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Finite duals in Grothendieck categories and coalgebra objects

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Abstract

We develop a theory of coalgebra objects and comodules that is internal to any k-linear Grothendieck category, where k is a commutative noetherian ring. We begin with a counterpart in k-linear Grothendieck categories for the finite dual construction of a k-algebra and the comodules over it. In the second part of the paper, we construct "coalgebra objects" inside a Grothendieck category. These are not coalgebras in an explicit sense, but enjoy several categorical properties arising in the classical theory of coalgebras, such as those of semiperfect or quasi-co-Frobenius coalgebras. In particular, this construction works in any Grothendieck category and there is no need for a monoidal structure in order to define these coalgebra objects.

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1. Introduction

Much of the development of notions such as abelian categories or Grothendieck categories is motivated by the study of module categories. Further, their categorical properties often correspond to ring-theoretic notions. For instance, the modules over a ring form a locally noetherian category if and only if the ring itself is noetherian. This approach allows us to replace a ring by the study of its category of modules. Since a Grothendieck category is like a generalized module category, it has a number of similar properties. For example, every Grothendieck category has a well-behaved theory of essential extensions and injective envelopes, every injective in a locally noetherian Grothendieck category splits as a direct sum of indecomposable injectives, and so on.

The aim of this paper is to construct categories that behave like the category of comodules over a coalgebra. In fact, we will show that there is a parallel for the classical theory of coalgebras and

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comodules that is internal to any Grothendieck category. Since Grothendieck categories abound in the literature, the resulting theory is very general and applies to numerous situations, such as abelian sheaves over a site, quasi-coherent sheaves over a scheme or comodules over a flat Hopf algebroid. Our starting point is as follows. If A is an algebra over a field k, an A-module M is said to be locally finite if any cyclic submodule of M is finite dimensional as a k-vector space. Quite strikingly, these locally finite A-modules can be expressed as comodules over a certain coalgebra A^0 known as the finite dual of A (see, for instance, [6, § 1.5]). In other words, any such module category has a category of comodules that is canonically associated to it and embedded inside it. We first ask if there is a counterpart of this construction when we replace the module category by a general Grothendieck category.

Let k be a commutative noetherian ring and let $\mathscr{A}$ be a k-linear Grothendieck category. The first step is to determine what it means for an object of $\mathscr{A}$ to be finite over k. Towards this, we say that $X \in \mathscr{A}$ is k-finite if the collection of morphisms $\mathscr{A}(F, X)$ is a finitely generated k-module for any finitely generated object F of $\mathscr{A}$. We say that X is strongly k-finite if every quotient of X is k-finite. The property of being strongly k-finite may be established by looking at the collections of morphisms with respect to any set of generators for $\mathscr{A}$. For other notions of finiteness in abelian categories, see for instance, [21, § 4]. We also show that strongly k-finite objects are closed under subobjects, quotients and extensions in $\mathscr{A}$.

We denote by $\mathscr{A}^{\mathcal{T}_k}$ the full subcategory of locally strongly k-finite objects in $\mathscr{A}$, i.e., those which can be written as a sum of their strongly k-finite subobjects. As with locally finite modules in module categories, $\mathscr{A}^{\mathcal{T}_k}$ is a coreflective subcategory of $\mathscr{A}$, i.e., the inclusion $\mathscr{A}^{\mathcal{T}_k} \hookrightarrow \mathscr{A}$ has a right adjoint $R^{\mathcal{T}_k} : \mathscr{A} \longrightarrow \mathscr{A}^{\mathcal{T}_k}$. We show that $\mathscr{A}^{\mathcal{T}_k}$ is a Grothendieck category. If $\mathscr{A}$ is locally finitely generated (resp. locally noetherian), so is $\mathscr{A}^{\mathcal{T}_k}$.

Our idea is that even though $\mathscr{A}^{\mathcal{T}_k}$ is not necessarily a category of comodules over a coalgebra, a number of properties of comodule categories have counterparts in $\mathscr{A}^{\mathcal{T}_k}$, especially with respect to injective objects and essential monomorphisms. For instance, any injective in $\mathscr{A}^{\mathcal{T}_k}$ is a direct summand of an injective of the form $R^{\mathcal{T}_k}(E)$ for some injective $E \in \mathscr{A}$. This corresponds to the fact that any injective comodule over a coalgebra C over a field k is a direct summand of a direct sum of copies of C. Our first main objective is to give conditions for $\mathscr{A}^{\mathcal{T}_k}$ to be closed under certain kinds of essential extensions in $\mathscr{A}$. For this, as with comodule categories, one needs an understanding of the simple objects in $\mathscr{A}^{\mathcal{T}_k}$. We fix a finitely generated object $F \in \mathscr{A}$ and set $el_F(M) := \mathscr{A}(F, M)$ for any $M \in \mathscr{A}$, which we think of as "elements of M over F." We say that an object $M \in \mathscr{A}$ is F-simple if $el_F(M) \neq 0$ and any $0 \neq N \subseteq M$ satisfies $el_F(N) = el_F(M)$. We say that $M \in \mathscr{A}$ is F-saturated if $el_F(N) = 0$ for a subobject $N \subseteq M$ implies $N = 0$. When k is a field, we show that any F-saturated $M \in \mathscr{A}^{\mathcal{T}_k}$ is an essential extension of its F-socle $soc_F(M)$. If $\mathscr{A}$ is locally noetherian, we show that the injective envelope $\mathcal{E}(M)$ of M in $\mathscr{A}$ decomposes as a direct sum of injective envelopes of F-simple subobjects. Using this, it is proved here that

Theorem 1.1. *Let k be a field and let $\mathscr{A}$ be a locally noetherian k-linear Grothendieck category. Let $\mathscr{C} \subseteq \mathscr{A}^{\mathcal{T}_k}$ be a closed subcategory. Let $F \in \mathscr{A}$ be a finitely generated object. Then, the following are equivalent*

(a) For any essential extension $M \subseteq N$ with $M \in \mathscr{C}$ and N an F-saturated object, we must have $N \in \mathscr{C}$.

(b) The injective envelope in $\mathscr{A}$ of any F-simple and strongly k-finite object $X \in \mathscr{C}$ lies in $\mathscr{C}$.

If we suppose that F is a generator for $\mathscr{A}$ and take $\mathscr{C} = \mathscr{A}^{\mathcal{T}_k}$ in Theorem 1.1, we obtain that

$\mathscr{A}^{\mathcal{T}_k}$ is essentially closed if and only if every the injective envelope of every simple and strongly k-finite object $X \in \mathscr{A}^{\mathcal{T}_k}$ is locally strongly k-finite. In [14, § 2] , Hatipoğlu and Lomp showed that the locally finite modules over a noetherian algebra A are closed under essential extensions if and only if the injective envelope of any finite dimensional simple A-module is locally finite. We note that the property of locally finite modules being essentially closed is related to a number of classical results on solvable groups, solvable Lie algebras and Weyl algebras (see, for instance, [7], [11], [12]). For more on injective envelopes of simple modules and their relations to essential closedness, we refer the reader to [3], [4], [5], [13], [18]).

In the second part of the paper, we construct "coalgebra objects" in $\mathscr{A}$. These are not coalgebras in an explicit sense, but enjoy several categorical properties arising in the classical theory of coalgebras. If C is an ordinary k-coalgebra, the forgetful functor from C-comodules to k-modules has a right adjoint that takes a k-module V to the comodule $C \otimes_k V$. This means that the coalgebra C may be recovered by evaluating this functor at k. Motivated by this, we fix $F \in \mathscr{A}$ that is finitely generated and projective. Then, we observe that the functor

$$\mathscr{A}(F,-) : \mathscr{A}^{\mathcal{T}_k} \longrightarrow Mod_k \tag{1}$$

preserves all colimits. Since $\mathscr{A}^{\mathcal{T}_k}$ and Mod_k are Grothendieck categories, this functor must have a right adjoint $R_F^{\mathcal{T}_k} : Mod_k \longrightarrow \mathscr{A}^{\mathcal{T}_k}$. We refer to $C(F) := R_F^{\mathcal{T}_k}(k)$ as the coalgebra object associated to F. We note that this construction works in any Grothendieck category and there is no need for any monoidal structure on $\mathscr{A}$ in order to define coalgebra objects.

The full subcategory $Sub^{\oplus}(C(F))$ consisting of all objects in $\mathscr{A}$ that embed into a direct sum of copies of $C(F)$ now plays the role of comodules over the coalgebra object $C(F)$. This corresponds to the fact that any comodule over a coalgebra C over a field embeds into a direct sum of copies of C. We show that any object in $Sub^{\oplus}(C(F))$ is F-saturated. Further, if k is a field, we show that $C(F)$ contains the injective envelope $\mathcal{E}^{\mathcal{T}_k}(M)$ in $\mathscr{A}^{\mathcal{T}_k}$ of any F-simple object $M \in Sub^{\oplus}(C(F))$ and that $C(F)$ itself decomposes as a direct sum of such injective envelopes. We also notice that the right adjoint of the functor in (1) gives for any $M \in \mathscr{A}^{\mathcal{T}_k}$

$$el_F(M)^* := Mod_k(el_F(M),k) = Mod_k(\mathscr{A}(F,M),k) = \mathscr{A}^{\mathcal{T}_k}(M, R_F^{\mathcal{T}_k}(k)) = \mathscr{A}(M, C(F)) \tag{2}$$

In particular, $el_F(M)^* = \mathscr{A}(M, C(F))$ is a module over $\mathscr{A}(C(F), C(F))$ in a canonical manner and we write $C(F)^*$ for the k-algebra $\mathscr{A}(C(F), C(F))$. This is a counterpart of the fact that for any k-coalgebra C, the dual $C^* = Mod_k(C,k)$ is a k-algebra and that the dual $N^* = Mod_k(N,k)$ of any C-comodule N carries the structure of a C^*-module. We now show that

Theorem 1.2. *Let k be a field and let $\mathscr{A}$ be a locally noetherian k-linear Grothendieck category. Let F be a finitely generated projective object in $\mathscr{A}$ and let $\mathscr{B} \subseteq {}_{C(F)^*}Mod$ be a closed subcategory. Let $t_{\mathscr{B}}$ be the functor that associates a $C(F)^*$-module to the sum of its submodules which lie in $\mathscr{B}$. Then, the following are equivalent:*

(a) $t_{\mathscr{B}}(C(F)^*)$ is dense in $C(F)$.

(b) If $M \in Sub^{\oplus}(C(F))$ is F-simple, then $t_{\mathscr{B}}(el_F(\mathcal{E}^{\mathcal{T}_k}(M))^*)$ is dense in $\mathcal{E}^{\mathcal{T}_k}(M)$.

(c) If $M \in Sub^{\oplus}(C(F))$ is injective in $\mathscr{A}^{\mathcal{T}_k}$, then $t_{\mathscr{B}}(el_F(M)^*)$ is dense in M.

(d) $t_{\mathscr{B}}(el_F(M)^*)$ is dense in M for any $M \in Sub^{\oplus}(C(F))$.

If we take $\mathscr{B}$ to be the subcategory of locally finite $C(F)^*$-modules, Theorem 1.2 gives a number of equivalent conditions for the locally finite submodule of $C(F)^*$ to be dense in $C(F)$ (see

Definition 5.10). We see that these are parallel to the usual equivalent conditions for a coalgebra over a field to be semiperfect (see [6, § 3.2.1], [16]). Our final step is to give a number of sufficient conditions for a coalgebra object $C(F)$ to be "semiperfect" in this sense. Among these is requiring $C(F)$ to be projective in $\mathscr{A}^{\mathcal{T}_k}$, which we show to be equivalent to all injectives in $Sub^{\oplus}(C(F)) \subseteq \mathscr{A}^{\mathcal{T}_k}$ also being projective. We note that this last condition is the counterpart of the classical notion of a quasi-co-Frobenius coalgebra.

2. Finite objects and strongly finite objects

We let k be a commutative noetherian ring and let Mod_k be the category of k-modules. Let $\mathscr{A}$ be a k-linear Grothendieck category. We recall that an object $X \in \mathscr{A}$ is said to be finitely generated if the functor $\mathscr{A}(X, -)$ preserves filtered colimits of monomorphisms. Using the fact that the morphism sets in $\mathscr{A}$ are modules over k, we now consider the following two notions.

Definition 2.1. Let k be a commutative noetherian ring and let $\mathscr{A}$ be a k-linear Grothendieck category. We will say that:

(a) An object $X \in \mathscr{A}$ is k-finite if $\mathscr{A}(F, X)$ is a finitely generated k-module for each finitely generated object $F \in \mathscr{A}$.

(b) An object $X \in \mathscr{A}$ is strongly k-finite if $\mathscr{A}(F, Y)$ is a finitely generated k-module for each finitely generated object $F \in \mathscr{A}$ and each quotient $X \twoheadrightarrow Y$ of X.

We let $\mathcal{S}_k(\mathscr{A})$ (resp. $\mathcal{T}_k(\mathscr{A})$) denote the collection of k-finite objects (resp. strongly k-finite objects) in $\mathscr{A}$.

Lemma 2.2. *(a) The collection $\mathcal{S}_k(\mathscr{A})$ of k-finite objects in $\mathscr{A}$ is closed under subobjects.*

(b) The collection $\mathcal{T}_k(\mathscr{A})$ of strongly k-finite objects in $\mathscr{A}$ is closed under subobjects and quotients.

Proof. Let $F \in \mathscr{A}$ be a finitely generated object and let $X' \hookrightarrow X$ be an inclusion in $\mathscr{A}$. Suppose that X is a k-finite object. Then, we have an induced monomorphism $\mathscr{A}(F, X') \hookrightarrow \mathscr{A}(F, X)$ of k-modules. By assumption, $\mathscr{A}(F, X)$ is finitely generated and k is noetherian, and hence $\mathscr{A}(F, X')$ is finitely generated as a k-module. This proves (a).

To prove (b), suppose that X is a strongly k-finite object. It is immediate from Definition 2.1 that $\mathcal{T}_k(\mathscr{A})$ is closed under quotients. Now let $X' \hookrightarrow X$ be a subobject and let $X' \twoheadrightarrow Y'$ be an epimorphism having kernel K. Then, $Y' = X'/K \hookrightarrow X/K$, which gives us an inclusion $\mathscr{A}(F, Y') \hookrightarrow \mathscr{A}(F, X/K)$ of k-modules. Since X is strongly k-finite, $\mathscr{A}(F, X/K)$ is a finitely generated k-module. Since k is noetherian, it follows that $\mathscr{A}(F, Y')$ is finitely generated as a k-module. This shows that the quotient $X' \in \mathcal{T}_k(\mathscr{A})$. □

Before we proceed further, we show that the properties of being k-finite or strongly k-finite may be established by testing with respect to a set of generators of $\mathscr{A}$.

Lemma 2.3. *Let $X \in \mathscr{A}$. Suppose there exists a set of generators $\{G_i\}_{i \in I}$ for $\mathscr{A}$ such that each $\mathscr{A}(G_i, X)$ is finitely generated as a k-module. Then, $X \in \mathscr{A}$ is k-finite. The converse holds if $\mathscr{A}$ is locally finitely generated.*

Proof. Let $F \in \mathscr{A}$ be a finitely generated object. Then, we can find a finite direct sum $\bigoplus\limits_{j \in J} G_j$ of copies of objects in $\{G_i\}_{i \in I}$ along with an epimorphism $\bigoplus\limits_{j \in J} G_j \twoheadrightarrow F$ in $\mathscr{A}$. This induces an inclusion $\mathscr{A}(F,X)\hookrightarrow \mathscr{A}(\bigoplus_{j\in J} G_j,X)=\bigoplus_{j\in J}\mathscr{A}(G_j,X)$. By assumption, each $\mathscr{A}(G_j,X)$ is finitely generated as a k-module. Since J is finite and k is noetherian, it follows that $\mathscr{A}(F,X)$ is finitely generated as a k-module. Hence, $X\in\mathcal{S}_k(\mathscr{A})$.

Conversely, suppose that $\mathscr{A}$ is locally finitely generated. By definition, we may choose a set $\{G_i\}_{i\in I}$ of finitely generated generators for $\mathscr{A}$. Then if $X\in\mathscr{A}$ is k-finite, each $\mathscr{A}(G_i,X)$ is finitely generated as a k-module. □

Lemma 2.4. *Let $X\in\mathscr{A}$. Suppose there exists a set of generators $\{G_i\}_{i\in I}$ for $\mathscr{A}$ such that each $\mathscr{A}(G_i,Y)$ is a finitely generated k-module for every quotient Y of X. Then, $X\in\mathscr{A}$ is strongly k-finite. The converse holds if $\mathscr{A}$ is locally finitely generated.*

Proof. This result follows as in the proof of Lemma 2.3, by using quotients of X in place of X. □

Proposition 2.5. *Suppose that $\mathscr{A}$ has a set $\{G_i\}_{i\in I}$ of finitely generated projective generators. Then, $X\in\mathscr{A}$ is k-finite if and only if $X\in\mathscr{A}$ is strongly k-finite.*

Proof. Let $X\in\mathcal{S}_k(\mathscr{A})$ and let $X\twoheadrightarrow Y$ be a quotient of X. Each $\mathscr{A}(G_i,X)$ is a finitely generated k-module. Additionally, since each G_i is projective, we have an epimorphism $\mathscr{A}(G_i,X)\twoheadrightarrow\mathscr{A}(G_i,Y)$ which shows that $\mathscr{A}(G_i,Y)$ is finitely generated. Since $\mathscr{A}$ is locally finitely generated, it now follows by applying Lemma 2.4 that $X\in\mathcal{T}_k(\mathscr{A})$. □

Proposition 2.6. *Suppose that $\mathscr{A}$ has a finitely generated generator G. Then, any $X\in\mathcal{S}_k(\mathscr{A})$ is also finitely generated as an object of $\mathscr{A}$.*

Proof. Let $X\in\mathcal{S}_k(\mathscr{A})$ and let $\{X_j\}_{j\in J}$ be a filtered system of finitely generated subobjects of X whose direct union is X. Since $G\in\mathscr{A}$ is finitely generated, we obtain $\varinjlim_{j\in J}\mathscr{A}(G,X_j)=\mathscr{A}(G,X)$ in the category of k-modules. Since $\mathscr{A}(G,X)$ is finitely generated and k is noetherian, it follows that there exists $j_0\in J$ such that the inclusion $X_{j_0}\hookrightarrow X$ induces an isomorphism $\mathscr{A}(G,X_{j_0})\xrightarrow{\cong}\mathscr{A}(G,X)$. Finally, since G is a generator for $\mathscr{A}$, this shows that $X_{j_0}=X$. □

Remark 2.7. *(a) Let A be a k-algebra and let $\mathscr{A}$ be the category Mod_A of (right) A-modules. Then, Mod_A is locally finitely generated, having A as a finitely generated generator. By Lemma 2.3, it follows that the k-finite objects in Mod_A are the A-modules which are finitely generated as k-modules.*

(b) We know (see, for instance, [8, § 2]) that a Grothendieck category which has a set of finitely generated projective generators is identical to the category of modules over a "ring with several objects" in the sense of Mitchell [17]. In other words, $\mathscr{A}=Mod-\mathscr{R}$, the category of additive functors $\mathscr{R}^{op}\longrightarrow Ab$ from a small preadditive category $\mathscr{R}$ to the category of abelian groups. Then, Proposition 2.5 shows that the concepts of being k-finite and strongly k-finite are equivalent in a category of modules over a ring with several objects.

(c) In Proposition 2.6, we assume that the Grothendieck category $\mathscr{A}$ has a finitely generated generator. For a Gabriel-Popescu type characterization of such Grothendieck categories, we refer the reader to Albu [1, Proposition 5.4.7].

For any $M\in\mathscr{A}$, we let $\mathcal{S}_k(M)$ denote the collection of k-finite subobjects of M. Similarly, we denote by $\mathcal{T}_k(M)$ the collection of strongly k-finite subobjects of M. Since $\mathscr{A}$ is a Grothendieck category, it is well-powered, and it follows that $\mathcal{S}_k(M)$, $\mathcal{T}_k(M)$ are both sets for any $M \in \mathscr{A}$. We now let $\mathscr{A}^{\mathcal{T}_k}$ denote the full subcategory of $\mathscr{A}$ given by

$$Ob(\mathscr{A}^{\mathcal{T}_k}) := \left\{ M \in \mathscr{A} \mid M = \sum_{Y \in \mathcal{T}_k(M)} Y \right\} \tag{3}$$

We will refer to $\mathscr{A}^{\mathcal{T}_k}$ as the subcategory of locally strongly k-finite objects in $\mathscr{A}$. We also define

$$R^{\mathcal{T}_k} : \mathscr{A} \longrightarrow \mathscr{A}^{\mathcal{T}_k} \qquad M \mapsto \sum_{Y \in \mathcal{T}_k(M)} Y \tag{4}$$

As with locally finite modules in a module category, we show that $\mathscr{A}^{\mathcal{T}_k}$ is a coreflective subcategory of $\mathscr{A}$.

Theorem 2.8. *Let k be a commutative noetherian ring and let $\mathscr{A}$ be a k-linear Grothendieck category. Then, the functor $R^{\mathcal{T}_k} : \mathscr{A} \longrightarrow \mathscr{A}^{\mathcal{T}_k}$ is right adjoint to the inclusion $\mathscr{A}^{\mathcal{T}_k} \hookrightarrow \mathscr{A}$. In other words, we have natural isomorphisms*

$$\mathscr{A}(M, M') \cong \mathscr{A}^{\mathcal{T}_k}(M, R^{\mathcal{T}_k}(M')) \tag{5}$$

for $M \in \mathscr{A}^{\mathcal{T}_k}$ and $M' \in \mathscr{A}$.

Proof. We begin with $\phi : M \longrightarrow M'$ in $\mathscr{A}$. If $Y \subseteq M$ is strongly k-finite, it follows by Lemma 2.2 and the definition in (4) that the quotient $\phi(Y) \subseteq M'$ of Y lies in $R^{\mathcal{T}_k}(M')$. Since $M \in \mathscr{A}^{\mathcal{T}_k}$ is the sum of its strongly k-finite subobjects, we see that $\phi : M \longrightarrow M'$ factors through $R^{\mathcal{T}_k}(M') \hookrightarrow M'$. Conversely, if we have $\psi : M \longrightarrow R^{\mathcal{T}_k}(M')$ in $\mathscr{A}^{\mathcal{T}_k}$, we compose with the inclusion $R^{\mathcal{T}_k}(M') \hookrightarrow M'$ to obtain a morphism $M \longrightarrow M'$ in $\mathscr{A}$. It is easily verified that these two associations are inverse to each other. □

3. Generators for locally strongly k-finite objects

We continue with k being a commutative noetherian ring and $\mathscr{A}$ being a k-linear Grothendieck category. From the definition in (3), we note that $\mathscr{A}^{\mathcal{T}_k}$ is closed under direct sums in $\mathscr{A}$. We begin this section by showing that the functor $R^{\mathcal{T}_k} : \mathscr{A} \longrightarrow \mathscr{A}^{\mathcal{T}_k}$ preserves direct sums.

Proposition 3.1. *Let k be a commutative noetherian ring and let $\mathscr{A}$ be a k-linear Grothendieck category. Then, the functor $R^{\mathcal{T}_k} : \mathscr{A} \longrightarrow \mathscr{A}^{\mathcal{T}_k}$ preserves direct sums.*

Proof. We consider a family $\{M_i\}_{i \in I}$ of objects in $\mathscr{A}$ and set $M := \bigoplus_{i \in I} M_i$. If $Y \in \mathcal{T}_k(M_i)$ for some $i \in I$, it is clear that $Y \in \mathcal{T}_k(M)$. Accordingly, we must have $\bigoplus_{i \in I} R^{\mathcal{T}_k}(M_i) \subseteq R^{\mathcal{T}_k}(M) \subseteq M$. For the reverse inclusion, we consider $Y \in \mathcal{T}_k(M)$ and let Y_i denote the image of the morphism $Y \hookrightarrow M = \bigoplus_{i \in I} M_i \twoheadrightarrow M_i$. Since Y is strongly k-finite, it follows by Lemma 2.2 that each quotient $Y_i \subseteq M_i$ of Y is also strongly k-finite. The morphisms $\{Y \longrightarrow Y_i\}_{i \in I}$ together induce a map $Y \longrightarrow \prod_{i \in I} Y_i$ and we now see that the following two compositions are equal

$$Y \hookrightarrow \bigoplus_{i \in I} M_i \longrightarrow \prod_{i \in I} M_i \qquad Y \longrightarrow \prod_{i \in I} Y_i \longrightarrow \prod_{i \in I} M_i \tag{6}$$

It follows from (6) that $Y \subseteq \bigoplus_{i\in I} Y_i = lim(\bigoplus_{i\in I} M_i \longrightarrow \prod_{i\in I} M_i \longleftarrow \prod_{i\in I} Y_i)$. But each Y_i is strongly k-finite, which gives $Y_i \subseteq R^{\mathcal{T}_k}(M_i)$. Hence, $Y \subseteq \bigoplus_{i\in I} R^{\mathcal{T}_k}(M_i)$ for any $Y \in \mathcal{T}_k(M)$. It follows that the sum $R^{\mathcal{T}_k}(M) = \sum_{Y\in\mathcal{T}_k(M)} Y \subseteq \bigoplus_{i\in I} R^{\mathcal{T}_k}(M_i)$. □

Lemma 3.2. *(a) The collection $\mathcal{S}_k(\mathscr{A})$ is closed under extensions.*

(b) The collection $\mathcal{T}_k(\mathscr{A})$ is closed under extensions.

Proof. (a) Let $0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0$ be a short exact sequence in $\mathscr{A}$ with X', $X'' \in \mathcal{S}_k(\mathscr{A})$. If $F \in \mathscr{A}$ is finitely generated, we see that $0 \longrightarrow \mathscr{A}(F, X') \longrightarrow \mathscr{A}(F, X) \longrightarrow \mathscr{A}(F, X'')$ is exact in Mod_k. Since k is noetherian and both $\mathscr{A}(F, X')$, $\mathscr{A}(F, X'')$ are finitely generated, so is $\mathscr{A}(F, X)$.

(b) By definition, an object is strongly k-finite if all its quotients are k-finite. Let $0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0$ be a short exact sequence in $\mathscr{A}$ with X', $X'' \in \mathcal{T}_k(\mathscr{A})$. Then if $Y := X/K$ is a quotient of X, we have the following commutative diagram where the horizontal rows are exact.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X' & \longrightarrow & X & \longrightarrow & X'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & Y' = X'/X' \cap K & \longrightarrow & Y = X/K & \longrightarrow & Y'' & \longrightarrow & 0 \end{array} \tag{7}$$

Since X', $X'' \in \mathcal{T}_k(\mathscr{A})$, the quotients Y', Y'' appearing in (7) lie in $\mathcal{S}_k(\mathscr{A})$. Using part (a), the quotient $Y \in \mathcal{S}_k(\mathscr{A})$. This shows that X is strongly k-finite. □

Lemma 3.3. *The subcategory $\mathscr{A}^{\mathcal{T}_k}$ is closed under subobjects and quotients in $\mathscr{A}$.*

Proof. Let $M \in \mathscr{A}^{\mathcal{T}_k}$ and let $\phi : M \twoheadrightarrow N$ be a quotient of M in $\mathscr{A}$. We know that $M = \sum_{X\in\mathcal{T}_k(M)} X$. Then, each $\phi(X) \in \mathcal{T}_k(N)$ and $N = \sum_{X\in\mathcal{T}_k(M)} \phi(X)$. Hence, $N \in \mathscr{A}^{\mathcal{T}_k}$. On the other hand, consider a subobject $L \subseteq M$. Let L' be the preimage of $L \subseteq M$ under the epimorphism $\bigoplus_{X\in\mathcal{T}_k(M)} X \longrightarrow M$. Since $\mathscr{A}^{\mathcal{T}_k}$ is closed under quotients, it suffices to show that $L' \in \mathscr{A}^{\mathcal{T}_k}$. Now, L' may be expressed as a colimit over all $L' \cap X_I$, where $X_I = \bigoplus_{i\in I} X_i$ for a finite subset $\{X_i\}_{i\in I} \subseteq \mathcal{T}_k(M)$. By Lemma 3.2(b), each finite direct sum $X_I \in \mathcal{T}_k(\mathscr{A})$ and hence each subobject $L' \cap X_I \in \mathcal{T}_k(\mathscr{A})$. Then, the direct sum of all $L' \cap X_I$ as $\{X_i\}_{i\in I}$ varies over all finite subsets of $\mathcal{T}_k(M)$ lies in $\mathscr{A}^{\mathcal{T}_k}$. The colimit L' is a quotient of this direct sum and $\mathscr{A}^{\mathcal{T}_k}$ is closed under quotients, which shows that $L' \in \mathscr{A}^{\mathcal{T}_k}$. □

We now recall (see, for instance, [19, Tag 002P]) the following fact. Let $\mathscr{C}$ be an abelian category that contains all direct sums and let $M : \mathscr{I} \longrightarrow \mathscr{C}$ be a system of objects in $\mathscr{C}$ indexed by a small category $\mathscr{I}$. Let $I_0 := Ob(\mathscr{I})$, $I_1 := Mor(\mathscr{I})$ and let $s, t : I_1 \longrightarrow I_0$ be the source and target maps respectively. Accordingly, for each $\xi \in I_1$, we have a morphism $M(\xi) : M(s(\xi)) \longrightarrow M(t(\xi))$ in $\mathscr{C}$. Then, the colimit of the system $M : \mathscr{I} \longrightarrow \mathscr{C}$ is given by the coequalizer

$$Coeq\left(\bigoplus_{\xi\in I_1} M(s(\xi)) \underset{\psi}{\overset{\phi}{\rightrightarrows}} \bigoplus_{i\in I_0} M(i)\right) = Coker(\phi - \psi) \tag{8}$$

In (8), the morphism ϕ is determined by mapping the component $M(s(\xi))$ to $M(t(\xi))$ via the morphism $M(\xi)$ for each $\xi \in I_1$. The morphism ψ is determined by mapping the component

$M(s(\xi))$ to $M(s(\xi))$ using the identity morphism for each $\xi \in I_1$. A dual result holds for computing limits in $\mathscr{C}$.

Proposition 3.4. *The category $\mathscr{A}^{\mathcal{T}_k}$ is abelian, cocomplete, and contains all finite limits. Moreover, colimits and finite limits in $\mathscr{A}^{\mathcal{T}_k}$ may be computed in $\mathscr{A}$.*

Proof. Applying Lemma 3.3, it follows that $\mathscr{A}^{\mathcal{T}_k}$ is closed under kernels and cokernels in $\mathscr{A}$, making $\mathscr{A}^{\mathcal{T}_k}$ an abelian category. We have noted before that $\mathscr{A}^{\mathcal{T}_k}$ is closed under direct sums in $\mathscr{A}$. From (8), we see that for any system $M : \mathscr{I} \longrightarrow \mathscr{A}^{\mathcal{T}_k}$ of objects in $\mathscr{A}^{\mathcal{T}_k}$ indexed by a small category $\mathscr{I}$, its colimit in $\mathscr{A}$ may be expressed in terms of cokernels and direct sums in $\mathscr{A}$. Since $\mathscr{A}^{\mathcal{T}_k}$ is closed under both cokernels and direct sums in $\mathscr{A}$, this colimit lies in $\mathscr{A}^{\mathcal{T}_k}$. It follows that $\mathscr{A}^{\mathcal{T}_k}$ is cocomplete, with all colimits computed in $\mathscr{A}$. Similarly, the limit of any finite system in $\mathscr{A}^{\mathcal{T}_k}$ may be expressed in terms of kernels and finite products in $\mathscr{A}$. Since $\mathscr{A}^{\mathcal{T}_k}$ is closed under both kernels and finite products in $\mathscr{A}$, it follows that $\mathscr{A}^{\mathcal{T}_k}$ contains all finite limits, and these are computed in $\mathscr{A}$. □

Theorem 3.5. *Let G be a generator for the Grothendieck category $\mathscr{A}$. Then, the collection*

$$\{G^n/K \mid K \subseteq G^n \textit{ is such that } G^n/K \in \mathscr{A}^{\mathcal{T}_k} \textit{ and } n \geq 1\} \tag{9}$$

is a set of generators for $\mathscr{A}^{\mathcal{T}_k}$, making $\mathscr{A}^{\mathcal{T}_k}$ a Grothendieck category. Additionally, if $\mathscr{A}$ is locally finitely generated, so is $\mathscr{A}^{\mathcal{T}_k}$.

Proof. By Proposition 3.4, both filtered colimits and finite limits exist in $\mathscr{A}^{\mathcal{T}_k}$ and are computed in $\mathscr{A}$. Accordingly, since $\mathscr{A}$ satisfies (AB5), so does $\mathscr{A}^{\mathcal{T}_k}$. We now consider some $M \in \mathscr{A}^{\mathcal{T}_k}$. Since G is a generator for $\mathscr{A}$, we can choose an epimorphism $\pi : G^{(I)} \longrightarrow M$ in $\mathscr{A}$ from a direct sum of copies of G.

For any finite subset $I_0 \subseteq I$, we write $M_{I_0} \subseteq M$ for the image $\pi(G^{(I_0)})$ in M. Then, we see that $\sum M_{I_0} = M$, where I_0 varies over all finite subsets of I. By Lemma 3.3, $\mathscr{A}^{\mathcal{T}_k}$ is closed under subobjects and hence each $M_{I_0} \in \mathscr{A}^{\mathcal{T}_k}$. Since $M_{I_0} = \pi(G^{(I_0)})$, it is now clear that M_{I_0} is isomorphic to an object in the collection (9). This makes the collection in (9) a set of generators for $\mathscr{A}^{\mathcal{T}_k}$.

Additionally, if $\{G_j\}_{j\in J}$ is a set of finitely generated generators for $\mathscr{A}$, it follows by similar reasoning that the following

$$\bigcup_{j\in J}\bigcup_{n\geq 1}\{G_j^n/K \mid K \subseteq G_j^n \text{ is such that } G_j^n/K \in \mathscr{A}^{\mathcal{T}_k}\} \tag{10}$$

is a set of generators for $\mathscr{A}^{\mathcal{T}_k}$. Each G_j^n/K is finitely generated as an object of $\mathscr{A}$. Since filtered colimits and kernels in $\mathscr{A}^{\mathcal{T}_k}$ are both computed in $\mathscr{A}$, it follows that each G_j^n/K appearing in (10) is also finitely generated in $A^{\mathcal{T}_k}$. □

We will say that a subobject $N \subseteq M$ in $\mathscr{A}$ is k-cofinite (resp. strongly k-cofinite) in M if the quotient $M/N \in \mathcal{S}_k(\mathscr{A})$ (resp. $M/N \in \mathcal{T}_k(\mathscr{A})$).

Theorem 3.6. *Let $\mathscr{A}$ be a k-linear Grothendieck category and let $M \in \mathscr{A}^{\mathcal{T}_k}$. Then, any finitely generated subobject of M is strongly k-finite.*

Additionally, suppose that $\mathscr{A}$ is locally finitely generated, with $\{G_i\}_{i\in I}$ being a set of finitely generated generators. Then for any $M \in \mathscr{A}$, the following statements are equivalent.

(a) M is locally strongly k-finite, i.e., $M \in \mathscr{A}^{\mathcal{T}_k}$.

(b) Every finitely generated subobject of M is strongly k-finite.

(c) For any $i \in I$ and any morphism $\phi : G_i \longrightarrow M$, the subobject $Ker(\phi)$ is strongly k-cofinite in G_i.

Proof. Let $M \in \mathscr{A}^{\mathcal{T}_k}$ and let $Y \subseteq M$ be a finitely generated subobject. We know that $M = \sum\limits_{Z \in \mathcal{T}_k(M)} Z$. Since $\mathcal{T}_k(\mathscr{A})$ is closed under quotients, and also under finite direct sums by Lemma 3.2, we see that the $\mathcal{T}_k(M)$ is a filtered system of subobjects of M. Since $Y \subseteq M = \sum\limits_{Z \in \mathcal{T}_k(M)} Z$ is finitely generated, it follows that we can find $Z \in \mathcal{T}_k(M)$ such that $Y \subseteq Z$. Applying Lemma 2.2, it now follows that Y is also strongly k-finite. We now suppose that $\mathscr{A}$ is locally finitely generated. We have already shown that (a) $\Rightarrow$ (b). We note that (b) $\Rightarrow$ (c) is also clear.

(b) $\Rightarrow$ (a) : Since $\mathscr{A}$ is locally finitely generated, every object in $\mathscr{A}$ is the sum of its finitely generated subobjects. By assumption, every finitely generated subobject of M is also strongly k-finite. Accordingly, M is the sum of its strongly k-finite subobjects, i.e., $M \in \mathscr{A}^{\mathcal{T}_k}$.

(c) $\Rightarrow$ (a) Since $\{G_i\}_{i \in I}$ is a set of generators for $\mathscr{A}$, the object M can be expressed as a sum of images of morphisms having source G_i for some $i \in I$. By assumption, the image of any such $\phi : G_i \longrightarrow M$ is strongly k-finite, and hence $M \in \mathscr{A}^{\mathcal{T}_k}$. ◻

Corollary 3.7. Let A be an algebra over a field k and let $\mathscr{A}$ be the category Mod_A of right A-modules. Then, we have

(a) A right A-module $M \in \mathscr{A} = Mod_A$ is strongly k-finite if and only if M is finite dimensional as a k-vector space.

(b) The subcategory $\mathscr{A}^{\mathcal{T}_k}$ of locally strongly k-finite objects in $\mathscr{A}$ can be identified with the category of left A^0-comodules, where A^0 is the finite dual coalgebra of A.

Proof. (a) Let $M \in \mathscr{A} = Mod_A$ be strongly k-finite. Since A is a finitely generated object in Mod_A, it follows from Definition 2.1 that $Mod_A(A, M) \cong M$ is a finite dimensional k-vector space. Conversely, let $M \in Mod_A$ be a right A-module such that M is finite dimensional as a k-vector space. It is clear that any quotient N of M in Mod_A is also finite dimensional as a k-vector space. Since any such quotient N satisfies $N \cong Mod_A(A, N)$ and A is a generator for the category Mod_A, it now follows from Lemma 2.4 that $M \in \mathscr{A} = Mod_A$ is strongly k-finite.

(b) We know that Mod_A is a locally finitely generated Grothendieck category having A as a finitely generated generator. Applying Theorem 3.6, it follows that $M \in \mathscr{A} = Mod_A$ lies in the subcategory $\mathscr{A}^{\mathcal{T}_k}$ if and only if for any morphism $\phi : A \longrightarrow M$ of A-modules, the quotient $A/Ker(\phi)$ is strongly k-finite. Using part (a), this means that $A/Ker(\phi)$ is finite dimensional as a k-vector space. It is now clear that $M \in Mod_A$ is locally strongly k-finite if and only if for each element $m \in M$, the cyclic submodule generated by m is a finite dimensional k-vector space. From [14, § 1], we know that A-modules having this property can be identified with comodules over the finite dual coalgebra A^0 of A. ◻

4. Injective objects and essential monomorphisms

We have already shown that the subcategory $\mathscr{A}^{\mathcal{T}_k}$ of locally strongly k-finite objects in $\mathscr{A}$ is a Grothendieck category. In [14], Hatipoğlu and Lomp have given conditions for locally finite modules over a noetherian algebra over a field to be closed under essential extensions. This motivates us to study closely the properties of essential extensions and injectives in $\mathscr{A}^{\mathcal{T}_k}$. We denote by $Inj(\mathscr{A})$ (resp. $Inj(\mathscr{A}^{\mathcal{T}_k})$) the collection of injectives in $\mathscr{A}$ (resp. $\mathscr{A}^{\mathcal{T}_k}$).

Lemma 4.1. *(a) If E is an injective object in $\mathscr{A}$, then $R^{\mathcal{T}_k}(E)$ is an injective object in $\mathscr{A}^{\mathcal{T}_k}$.*

(b) An object $M \in \mathscr{A}^{\mathcal{T}_k}$ lies in $Inj(\mathscr{A}^{\mathcal{T}_k})$ if and only if it is a direct summand of an injective of the form $R^{\mathcal{T}_k}(E)$ for some $E \in Inj(\mathscr{A})$.

Proof. (a) We know that the inclusion functor $\mathscr{A}^{\mathcal{T}_k} \hookrightarrow \mathscr{A}$ is exact. Hence, its right adjoint $R^{\mathcal{T}_k} : \mathscr{A} \longrightarrow \mathscr{A}^{\mathcal{T}_k}$ preserves injectives.

(b) We consider some $M \in \mathscr{A}^{\mathcal{T}_k}$ and choose an embedding $M \hookrightarrow E$ into an injective $E \in \mathscr{A}$. By Theorem 2.8, since M lies in $\mathscr{A}^{\mathcal{T}_k}$, we must have $M \subseteq R^{\mathcal{T}_k}(E) \subseteq E$. In particular, if $M \in Inj(\mathscr{A}^{\mathcal{T}_k})$, then M becomes a direct summand of $R^{\mathcal{T}_k}(E)$. On the other hand, we know from part (a) that $R^{\mathcal{T}_k}(E)$ is injective in $\mathscr{A}^{\mathcal{T}_k}$ for any $E \in Inj(\mathscr{A})$. Hence, any direct summand of $R^{\mathcal{T}_k}(E)$ in $\mathscr{A}^{\mathcal{T}_k}$ is injective in $\mathscr{A}^{\mathcal{T}_k}$. □

Proposition 4.2. *Suppose that $\mathscr{A}$ is locally noetherian. Then, $\mathscr{A}^{\mathcal{T}_k}$ is locally noetherian.*

Proof. Since $\mathscr{A}$ is locally notherian (and in particular, locally finitely generated), it follows from Theorem 3.5 that $\mathscr{A}^{\mathcal{T}_k}$ is locally finitely generated. In order to show that $\mathscr{A}^{\mathcal{T}_k}$ is locally noetherian, it therefore suffices by [20, V.4.3] to show that a direct sum of injectives in $\mathscr{A}^{\mathcal{T}_k}$ is injective.

Accordingly, let $\{E_i\}_{i\in I}$ be a family of injective objects in $\mathscr{A}^{\mathcal{T}_k}$. Applying Lemma 4.1, we can choose for each $i \in I$ an object $E'_i \in Inj(\mathscr{A})$ such that E_i is a direct summand of $R^{\mathcal{T}_k}(E'_i)$. Then, $\bigoplus\limits_{i\in I} E_i$ is a direct summand of $\bigoplus\limits_{i\in I} R^{\mathcal{T}_k}(E'_i)$. By Proposition 3.1, we have $\bigoplus\limits_{i\in I} R^{\mathcal{T}_k}(E'_i) = R^{\mathcal{T}_k}(\bigoplus\limits_{i\in I} E'_i)$. Since $\mathscr{A}$ is locally noetherian, $\bigoplus\limits_{i\in I} E'_i$ is injective. Applying Lemma 4.1 again, it follows that the direct summand $\bigoplus\limits_{i\in I} E_i$ of $R^{\mathcal{T}_k}(\bigoplus\limits_{i\in I} E'_i)$ is injective in $\mathscr{A}^{\mathcal{T}_k}$. □

The next result will show when $\mathscr{A}^{\mathcal{T}_k}$ is closed under essential extensions.

Proposition 4.3. *Let $\mathscr{C}$ be a Grothendieck category. Let $\mathscr{D} \subseteq \mathscr{C}$ be a full subcategory that satisfies the following*

(i) the inclusion $i : \mathscr{D} \hookrightarrow \mathscr{C}$ is exact and has a right adjoint $R : \mathscr{C} \longrightarrow \mathscr{D}$.

(ii) $\mathscr{D}$ is closed under subobjects in $\mathscr{C}$

(iii) $\mathscr{D}$ is a Grothendieck category.

Then, the following are equivalent.

(a) $\mathscr{D}$ is closed under essential extensions in $\mathscr{C}$.

(b) For any injective object $E \in \mathscr{C}$, the object $R(E)$ is a direct summand of E.

Proof. Since $\mathscr{D}$ is closed under subobjects in $\mathscr{C}$, we can show that $R(M) \subseteq M$ for any $M \in \mathscr{C}$.

(a) ⇒ (b): Let $E \in \mathscr{C}$ be injective and let $\mathcal{E}(R(E))$ be the injective envelope of $R(E)$ in $\mathscr{C}$. Since $R(E) \subseteq E$ and E is injective, it follows that $\mathcal{E}(R(E))$ is a direct summand of E. By assumption, $\mathscr{D}$ is closed under essential extensions, and hence the essential extension $\mathcal{E}(R(E))$ of $R(E) \in \mathscr{D}$ lies in $\mathscr{D}$. Since the inclusion $\mathscr{D} \hookrightarrow \mathscr{C}$ is exact, its right adjoint $R : \mathscr{C} \longrightarrow \mathscr{D}$ preserves injectives. Hence, $R(E)$ is injective in $\mathscr{D}$. Since $\mathscr{D}$ is a Grothendieck category, the injective $R(E) \in \mathscr{D}$ can have no non-trivial essential extensions in $\mathscr{D}$. This gives $R(E) = \mathcal{E}(R(E))$.

(b) $\Rightarrow$ (a): Let $M \in \mathscr{D}$ and let E be an injective envelope of M in $\mathscr{C}$. Then, $M \subseteq R(E) \subseteq E$. Hence, $R(E) \subseteq E$ is an essential subobject. By assumption, $R(E)$ is also a direct summand of E. Hence, $R(E) = E$, which gives $E \in \mathscr{D}$. If $M \subseteq N$ is an essential extension in $\mathscr{C}$, then N may be expressed as a subobject of the injective envelope E of M. Since $E \in \mathscr{D}$ and $\mathscr{D}$ is closed under subobjects, we get $N \in \mathscr{D}$. □

Corollary 4.4. The following are equivalent:

(a) $\mathscr{A}^{\mathcal{T}_k}$ is closed under essential extensions in $\mathscr{A}$.

(b) For any injective $E \in \mathscr{A}$, the object $R^{\mathcal{T}_k}(E)$ is a direct summand of E.

Proof. This follows from the fact that the subcategory $\mathscr{A}^{\mathcal{T}_k} \subseteq \mathscr{A}$ satisfies all the conditions in Proposition 4.3. □

For any $M \in \mathscr{A}$ and any finitely generated object $F \in \mathscr{A}$, we now set

$$el_F(M) := \mathscr{A}(F, M) \tag{11}$$

Indeed, the purpose of this notation is to suggest that the morphisms in $\mathscr{A}(F, M)$ from a finitely generated object $F \in \mathscr{A}$ should be seen as the "elements of M over F." This motivates the following definitions.

Definition 4.5. Let $F \in \mathscr{A}$ be finitely generated. Then:

(a) An object $M \in \mathscr{A}$ is F-simple if $el_F(M) \neq 0$ and for any $0 \neq N \subseteq M$ we have $el_F(N) = el_F(M)$.

(b) An object $M \in \mathscr{A}$ is F-saturated if $el_F(N) = 0$ for a subobject $N \subseteq M$ implies $N = 0$.

For the rest of this section, we suppose that k is a field.

Lemma 4.6. *Let k be a field. Let $F \in \mathscr{A}$ be finitely generated and let $0 \neq M \in \mathscr{A}^{\mathcal{T}_k}$ be F-saturated. Then, M contains an F-simple subobject that is strongly k-finite.*

Proof. Since $M \in \mathscr{A}^{\mathcal{T}_k}$, we can express M as a sum $M = \sum\limits_{X \in \mathcal{T}_k(M)} X$. We have noted before that the system $\mathcal{T}_k(M)$ is filtered when ordered by inclusion and we may write $M = \varinjlim\limits_{X \in \mathcal{T}_k(M)} X$. If $el_F(M) = 0$, then $M = 0$ since M is F-saturated. Hence, $el_F(M) = \mathscr{A}(F, M) \neq 0$. Since F is finitely generated, we now have

$$0 \neq el_F(M) = \mathscr{A}(F, M) = \varinjlim_{X \in \mathcal{T}_k(M)} \mathscr{A}(F, X) \tag{12}$$

Accordingly, there exists $X \in \mathcal{T}_k(M)$ such that $\mathscr{A}(F, X) \neq 0$. We now choose $X_0 \in \mathcal{T}_k(M)$ such that $\mathscr{A}(F, X_0)$ has the minimum possible finite non-zero dimension as a k-vector space. Then, if we consider any $Y \subseteq X_0$, we must have $Y \in \mathcal{T}_k(M)$ because $\mathcal{T}_k(\mathscr{A})$ is closed under subobjects. We also have a canonical inclusion $\mathscr{A}(F, Y) \subseteq \mathscr{A}(F, X_0)$. By the property of X_0, this means that either $\mathscr{A}(F, Y) = \mathscr{A}(F, X_0)$ or $\mathscr{A}(F, Y) = 0$. But if $\mathscr{A}(F, Y) = 0$, we again have $Y = 0$ since M is F-saturated. It is now clear that X_0 is F-simple. □

For any $M \in \mathscr{A}$ and any finitely generated object $F \in \mathscr{A}$, we now define the F-socle $soc_F(M)$ of M to be

$$soc_F(M) := \{\text{sum of all } F\text{-simple subobjects of } M\} \tag{13}$$

Lemma 4.7. *Let k be a field. Let $F \in \mathscr{A}$ be finitely generated. Then, for any $0 \neq M \in \mathscr{A}^{\mathcal{T}_k}$ that is F-saturated, the inclusion of the F-socle $soc_F(M) \subseteq M$ is essential.*

Proof. Suppose that the inclusion $soc_F(M) \subseteq M$ is not essential. Then, there exists a subobject $N' \subseteq M$ such that $N' \cap soc_F(M) = 0$ and $N' \neq 0$. Since $\mathscr{A}^{\mathcal{T}_k}$ is closed under subobjects and $N' \subseteq M$, we have $N' \in \mathscr{A}^{\mathcal{T}_k}$. It is also clear that $N' \subseteq M$ is F-saturated. Applying Lemma 4.6, we can now find $N'' \subseteq N'$ that is F-simple. Then, $N'' \neq 0$ and by the definition in (13), we must have $N'' \subseteq soc_F(M)$. This contradicts the fact that $N' \cap soc_F(M) = 0$. □

Lemma 4.8. *Let k be a field. Let $F \in \mathscr{A}$ be finitely generated. Let $0 \neq M \in \mathscr{A}^{\mathcal{T}_k}$ be F-saturated. Then, the F-socle $soc_F(M)$ of M is an essential extension of a direct sum of F-simple subobjects of M.*

Proof. We consider families $\{S_i\}_{i\in I}$ of subobjects of $soc_F(M)$ satisfying the following conditions:

(a) Each S_i is F-simple

(b) The sum $\sum_{i\in I} S_i$ is direct.

Since $\mathscr{A}$ is a Grothendieck category, we note that the sum $\sum_{i\in I} S_i$ is direct if and only if $S_{i_0} \cap (\sum_{j\in J} S_j) = 0$ for any $i_0 \in I$ and any finite subset $J \subseteq I - \{i_0\}$. Applying Zorn's lemma, we may therefore choose a maximal such family $\{S_i\}_{i\in I_0}$ of F-simple subobjects of $soc_F(M)$ whose sum is direct.

We claim that $\bigoplus_{i\in I_0} S_i$ is an essential subobject of $soc_F(M)$. Otherwise, we can find $0 \neq N \subseteq soc_F(M)$ such that N does not intersect $\bigoplus_{i\in I_0} S_i$. Then, $0 \neq N \in \mathscr{A}^{\mathcal{T}_k}$ and N is F-saturated. Applying Lemma 4.6, we can find $0 \neq N' \subseteq N$ that is F-simple. Then, $N' \cap (\bigoplus_{i\in I_0} S_i) = 0$ and the family $\{N'\} \cup \{S_i\}_{i\in I_0}$ contradicts the maximality of $\{S_i\}_{i\in I_0}$. This proves the result. □

Proposition 4.9. *Let k be a field and let $\mathscr{A}$ be a locally noetherian k-linear Grothendieck category. Let $F \in \mathscr{A}$ be finitely generated and let $0 \neq M \in \mathscr{A}^{\mathcal{T}_k}$ be F-saturated. Then, the injective envelope $\mathcal{E}(M)$ of M in $\mathscr{A}$ can be expressed as a direct sum of injective envelopes of F-simple subobjects of M.*

Proof. Combining Lemma 4.7 and Lemma 4.8, we see that M is an essential extension of a direct sum $\bigoplus_{i\in I} S_i$ of F-simple subobjects of M. For each $i \in I$, we let $\mathcal{E}(S_i)$ be an injective envelope of S_i in $\mathscr{A}$. We know that essential extensions are preserved by finite direct sums. Further, since $\mathscr{A}$ is a locally noetherian category, the filtered colimit of essential monomorphisms is an essential monomorphism (see [2, Lemma 2.13]). Hence, $\bigoplus_{i\in I} \mathcal{E}(S_i)$ is an essential extension of $\bigoplus_{i\in I} S_i$. Again, since $\mathscr{A}$ is locally noetherian, the direct sum $\bigoplus_{i\in I} \mathcal{E}(S_i)$ is injective. It follows therefore that $\bigoplus_{i\in I} \mathcal{E}(S_i)$ is an injective envelope of $\bigoplus_{i\in I} S_i$ and hence an injective envelope of M. □

We now recall that a full subcategory $\mathscr{C}$ of a Grothendieck category is said to be closed if it is closed under subobjects, quotients and direct sums. We can now give conditions for a closed $\mathscr{C} \subseteq \mathscr{A}^{\mathcal{T}_k}$ to contain F-saturated essential extensions.

Theorem 4.10. *Let k be a field and let $\mathscr{A}$ be a locally noetherian k-linear Grothendieck category. Let $\mathscr{C} \subseteq \mathscr{A}^{\mathcal{T}_k}$ be a closed subcategory. Let $F \in \mathscr{A}$ be a finitely generated object. Then, the following are equivalent* *(a) For any essential extension $M \subseteq N$ with $M \in \mathscr{C}$ and N an F-saturated object, we must have $N \in \mathscr{C}$.*

(b) The injective envelope in $\mathscr{A}$ of any F-simple and strongly k-finite object $X \in \mathscr{C}$ lies in $\mathscr{C}$.

Proof. (a) $\Rightarrow$ (b): We already have $X \in \mathscr{C}$. We consider an injective envelope $X \hookrightarrow \mathcal{E}(X)$ of X in $\mathscr{A}$. We claim that $\mathcal{E}(X)$ is F-saturated. For this, we consider $0 \neq L \subseteq \mathcal{E}(X)$. Since $X \hookrightarrow \mathcal{E}(X)$ is essential, we know that $X \cap L \neq 0$. But since X is F-simple, we now have $el_F(X \cap L) = el_F(X) \neq 0$. Because $el_F(L) \supseteq el_F(X \cap L)$, we see that $el_F(L) \neq 0$. By assumption (a), it now follows that $\mathcal{E}(X) \in \mathscr{C}$.

(b) $\Rightarrow$ (a): Since $\mathscr{A}$ is in particular locally finitely generated, we can express N as the colimit over its finitely generated non-zero subobjects $\{N_i\}_{i\in I}$. Then, for each $i \in I$, we know that

1. $0 \neq M \cap N_i \subseteq N_i$ is an essential extension.
2. $M \cap N_i \in \mathscr{C}$ since $M \cap N_i \subseteq M$ and $\mathscr{C}$ is closed under subobjects.
3. $M \cap N_i$ is F-saturated since N is F-saturated
4. Since $\mathscr{A}$ is locally noetherian, $M \cap N_i \subseteq N_i$ is finitely generated. Now since $M \cap N_i \subseteq M$ and $M \in \mathscr{C} \subseteq \mathscr{A}^{\mathcal{T}_k}$, it follows by Theorem 3.6 that $M \cap N_i \in \mathcal{T}_k(\mathscr{A})$.

Applying Proposition 4.9, it now follows that the injective envelope $\mathcal{E}(M \cap N_i)$ may be expressed as a direct sum $\bigoplus_{j\in J_i} \mathcal{E}(X_j)$ of injective envelopes $\{\mathcal{E}(X_j)\}_{j\in J_i}$ of F-simple subobjects $\{X_j\}_{j\in J_i}$ of $M \cap N_i$. Since $M \cap N_i \in \mathcal{T}_k(\mathscr{A})$, each subobject $X_j \subseteq M \cap N_i$ also lies in $\mathcal{T}_k(\mathscr{A})$. Also since $\mathscr{C}$ is closed under subobjects, we see that $X_j \subseteq M \cap N_i$ lies in $\mathscr{C}$.

By assumption (b), the injective envelope $\mathcal{E}(X_j)$ of the F-simple object $X_j \in \mathcal{T}_k(\mathscr{A}) \cap \mathscr{C}$ lies in $\mathscr{C}$. Since $\mathscr{C}$ is closed under direct sums, it follows that $\mathcal{E}(M \cap N_i) = \bigoplus_{j\in J_i} \mathcal{E}(X_j) \in \mathscr{C}$. Since $M \cap N_i \subseteq N_i$ is an essential extension, we can embed N_i into the injective envelope $\mathcal{E}(M \cap N_i)$. Since $\mathscr{C}$ is closed under subobjects, this now shows that each $N_i \in \mathscr{C}$. Applying Proposition 3.4 and using the fact that $\mathscr{C}$ contains both direct sums and quotients it now follows that the colimit N of $\{N_i\}_{i\in I}$ lies in $\mathscr{C}$. □

Theorem 4.11. *Let k be a field and let $\mathscr{A}$ be a locally noetherian k-linear Grothendieck category having a finitely generated generator. Let $\mathscr{C} \subseteq \mathscr{A}^{\mathcal{T}_k}$ be a closed subcategory. Then, the following are equivalent*

(a) $\mathscr{C}$ is closed under essential extensions in $\mathscr{A}$.

(b) The injective envelope of any simple and strongly k-finite object $X \in \mathscr{C}$ lies in $\mathscr{C}$.

Proof. Let G be a finitely generated generator for $\mathscr{A}$. By Definition 4.5, it is clear that every object in $\mathscr{A}$ is G-saturated. Also, $0 \neq X \in \mathscr{A}$ is G-simple if and only if it is simple. The result now follows from Theorem 4.10. □

5. Coalgebra objects in $\mathscr{A}$

We let F be a finitely generated and projective object in $\mathscr{A}$. Corresponding to F, we will construct an object $C(F) \in \mathscr{A}^{\mathcal{T}_k}$ that behaves in many ways like an ordinary coalgebra. We will also consider a subcategory of $\mathscr{A}^{\mathcal{T}_k}$ that behaves like a category of comodules over $C(F)$. By imposing additional conditions on $C(F)$, we will see that these "comodules" satisfy properties similar to those of comodules over certain special classes of coalgebras, such as semiperfect coalgebras or quasi-co-Frobenius coalgebras.

Lemma 5.1. *Let F be a finitely generated and projective object in $\mathscr{A}$. Then, the functor $\mathscr{A}(F,-) : \mathscr{A}^{\mathcal{T}_k} \longrightarrow Mod_k$ has a right adjoint $R_F^{\mathcal{T}_k} : Mod_k \longrightarrow \mathscr{A}^{\mathcal{T}_k}$.*

Proof. Since F is finitely generated, the functor $\mathscr{A}(F,-)$ preserves filtered colimits of monomorphisms, and in particular, preserves direct sums. Additionally, since $F \in \mathscr{A}$ is projective, $\mathscr{A}(F,-)$ preserves cokernels. Accordingly, $\mathscr{A}(F,-)$ preserves all colimits. We know that both $\mathscr{A}^{\mathcal{T}_k}$ and Mod_k are Grothendieck categories, and it now follows by [15, Proposition 8.3.27(iii)] that $\mathscr{A}(F,-)$ has a right adjoint $R_F^{\mathcal{T}_k} : Mod_k \longrightarrow \mathscr{A}^{\mathcal{T}_k}$. □

Given the right adjoint $R_F^{\mathcal{T}_k} : Mod_k \longrightarrow \mathscr{A}^{\mathcal{T}_k}$ as in Lemma 5.1, we now set $C(F) := R_F^{\mathcal{T}_k}(k) \in \mathscr{A}^{\mathcal{T}_k}$. We will say that $C(F)$ is the coalgebra object associated to F. If $\mathscr{A} = Mod_A$ for an algebra A over a field k, we know from Corollary 3.7 that $\mathscr{A}^{\mathcal{T}_k}$ is the category of A^0-comodules, where A^0 is the finite dual of A. Then if we take $F = A$, we note that $\mathscr{A}(F,-) : \mathscr{A}^{\mathcal{T}_k} \longrightarrow Mod_k$ becomes the forgetful functor from A^0-comodules to k-vector spaces. Its right adjoint is the functor that takes a vector space V to $A^0 \otimes_k V$ (see, for instance, [6, § 2.3.8]). As such, we get $C(F) = A^0$, which is a k-coalgebra in the usual sense.

Lemma 5.2. *Let k be a quasi-Frobenius ring. Then, $C(F) \in Inj(\mathscr{A}^{\mathcal{T}_k})$.*

Proof. Since F is finitely generated and projective, we see that the left adjoint functor $\mathscr{A}(F,-) : \mathscr{A}^{\mathcal{T}_k} \longrightarrow Mod_k$ is exact. Accordingly, its right adjoint $R_F^{\mathcal{T}_k} : Mod_k \longrightarrow \mathscr{A}^{\mathcal{T}_k}$ preserves injectives. Since k is a quasi-Frobenius ring, we know that k is injective as a k-module. Hence, $C(F) = R_F^{\mathcal{T}_k}(k)$ is injective in $\mathscr{A}^{\mathcal{T}_k}$. □

Since $\mathscr{A}^{\mathcal{T}_k}$ is a Grothendieck category, it contains injective envelopes. For an object $M \in \mathscr{A}^{\mathcal{T}_k}$, we will use $\mathcal{E}^{\mathcal{T}_k}(M)$ to denote its injective envelope in $\mathscr{A}^{\mathcal{T}_k}$, in order to distinguish it from its injective envelope $\mathcal{E}(M)$ in $\mathscr{A}$. It may be easily verified that $\mathcal{E}^{\mathcal{T}_k}(M) = R^{\mathcal{T}_k}(\mathcal{E}(M))$.

Proposition 5.3. *Suppose that $\mathscr{A}$ is locally finitely generated. Let F be a finitely generated and projective object in $\mathscr{A}$. Then, the functor $R_F^{\mathcal{T}_k} : Mod_k \longrightarrow \mathscr{A}^{\mathcal{T}_k}$ preserves direct sums.*

Proof. We consider a family $\{V_i\}_{i\in I}$ of k-modules and set $V := \bigoplus\limits_{i\in I} V_i$. We choose some $M \in \mathscr{A}^{\mathcal{T}_k}$. Since $\mathscr{A}$ is locally finitely generated, we can express M as a filtered colimit of finitely generated subobjects $\{M_j\}_{j\in J}$. Since $\mathscr{A}^{\mathcal{T}_k}$ is closed under subobjects, we note that each $M_j \in \mathscr{A}^{\mathcal{T}_k}$. Again since $\mathscr{A}$ is locally finitely generated, it follows by Theorem 3.6 that each M_j is strongly k-finite. We now have for each $j \in J$:

$$\begin{aligned}
&\mathscr{A}^{\mathcal{T}_k}(M_j, R_F^{\mathcal{T}_k}(V)) \\
&= Mod_k(\mathscr{A}(F, M_j), V) \\
&= \bigoplus_{i\in I} Mod_k(\mathscr{A}(F, M_j), V_i) && \text{(since } M_j \in \mathcal{T}_k(\mathscr{A}) \Rightarrow \mathscr{A}(F, M_j) \text{ is a f.g. } k\text{-module)} \\
&= \bigoplus_{i\in I} \mathscr{A}^{\mathcal{T}_k}(M_j, R_F^{\mathcal{T}_k}(V_i)) \\
&= \mathscr{A}^{\mathcal{T}_k}(M_j, \bigoplus_{i\in I} R_F^{\mathcal{T}_k}(V_i)) && \text{(since } M_j \in \mathscr{A} \text{ is finitely generated)}
\end{aligned} \tag{14}$$

Finally, writing M as a colimit over all $\{M_j\}_{j\in J}$, it follows from (14) that $\mathscr{A}^{\mathcal{T}_k}(M, R_F^{\mathcal{T}_k}(V)) = \mathscr{A}^{\mathcal{T}_k}(M, \bigoplus\limits_{i\in I} R_F^{\mathcal{T}_k}(V_i))$ for each $M \in \mathscr{A}^{\mathcal{T}_k}$. This proves the result. □

We know (see, for instance, [6, Theorem 2.4.16] that a coalgebra over a field decomposes as a direct sum of injective envelopes of finite dimensional simple comodules. This motivates us to proceed as follows. We consider the full subcategory $Sub^{\oplus}(C(F)) \subseteq \mathscr{A}^{\mathcal{T}_k}$ consisting of all objects that embed into a direct sum of copies of $C(F)$. We will see that $Sub^{\oplus}(C(F))$ behaves like a category of comodules over the coalgebra object $C(F)$. We recall here that if C is an ordinary coalgebra over a field k, then any C-comodule admits an embedding into a direct sum of copies of C.

Lemma 5.4. *Every object in $Sub^{\oplus}(C(F))$ is F-saturated.*

Proof. We consider $M \in Sub^{\oplus}(C(F))$ which embeds into a direct sum $C(F)^{(I)}$ of copies of $C(F)$. Suppose we have $N \subseteq M$ with $\mathscr{A}(F,N) = el_F(N) = 0$. Then, $0 = Mod_k(\mathscr{A}(F,N), k^{(I)}) = \mathscr{A}^{\mathcal{T}_k}(N, R_F^{\mathcal{T}_k}(k^{(I)}))$. By Proposition 5.3, the functor $R_F^{\mathcal{T}_k}$ preserves direct sums, which gives $R_F^{\mathcal{T}_k}(k^{(I)}) = R_F^{\mathcal{T}_k}(k)^{(I)} = C(F)^{(I)}$. Hence, $\mathscr{A}^{\mathcal{T}_k}(N, C(F)^{(I)}) = 0$. Since $N \subseteq M \subseteq C(F)^{(I)}$, this shows that $N = 0$.

□

Lemma 5.5. *Let $M \in \mathscr{A}^{\mathcal{T}_k}$ be F-simple. Then, $el_F(M)$ is a finitely generated k-module.*

Proof. Since $0 \neq M \in \mathscr{A}^{\mathcal{T}_k}$, we may choose $0 \neq X \subseteq M$ such that X is strongly k-finite. Then, $el_F(X) = \mathscr{A}(F,X)$ is a finitely generated k-module. Since M is F-simple, it follows that $el_F(M) = el_F(X)$ is a finitely generated k-module. □

For the rest of this section, we suppose that k is a field.

Lemma 5.6. *Let $M \in \mathscr{A}^{\mathcal{T}_k}$ be F-simple. Then, every non-zero subobject of M is essential.*

Proof. We consider $0 \neq N \subseteq M$. If N is not an essential subobject, we can find $0 \neq N' \subseteq M$ such that $N \oplus N' \subseteq M$. Since M is F-simple, it follows that $el_F(N) \oplus el_F(N') \cong el_F(N \oplus N') = el_F(N) = el_F(N') = el_F(M)$. By Lemma 5.5, we also know that $el_F(M) \neq 0$ is a finite dimensional vector space, which gives a contradiction. □

Proposition 5.7. *Let $\mathscr{A}$ be locally finitely generated and let $M \in Sub^{\oplus}(C(F))$ be F-simple. Then, the injective envelope $\mathcal{E}^{\mathcal{T}_k}(M)$ of M in $\mathscr{A}^{\mathcal{T}_k}$ is a subobject of $C(F)$.*

Proof. Since $\mathscr{A}$ is locally finitely generated, we choose a non-zero subobject $N \subseteq M$ that is finitely generated. Since M is F-simple, it follows by Lemma 5.6 that N is essential in M. From Definition 4.5, it is also clear that N is F-simple. Since $M \in Sub^{\oplus}(C(F))$, we know that M embeds into a direct sum of copies of $C(F)$ and hence so does $N \subseteq M$. Additionally since N is finitely generated, we may write $N \subseteq C(F)^n$ for some finite $n > 0$. We let $\pi : C(F)^n \longrightarrow C(F)^{n-1}$ denote the quotient over one of the copies of $C(F)$ and consider the short exact sequence

$$0 \longrightarrow N \cap C(F) \longrightarrow N \longrightarrow N' := Im(\pi|N : N \longrightarrow C(F)^{n-1}) \longrightarrow 0 \tag{15}$$

Suppose that $N \cap C(F) \neq 0$. Then $N \cap C(F) \subseteq N$ is essential by Lemma 5.6. By Lemma 5.2, we know that $C(F)$ is injective in $\mathscr{A}^{\mathcal{T}_k}$. Accordingly, we have $\mathcal{E}^{\mathcal{T}_k}(M) = \mathcal{E}^{\mathcal{T}_k}(N) = \mathcal{E}^{\mathcal{T}_k}(N \cap C(F)) \subseteq C(F)$ and the result is proved.

Otherwise, suppose that $N \cap C(F) = 0$. Then, the short exact sequence (15) shows that N is isomorphic to a subobject of $C(F)^{n-1}$. By repeating the argument, the result is now clear. □

Theorem 5.8. *Let k be a field and let $\mathscr{A}$ be a locally noetherian k-linear Grothendieck category. If $F \in \mathscr{A}$ is a finitely generated and projective object such that $C(F) \neq 0$, then $C(F)$ can be expressed as a direct sum of injective envelopes in $\mathscr{A}^{\mathcal{T}_k}$ of F-simple subobjects of $C(F)$.*

Proof. By Lemma 5.4, $C(F)$ is F-saturated. Then, it follows by Proposition 4.9 that the injective envelope $\mathcal{E}(C(F))$ in $\mathscr{A}$ is a direct sum of injective envelopes in $\mathscr{A}$ of F-simple subobjects of $C(F)$. Since $C(F)$ is injective in $\mathscr{A}^{\mathcal{T}_k}$, the functor $R^{\mathcal{T}_k} : \mathscr{A} \longrightarrow \mathscr{A}^{\mathcal{T}_k}$ preserves direct sums and $\mathcal{E}^{\mathcal{T}_k}(M) = R^{\mathcal{T}_k}(\mathcal{E}(M))$ for any $M \in \mathscr{A}^{\mathcal{T}_k}$, the result is now clear. □

From now onwards, we fix F such that $C(F) \neq 0$. For any $M \in \mathscr{A}^{\mathcal{T}_k}$, we note that

$$el_F(M)^* := Mod_k(el_F(M), k) = Mod_k(\mathscr{A}(F, M), k) = \mathscr{A}^{\mathcal{T}_k}(M, C(F)) = \mathscr{A}(M, C(F)) \quad (16)$$

It is also clear that $el_F(M)^* = \mathscr{A}(M, C(F))$ is a left module over the endomorphism ring $\mathscr{A}(C(F), C(F))$. We will simply write $C(F)^* = \mathscr{A}(C(F), C(F))$. We note that if we take $\mathscr{A} = Mod_A$ for an algebra A over a field k and set $F = A$, then the ring $\mathscr{A}(C(F), C(F))$ is the linear dual of the k-coalgebra $C(F) = A^0$.

Lemma 5.9. *Let $M \in Sub^{\oplus}(C(F))$. Then,* $\bigcap\limits_{f \in el_F(M)^*} Ker(f) = 0$.

Proof. Since $M \in Sub^{\oplus}(C(F))$, we consider an embedding $M \hookrightarrow C(F)^{(I)}$ into a direct sum of copies of $C(F)$. For each $i \in I$, let $C(F)^{(I)} \xrightarrow{\pi_i} C(F)$ denote the canonical projection to the i-th component. If $K \subseteq M$, we must have $K \subseteq \bigoplus\limits_{i \in I} K_i$, where $\{K_i = Im(K \hookrightarrow M \hookrightarrow C(F)^{(I)} \xrightarrow{\pi_i} C(F)\}_{i \in I}$. This means that if $K \subseteq \bigcap\limits_{f \in el_F(M)^*} Ker(f)$, each $K_i = 0$ and hence $K = 0$. □

Definition 5.10. Let $M \in Sub^{\oplus}(C(F))$. A subspace $W \subseteq el_F(M)^*$ is dense in M if $\bigcap\limits_{f \in W} Ker(f) = 0$.

Lemma 5.11. *(a) Let $\{M_i\}_{i \in I}$ be a family of objects in $Sub^{\oplus}(C(F))$ and let $M := \bigoplus\limits_{i \in I} M_i$. For each $i \in I$, let $W_i \subseteq el_F(M_i)^*$ be dense in M_i. Then, $W := \bigoplus\limits_{i \in I} W_i$ is dense in M.*

(b) Let $M \in Sub^{\oplus}(C(F))$. Let $\phi : N \hookrightarrow M$ be an inclusion and consider the induced morphism $el_F(\phi)^* : el_F(M)^* \longrightarrow el_F(N)^*$. Then if $W \subseteq el_F(M)^*$ is dense in M, the image $el_F(\phi)^*(W) \subseteq el_F(N)^*$ is dense in N.

Proof. (a) For each $i \in I$, let $\pi_i : M = \bigoplus\limits_{i \in I} M_i \longrightarrow M_i$ denote the canonical projection. Let $K \subseteq \bigcap\limits_{f \in W} Ker(f) \subseteq M$ and set $K_i := \pi_i(K) \subseteq M_i$ for each $i \in I$. Then, we see that $K_i \subseteq \bigcap\limits_{g \in W_i} Ker(g) \subseteq M_i$. Since W_i is dense in M_i, we get $K_i = 0$ for each $i \in I$. Since $K \subseteq \bigoplus\limits_{i \in I} K_i$, we conclude that $K = 0$.

(b) Suppose $K \subseteq N \subseteq M$ is such that $K \subseteq Ker(g)$ for each $g \in el_F(\phi)^*(W)$. Then, we see that $\phi(K) \subseteq Ker(f)$ for each $f \in W \subseteq el_F(M)^*$. Since W is dense in M, this gives $K \cong \phi(K) = 0$. The result is now clear. □

Lemma 5.12. *Let $\mathscr{A}$ be locally noetherian and let $0 \neq M \in Sub^{\oplus}(C(F))$. Then, the injective envelope $\mathcal{E}^{\mathcal{T}_k}(M)$ of M in $\mathscr{A}^{\mathcal{T}_k}$ also lies in $Sub^{\oplus}(C(F))$.*

Proof. Since $M \in Sub^{\oplus}(C(F))$, it follows by Lemma 5.4 that M is F-saturated. Applying Proposition 4.9, it follows that the injective envelope $\mathcal{E}(M)$ in $\mathscr{A}$ is a direct sum of injective envelopes in $\mathscr{A}$ of F-simple subobjects of M. Since $R^{\mathcal{T}_k} : \mathscr{A} \longrightarrow \mathscr{A}^{\mathcal{T}_k}$ preserves direct sums and $\mathcal{E}^{\mathcal{T}_k}(N) = R^{\mathcal{T}_k}(\mathcal{E}(N))$ for any $N \in \mathscr{A}^{\mathcal{T}_k}$, we can now write $\mathcal{E}^{\mathcal{T}_k}(M) = \bigoplus_{i\in I}\mathcal{E}^{\mathcal{T}_k}(M_i)$, where each M_i is an F-simple subobject of M. Since each $M_i \subseteq M$, we see that $M_i \in Sub^{\oplus}(C(F))$. Since $\mathscr{A}$ is also locally finitely generated, it follows by Proposition 5.7 that each $\mathcal{E}^{\mathcal{T}_k}(M_i)$ is a subobject of $C(F)$. Hence, we have $\mathcal{E}^{\mathcal{T}_k}(M) = \bigoplus_{i\in I}\mathcal{E}^{\mathcal{T}_k}(M_i) \in Sub^{\oplus}(C(F))$. □

We now consider a closed subcategory $\mathscr{B}$ of the category ${}_{C(F)^*}Mod$ of left $C(F)^*$-modules. Then, the inclusion $\mathscr{B} \hookrightarrow {}_{C(F)^*}Mod$ has a right adjoint $t_{\mathscr{B}} : {}_{C(F)^*}Mod \longrightarrow \mathscr{B}$ that takes a $C(F)^*$-module V to the sum of its subobjects contained in $\mathscr{B}$. Since $\mathscr{B}$ is closed, the right adjoint $t_{\mathscr{B}}$ preserves direct sums. Further, a monomorphism in $\mathscr{B}$ is still a monomorphism in ${}_{C(F)^*}Mod$.

Theorem 5.13. *Let k be a field and let $\mathscr{A}$ be a locally noetherian k-linear Grothendieck category. Let F be a finitely generated projective object in $\mathscr{A}$ and let $\mathscr{B} \subseteq {}_{C(F)^*}Mod$ be a closed subcategory. Then, the following are equivalent:*

(a) $t_{\mathscr{B}}(C(F)^*)$ is dense in $C(F)$.

(b) If $M \in Sub^{\oplus}(C(F))$ is F-simple, then $t_{\mathscr{B}}(el_F(\mathcal{E}^{\mathcal{T}_k}(M))^*)$ is dense in $\mathcal{E}^{\mathcal{T}_k}(M)$.

(c) If $M \in Sub^{\oplus}(C(F))$ is injective in $\mathscr{A}^{\mathcal{T}_k}$, then $t_{\mathscr{B}}(el_F(M)^*)$ is dense in M.

(d) $t_{\mathscr{B}}(el_F(M)^*)$ is dense in M for any $M \in Sub^{\oplus}(C(F))$.

Proof. (a) ⇒ (b) : If $M \in Sub^{\oplus}(C(F))$ is F-simple, it follows by Proposition 5.7 that $\mathcal{E}^{\mathcal{T}_k}(M) \subseteq C(F)$. Since $\mathcal{E}^{\mathcal{T}_k}(M)$ is injective, we may write $C(F) = \mathcal{E}^{\mathcal{T}_k}(M) \oplus L$. This gives us $t_{\mathscr{B}}(C(F)^*) = t_{\mathscr{B}}(el_F(\mathcal{E}^{\mathcal{T}_k}(M))^*) \oplus t_{\mathscr{B}}(el_F(L)^*)$. Since $t_{\mathscr{B}}(C(F)^*)$ is dense in $C(F)$, it is now clear from Definition 5.10 that $t_{\mathscr{B}}(el_F(\mathcal{E}^{\mathcal{T}_k}(M))^*)$ is dense in $\mathcal{E}^{\mathcal{T}_k}(M)$.

(b) ⇒ (c) : Let $M \in Sub^{\oplus}(C(F))$ be injective in $\mathscr{A}^{\mathcal{T}_k}$. By Lemma 5.4, we know that M is F-saturated. Applying Proposition 4.9, it follows that the injective envelope $\mathcal{E}(M)$ in $\mathscr{A}$ is a direct sum of injective envelopes in $\mathscr{A}$ of F-simple subobjects of M. Since $R^{\mathcal{T}_k} : \mathscr{A} \longrightarrow \mathscr{A}^{\mathcal{T}_k}$ preserves direct sums and $\mathcal{E}^{\mathcal{T}_k}(N) = R^{\mathcal{T}_k}(\mathcal{E}(N))$ for any $N \in \mathscr{A}^{\mathcal{T}_k}$, we can now write $M = \mathcal{E}^{\mathcal{T}_k}(M) = \bigoplus_{i\in I}\mathcal{E}^{\mathcal{T}_k}(M_i)$, where each M_i is an F-simple subobject of M.

By assumption, each $t_{\mathscr{B}}(el_F(\mathcal{E}^{\mathcal{T}_k}(M_i))^*)$ is dense in $\mathcal{E}^{\mathcal{T}_k}(M_i)$. Using Lemma 5.11(a), we see that $\bigoplus_{i\in I}t_{\mathscr{B}}(el_F(\mathcal{E}^{\mathcal{T}_k}(M_i))^*)$ is dense in M. Since $t_{\mathscr{B}}$ preserves direct sums and monomorphisms in $\mathscr{B}$ are inclusions of $C(F)^*$-modules we consider

$$\bigoplus_{i\in I}t_{\mathscr{B}}(el_F(\mathcal{E}^{\mathcal{T}_k}(M_i))^*) = t_{\mathscr{B}}\left(\bigoplus_{i\in I}\mathscr{A}(\mathcal{E}^{\mathcal{T}_k}(M_i), C(F))\right) \begin{array}{l} \subseteq t_{\mathscr{B}}\left(\prod_{i\in I}\mathscr{A}(\mathcal{E}^{\mathcal{T}_k}(M_i), C(F))\right) \\ = t_{\mathscr{B}}(el_F(M)^*) \end{array} \tag{17}$$

It now follows from (17) that $t_{\mathscr{B}}(el_F(M)^*)$ is dense in M.

(c) ⇒ (d) : We take $M \in Sub^{\oplus}(C(F))$ and consider the inclusion $M \subseteq \mathcal{E}^{\mathcal{T}_k}(M)$. Applying Lemma 5.12, we see that $\mathcal{E}^{\mathcal{T}_k}(M) \in Sub^{\oplus}(C(F))$. Since $\mathcal{E}^{\mathcal{T}_k}(M) \in Sub^{\oplus}(C(F))$ is injective in $\mathscr{A}^{\mathcal{T}_k}$, we know that $t_{\mathscr{B}}(el_F(\mathcal{E}^{\mathcal{T}_k}(M))^*)$ is dense in $\mathcal{E}^{\mathcal{T}_k}(M)$. Using Lemma 5.11(b), it follows that the image of $t_{\mathscr{B}}(el_F(\mathcal{E}^{\mathcal{T}_k}(M))^*) \hookrightarrow el_F(\mathcal{E}^{\mathcal{T}_k}(M))^* \longrightarrow el_F(M)^*$ is dense in M. But this image lies in $t_{\mathscr{B}}(el_F(M)^*)$. The part (d) ⇒ (a) is obvious. □

If R is any algebra over a field k, the locally finite R-modules form a closed subcategory of the category of R-modules. For any R-module V, we denote by $Loc_R(V)$ its locally finite submodule. Then, we have the following consequence of Theorem 5.13, which should be compared to the equivalent conditions for a coalgebra over a field to be semiperfect (see, for instance, [6, § 3.2.1]).

Theorem 5.14. *Let k be a field and let $\mathscr{A}$ be a locally noetherian k-linear Grothendieck category. Let F be a finitely generated projective object in $\mathscr{A}$. Then, the following are equivalent:*

(a) $Loc_{C(F)^*}(C(F)^*)$ is dense in $C(F)$.

(b) If $M \in Sub^{\oplus}(C(F))$ is F-simple, then $Loc_{C(F)^*}(el_F(\mathcal{E}^{\mathcal{T}_k}(M))^*)$ is dense in $\mathcal{E}^{\mathcal{T}_k}(M)$.

(c) If $M \in Sub^{\oplus}(C(F))$ is injective in $\mathscr{A}^{\mathcal{T}_k}$, then $Loc_{C(F)^*}(el_F(M)^*)$ is dense in M.

(d) $Loc_{C(F)^*}(el_F(M)^*)$ is dense in M for any $M \in Sub^{\oplus}(C(F))$.

We conclude by giving a number of sufficient conditions for a coalgebra object $C(F)$ to satisfy the equivalent conditions in Theorem 5.14. The first is given in terms of finite dimensionality of $el_F(\mathcal{E}^{\mathcal{T}_k}(M))$, where $M \in Sub^{\oplus}(C(F))$ is F-simple. We recall from Lemma 5.5 that $el_F(M)$ is already finite dimensional for any F-simple object $M \in Sub^{\oplus}(C(F))$.

Proposition 5.15. *Let k be a field and let $\mathscr{A}$ be a locally noetherian k-linear Grothendieck category. Let F be a finitely generated projective object in $\mathscr{A}$. Suppose that any one of the following conditions hold:*

(a) For any $M \in Sub^{\oplus}(C(F))$ that is F-simple, $el_F(\mathcal{E}^{\mathcal{T}_k}(M))$ is finite dimensional as a k-vector space.

(b) $Loc_{C(F)^*}(-)$ is an exact functor on the category of left $C(F)^*$-modules.

Then, $Loc_{C(F)^*}(C(F)^*)$ is dense in $C(F)$.

Proof. (a) If $el_F(\mathcal{E}^{\mathcal{T}_k}(M))$ is finite dimensional, so is its dual $el_F(\mathcal{E}^{\mathcal{T}_k}(M))^*$. Hence, we see that $Loc_{C(F)^*}(el_F(\mathcal{E}^{\mathcal{T}_k}(M))^*) = el_F(\mathcal{E}^{\mathcal{T}_k}(M))^*$. By Proposition 5.7, $\mathcal{E}^{\mathcal{T}_k}(M) \in Sub^{\oplus}(C(F))$. By Lemma 5.9, $Loc_{C(F)^*}(el_F(\mathcal{E}^{\mathcal{T}_k}(M))^*) = el_F(\mathcal{E}^{\mathcal{T}_k}(M))^*$ is dense in $\mathcal{E}^{\mathcal{T}_k}(M)$. Applying Theorem 5.14, it follows that $Loc_{C(F)^*}(C(F)^*)$ is dense in $C(F)$.

(b) We consider some $0 \neq M \in Sub^{\oplus}(C(F))$. Since $M \in \mathscr{A}^{\mathcal{T}_k}$, we can choose some strongly k-finite $0 \neq N \subseteq M$. This induces an epimorphism $el_F(M)^* \longrightarrow el_F(N)^*$. Applying the functor $Loc_{C(F)^*}(-)$ which is exact by assumption, we have an epimorphism $Loc_{C(F)^*}(el_F(M)^*) \twoheadrightarrow Loc_{C(F)^*}(el_F(N)^*)$. But since N is strongly k-finite, $el_F(N)$ is finite dimensional and hence so is $el_F(N)^*$. Thus we have an epimorphism $Loc_{C(F)^*}(el_F(M)^*) \twoheadrightarrow Loc_{C(F)^*}(el_F(N)^*) = el_F(N)^*$. Since N is F-saturated, $el_F(N) \neq 0$ and hence we can choose some $0 \neq f \in el_F(N)^* = \mathscr{A}(N, C(F))$. The epimorphism $Loc_{C(F)^*}(el_F(M)^*) \twoheadrightarrow el_F(N)^*$ now shows that we can lift f to some $g : M \longrightarrow C(F)$ in $Loc_{C(F)^*}(el_F(M)^*)$. Hence, $N \not\subseteq Ker(g)$. It follows that $Loc_{C(F)^*}(el_F(M)^*)$ is dense in M. □

The condition appearing in the next result should be compared to the notion of a quasi-co-Frobenius coalgebra (see [9], [10]), as we shall see in Proposition 5.17.

Proposition 5.16. *Let k be a field and let $\mathscr{A}$ be a locally noetherian k-linear Grothendieck category. Let F be a finitely generated projective object in $\mathscr{A}$. If $P \in Sub^{\oplus}(C(F))$ is a projective object in $\mathscr{A}^{\mathcal{T}_k}$, then $Loc_{C(F)^*}(el_F(P)^*)$ is dense in P. In particular, if $C(F)$ is projective in $\mathscr{A}^{\mathcal{T}_k}$, then $Loc_{C(F)^*}(C(F)^*)$ is dense in $C(F)$.*

Proof. Since $P \in Sub^{\oplus}(C(F)) \subseteq \mathscr{A}^{\mathcal{T}_k}$, we know that P can be written as a sum of strongly k-finite subobjects $\{Q_i\}_{i\in I}$. This yields an epimorphism $\pi : \bigoplus_{i\in I} Q_i \longrightarrow P$ in $\mathscr{A}^{\mathcal{T}_k}$. Since P is projective in $\mathscr{A}^{\mathcal{T}_k}$, this allows us to write $\bigoplus_{i\in I} Q_i = P \oplus T$ for $T = Ker(\pi)$. We note that each $Q_i \in Sub^{\oplus}(C(F))$ and hence $T = Ker(\pi) \subseteq \bigoplus_{i\in I} Q_i$ lies in $Sub^{\oplus}(C(F))$.

We now set $Q := \bigoplus_{i\in I} Q_i$, which gives $Q = P \oplus T$ and hence we have $Loc_{C(F)^*}(el_F(Q)^*) = Loc_{C(F)^*}(el_F(P)^*) \oplus Loc_{C(F)^*}(el_F(T)^*)$. Using Lemma 5.11(a), we see that $\bigoplus_{i\in I} el_F(Q_i)^*$ is dense in Q. By definition, we have $el_F(Q_i)^* = Mod_k(\mathscr{A}(F, Q_i), k)$. Since Q_i is strongly k-finite, $\mathscr{A}(F, Q_i)$ is finite dimensional over k and so is its dual $el_F(Q_i)^*$. Then, $\bigoplus_{i\in I} el_F(Q_i)^*$ lies in the locally finite part of $el_F(Q)^*$, i.e., $\bigoplus_{i\in I} el_F(Q_i)^* \subseteq Loc_{C(F)^*}(el_F(Q)^*)$. Hence, $Loc_{C(F)^*}(el_F(P)^*)$ is dense in P. □

Proposition 5.17. *Let k be a field and let $\mathscr{A}$ be a locally noetherian k-linear Grothendieck category. Let F be a finitely generated projective object in $\mathscr{A}$. Then, the following are equivalent:*

(a) $C(F)$ is projective in $\mathscr{A}^{\mathcal{T}_k}$

(b) Every $M \in Sub^{\oplus}(C(F))$ that is injective in $\mathscr{A}^{\mathcal{T}_k}$ is also projective in $\mathscr{A}^{\mathcal{T}_k}$.

In particular, if any one of these statements hold, then $Loc_{C(F)^*}(C(F)^*)$ is dense in $C(F)$.

Proof. (b) ⇒ (a) is obvious from Lemma 5.2. To show that (a) ⇒ (b), we consider some $M \in Sub^{\oplus}(C(F))$ that is injective in $\mathscr{A}^{\mathcal{T}_k}$. As in the proof of Theorem 5.13, we can write $M = \mathcal{E}^{\mathcal{T}_k}(M) = \bigoplus_{i\in I} \mathcal{E}^{\mathcal{T}_k}(M_i)$, where each M_i is an F-simple subobject of M. By Proposition 5.7, each $\mathcal{E}^{\mathcal{T}_k}(M_i) \subseteq C(F)$ and hence a direct summand of $C(F)$. By assumption, $C(F) \in \mathscr{A}^{\mathcal{T}_k}$ is projective, and hence so is each $\mathcal{E}^{\mathcal{T}_k}(M_i)$. It now follows that the direct sum $M = \bigoplus_{i\in I} \mathcal{E}^{\mathcal{T}_k}(M_i)$ is projective in $\mathscr{A}^{\mathcal{T}_k}$. The last statement now follows from Proposition 5.16. □

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