the previous proposition. To get the result, using Proposition 4.2.4, it thus suffices to produce a strong Steiner polygraph S that satisfies these two properties. We could prove that the polygraph defining $\mathcal{O}_n$ does the job but it is simpler to refer to Steiner: the polygraph associated to the complex $\Delta[n]$ of [16, Example 3.8] satisfies these conditions. □

Theorem 4.3.4. *The cosimplicial object $\mathcal{O} : \Delta \to \mathbf{Cat}_\omega$ of Definition 2.3.6 is canonically isomorphic to the cosimplicial objects of orientals as introduced by Street in [18].*

Proof. By [18, Section 3 and Corollary 4.2], the n-th oriental defined by Street satisfies the conditions of the previous proposition. This shows that the two cosimplicial objects agree on objects. To show that they also agree on morphisms, by using Theorem 4.2.3, it suffices to show that they agree after applying $\lambda : \mathbf{Cat}_\omega \to \mathbf{ADC}$. This follows from Proposition 3.3.4 and [18, Section 5]. □

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