HIGHER STRUCTURES

Gray tensor product and saturated N-complicial sets

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Abstract

We show that the pretensor and tensor products of simplicial sets with marking are compatible with the homotopy theory of saturated N-complicial sets (which are a proposed model of (∞, N)-categories), in the form of a Quillen bifunctor and a homotopical bifunctor, respectively.

Overview

Higher category theory is becoming increasingly important as a unifying language for various areas of mathematics, most notably for algebraic topology and algebraic geometry, where many relevant structures occur naturally as (∞, N)-*categories*, rather than strict N-categories. In this article, we are concerned with an (∞, N)-categorical version of the *Crans–Gray tensor product* [5, 3], originally defined for strict N-categories in order to encode different flavors of lax natural transformations.

The model of (∞, N)-categories that we consider, due to the third-named author, is that of *saturated N-complicial sets*. A saturated N-complicial set is a simplicial set with marking satisfying extra conditions that guarantee that the marked simplices behave as higher equivalences. In [14], he constructed two pointset models of the Gray tensor product of simplicial

sets with marking: the tensor $\otimes$ and the pretensor $\boxtimes$, homotopically equivalent but each with different valuable properties, and showed that they are compatible with the homotopy theory of (non-saturated) N-complicial sets.

In this note, we provide the extra verification that enables us to conclude that the pretensor and the tensor products $\boxtimes$ and $\otimes$ are in fact also compatible with the model structure for *saturated N-complicial sets*, in a sense that will be made precise by Corollaries 2.3 and 2.6.

Main Theorem. *For any $N \in \mathbb{N}$, the bifunctors $\boxtimes$ and $\otimes$ are homotopical with respect to the model structure on simplicial sets with marking for saturated N-complicial sets, which model (∞, N)-categories.*

The theorem was proven for $N = 1$ by Joyal [6, Thm 6.1] in the context of quasi-categories and by Lurie [9, Cor. 3.1.4.3] in the context of marked simplicial sets. During the final work on the completion of this paper, analogous result was shown for $N = 2$ by Gagna–Harpaz–Lanari [4] in the context of scaled simplicial sets. For general N, the result was previously obtained by the third-named author, and recently rediscovered by the first two authors.

Beside for its own interest, the result would play a role in work by Campion–Kapulkin–Maehara, in comparing cubical models of (∞, N)-categories to saturated N-complicial sets, as indicated in [2, Rmk 7.3, Conj. 7.4].

1. Background on simplicial sets with marking

We recall in this section the background material on simplicial sets with marking, saturated complicial sets, and on the pretensor and tensor product, $\boxtimes$ and $\otimes$.

Definition 1.1. A *simplicial set with marking*[1] is a simplicial set with a designated subset of *marked* or *thin* positive-dimensional simplices that includes all degenerate simplices. A *map of simplicial sets with marking* is a simplicial map that preserves the marking. We denote by $ms\mathcal{S}et$ the category of simplicial sets with marking and maps of simplicial sets with marking.

1.1 The model structures on simplicial sets with marking The following notational conventions will be used to define saturated N-complicial sets and to describe the model structure for N-complicial sets on $ms\mathcal{S}et$. The material is mostly drawn from [15, §§2.1-2.2], [12] and [10, §1], and we refer the reader to these references for a more detailed account.

Notation 1.2. We denote

- by $\Delta[-1]$ the empty simplicial set.
- by $\Delta[m]$ the simplicial set with marking whose underlying simplicial set is $\Delta[m]$ and in which only degenerate simplices are marked.
- by $\partial\Delta[m]$ the simplicial set with marking whose underlying simplicial set is $\partial\Delta[m]$ and in which only degenerate simplices are marked.
- by $\Delta[m]_t$ the simplicial set with marking whose underlying simplicial set is $\Delta[m]$ and in which only degenerate simplices and the top m-simplex are marked.

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[1]This notion is the same as *stratified simplicial set* in the sense of Verity [14], and is different from (but related to) *marked simplicial set* in the sense of Lurie [9].

- by $\Delta^k[m]$, for $0 \leq k \leq m$, the simplicial set with marking whose underlying simplicial set is $\Delta[m]$ and in which a non-degenerate simplex is marked if and only if it contains the vertices $\{k-1, k, k+1\} \cap [m]$.
- by $\Delta^k[m]'$, for $0 \leq k \leq m$, the simplicial set with marking obtained from $\Delta^k[m]$ by additionally marking the $(k-1)$-st and $(k+1)$-st face of $\Delta[m]$.
- by $\Delta^k[m]''$, for $0 \leq k \leq m$, the simplicial set with marking obtained from $\Delta^k[m]'$ by additionally marking the k-th face of $\Delta[m]$.
- by $\Lambda^k[m]$, for $0 \leq k \leq m$, the simplicial set with marking whose underlying simplicial set is the k-horn $\Lambda^k[m]$ and whose simplex is marked if and only if it is marked in $\Delta^k[m]$.
- by $\Delta[3]_{eq}$ the simplicial set with marking whose underlying simplicial set is $\Delta[3]$, and the non-degenerate marked simplices consist of all 2- and 3-simplices, as well as 1-simplices $[02]$ and $[13]$.
- by $\Delta[3]_\sharp$ the simplicial set with marking whose underlying simplicial set is $\Delta[3]$, and all simplices in positive dimensions are marked.
- by $\Delta[\ell|3_{eq}]$, for $\ell \geq -1$, the simplicial set with marking $\Delta[\ell] \star \Delta[3]_{eq}$.
- by $\Delta[\ell'|3_{eq}|\ell]$, for $\ell, \ell' \geq -1$, the simplicial set with marking $\Delta[\ell'] \star \Delta[3]_{eq} \star \Delta[\ell]$.
- by $\Delta[\ell|3_\sharp]$ for $\ell \geq -1$, the simplicial set with marking $\Delta[\ell] \star \Delta[3]_\sharp$.
- by $\Delta[\ell'|3_\sharp|\ell]$, for $\ell, \ell' \geq -1$, the simplicial set with marking $\Delta[\ell'] \star \Delta[3]_\sharp \star \Delta[\ell]$.

Here, $\star$ denotes the join of simplicial sets with marking, which can be found in [15, Observation 34] or [12, Def. 3.2.5], and which we recall for the reader's convenience.

Definition 1.3. Given simplicial sets with marking X and Y, the *join* $X \star Y$ is a simplicial set with marking whose underlying simplicial set is the join of the underlying simplicial sets, and in which an r-simplex $x \star y \colon \Delta[k] \star \Delta[r-k-1] \to X \star Y$ for $-1 \leq k \leq r$ is marked if and only if the simplex x is marked in X or the simplex y is marked in Y (or both).

Definition 1.4. For $N \in \mathbb{N} \cup \{\infty\}$, an *elementary (∞, N)-anodyne extension* is one of the following.

(1) The *complicial horn extension*, i.e., the canonical map

$$\Lambda^k[m] \to \Delta^k[m] \text{ for } m \geq 1 \text{ and } 0 \leq k \leq m,$$

which is the ordinary horn inclusion on the underlying simplicial sets.

(1') The *complicial thinness extension*, i.e., the canonical map

$$\Delta^k[m]' \to \Delta^k[m]'' \text{ for } m \geq 2 \text{ and } 0 \leq k \leq m,$$

which is the identity on the underlying simplicial set.

(2) The *left saturation extension*, i.e., the canonical map

$$\Delta[\ell|3_{eq}] \to \Delta[\ell|3_\sharp] \text{ for } \ell \geq -1,$$

which is the identity on the underlying simplicial set.

(3) The *triviality extension* map, i.e., the canonical map

$$\Delta[p] \to \Delta[p]_t \text{ for } p > N,$$

which is the identity on the underlying simplicial set.

Remark 1.5. We point out that the parameter N only plays a role in the triviality anodyne extension in (3). In particular, complicial horn extensions, thinness extensions and saturation anodyne extensions are (∞, N)-anodyne for every $N \in \mathbb{N} \cup \{\infty\}$.

Definition 1.6. Let X be a simplicial set with marking, and $N \in \mathbb{N} \cup \{\infty\}$.

1. X is a *complicial set*, also called a *weak complicial set*, if it has the right lifting property with respect to the complicial horn anodyne extensions $\Lambda^k[m] \to \Delta^k[m]$ and the thinness anodyne extensions $\Delta^k[m]' \to \Delta^k[m]''$ for $m \geq 1$ and $0 \leq k \leq m$.
2. X is a *saturated complicial set* if it is a complicial set and it has the right lifting property with respect to the left saturation anodyne extensions $\Delta[\ell|3_{eq}] \to \Delta[\ell|3_\sharp]$ for $\ell \geq -1$.
3. X is a *saturated N-complicial set* if it is a saturated complicial set and it has the right lifting property with respect to the triviality anodyne extensions $\Delta[p] \to \Delta[p]_t$ for $p > N$.

For any $N \in \mathbb{N}$, saturated N-complicial sets are a proposed model for (∞, N)-categories[2], and we refer the reader to [14, 12, 10] for a description of the intuition behind this combinatorics.

Roughly speaking, according to the intuition that the r-simplices of a simplicial set with marking represent r-morphisms and that the marked simplices represent r-equivalences, we can rephrase as follows.

1. In a complicial set r-morphisms can be composed, and composite of r-equivalences is an r-equivalence.
2. In a saturated complicial set r-equivalences satisfy the two-out-of-six property.
3. In a saturated N-complicial set all r-morphisms are equivalences in dimension $r > N$.

There is a model structure on $ms\mathcal{S}et$ for saturated N-complicial sets.

Theorem 1.7 ([14, 12, 10]). *Let $N \in \mathbb{N} \cup \{\infty\}$. There is a cofibrantly generated model structure on $ms\mathcal{S}et$ in which*

- *the cofibrations are precisely the monomorphisms;*
- *the fibrant objects are precisely the saturated N-complicial sets;*
- *all elementary anodyne extensions are acyclic cofibrations.*

We call this model structure the model structure for (∞, N)-categories, or the model structure for saturated N-complicial sets, we denote it by $ms\mathcal{S}et_{(\infty,N)}$, and we call the acyclic cofibrations (∞, N)-acyclic cofibrations.

Remark 1.8. As discussed in [15, Example 21], the generating cofibrations for the model structure for (∞, N)-categories are

- the *boundary inclusions*
$$\partial\Delta[m] \to \Delta[m] \text{ for } m \geq 0,$$

- and the *marking inclusions*
$$\Delta[m] \to \Delta[m]_t \text{ for } m \geq 1.$$

We mentioned that, by construction, all left saturation extensions $\Delta[\ell|3_{eq}] \to \Delta[\ell|3_\sharp]$ for $\ell \geq -1$ are acyclic cofibrations. In fact, even the saturation extensions of the more general form $\Delta[\ell'|3_{eq}|\ell] \to \Delta[\ell'|3_\sharp|\ell]$ for $\ell, \ell' \geq -1$ are acyclic cofibrations.

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[2]The case $N = \infty$ is subtle, since there are at least two different viewpoints on what an (∞, ∞)-category should be.

Lemma 1.9. *The* saturation extension

$$\Delta[\ell'|3_{eq}|\ell] \to \Delta[\ell'|3_\sharp|\ell] \quad for \quad \ell, \ell' \geq -1$$

is acyclic cofibration.

Proof. The saturation extensions $\Delta[\ell'|3_{eq}|\ell] \to \Delta[\ell'|3_\sharp|\ell]$ have the left lifting property with respect to all saturated N-complicial sets, as shown [13, §D.7], and since they are isomorphisms on the underlying simplicial sets they must also have the right lifting property with respect to all fibrations between saturated N-complicial sets. We then conclude that they are acyclic cofibrations as an instance of [7, Lemma 7.14]. □

Proposition 1.10 ([11, Lemma 1.8])**.** *Let $\mathcal{M}$ be a model category. A left adjoint functor $F \colon ms\mathcal{S}et_{(\infty,N)} \to \mathcal{M}$ is left Quillen if and only if it respects cofibrations and sends all elementary anodyne extensions to weak equivalences of $\mathcal{M}$.*

1.2 Pretensor and tensor product of simplicial sets with marking Inspired[3] by the Crans–Gray tensor product of ω-categories from [5, 3], which can be thought as strict ∞-categories, Verity defined two models of Gray tensor products of simplicial sets with marking: the pretensor $\boxtimes$ and the tensor $\otimes$. In this paper, we will work with the definition of the tensor product $\otimes$, while the pretensor product $\boxtimes$ plays a more indirect role. For completeness, we recall both definitions.

Notation 1.11 ([14, Notation 5])**.** For any $p, q \geq 0$,

- the *degeneracy partition operator* is the map in Δ

$$\Pi_1^{p,q} \colon [p+q] \to [p] \quad \text{and} \quad \Pi_2^{p,q} \colon [p+q] \to [q]$$

defined by

$$i \mapsto \begin{cases} i & \text{if } i \leq p \\ p & \text{if } i > p \end{cases} \quad \text{and} \quad i \mapsto \begin{cases} 0 & \text{if } i < p \\ i - p & \text{if } i \geq p \end{cases}$$

- the *face partition operator* is the map in Δ

$$\amalg_1^{p,q} \colon [p] \to [p+q] \quad \text{and} \quad \amalg_2^{p,q} \colon [q] \to [p+q]$$

defined by

$$i \mapsto i \quad \text{and} \quad i \mapsto p + i.$$

Remark 1.12. As explained in [14, §1.6], any non-degenerate r-simplex of $\Delta[r] \to \Delta[p] \times \Delta[q]$ can be pictured as a path of length r in a rectangular grid of size $p \times q$. According to this interpretation, the $(p+q)$-simplex given by $(\Pi_1^{p,q}, \Pi_2^{p,q}) \colon \Delta[p+q] \to \Delta[p] \times \Delta[q]$ is the path with "first all to the right, then all up", as shown in the following picture for $p = 3$ and $q = 2$.

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[3]In [15, Observation 62], Verity states the relationship between the Crans–Gray tensor product of ω-categories and the tensor product of simplicial sets with marking, using the fact that ω-categories (in the form of strict complicial sets) form a reflective subcategory of simplicial sets with marking. Given two ω-categories, their Crans–Gray tensor product can be obtained by reflecting their tensor product as simplicial sets with marking.
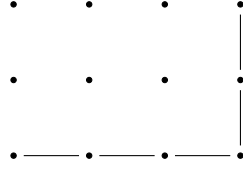
Definition 1.13 ([15, Def. 135]). Given simplicial sets with marking X and Y, the *pretensor* $X \boxtimes Y$ is formed by taking the product of underlying simplicial sets and endowing it with a marking under which a non-degenerate r-simplex $(x, y)\colon \Delta[r] \to X \times Y$ is marked if either

- it is a *mediator*, i.e., there exists $0 < k < r$ and $(r-1)$-simplices $x'\colon \Delta[r-1] \to X$ and $y'\colon \Delta[r-1] \to Y$ such that $x = s_{k-1}x' = x' \circ s^{k-1}$ and $y = s_k y' = y' \circ s^k$.
- it is a *crushed cylinder*, i.e., there exists a partition p, q of $r = p+q$ and simplices $x'\colon \Delta[p] \to X$ and $y'\colon \Delta[q] \to Y$ such that $x = x' \circ \Pi_1^{p,q}$ and $y = y' \circ \Pi_2^{p,q}$, and either the simplex x' is marked in X or the simplex y' is marked in Y (or both).

It is proven in [15, Lemma 142] that $\boxtimes$ is a bifunctor that preserves colimits in each variable. We then obtain the following adjunctions. Regarding the terminology of lax and oplax, we follow the same convention as e.g. [8, 1].

Proposition 1.14 ([14, Cor. 144]). *For any simplicial set with marking S there are adjunctions*

$$- \boxtimes S\colon ms\mathcal{S}et \rightleftarrows ms\mathcal{S}et \colon [S, -]_{oplax}$$

and

$$S \boxtimes -\colon ms\mathcal{S}et_{(\infty,N)} \rightleftarrows ms\mathcal{S}et_{(\infty,N)} \colon [S, -]_{lax}.$$

However, the pretensor $\boxtimes$ is not associative, so it cannot be used to build a monoidal structure on $ms\mathcal{S}et$. For this purpose, one can instead consider the tensor product $\otimes$ (which however does not preserve colimits).

Definition 1.15 ([14, Def. 128]). Given simplicial sets with marking X and Y, the *tensor $X \otimes Y$* is formed by taking the product of underlying simplicial sets and endowing it with a marking under which a non-degenerate r-simplex $(x, y)\colon \Delta[r] \to X \times Y$ is marked if for each $p, q \geq 0$ the partition $r = p + q$ *cleaves* the simplex (x, y), i.e., the p-simplex $x \circ \Pi_1^{p,q}$ is marked in X or the q-simplex $y \circ \Pi_2^{p,q}$ is marked in Y.

Pretensor and tensor are equivalent in the following sense.

Proposition 1.16 ([14, Lemma 149]). *For any simplicial sets with marking X and Y the canonical inclusion*

$$X \boxtimes Y \hookrightarrow X \otimes Y$$

is an (∞, N)-acyclic cofibration for any $N \in \mathbb{N} \cup \{\infty\}$. In particular there is an objectwise weak equivalence

$$- \boxtimes - \simeq - \otimes -\colon ms\mathcal{S}et_{(\infty,N)} \times ms\mathcal{S}et_{(\infty,N)} \to ms\mathcal{S}et_{(\infty,N)}.$$

To highlight the difference between the pretensor and the tensor, we briefly discuss an example. We refer the reader to [14, §6.3] for a deeper treatment and for more details and examples.

Example 1.17. We consider the case of $X = \Delta[2]_t$ and $Y = \Delta[1]$.

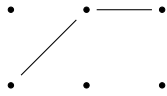- the simplex $\Delta[2] \to \Delta[2]_t \times \Delta[1]$ depicted as
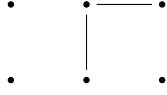
  is a mediator, and is therefore marked in both $\Delta[2]_t \boxtimes \Delta[1]$ and $\Delta[2]_t \otimes \Delta[1]$.
- the simplex $\Delta[3] \to \Delta[2]_t \times \Delta[1]$ depicted as
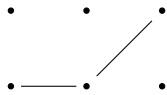

  is a crushed cylinder, and is therefore marked in both $\Delta[2]_t \boxtimes \Delta[1]$ and $\Delta[2]_t \otimes \Delta[1]$.
- the simplex $\Delta[2] \to \Delta[2]_t \times \Delta[1]$ depicted as

  is cleaved by every partition, and is therefore marked in $\Delta[2]_t \otimes \Delta[1]$, but it is not marked in $\Delta[2]_t \boxtimes \Delta[1]$.
- the simplex $\Delta[2] \to \Delta[2]_t \times \Delta[1]$ depicted as

  is cleaved by the partitions $(2,0)$ and $(0,2)$, but not by the partition $(1,1)$, and is therefore not marked neither in $\Delta[2]_t \boxtimes \Delta[1]$ nor in $\Delta[2]_t \otimes \Delta[1]$.

2. The main theorem

The main result is the following.

Theorem 2.1. *Let $N \in \mathbb{N} \cup \{\infty\}$. For any simplicial set with marking S the adjunction*

$$- \boxtimes S \colon ms\mathcal{S}et_{(\infty,N)} \rightleftarrows ms\mathcal{S}et_{(\infty,N)} \colon [S,-]_{oplax}$$

is a Quillen pair. In particular, the functor

$$- \boxtimes S \colon ms\mathcal{S}et_{(\infty,N)} \to ms\mathcal{S}et_{(\infty,N)}$$

is homotopical.

The theorem admits many essentially equivalent reformulations or direct consequences, which we collect as corollaries.

Using Proposition 1.16 we obtain the following corollary.

Corollary 2.2. *Let $N \in \mathbb{N} \cup \{\infty\}$. For any simplicial set with marking S the functor*

$$- \otimes S \colon ms\mathcal{S}et_{(\infty,N)} \to ms\mathcal{S}et_{(\infty,N)}$$

is homotopical.

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The statement can then be strengthened as follows.

Corollary 2.3. *Let $N \in \mathbb{N} \cup \{\infty\}$. The functor*

$$- \otimes - : ms\mathcal{S}et_{(\infty,N)} \times ms\mathcal{S}et_{(\infty,N)} \to ms\mathcal{S}et_{(\infty,N)}$$

is homotopical.

Lemma 2.4. *Let $f \colon X \to Y$ be a map of simplicial sets with marking. Then f is a weak equivalence in the model structure for saturated N-complicial sets if and only if f^{op} is one.*

Proof of Lemma 2.4. We argue that $(-)^{\mathrm{op}}$ is left Quillen, so in particular homotopical, and hence respects weak equivalences. Given the canonical isomorphism $(X^{\mathrm{op}})^{\mathrm{op}} \cong X$ from [15, Observation 38], we also obtain that $(-)^{\mathrm{op}}$ reflects weak equivalences, concluding the proof.

To see that $(-)^{\mathrm{op}} \colon ms\mathcal{S}et_{(\infty,N)} \to ms\mathcal{S}et_{(\infty,N)}$ is a left Quillen functor, we observe the following.

(0) Since $(-)^{\mathrm{op}}$ is an isomorphism, if $X \to Y$ is a monomorphism, then $X^{\mathrm{op}} \to Y^{\mathrm{op}}$ is a monomorphism, so $(-)^{\mathrm{op}}$ preserves cofibrations.

(1) By [14, Observation 157], for $m \geq 0$ and $0 \leq k \leq m$ the map $\Lambda^k[m]^{\mathrm{op}} \to \Delta^k[m]^{\mathrm{op}}$ is the map $\Lambda^{m-k}[m] \to \Delta^{m-k}[m]$, which is a weak equivalence in the model structure for saturated N-complicial sets. In particular, $(-)^{\mathrm{op}}$ sends complicial horn extensions to weak equivalences.

(1') By [14, Observation 125], for $m \geq 0$ and $0 \leq k \leq m$ the map $\Delta^k[m]'^{\mathrm{op}} \to \Delta^k[m]''^{\mathrm{op}}$ is the map $\Delta^{m-k}[m]' \to \Delta^{m-k}[m]''$, which is a weak equivalence in the model structure for saturated N-complicial sets. In particular, $(-)^{\mathrm{op}}$ sends thinness extensions to weak equivalences.

(2) By [14, Observation 107], for $p > N$ the map $\Delta[p]^{\mathrm{op}} \to \Delta[p]_t^{\mathrm{op}}$ is $\Delta[p] \to \Delta[p]_t$, which is a weak equivalence in the model structure for saturated N-complicial sets. In particular $(-)^{\mathrm{op}}$ sends triviality extensions for $p > N$ to weak equivalences.

(3) For $\ell \geq -1$, one can use [15, Observation 36] to show that the map $\Delta[\ell|3_{eq}]^{\mathrm{op}} \to \Delta[\ell|3_{\sharp}]^{\mathrm{op}}$ is the map $\Delta[3_{eq}|\ell] \to \Delta[3_{\sharp}|\ell]$, which was shown in Lemma 1.9 to be a weak equivalence in the model structure for saturated N-complicial sets. In particular, $(-)^{\mathrm{op}}$ sends left saturation extensions to weak equivalences.

By Proposition 1.10, we then conclude that $(-)^{\mathrm{op}}$ is a left Quillen functor, as desired. $\square$

Proof of Corollary 2.3. We already know from Theorem 2.1 that the functor $\otimes$ respects weak equivalences in the first variable, and we now check that it respects weak equivalences in the second variable, too. If $X \to Y$ is a weak equivalence, by Lemma 2.4 the map $X^{\mathrm{op}} \to Y^{\mathrm{op}}$ is a weak equivalence. By Theorem 2.1 the map $X^{\mathrm{op}} \boxtimes S^{\mathrm{op}} \to Y^{\mathrm{op}} \boxtimes S^{\mathrm{op}}$ is a weak equivalence. By Proposition 1.16 the map $X^{\mathrm{op}} \otimes S^{\mathrm{op}} \to Y^{\mathrm{op}} \otimes S^{\mathrm{op}}$, which is by [14, Lemma 131] the map $(S \otimes X)^{\mathrm{op}} \to (S \otimes Y)^{\mathrm{op}}$, is a weak equivalence. Using Lemma 2.4, the map $S \otimes X \to S \otimes Y$ is then a weak equivalence, as desired. $\square$

Using again Proposition 1.16 we obtain the following corollary.

Corollary 2.5. *Let $N \in \mathbb{N} \cup \{\infty\}$. The functor*

$$- \boxtimes - : ms\mathcal{S}et_{(\infty,N)} \times ms\mathcal{S}et_{(\infty,N)} \to ms\mathcal{S}et_{(\infty,N)}$$

is homotopical.

Since cofibrations in the model category $ms\mathcal{S}et_{(\infty,N)}$ are checked on the underlying simplicial set, we obtain the following corollary.

Corollary 2.6. *Let $N \in \mathbb{N} \cup \{\infty\}$. The functor*

$$- \boxtimes - : ms\mathcal{S}et_{(\infty,N)} \times ms\mathcal{S}et_{(\infty,N)} \to ms\mathcal{S}et_{(\infty,N)}$$

is a left Quillen bifunctor. In particular, for any simplicial set with marking S the adjunction

$$S \boxtimes - : ms\mathcal{S}et_{(\infty,N)} \rightleftarrows ms\mathcal{S}et_{(\infty,N)} : [S,-]_{lax}$$

is a Quillen pair.

2.1 The formal part of the proof In this subsection we prove Theorem 2.1 building on existing work of the third-named author and on a technical fact (Proposition 2.10) whose proof will be postponed until the last subsection.

Proposition 2.7. *Let $N \in \mathbb{N} \cup \{\infty\}$. For any $m \geq 0$ the pushout-pretensor*

$$(J \boxtimes \Delta[m]) \underset{I \boxtimes \Delta[m]}{\amalg} (I \boxtimes \Delta[m]_t) \to J \boxtimes \Delta[m]_t$$

of an (∞, N)-elementary anodyne extension $I \to J$ with the canonical map $\Delta[m] \hookrightarrow \Delta[m]_t$ is an (∞, ∞)-acyclic cofibration.

Proof. By [14, Lemma 140] the pushout-pretensor of two *entire* maps in the sense of [14, Notation 100], namely maps that are an isomorphism on the underlying simplicial sets, is an isomorphism. Hence, in particular the pushout-pretensor of a complicial thinness extension $\Delta^k[m]' \hookrightarrow \Delta^k[m]''$ with the canonical map $\Delta[m] \hookrightarrow \Delta[m]_t$ is an isomorphism. Moreover, it is explained in the proof of [14, Lemma 169] that the pushout-pretensor of a complicial horn extension $\Lambda^k[m] \hookrightarrow \Delta^k[m]$ with the canonical map $\Delta[m] \hookrightarrow \Delta[m]_t$ is an (∞, ∞)-acyclic cofibration. □

Proposition 2.8. *Let $N \in \mathbb{N} \cup \{\infty\}$. For any $m \geq 0$ the pushout-pretensor*

$$(J \boxtimes \partial\Delta[m]) \underset{I \boxtimes \partial\Delta[m]}{\amalg} (I \boxtimes \Delta[m]) \to J \boxtimes \Delta[m]$$

of an elementary (∞, N)-anodyne extension $I \to J$ with a boundary inclusion $\partial\Delta[m] \hookrightarrow \Delta[m]$ is an (∞, N)-acyclic cofibration.

Proof. We treat each type of elementary anodyne extension.

1. It is explained in the proof of [14, Lemma 143] that the pushout-pretensor of a thinness elementary anodyne extension $\Delta^k[m]' \hookrightarrow \Delta^k[m]''$ with a boundary inclusion is an (∞, ∞)-acyclic cofibration.
2. It is explained in the proof of [14, Lemma 169] that the pushout-pretensor of a complicial horn extension $\Lambda^k[m] \hookrightarrow \Delta^k[m]$ with a boundary inclusion is an (∞, ∞)-acyclic cofibration.
3. We will show in Proposition 2.10 that the pushout-tensor of a left saturation extension $\Delta[\ell|3_{eq}] \to \Delta[\ell|3_\sharp]$ with a boundary inclusion is an (∞, ∞)-acyclic cofibration. By Proposition 1.16 (together with the fact that the pushout that needs to be analyzed is in fact a homotopy pushout), this implies that also that the pushout-pretensor of a left saturation extension $\Delta[\ell|3_{eq}] \hookrightarrow \Delta[\ell|3_\sharp]$ with a boundary inclusion is an (∞, ∞)-acyclic cofibration.

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4. We will show in Proposition 2.9 that the pushout-pretensor of a triviality extension $\Delta[p] \to \Delta[p]_t$ for $p > N$ with a boundary inclusion is an (∞, N)-acyclic cofibration. $\square$

The proof above made use of the following two propositions.

Proposition 2.9. *Let $N \in \mathbb{N} \cup \{\infty\}$. For any $m \geq 0$ and $p > N$ the pushout-pretensor*

$$(\Delta[p]_t \otimes \partial\Delta[m]) \underset{\Delta[p] \otimes \partial\Delta[m]}{\amalg} (\Delta[p] \otimes \Delta[m]) \to \Delta[p]_t \otimes \Delta[m]$$

of an (∞, N)-triviality anodyne extension $\Delta[p] \to \Delta[p]_t$ with a boundary inclusion $\partial\Delta[m] \hookrightarrow \Delta[m]$ is an (∞, N)-acyclic cofibration.

Proof. The simplicial sets with marking $(\Delta[p]_t \otimes \partial\Delta[m]) \amalg_{\Delta[p] \otimes \partial\Delta[m]} (\Delta[p] \otimes \Delta[m])$ and $\Delta[p]_t \otimes \Delta[m]$ have the same underlying simplicial set, isomorphic to $\Delta[p] \times \Delta[m]$. We observe that they also have the same set of marked r-simplices for $r < p$. Indeed, the set of marked simplices in dimension $r < p$ is already contained in $\partial\Delta[p] \otimes \Delta[m]$. Moreover, for any r-simplex $\sigma \colon \Delta[r] \to \Delta[p]_t \otimes \Delta[m]$ for $r \geq p$ we can consider the map of simplicial sets

$$\Delta[r] \to (\Delta[p]_t \otimes \partial\Delta[m]) \underset{\Delta[p] \otimes \partial\Delta[m]}{\amalg} (\Delta[p] \otimes \Delta[m]),$$

and realize $\Delta[p]_t \otimes \Delta[m]$ as the pushout along the union of many triviality anodyne extensions:

$$
\begin{array}{ccc}
\coprod_\sigma \Delta[r] & \longrightarrow & \coprod_\sigma \Delta[r]_t \\
\downarrow & & \downarrow \\
(\Delta[p]_t \otimes \partial\Delta[m]) \underset{\Delta[p] \otimes \partial\Delta[m]}{\amalg} (\Delta[p] \otimes \Delta[m]) & \longrightarrow & \Delta[p]_t \otimes \Delta[m]
\end{array}
$$

In particular, the inclusion in question is an (∞, N)-acyclic cofibration, as desired. $\square$

Proposition 2.10. *Let $N \in \mathbb{N} \cup \{\infty\}$. For any $m \geq 0$ and $\ell \geq -1$ the pushout-tensor*

$$(\Delta[\ell|3_\sharp] \otimes \partial\Delta[m]) \underset{\Delta[\ell|3_{eq}] \otimes \partial\Delta[m]}{\amalg} (\Delta[\ell|3_{eq}] \otimes \Delta[m]) \to \Delta[\ell|3_\sharp] \otimes \Delta[m]$$

of a saturation anodyne extension $\Delta[\ell|3_{eq}] \to \Delta[\ell|3_\sharp]$ with a boundary inclusion $\partial\Delta[m] \hookrightarrow \Delta[m]$ is an (∞, ∞)-acyclic cofibration.

The proof of this proposition is postponed until the last section.

We can now prove the theorem.

Proof of Theorem 2.1. To see that $- \boxtimes S \colon ms\mathcal{S}et_{(\infty, N)} \to ms\mathcal{S}et_{(\infty, N)}$ is a left Quillen functor, we observe the following.

- Since the underlyng simplicial set of the pretensor of simplicial sets with marking is product of the underlying simplicial sets, if $X \to Y$ is a monomorphism, then $X \boxtimes S \to Y \boxtimes S$ is a monomorphism at the level of underlying simplicial sets. In particular, $- \boxtimes S$ preserves cofibrations.

- If $I \to J$ is an elementary anodyne extension, the map $I \boxtimes S \to J \boxtimes S$ can be written as the pushout product

$$J \boxtimes \Delta[-1] \underset{I \boxtimes \Delta[-1]}{\amalg} I \boxtimes S \to J \boxtimes S.$$

It can then be deduced from Propositions 2.7 and 2.8 using the compatibility of pushouts and pretensor product with colimits that the functor $- \boxtimes S$ sends all elementary anodyne extensions to weak equivalences.

By Proposition 1.10, we then conclude that the functor $- \boxtimes S$ is a left Quillen functor, as desired. $\qquad \square$

2.2 Proof of Proposition 2.10
In this subsection we provide the last missing verification.

Remark 2.11. A non-degenerate r-simplex $\sigma\colon \Delta[r] \to \Delta[\ell+4] \times \Delta[m]$ is marked in $\Delta[\ell|3_{eq}] \otimes \Delta[m]$ (resp. $\Delta[\ell|3_\sharp] \otimes \Delta[m]$) if and only if[4]

- the second projection $\mathrm{pr}_2 \sigma$ is degenerate (in particular there exists a maximal $1 \leq h \leq r$ such that $\mathrm{pr}_2 \sigma(h-1) = \mathrm{pr}_2 \sigma(h)$ and we call this h the *degeneracy index* of σ), and
- the partition face $\mathrm{II}_1^{h,r-h}$ of the first projection $(\mathrm{pr}_1 \sigma) \circ \mathrm{II}_1^{h,r-h}$ is marked in $\Delta[\ell|3_{eq}]$ (resp. $\Delta[\ell|3_\sharp]$).

Informally speaking, the degeneracy index h of a simplex σ is the maximal value for which $\sigma(h)$ is the final point of a horizontal piece in the path that describes the simplex σ.

Proof of Proposition 2.10. For simplicity of notation, we write

$$S_0 := (\Delta[\ell|3_\sharp] \otimes \partial\Delta[m]) \underset{\Delta[\ell|3_{eq}] \otimes \partial\Delta[m]}{\amalg} (\Delta[\ell|3_{eq}] \otimes \Delta[m]).$$

and we show by induction on ℓ that the map $S_0 \to \Delta[\ell|3_\sharp] \otimes \Delta[m]$ is an acyclic cofibration for any $m \geq 0$ and any $\ell \geq -1$.

The simplicial sets with marking S_0 and $\Delta[\ell|3_\sharp] \otimes \Delta[m]$ have the same underlying simplicial set, isomorphic to $\Delta[\ell+4] \times \Delta[m]$. By Remark 2.11, the r-simplices of $\Delta[\ell|3_\sharp] \otimes \Delta[m]$ that are not marked in S_0 are then characterized as follows: An r-simplex is marked in $\Delta[\ell|3_\sharp] \otimes \Delta[m]$ and not in S_0 if and only if

- the second projection $\mathrm{pr}_2 \sigma$ is surjective, so in particular $r \geq m$ and

$$(\mathrm{pr}_2 \sigma) \circ \mathrm{II}_2^{h,r-h} = \mathrm{id}_{\Delta[r-h]}\colon \Delta[r-h] \to \Delta[r-h],$$

and
- the partition face $\mathrm{II}_1^{h,r-h}$ of the first component $\mathrm{pr}_1 \sigma$ is of the form

$$(\mathrm{pr}_1 \sigma) \circ \mathrm{II}_1^{h,r-h} = \sigma' \star \sigma''\colon \Delta[h-2] \star \Delta[1] \to \Delta[\ell] \star \Delta[3]$$

with $\sigma'' \in \{[01], [03], [12], [23]\}$ and σ' non-degenerate.

We will now mark all simplices σ marked in $\Delta[\ell|3_\sharp] \otimes \Delta[m]$ and not in S_0 by constructing a sequence of entire acyclic cofibrations

$$S_0 \hookrightarrow S_1 \hookrightarrow S_2 \hookrightarrow S_3 \hookrightarrow S_4 \hookrightarrow S_5 \hookrightarrow S_6 \cong \Delta[\ell|3_\sharp] \otimes \Delta[m],$$

which will prove the lemma. More precisely, we will mark

---

[4]This reasoning is inspired by [14, Lemma 129].

1. in S_1 exactly all simplices σ marked in $\Delta[\ell|3_\sharp] \otimes \Delta[m]$ and not in S_0 that are contained in a copy of $\Delta[\ell-1|3_\sharp] \otimes \Delta[m] \hookrightarrow \Delta[\ell|3_\sharp] \otimes \Delta[m]$ by means of induction hypothesis if $\ell > -1$.

2. in S_2 all simplices σ marked in $\Delta[\ell|3_\sharp] \otimes \Delta[m]$ and not in S_1 with $\sigma'' \in \{[03],[23]\}$ (as well as other simplices) by means of saturation extensions. The generic simplex σ that is being marked in S_2 can be depicted as follows.
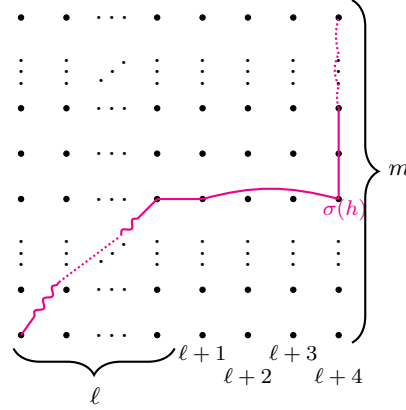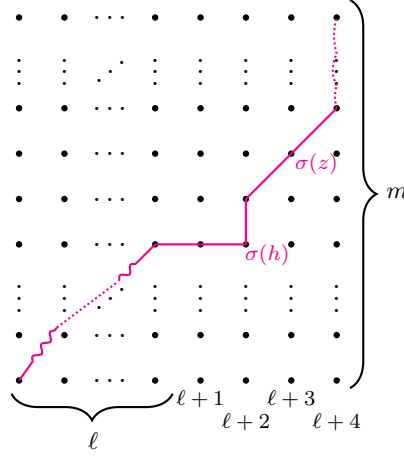
3. in S_3 exactly all simplices σ marked in $\Delta[\ell|3_\sharp] \otimes \Delta[m]$ and not in S_2 with $\sigma'' = [12]$ by means of thinness extensions. The generic simplex σ that is being marked in S_3 can be depicted as follows.
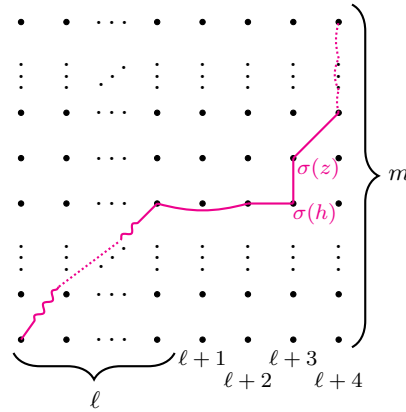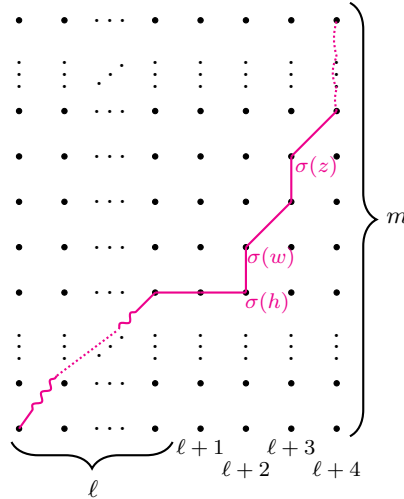
4. in S_4 exactly all simplices σ marked in $\Delta[\ell|3_\sharp] \otimes \Delta[m]$ and not in S_3 with $\sigma'' = [01]$ and $\mathrm{pr}_1\,\sigma$ hitting at most one of the values $l+3$ and $l+4$ by means of thinness extensions. The generic simplex σ that is being marked in S_4 can be depicted as follows.
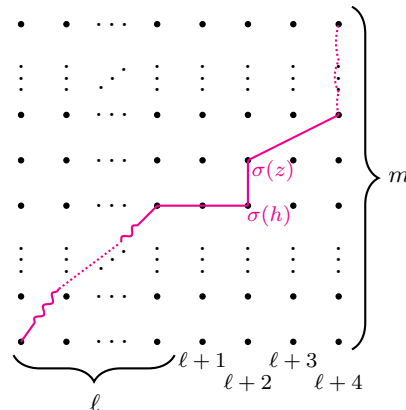
5. in S_5 exactly all simplices σ marked in $\Delta[\ell|3_\sharp] \otimes \Delta[m]$ and not in S_4 with $\sigma'' = [01]$ and $\mathrm{pr}_1\,\sigma$ hitting both $l+3$ and $l+4$, with last appearances of $l+2$ and $l+3$ in consecutive positions by means of thinness extensions. The generic simplex σ that is being marked in S_5 can be depicted as follows.

6. in S_6 exactly all simplices σ marked in $\Delta[\ell|3_\sharp] \otimes \Delta[m]$ and not in S_5 (which in particular have $\sigma'' = [01]$ and $\mathrm{pr}_1\,\sigma$ hitting both $\ell+3$ and $\ell+4$, with last appearances of $\ell+2$ and $\ell+3$ not in consecutive positions) by means of thinness extensions. The generic simplex σ that is being marked in S_6 can be depicted as follows.

We now proceed to explaining how to build the desired filtrations.

1. For $\ell = -1$ we set $S_1 = S_0$, and for $\ell > -1$ we will obtain S_1 from S_0 by marking exactly all simplices σ marked in $\Delta[\ell|3_\sharp] \otimes \Delta[m]$ and not in S_0 that are contained in a copy of $\Delta[\ell-1|3_\sharp] \otimes \Delta[m] \hookrightarrow \Delta[\ell|3_\sharp] \otimes \Delta[m]$. For each $0 \le i \le \ell$, we consider the map of simplicial sets with marking $\Delta[\ell-1|3_{eq}] \otimes \Delta[m] \to S_0$ induced by the i-th face, and we can then express the inclusions $S_0 \hookrightarrow S_1$ as the pushout with a disjoint union of the inclusion

$$\Delta[\ell-1|3_\sharp] \otimes \partial\Delta[m] \underset{\Delta[\ell-1|3_{eq}] \otimes \partial\Delta[m]}{\amalg} \Delta[\ell-1|3_{eq}] \otimes \Delta[m] \hookrightarrow \Delta[\ell-1|3_\sharp] \otimes \Delta[m]$$

which are acyclic cofibrations given by the induction hypothesis:

$$\coprod_{i\in[l]} \Delta[\ell-1|3_\sharp] \otimes \partial\Delta[m] \underset{\Delta[\ell-1|3_{eq}]\otimes\partial\Delta[m]}{\coprod} \Delta[\ell-1|3_{eq}] \otimes \Delta[m] \longrightarrow \coprod_{i\in[l]} \Delta[\ell-1|3_\sharp] \otimes \Delta[m]$$

$$\downarrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow$$

$$S_0 \xrightarrow{\hspace{6cm}} S_1.$$

In particular, $S_0 \hookrightarrow S_1$ is an acyclic cofibration. Moreover, we have an induced inclusion

$$S_1 \hookrightarrow \Delta[\ell|3_\sharp] \otimes \Delta[m].$$

2. We obtain S_2 from S_1 by marking in particular for $m \le r \le \ell+4+m$ all r-simplices σ marked in $\Delta[\ell|3_\sharp] \otimes \Delta[m]$ and not in S_1 with $\sigma'' \in \{[03],[23]\}$, as well as some additional simplices. For any $m \le r \le \ell+4+m$ and any

$$\vec{b} := (b_0 \le \ldots \le b_\ell)$$

an increasing sequence in $[0, \ell+4+m-r]$, we argue that the simplicial map

$$\varphi \colon \Delta[l] \star \Delta[3] \star \Delta[r-\ell-5] \to \Delta[\ell+4] \times \Delta[m]$$

defined by the formula

$$i \mapsto \begin{cases} (i, b_i) & \text{if } 0 \le i \le \ell \\ (i, \ell+4+m-r) & \text{if } \ell+1 \le i \le \ell+4 \\ (\ell+4, m-r+i) & \text{if } \ell+5 \le i \le r \end{cases}$$

is in particular a map of simplicial sets with marking $\Delta[\ell|3_{eq}|r-\ell-5] \to S_1$.
To see this, we suppose that

$$\gamma_1 \star \gamma_2 \star \gamma_3 \colon \Delta[r_1] \star \Delta[r_2] \star \Delta[r_3] \to \Delta[\ell] \star \Delta[3] \star \Delta[r-\ell-5]$$

is a generic marked and non-degenerate $(r_1+1+r_2+1+r_3)$-simplex of $\Delta[\ell]\star\Delta[3]\star\Delta[r-\ell-5]$ and we prove that the $(r_1+1+r_2+1+r_3)$-simplex of $\Delta[\ell+4]\times\Delta[m]$ defined by the composite of maps of simplicial sets

$$\Delta[r_1] \star \Delta[r_2] \star \Delta[r_3] \xrightarrow{\gamma_1\star\gamma_2\star\gamma_3} \Delta[\ell] \star \Delta[3] \star \Delta[r-\ell-5] \xrightarrow{\varphi} \Delta[\ell+4] \times \Delta[m]$$

is marked in $\Delta[\ell|3_{eq}] \otimes \Delta[m]$. Since $\gamma_1 \star \gamma_2 \star \gamma_3$ is marked, one amongst the γ_i's must be marked, and since moreover $\gamma_1 \star \gamma_2 \star \gamma_3$ is non-degenerate the simplex γ_2 must be marked in $\Delta[3]_{eq}$. By construction, the degeneracy index of the composite $\varphi \circ (\gamma_1 \star \gamma_2 \star \gamma_3)$ is $r_3 + 1$. Moreover, we see that the partition face $\text{II}_1^{r_1+1+r_2,r_3+1}$ of the first component of $\varphi\circ(\gamma_1\star\gamma_2\star\gamma_3)$ is of the form

$$\Delta[r_1] \star \Delta[r_2] \xrightarrow{\gamma_1\star\gamma_2} \Delta[\ell] \star \Delta[3] \to \Delta[\ell|3_{eq}]$$

and it is marked because it is the join of the marked simplex $\gamma_2 \colon \Delta[r_2] \to \Delta[3]_{eq}$ with another simplex of the form $\Delta[r_1] \to \Delta[\ell]$. This proves that the simplicial map φ does indeed preserve the marking.

We then define the inclusion $S_1 \hookrightarrow S_2$ as the pushout with the union of a family of saturation extensions (which are acyclic cofibrations by Lemma 1.9) of the form $\Delta[\ell|3_{eq}|r - \ell - 5] \to \Delta[\ell|3_\sharp|r - \ell - 5]$:

$$
\begin{array}{ccc}
\coprod_r \coprod_{\vec{b}} \Delta[\ell|3_{eq}|r - \ell - 5] & \longrightarrow & \coprod_r \coprod_{\vec{b}} \Delta[\ell|3_\sharp|r - \ell - 5] \\
\downarrow & & \downarrow \\
S_1 & \longrightarrow & S_2.
\end{array}
$$

In particular, $S_1 \hookrightarrow S_2$ is an acyclic cofibration and we have added all simplices σ marked in $\Delta[\ell|3_\sharp] \otimes \Delta[m]$ and not in S_1 with $\sigma'' \in \{[03], [23]\}$. Moreover, with a reasoning similar to the one producing the map $\Delta[\ell|3_{eq}|r - \ell - 5] \to S_1$, one can show that there is an induced map

$$
S_2 \hookrightarrow \Delta[\ell|3_\sharp] \otimes \Delta[m].
$$

3. We obtain S_3 from S_2 by marking for $m \le r \le \ell + 4 + m$ all r-simplices σ marked in $\Delta[\ell|3_\sharp] \otimes \Delta[m]$ and not in S_2 with $\sigma'' = [12]$. For any such σ in question there is a degeneracy index $h > 1$ and a unique maximal $h \le z \le r$ so that $\mathrm{pr}_1 \sigma(z) = \ell + 3$. In particular, $r - m \le z \le r$. The new markings will be added by constructing a sequence of acyclic cofibrations

$$
S_2 =: S_2^{(0)} \hookrightarrow S_2^{(1)} \hookrightarrow \cdots \hookrightarrow S_2^{(z)} \hookrightarrow S_2^{(z+1)} \hookrightarrow \cdots \hookrightarrow S_2^{(\ell+4+m)} =: S_3
$$

such that $S_2^{(z)}$ contains all missing markings for simplices of a given z. For any σ with a given z, the simplicial map

$$
\psi \colon \Delta[r + 1] \to \Delta[\ell + 4] \times \Delta[m]
$$

defined by the formula

$$
i \mapsto \begin{cases} \sigma(i) & \text{if } 0 \le i \le z \\ (\ell + 4, m - r + z) & \text{if } i = z + 1 \\ \sigma(i - 1) & \text{if } z + 1 < i \le r + 1 \end{cases}
$$

is in particular a map of simplicial sets with marking $\Delta^{z+1}[r + 1]' \to S_2^{(z-1)}$.
To see this, we consider a non-degenerate marked s-simplex $\tau \colon \Delta[s] \to \Delta[r+1]$ of $\Delta^{z+1}[r+1]'$, and we prove that the s-simplex of $\Delta[\ell+4] \times \Delta[m]$ defined by the composite of map of simplicial sets

$$
\psi \circ \tau \colon \Delta[s] \xrightarrow{\tau} \Delta[r + 1] \xrightarrow{\psi} \Delta[\ell + 4] \times \Delta[m]
$$

is marked in $S_2^{(z-1)}$.
- If τ contains $\{z, z + 1, z + 2\} \cap [r + 1]$, by construction the second projection of $\psi \circ \tau$ is degenerate, with degeneracy index being the preimage of $z + 1$ in $\Delta[s]$. Moreover, the face partition of the first component of $\psi \circ \tau$ contains the edge $[(\ell + 3)(\ell + 4)]$ in $\Delta[\ell + 4]$ and so the simplex $\psi \circ \tau$ is marked in S_2.
- If $\tau = d^{z+2}$, by construction the second projection of $\psi \circ \tau$ is not surjective (as it misses the value $m - r + z + 1$) and moreover degenerate, with degeneracy index being the preimage of $z + 1$ in $\Delta[r]$. Moreover, the face partition of the first component of $\psi \circ \tau$ hits at least a 1-dimensional simplex of $\Delta[3]$. In particular, $\psi \circ \tau$ is marked already in $\Delta[\ell|3_\sharp] \otimes \partial \Delta[m]$.

- If $\tau = d^z$, we distinguish two cases. If $z = h$, by construction the second projection of $\psi \circ \tau$ is degenerate, with degeneracy index $z = h$. Moreover, the face partition of the first component of $\psi \circ \tau$ contains the edge $[(\ell+2)(\ell+4)]$ in $\Delta[\ell+4]$ and so the simplex $\psi \circ \tau$ is marked in S_2. If $h < z$, by construction the second projection of $\psi \circ \tau$ is degenerate, with degeneracy index h. Moreover, the face partition of the first component of $\psi \circ \tau$ contains the edge $[(\ell+2)(\ell+3)]$ in $\Delta[\ell+4]$ and in fact the marking of the simplex $\psi \circ \tau$ was added in $S_2^{(z-1)}$.

This proves that the simplicial map ψ does indeed preserve the marking.

We then define the inclusion $S_2^{(z-1)} \hookrightarrow S_2^{(z)}$ as the pushout with several thinness extensions $\Delta^{z+1}[r+1]' \to \Delta^{z+1}[r+1]''$ (as many as r-simplices σ as z varies):

$$
\begin{array}{ccc}
\coprod_r \coprod_z \coprod_\sigma \Delta^{z+1}[r+1]' & \longrightarrow & \coprod_r \coprod_z \coprod_\sigma \Delta^{z+1}[r+1]'' \\
\downarrow & & \downarrow \\
S_2^{(z-1)} & \longrightarrow & S_2^{(z)}.
\end{array}
$$

In particular $S_2^{(z-1)} \hookrightarrow S_2^{(z)}$ is an acyclic cofibration. Moreover, by construction there is an induced map

$$S_2^{(z)} \hookrightarrow \Delta[\ell|3_\sharp] \otimes \Delta[m].$$

We then set $S_3 := S_2^{(r)}$, so that in particular $S_2 \hookrightarrow S_3$ is an acyclic cofibration and we have marked all simplices σ marked in $\Delta[\ell|3_\sharp] \otimes \Delta[m]$ and not in S_2 with $\sigma'' = [12]$. Moreover, by construction we have an induced map

$$S_3 \hookrightarrow \Delta[\ell|3_\sharp] \otimes \Delta[m]$$

4. We obtain S_4 from S_3 by marking for $m \le r \le \ell + 4 + m$ all r-simplices σ marked in $\Delta[\ell|3_\sharp] \otimes \Delta[m]$ and not in S_3 with $\sigma'' = [01]$ and $\mathrm{pr}_1 \sigma$ hitting at most one of the values $\ell + 3$ and $\ell + 4$. For any such σ in question there is a unique maximal $h \le z \le r$ so that $\mathrm{pr}_1 \sigma(z) = \ell + 2$. In particular, $r - m \le z \le r$. We will add the missing simplices by constructing a sequence of anodyne extensions

$$S_3 =: S_3^{(0)} \hookrightarrow S_3^{(1)} \hookrightarrow \cdots \hookrightarrow S_3^{(z-1)} \hookrightarrow S_3^{(z)} \hookrightarrow \cdots \hookrightarrow S_3^{(\ell+4+m)} =: S_4$$

such that $S_3^{(z)}$ contains all missing simplices for a given z. For any σ with a given z, the simplicial map

$$\psi \colon \Delta[r+1] \to \Delta[\ell+4] \times \Delta[m]$$

defined by the formula

$$
i \mapsto \begin{cases}
\sigma(i) & \text{if } 0 \le i \le z \\
(\ell+4, z) & \text{if } i = z+1 \text{ and } \mathrm{pr}_1 \sigma(z+1) = \ell+4 \text{ or } z = r, \\
(\ell+3, z) & \text{if } i = z+1 \text{ and } \mathrm{pr}_1 \sigma(z+1) = \ell+3, \\
\sigma(i-1) & \text{if } z+1 < i \le r+1
\end{cases}
$$

is in particular a map of simplicial sets with marking $\Delta^{z+1}[r+1]' \to S_3^{(z-1)}$.

To see this, we consider a non-degenerate marked s-simplex $\tau \colon \Delta[s] \to \Delta[r+1]$ of $\Delta^{z+1}[r+1]'$, and we prove that the s-simplex of $\Delta[\ell+4] \times \Delta[m]$ defined by the composite of maps of simplicial sets

$$\psi \circ \tau \colon \Delta[s] \xrightarrow{\tau} \Delta[r+1] \xrightarrow{\psi} \Delta[\ell+4] \times \Delta[m]$$

is marked in $S_3^{(z-1)}$.

- If τ contains $\{z, z+1, z+2\} \cap [r+1]$, by construction the second projection of $\psi \circ \tau$ is degenerate, with degeneracy index being the preimage of $z+1$ in $\Delta[s]$. Moreover, the face partition of the first component of $\psi \circ \tau$ contains the edge $[(\ell+2)(\ell+3)]$ or $[(\ell+2)(\ell+4)]$ in $\Delta[\ell+4]$ and so the simplex $\psi \circ \tau$ is marked in S_3.
- If $\tau = d^{z+2}$, by construction the second projection of $\psi \circ \tau$ is not surjective (as it misses the value $m - r + z + 1$) and moreover degenerate, with degeneracy index being the preimage of $z+1$ in $\Delta[r]$. Moreover, the face partition of the first component of $\psi \circ \tau$ hits at least a 1-dimensional simplex of $\Delta[3]$. In particular, $\psi \circ \tau$ is marked already in $\Delta[\ell|3_\sharp] \otimes \partial\Delta[m]$.
- If $\tau = d^z$, we distinguish two cases. If $h = z$, by construction the second projection of $\psi \circ \tau$ is degenerate, with degeneracy index $h = z$. Moreover, the face partition of the first component of $\psi \circ \tau$ contains the edge $[(\ell+1)(\ell+3)]$ or $[(\ell+1)(\ell+4)]$ in $\Delta[\ell+4]$ and so the simplex $\psi \circ \tau$ is marked in S_2. If $h < z$, by construction the second projection of $\psi \circ \tau$ is degenerate, with degeneracy index h. Moreover, the face partition of the first component of $\psi \circ \tau$ contains the edge $[(\ell+1)(\ell+2)]$ in $\Delta[\ell+4]$ and in fact the marking of the simplex $\psi \circ \tau$ was added in $S_3^{(z-1)}$.

This proves that the simplicial map ψ does indeed preserve the marking.

We then define the inclusion $S_3^{(z-1)} \hookrightarrow S_3^{(z)}$ as the pushout with many thinness anodyne extensions $\Delta^{z+1}[r+1]' \to \Delta^{z+1}[r+1]''$ (as many as r-simplices σ as z varies):

$$
\begin{array}{ccc}
\coprod_r \coprod_z \coprod_\sigma \Delta^{z+1}[r+1]' & \longrightarrow & \coprod_r \coprod_z \coprod_\sigma \Delta^{z+1}[r+1]'' \\
\downarrow & & \downarrow \\
S_3^{(z-1)} & \longrightarrow & S_3^{(z)}.
\end{array}
$$

In particular $S_3^{(z-1)} \hookrightarrow S_3^{(z)}$ is an acyclic cofibration. Moreover, we have an induced map

$$
S_3^{(z)} \hookrightarrow \Delta[\ell|3_\sharp] \otimes \Delta[m]
$$

We then set $S_4 := S_3^{(\ell+4+m)}$, so that in particular $S_3 \hookrightarrow S_4$ is an acyclic cofibration and we have marked all simplices σ marked in $\Delta[\ell|3_\sharp] \otimes \Delta[m]$ and not in S_3 with $\sigma'' = [01]$ and $\mathrm{pr}_1 \sigma$ hitting at most one of the values $\ell+3$ and $\ell+4$. Moreover, we have an induced map

$$
S_4 \hookrightarrow \Delta[\ell|3_\sharp] \otimes \Delta[m].
$$

5. We obtain S_5 from S_4 by marking for $m \leq r \leq \ell+4+m$ all r-simplices σ marked in $\Delta[\ell|3_\sharp] \otimes \Delta[m]$ and not in S_4 with $\sigma'' = [01]$ and $\mathrm{pr}_1 \sigma$ hitting both $\ell+3$ and $\ell+4$, with last appearances of $\ell+2$ and $\ell+3$ in consecutive positions. More precisely, for any such σ in question there is a unique maximal $h+1 \leq z \leq r-1$ so that $\mathrm{pr}_1 \sigma(z) = \ell+3$, and by assumption $\mathrm{pr}_1 \sigma(z+1) = \ell+4$ and $\mathrm{pr}_2 \sigma(z-1) = \ell+2$. In particular, $r-m+1 \leq z \leq \ell+3+m$. For any σ with a given z, the simplicial map

$$
\psi \colon \Delta[r+1] \to \Delta[\ell+4] \times \Delta[m]
$$

defined by the formula

$$
i \mapsto \begin{cases} \sigma(i) & \text{if } 0 \leq i \leq z \\ (m-r+z, \ell+4) & \text{if } i = z+1 \\ \sigma(i-1) & \text{if } z+1 < i \leq r+1 \end{cases}
$$

is in particular a map of simplicial sets with marking $\Delta^{z+1}[r+1]' \to S_4$.

To see this, we consider a non-degenerate marked s-simplex $\tau\colon \Delta[s] \to \Delta[r+1]$ of $\Delta^{z+1}[r+1]'$, and we prove that the s-simplex of $\Delta[\ell+4] \times \Delta[m]$ defined by the composite of maps of simplicial sets

$$\psi \circ \tau\colon \Delta[s] \xrightarrow{\tau} \Delta[r+1] \xrightarrow{\psi} \Delta[\ell+4] \times \Delta[m]$$

is marked in S_4.

- If τ contains $\{z, z+1, z+2\} \cap [r+1]$, by construction the second projection of $\psi \circ \tau$ is degenerate, with degeneracy index being the preimage of $z+1$ in $\Delta[s]$. Moreover, the face partition of the first component of $\psi \circ \tau$ contains the edge $[(\ell+3)(\ell+4)]$ in $\Delta[\ell+4]$ and so the simplex $\psi \circ \tau$ is marked in S_2.

- If $\tau = d^{z+2}$, by construction the second projection of $\psi \circ \tau$ is not surjective (as it misses the value $m - r + z + 1$) and moreover degenerate, with degeneracy index being the preimage of $z+1$ in $\Delta[r]$. Moreover, the face partition of the second component of $\psi \circ \tau$ hits at least a 1-dimensional simplex of $\Delta[3]$. In particular, $\psi \circ \tau$ is marked already in $\Delta[\ell|3_\sharp] \otimes \partial\Delta[m]$.

- If $\tau = d^z$, by construction the second projection of $\psi \circ \tau$ is degenerate, with degeneracy index h. Moreover, the face partition of the first component of $\psi \circ \tau$ contains the edge $[(\ell+1)(\ell+2)]$ in $\Delta[\ell+4]$ and does not hit $\ell+3$, so the marking of the simplex $\psi \circ \tau$ was added in S_4.

This proves that the simplicial map ψ does indeed preserve the marking.

We define the inclusion $S_4 \hookrightarrow S_5$ as the pushout with several thinness extensions $\Delta^{z+1}[r+1]' \to \Delta^{z+1}[r+1]''$ (as many as r-simplices σ as z varies):

$$
\begin{array}{ccc}
\coprod_r \coprod_z \coprod_\sigma \Delta^{z+1}[r+1]' & \longrightarrow & \coprod_r \coprod_z \coprod_\sigma \Delta^{z+1}[r+1]'' \\
\downarrow & & \downarrow \\
S_4 & \longrightarrow & S_5.
\end{array}
$$

In particular $S_4 \hookrightarrow S_5$ is an acyclic cofibration and we have marked all simplices in $\Delta[\ell|3_\sharp] \otimes \Delta[m]$ and not in S_4 with $\sigma'' = [01]$ and $\mathrm{pr}_1\, \sigma$ hitting both the values $\ell+3$ and $\ell+4$, with last appearances of $\ell+2$ and $\ell+3$ in consecutive positions. Moreover, we have an induced map

$$S_5 \hookrightarrow \Delta[\ell|3_\sharp] \otimes \Delta[m].$$

6. We obtain S_6 from S_5 by marking for $m \leq r \leq \ell+4+m$ all missing r-simplices with $\sigma'' = [01]$ and $\mathrm{pr}_1\, \sigma$ hitting both $\ell+3$ and $\ell+4$, with last appearances of $\ell+2$ and $\ell+3$ not in consecutive positions. More precisely, for any such σ in question there is a unique maximal $h < z < r$ so that $\mathrm{pr}_1\, \sigma(z) = \ell+3$. In particular, $r - m \leq z \leq \ell+3+m$. We will add them by constructing a sequence of acyclic cofibrations

$$S_5 =: S_5^{(r-m)} \hookrightarrow S_5^{(r-m+1)} \hookrightarrow \cdots \hookrightarrow S_5^{(z-1)} \hookrightarrow S_5^{(z)} \hookrightarrow \cdots \hookrightarrow S_5^{(\ell+3+m)} =: S_6$$

such that $S_5^{(z)}$ contains all missing simplices for a given z. For any σ with a given z, the simplicial map

$$\psi\colon \Delta[r+1] \to \Delta[\ell+4] \times \Delta[m]$$

defined by the formula

$$
i \mapsto
\begin{cases}
\sigma(i) & \text{if } 0 \leq i \leq z \\
(\ell+4, m-r+z) & \text{if } i = z+1 \\
\sigma(i-1) & \text{if } z+1 < i \leq r+1
\end{cases}
$$

is in particular a map of simplicial sets with marking $\Delta^{z+1}[r+1]' \to S_5^{(z-1)}$.

To see this, we consider a non-degenerate marked s-simplex $\tau\colon \Delta[s] \to \Delta[r+1]$ of $\Delta^{z+1}[r+1]'$, and we prove that the s-simplex of $\Delta[\ell+4] \times \Delta[m]$ defined by the composite of maps of simplicial sets

$$\psi \circ \tau \colon \Delta[s] \xrightarrow{\tau} \Delta[r+1] \xrightarrow{\psi} \Delta[\ell+4] \times \Delta[m]$$

is marked in $S_5^{(z-1)}$.

- If τ contains $\{z, z+1, z+2\} \cap [r+1]$, by construction the second projection of $\psi \circ \tau$ is degenerate, with degeneracy index being the preimage of $z+1$ in $\Delta[s]$. Moreover, the face partition of the first component of $\psi \circ \tau$ contains the edge $[(\ell+3)(\ell+4)]$ in $\Delta[\ell+4]$ and so the simplex $\psi \circ \tau$ is marked in S_2.

- If $\tau = d^{z+2}$, by construction the second projection of $\psi \circ \tau$ is not surjective (as it misses the value $m - r + z + 1$) and moreover degenerate, with degeneracy index being the preimage of $z+1$ in $\Delta[r]$. Moreover, the face partition of the first component of $\psi \circ \tau$ hits at least a 1-dimensional simplex of $\Delta[3]$. In particular, $\psi \circ \tau$ is marked already in $\Delta[\ell|3_\sharp] \otimes \partial\Delta[m]$.

- If $\tau = d^z$, we distinguish two cases depending on the value of w, being the maximal value for which $\mathrm{pr}_1 \sigma(w) = \ell + 2$. By assumption, $h \le w < z - 1$. If $w = z - 2$, by construction the second projection of $\psi \circ \tau$ is degenerate, with degeneracy index h. Moreover, the face partition of the first component of $\psi \circ \tau$ contains the edge $[(\ell+1)(\ell+2)]$ in $\Delta[\ell+4]$ and hits $\ell+2$ and $\ell+3$ in consecutive positions for the last time and so the marking of $\psi \circ \tau$ was added in S_5. If $w < z - 2$, by construction the second projection of $\psi \circ \tau$ is degenerate, with degeneracy index h. Moreover, the face partition of the second component of $\psi \circ \tau$ contains the edge $[(\ell+1)(\ell+2)]$ in $\Delta[\ell+4]$ and in fact the marking of the simplex $\psi \circ \tau$ was added in $S_5^{(z-1)}$.

This proves that the simplicial map ψ does indeed preserve the marking.

We then define the inclusion $S_5^{(z-1)} \hookrightarrow S_5^{(z)}$ as the pushout with several thinness extensions $\Delta^{z+1}[r+1]' \to \Delta^{z+1}[r+1]''$ (as many as r-simplices σ as z varies):

$$\begin{array}{ccc}
\coprod_r \coprod_z \coprod_\sigma \Delta^{z+1}[r+1]' & \longrightarrow & \coprod_r \coprod_z \coprod_\sigma \Delta^{z+1}[r+1]'' \\
\downarrow & & \downarrow \\
S_5^{(z-1)} & \longrightarrow & S_5^{(z)}.
\end{array}$$

In particular $S_5^{(z-1)} \hookrightarrow S_5^{(z)}$ is an acyclic cofibration. Moreover, we have an induced map

$$S_5^{(z)} \hookrightarrow \Delta[\ell|3_\sharp] \otimes \Delta[m].$$

We then set $S_6 := S_5^{(\ell+3+m)}$, so that in particular $S_5 \hookrightarrow S_6$ is an acyclic cofibration and we have marked all simplices σ marked in $\Delta[\ell|3_\sharp] \otimes \Delta[m]$ and not in S_5 with $\sigma'' = [01]$ and $\mathrm{pr}_1 \sigma$ hitting both the values $\ell+3$ and $\ell+4$, with last appearances of $l+2$ and $l+3$ not in consecutive positions. In particular, we have an isomorphism

$$S_6 \cong \Delta[\ell|3_\sharp] \otimes \Delta[m].$$

This concludes the proof. $\qquad\square$

Acknowledgements

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