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Discrete 2-Fibrations

Michael Lambert[a]

[a]*Department of Mathematics, University of Massachusetts-Boston, Boston, MA, USA*

Abstract

This paper is a study of 2-dimensional discrete fibrations. A definition is proposed as a specialization of 2-fibrations. It is shown that discrete 2-fibrations correspond via a category of elements construction to contravariant category-valued 2-functors. The second part of the paper is dedicated to a monadicity result. It is shown that 2-fibrations are algebras for a monad given by an action of Bénabou's cylinders construction. This should be seen as categorifying the monad for which ordinary fibrations are algebras, namely, that given by an action of an ordinary arrow category. Monadicity of discrete 2-fibrations is recovered by restricting the cylinders monad. To support the correctness of this categorification, it is shown that cylinders are a cotensor in a 3-categorical structure of 2-categories, 2-functors, lax natural transformations and modifications. This is an example of a *lax 3-category* which is introduced here to describe this universality.

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1. Introduction

Fibration properties of the 2-functor taking a fibration to its codomain led to the definition of a 2-fibration in [19]. This definition was generalized to bicategories in [2] and refined and studied extensively in [7]. Roughly, a 2-fibration is a 2-functor that has enough suitably defined 2-cartesian arrows. The precise definition vertically categorifies in dimension 2 that of an ordinary fibration introduced in [17, §VI.6]. And so, just as fibrations generalize discrete fibrations, it is expected that there is an analogously specialized concept of a *discrete 2-fibration*. What, then, makes a 2-fibration discrete?

An ordinary discrete fibration is a functor $F\colon \mathscr{F} \to \mathscr{C}$ whose fibers are sets that vary functorially with respect to morphisms in the base category $\mathscr{C}$. Such a functor is *discrete* in the sense that the fibers are locally discrete as categories. That is, discrete fibrations are discrete *relative to* ordinary fibrations, which are functors $F\colon \mathscr{F} \to \mathscr{C}$ such that each fiber is an honest category,

Email address: michael.james.lambert@gmail.com (Lambert)

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not a mere set, and possesses enough so-called *cartesian arrows* having a certain lifting property. To follow the pattern, then, a discrete 2-fibration would be a 2-functor whose fibers are locally discrete 2-categories, that is, ordinary categories, and which satisfies further lifting properties.

Another way to look at the situation is using elements correspondences. As reviewed below, discrete fibrations correspond to set-valued contravariant functors $\mathscr{C}^{op} \to \mathbf{Set}$ and cloven fibrations correspond to category-valued contravariant pseudo-functors $\mathscr{C}^{op} \to \mathbf{Cat}$. The former is a special case of the latter in that the codomain values of the former are locally discrete relative to those of the latter. Both correspondences are achieved by a category of elements construction leading to what will be called *representation theorems* since fibrations are therefore equivalent to representations of categories. An elements correspondence for 2-fibrations leading to a representation theorem of the same form was developed in [7]. It was shown there that 2-fibrations correspond to 2-category valued functors on 2-categories. Under a restricted correspondence, following the pattern, discrete 2-fibrations ought correspond to category-valued 2-functors on a 2-category $\mathfrak{C}^{coop} \to \mathbf{Cat}$ since categories are locally discrete as 2-categories.

Now the goals of the paper can be stated. First are those of Section 2 and Section 3 concerning the definition, general theory, examples and relation to 2-fibrations. These are

1. to define a discrete 2-fibration (Definition 2.4), an elements construction (Construction 2.1), and to discuss examples, in particular, that of cod: $\mathbf{DFib} \to \mathbf{Cat}$, a *discrete* 2-fibration;
2. to justify the definition by exhibiting the pseudo-inverse (Construction 3.2) leading to a representation theorem in Theorem 3.7;
3. and finally to show in Theorem 3.23 how this equivalence restricts the established equivalence for 2-fibrations from [7] mentioned above.

Additionally, but no less importantly, there are several monadicity results. These are

1. to show in Theorem 4.13 that (discrete) 2-fibrations are monadic and that the monad is given by an action of Bénabou's cylinder construction (from [3, §8.2]);
2. and finally to develop a 3-categorical setting to describe universal constructions arising in the description of (discrete) 2-fibrations as algebras, namely, the notion of a *lax 3-category* in Definition 4.21. This is a category enriched in the cartesian monoidal category of 2-categories and lax functors. Since the external composition assignments are thus *lax* 2-functors, lax 3-categories are neither 2- nor 3-categories.

These are the topics of Section 4. In particular, lax 3-categories, discussed in Section 4.3, are introduced to described the cylinder construction as a certain limit, namely, a cotensor with **2**. This is meant to mimic the 1-dimensional case where ordinary fibrations are monadic relative to actions of ordinary arrow categories which are cotensors with **2** in the 2-category of categories. More discussion of the monadicity goals follows below in Section 1.3 and in Section 4.1.

Along the way, we will also prove some results clarifying the way in which 2-fibrations relate to long-standing characterizations of fibrations internal to 2-categories. This question was not addressed directly in [7]. That is, there are a number of equivalent *intrinsic characterizations* of internal fibrations and internal discrete fibrations. For 2-categories, these originate with [37, §2] which forms the basis for later (bicategorical) generalizations in for example [38], [39], and [21]. Roughly, in the 2-categorical setting, [37] defines fibrations as algebras for a certain lax-idempotent monad given by an action of a cotensor. This definition is then shown [37, Proposition 9] to be equivalent to an internalized *Chevalley criterion*. The take-away of our Section 3.4 is that the notion of a 2-fibration is *not* that of a fibration in a 2-category in the sense of [37] if by that one means the naive thing, namely, the 2-category **2Cat**. Likewise, we show directly that discrete 2-fibrations are not discrete fibrations internal to **2Cat**. All in all, this means that (discrete) 2-fibrations are not algebras for the cotensor action in **2Cat** as described in the reference. It was in fact the search for what type of action could be defined on a (discrete) 2-fibration that led to our monadicity results and the definition of a *lax 3-category*. More on this immediately below and in Section 1.3.

The rest of this introductory section is meant to review some basics on fibrations (Section 1.1) and monadicity (Section 1.3) to state the goals of the paper more precisely. Along the way we will discuss the importance and role of laxity (Section 1.2) in our results. Before all that, however, we need to make some remarks on our conventions, notation, and outlook.

Throughout the paper, 2-categories should be understood to be *strict* and distinguished from bicategories where composition is associative only up to coherent isomorphism. It will be stated explicitly whenever *bicategory* is meant and not *2-category*. Likewise 3-categories are always strict. In fact, this paper deals very little with bicategories and other up-to-isomorphism settings. For example, we will not be discussing fibrations *in* a bicategory [38], nor fibrations *of* bicategories [7, §3]. There are three reasons for this.

1. Owing to the comments on internalization immediately above, we view the present work as a kind of fresh start informed by the narratives of previous developments. As a purely practical matter, bicategories introduce significantly more complication in terms of coherence conditions [7, §3], and we believe simply setting out the results and analysis for the strict case clears the way for further developments in the up-to-isomorphism scenarios for those who have the interest, patience and energy to make the required calculations.
2. Secondly, we do not view bicategories simply as somehow *weak 2-categories*, or conversely 2-categories as somehow merely *strict bicategories*. Rather 2- and bi-categories are orthogonal concepts that are unified by the notion of a double category [13]. In particular, each double category has a vertical 2-category and a horizontal bicategory; every 2-category is a double category with a trivial horizontal bicategory and every bicategory is a double category with a trivial vertical 2-category. Viewed this way, morphisms in bicategories should be imagined as bimodules, spans, or profunctors. Not only do these behave differently from ordinary arrows (thought of as functions, homomorphisms, other structure-preserving maps), they should typically be thought of rather as more like objects[1]. So, from this perspective, to ask for cartesian-like lifting properties for objects is at least awkward and worse seems at odds with all the standard variations of the notion of fibration. Rather, work on *double fibrations* [8] would imply that, from this point of view, a good candidate for a fibration of bicategories is a double fibration that is vertically trivial, having a lifting property for cells but not the arrows of the bicategory.
3. Finally, our intended applications concern purely 2-categorical generalizations, in the setting of 2-toposes [40], of filtering and flatness results familiar from the theory of ordinary presheaves [32]. More on this potential application in Section 1.3 and Section 5.1.

Collaterally, these reasons together also justify our use of [7, §2] as the primary reference on 2-fibrations. It is the most recent publication on the matter, dealing in the cited section with the purely 2-categorical notions of interest. It also rectifies an omission in the original definition [19] which was replicated in the bicategorical definitions of [2] which anyway specialize to those of [19] in the case of strict 2-categories (as observed [2, p.46]). On these points see also the

[1] "...our philosophy [is] that the horizontal 1-cells [of a double category] are not 'morphisms', but rather objects in their own right which just happen to be 'labeled' by a pair of objects of another type." [34, p. 655]

commentary in [7, Remark 2.2.12].

References on 2- and 3-categories include, for example, [26], [16, I,2 and I,3], [6, Chapter 7], [22, Chapter B1], and [27]. For 2-categories, internal composition of 2-cells (that is, of 2-cells with a shared ordinary arrow) is denoted by juxtaposition $\beta\alpha$, while external composition (2-cells with a shared object) is denoted by '$*$' as in $\delta * \gamma$. By the term 'set' is meant a set or collection, naively construed. Assume throughout that there is a sufficient supply of so-called *universes* in the sense of [1], that is, a set of sets that is transitive under membership and closed under pairing, powersets, and unions indexed by elements of the universe. The **universe assumption** is that every set belongs to some universe. Throughout **Set** denotes a fixed category of distinguished sets that are members of some given, fixed universe. Thus, a given arbitrary set is potentially large, i.e., not in **Set**. Throughout assume the basic machinery of enriched categories as in [24]. Thus, reemphasizing the strictness assumption, a 2-category $\mathfrak{K}$ is a $|\mathbf{Cat}|$-enriched category where $|\mathbf{Cat}|$ is the 1-category of (small) categories and functors. In general the notation '$|\cdot|$' for higher categories as in '$|\mathbf{Cat}|$' will mean the $n-1$ dimensional structure obtained by discarding the top-dimensional cells of the structure enclosed in the '$|\cdot|$'. So, for example, '$||\mathbf{2Cat}||$' denotes the 1-category of 2-categories and 2-functors obtained from the 3-category **2Cat** by forgetting the 3-cells and the 2-cells.

1.1 Review of Fibrations and Discrete Fibrations From some background on fibrations, we will give the desiderata guiding the results of the paper. Standard references on (discrete) fibrations include, for example, [15, §2, §3], [4], [20, Chapter 1], [22, §B1.3], [40, §2.2] and [30]. Recall first that a **discrete fibration** over a small category $\mathscr{C}$ is a functor $F\colon \mathscr{F} \to \mathscr{C}$ such that for each morphism $f\colon C \to FX$ with $X \in \mathscr{F}$, there is a unique morphism $Y \to X$ of $\mathscr{F}$ above f. A functor $E\colon \mathscr{E} \to \mathscr{C}$ is a **discrete opfibration** if E^{op} is a discrete fibration. A morphism of discrete fibrations $F\colon \mathscr{F} \to \mathscr{C}$ and $G\colon \mathscr{G} \to \mathscr{C}$ is a functor $H\colon \mathscr{F} \to \mathscr{G}$ such that $GH = F$ holds. Let $\mathbf{DFib}(\mathscr{C})$ denote the category of discrete fibrations over $\mathscr{C}$ and $\mathbf{DOpf}(\mathscr{C})$ denote the category of discrete opfibrations over $\mathscr{C}$.

For each set-valued functor $F\colon \mathscr{C}^{op} \to \mathbf{Set}$, there is an associated category of elements detailed for example in [31, §II.6, §III.7], yielding a discrete fibration $\Pi\colon \mathbf{Elt}(E) \to \mathscr{C}$. The source category has as objects pairs (C, x) with $C \in \mathscr{C}_0$ and $x \in FC$ and as morphisms $(C, x) \to (D, y)$ those morphisms $f\colon C \to D$ of $\mathscr{C}$ with $x = Ff(y)$.

Theorem 1.1 (Representation Theorem I)**.** *The category of elements construction is one half of an equivalence of categories*

$$\mathbf{DFib}(\mathscr{C}) \simeq [\mathscr{C}^{op}, \mathbf{Set}].$$

between discrete fibrations and presheaves on $\mathscr{C}$*.*

Proof. The pseudo-inverse sends a discrete fibration $F\colon \mathscr{F} \to \mathscr{C}$ to the functor $\mathscr{C}^{op} \to \mathbf{Set}$ whose action on $C \in \mathscr{C}_0$ is to take the fiber of F above C. Notice that these fibers must be discrete categories by the uniqueness assumption. See [30, Theorem 2.1.2] for further details. □

Remark 1.2 (Desiderata 1)**.** The first main goal of the paper is an analogue of Theorem 1.1 for discrete 2-fibrations. The main result is given as Theorem 3.7. □

Recall that a functor $F\colon \mathscr{F} \to \mathscr{C}$ is a **fibration** if for each $x\colon X \to FA$ there is an arrow $f\colon B \to A$ of $\mathscr{F}$ such that $Ff = x$ and having the property that whenever $h\colon C \to A$ makes a commutative triangle $xu = Fh$ as below there is a unique F-lift $\hat{u}\colon C \to B$ over u making a commutative triangle in $\mathscr{F}$ as indicated in the following picture

$$\begin{array}{ccc} C & \xrightarrow{h} & A \\ {\scriptstyle \hat{u}}\downarrow & \nearrow_{f} & \\ B & & \end{array} \qquad \begin{array}{ccc} FC & \xrightarrow{Fh} & FA \\ {\scriptstyle u}\downarrow & \nearrow_{Ff=x} & \\ X & & \end{array}$$

Such a morphism f is **cartesian** over x. A morphism of $\mathscr{F}$ is F-vertical if its image under F is an identity. The **fiber** of F over an object $C \in \mathscr{C}$ is the subcategory of $\mathscr{F}$ of objects and vertical morphisms over C via F. A functor $E\colon \mathscr{E} \to \mathscr{C}$ is an **opfibration** if E^{op} is a fibration; in this case the morphisms of $\mathscr{E}$ having the special lifting property are called *opcartesian*. A **cleavage** ϕ for a fibration specifies a cartesian morphism in $\mathscr{F}$ for each such $f\colon X \to FA$ in $\mathscr{C}$. Denote the chosen cartesian morphism by $\phi(f, A)\colon f^*A \to A$. A fibration with a cleavage is said to be *cloven*. Notice that each discrete fibration is a cloven fibration.

Remark 1.3. A cleavage need not be functorial. That is, given composable arrows $f\colon X \to Y$ and $g\colon Y \to FB$ of $\mathscr{C}$, there is a diagram of chosen cartesian arrows in $\mathscr{F}$ of the form

$$\begin{array}{ccc} f^*g^*B & \xrightarrow{\phi(f,g^*B)} & g^*B \\ {\scriptstyle \cong}\downarrow & & \downarrow{\scriptstyle \phi(g,B)} \\ (gf)^*B & \xrightarrow[\phi(gf,B)]{} & B \end{array}$$

The dashed arrow exists since a composition of cartesian morphisms is again cartesian. It is an isomorphism by the uniqueness aspect of the definition. But in general this isomorphism is not an identity. When every such isomorphism is an identity, the fibration $F\colon \mathscr{F} \to \mathscr{C}$ is said to be **split**. The difference between cloven and split fibrations over a base category is essentially the difference between category-valued pseudo-functors and 2-functors indexed by the base. □

In this paper, fibrations will usually be split. Thus, let $\mathbf{Fib}(\mathscr{C})$ denote the 2-category of split fibrations over $\mathscr{C}$ with splitting-preserving functors as morphisms and natural transformations with vertical components as the 2-cells. Dually, $\mathbf{Opf}(\mathscr{C})$ is the 2-category of split opfibrations over $\mathscr{C}$ with appropriate morphisms and 2-cells.

Now, start with a 2-functor $F\colon \mathscr{C}^{op} \to \mathbf{Cat}$. Denote the transition functor associated to an arrow $f\colon C \to D$ by $f^*\colon ED \to EC$. As in the discrete case, there is an associated fibration arising as a category of elements construction $\Pi\colon \mathbf{Elt}(F) \to \mathscr{C}$ originating in [17]. The source category has objects pairs (C, X) with $X \in FC$ and as morphisms $(C, X) \to (D, Y)$ pairs (f, u) where $f\colon C \to D$ and $u\colon X \to f^*Y$ is a morphism of FC. Units and composition are well-known, but described for example in [22, §B1.3].

Theorem 1.4 (Representation Theorem II)**.** *The category of elements construction is one-half of an equivalence of 2-categories*

$$\mathbf{Fib}(\mathscr{C}) \simeq [\mathscr{C}^{op}, \mathbf{Cat}]$$

between split fibrations over $\mathscr{C}$ and contravariant category-valued 2-functors on $\mathscr{C}$.

Proof. Again the pseudo-inverse sends a split fibration F to the 2-functor that associates to each $C \in \mathscr{C}_0$ the fiber of F over it. For more see [22, Theorem B1.3.5] for example. □

Remark 1.5 (Desiderata 2)**.** Every discrete fibration is a split fibration. Additionally, the elements construction for a category-valued functor on $\mathscr{C}^{op}$ applied to one taking discrete categories as values reduces to the category of elements construction for presheaves. Thus, the equivalence in Theorem 1.4 restricts to that of Theorem 1.1 as in the commutative diagram

$$\begin{array}{ccc} \mathbf{Fib}(\mathscr{C}) & \xrightarrow{\simeq} & [\mathscr{C}^{op}, \mathbf{Cat}] \\ \uparrow & & \uparrow \\ \mathbf{DFib}(\mathscr{C}) & \xrightarrow[\simeq]{} & [\mathscr{C}^{op}, \mathbf{Set}]. \end{array}$$

[7, Theorem 2.2.11] gives a categorification of Theorem 1.4 as a 3-equivalence between 2-fibrations and 2-category-valued functors indexed by the base 2-category:

$$\mathbf{2Fib}(\mathfrak{B}) \simeq [\mathfrak{B}^{coop}, \mathbf{2Cat}]$$

More on the details of this correspondence in Section 3.3. For now, this is enough to state the requirement on subsequent developments that any equivalence between discrete 2-fibrations and category-valued functors should be a restriction of that for 2-fibrations above. □

1.2 Centrality of Lax Notions Lax transformations arise in Section 2 and especially in Section 4. This is partly owing to the nature of the universality of elements constructions. For example, recall that for any functor $F\colon \mathscr{C}^{op} \to \mathbf{Set}$, the associated category of elements fits into an ordinary comma square

$$\begin{array}{ccc} \mathbf{Elt}(F)^{op} & \xrightarrow{\Pi^{op}} & \mathscr{C}^{op} \\ \downarrow & \overset{\phi}{\Rightarrow} & \downarrow F \\ \mathbf{1} & \xrightarrow[*]{} & \mathbf{Set} \end{array}$$

where $*\colon \mathbf{1} \to \mathbf{Set}$ denotes the inclusion of the one-element set. Notice that Π^{op} is a discrete opfibration, hence that Π is a discrete fibration. And the content of Theorem 1.1 is that a functor $P\colon \mathscr{F} \to \mathscr{C}$ is a discrete fibration if, and only if, P^{op} is isomorphic over $\mathscr{C}^{op}$ to the opposite of the category of elements of a canonically constructed functor $F_P\colon \mathscr{C}^{op} \to \mathbf{Set}$. Going up a level to category-valued functors, the category of elements for a 2-functor $F\colon \mathscr{C}^{op} \to \mathbf{Cat}$ fits into not a 2-comma square, but instead a *lax* comma square of the same form

$$\begin{array}{ccc} \mathbf{Elt}(F)^{op} & \xrightarrow{\Pi^{op}} & \mathscr{C}^{op} \\ \downarrow & \overset{\phi}{\Rightarrow} & \downarrow F \\ \mathbf{1} & \xrightarrow[*]{} & \mathbf{Cat} \end{array}$$

where ϕ is a *lax* natural transformation. Below in Proposition 2.18 the elements construction associated to a category-valued 2-functor on a 2-category will be characterized as one leg of a comma square with a universal lax natural transformation between 2-functors. This result will be used in proving the main representation theorems in Section 3. But in fact laxity pervades the constructions and results of the paper and is in no small part responsible for the failure of the notion of a (discrete) 2-fibration to conform to the ordinary internal/representable definition of internal fibration from [37] where by *internally* we mean internally in **2Cat**. Rather what we shall see in Section 4 is that there is a different structure, namely, that of a *lax 3-category*, which again is not a 2- or 3-category, whose 2-cells are lax natural transformations that describes the universality of the cotensor action for which (discrete) 2-fibrations are algebras and that also gives a kind of recovered representability result in Proposition 4.29.

1.3 Monadicity Discrete fibrations over a category $\mathscr{C}$ are monadic over $\mathbf{Set}/\mathscr{C}_0$ [32, §V.7]. The monad is given by an action of the set of arrows $\mathscr{C}_1$. For an application of this result, recall first that the tensor product extension of a presheaf along the Yoneda embedding is left exact if, and only if, its category of elements is filtered [32, §VII.6]. Diaconescu gave an elementary version of this result in [10] and [11] replacing the category of sets by an arbitrary topos $\mathscr{E}$. Base-valued functors on an internal category $\mathbb{C}$ are replaced by algebras for an action of C_1 on objects of the slice $\mathscr{E}/C_0$. The main results is that an algebra is internally filtered if, and only if, its internal tensor product extension is left exact.

A tensor product of category-valued 2-functors was developed in [9] and [28] with a view to internalizing the results of Diaconescu discussed above to an arbitrary 2-topos in the sense of [40]. So, the natural question arising in this connection is that of the filtering conditions on the corresponding 2-category of elements that are equivalent to the exactness of the tensor product extension. This was answered in [9] where a notion of 2-filteredness was proposed and shown to be equivalent to exactness. Now, any elementary version of these results in a 2-topos along the pattern of Diaconescu's work would first require a repackaging of base-valued 2-functors as some kind of algebra. It is the second overall goal of this paper to show that the concept of discrete 2-fibration proposed here is 2-monadic over a slice of the 2-category of categories and that the monad is given by an action of the 2-category of cylinders associated to a 2- or bi-category in [3], reviewed below in Construction 2.9.

Fibrations over a base category $\mathscr{C}$ are algebras for a monad given by an action of $\mathscr{C}^{\mathbf{2}}$, the cotensor in **Cat** of $\mathscr{C}$ with the two-element ordinal category, that is, an action of the *arrow category* of the base category [15]. This classical development led to the intrinsic definition in [37] as algebras for the action of the corresponding cotensor with **2**. Since a discrete 2-fibration is at least a fibration, it might be expected that the corresponding action for such a monadicity result is by an arrow 2-category in **2Cat** consisting of arrows, commutative squares and pairs of 2-cells satisfying a compatibility condition, directly adding higher structure to the ordinary arrow category by vertical categorification. In fact, such a structure is the cotensor in the 3-category of 2-categories, 2-functors, 2-natural transformations and modifications (Proposition 2.18). However, because a discrete 2-fibration is also locally a discrete fibration, it admits an action from a more general structure where the commutative squares are replaced by squares with a mere 2-cell. And indeed to obtain a discrete 2-fibration from a functor admitting an action of some kind of 2-categorical arrow structure, both the commutative squares and the globular structure of the base 2-category are required. For the squares give the transition functors and the globular structure gives the transition 2-cells. The natural way to pack this information into a 1-category is to take the underlying 1-category of the 2-category of cylinders.

However, this cylinder construction is *not* a cotensor in **2Cat**. For it is universal not among 2-natural transformations but among *lax* natural transformations Corollary 4.27. Accordingly, the last subsection Section 4.3 of the paper is devoted to building up a 3-dimensional categorical structure to describe this universality, that is, a setting whose objects are 2-categories, whose morphisms are 2-functors, whose 2-cells are lax natural transformations, and whose 3-cells are modifications. This can be obtained as a category enriched in the 1-category of 2-categories and lax functors, giving what we call a *lax 3-category* (Definition 4.21) which is neither a 2- nor a 3-category. The cylinder construction is then the cotensor with **2** in this setting (Corollary 4.27). We believe that laxity partly explains the failure of (discrete) 2-fibrations to be recovered by the ordinary intrinsic/representable definition applied to **2Cat**. Again a recovered representability result in this context appears in Proposition 4.29.

2. Discrete 2-Fibrations

This section sets out the main definition of the paper, namely, that of a discrete 2-fibration in Section 2.1. Examples are discussed in Section 2.2. Finally Section 2.3 exhibits one aspect of the universality of the elements construction.

2.1 The Definition Properties of the 2-category of elements construction in Proposition 2.2 lead to the definition of a discrete 2-fibration. The elements construction is based on that for 2-category-valued functors but specialized to the case where these 2-categories are locally discrete, that is, ordinary categories. The construction originates with Bird's thesis [5] and appears for 2-fibrations in [7, §2.2.1].

As a matter of notation, if $F\colon \mathfrak{B}^{coop} \to \mathbf{Cat}$ is a 2-functor, write $f^*\colon FY \to FX$ and $\alpha^*\colon g^* \Rightarrow f^*$ for the transition functors and natural transformations in $\mathbf{Cat}$ associated to ordinary arrows $f\colon X \to Y$ and cells $\alpha\colon f \Rightarrow g$ of $\mathfrak{B}$.

Construction 2.1 (2-Category of Elements)**.** For any 2-functor $F\colon \mathfrak{B}^{coop} \to \mathbf{Cat}$ on a 2-category $\mathfrak{B}$, the **2-category of elements** of E is the 2-category whose

1. objects are pairs (B, X) with $B \in \mathfrak{B}_0$ and $X \in FB$;
2. arrows are pairs $(f, \bar{f})\colon (B, X) \to (C, Y)$ with $f\colon B \to C$ in $\mathfrak{B}$ and $\bar{f}\colon X \to f^*Y$ in the category FB;
3. and whose 2-cells : $(f, \bar{f}) \Rightarrow (g, \bar{g})$ are those $\alpha\colon f \Rightarrow g$ in $\mathfrak{B}$ making a commutative triangle

$$\begin{array}{ccc} X & \xrightarrow{\bar{f}} & f^*Y \\ \Vert & & \uparrow{\scriptstyle \alpha^*_Y} \\ X & \xrightarrow[\bar{g}]{} & g^*Y \end{array}$$

of arrows in the category FB.

Denote this 2-category by $\mathbf{Elt}(E)$. There is an evident projection 2-functor $\Pi\colon \mathbf{Elt}(E) \to \mathfrak{B}$. □

Up to isomorphism, every discrete fibration is the projection from a category of elements. So, the fibration properties of the above projection should suggest the 2-dimensional analogue.

Proposition 2.2. *Let $F\colon \mathfrak{B}^{coop} \to \mathbf{Cat}$ denote a 2-functor. The projection $\Pi\colon \mathbf{Elt}(F) \to \mathfrak{B}$ from the 2-category of elements (Construction 3.22) has the following fibration properties.*

1. *The ordinary functor $|\Pi|\colon |\mathbf{Elt}(F)| \to |\mathfrak{B}|$ of underlying 1-categories is a split fibration.*
2. *Locally Π is a discrete fibration.*

Proof. Since at the level of 1-categories, the 2-category of elements is the same as the ordinary 1-category of elements, the first point has been established. For the discrete fibration claim, start with a morphism $(f, \bar{f})\colon (C, X) \to (D, Y)$ and a cell $\alpha\colon f \Rightarrow g\colon C \rightrightarrows D$ from the image of $(f, \bar{f})$ under Π in $\mathfrak{B}$. The required lift is the cell

$$(C, X) \;\underset{(g,\bar{g})}{\overset{(f,\alpha^*_Y u)}{\Downarrow (\alpha,1)}}\; (D, Y)$$

This cell is evidently over α via the projection Π. And it is the unique such morphism since the values of F are ordinary categories, hence locally discrete as 2-categories. □

Remark 2.3. The 2-category of elements construction for a 2-functor $E\colon \mathfrak{B} \to \mathbf{Cat}$ will be a split opfibration at the level of underlying 1-categories and a discrete fibration locally. □

Definition 2.4. A **split discrete 2-fibration** is a 2-functor $E\colon \mathfrak{E} \to \mathfrak{B}$ such that

1. the underlying functor $|E|\colon |\mathfrak{E}| \to |\mathfrak{B}|$ is a split fibration;
2. E is locally a discrete fibration, in that each functor $E\colon \mathfrak{E}(X, Y) \to \mathfrak{B}(EX, EY)$ is a discrete fibration.

Dually, a discrete 2-opfibration is a 2-functor $E\colon \mathfrak{E} \to \mathfrak{B}$ whose underlying functor of 1-categories is a split opfibration and which is locally a discrete opfibration. Let $\mathbf{D2Fib}(\mathfrak{B})$ denote the 2-category of split discrete 2-fibrations, splitting-preserving 2-functors over $\mathfrak{B}$, and 2-natural transformations with vertical components. □

Remark 2.5 (Terminology)**.** It might be expected that representable functors $\mathfrak{B}(-, X)\colon \mathfrak{B}^{op} \to \mathbf{Cat}$ should correspond to prime examples of discrete 2-fibrations under the elements correspondence. However, the resulting 2-functors on the fibration side are discrete opfibrations locally since each $\mathfrak{B}(-, X)$ is contravariant only on arrows and not on 2-cells. The terminology of Hermida and Buckley is established, however, so it will be kept here. Rather those fibration structure that are fibrations globally and opfibrations locally will be called *(discrete) 2-cofibrations* indicating duality to the fibration concept at the level of 2-cells. □

Remark 2.6 (Why not *doubly discrete*?)**.** There might be some expectation that the underlying functor $|E|\colon |\mathfrak{E}| \to |\mathfrak{B}|$ in the definition would be a discrete fibration as well. From the point of view of the elements construction, such a functor would arise from one $\mathfrak{B}^{coop} \to \mathbf{Cat}$ such that the image of any object is a mere set. As a result there would be no meaningful cell assignment since $\mathbf{Set}$ is locally discrete as a 2-category. In this way, if $|E|\colon |\mathfrak{E}| \to |\mathfrak{B}|$ were taken to be discrete as well, the concept would just be one of an ordinary discrete fibration that happens to include into a 2-category. It would have no local lifting properties and not be a 2-fibration. For this reason, a discrete 2-fibration is not a kind of discrete double fibration [29]. It is either an orthogonal concept or a species of *double fibration*. More on this point in Section 5.2. □

2.2 Examples

Up to now the use of notation such as '**Set**', '**Cat**' and '**2Cat**' has been a bit cavalier with regard to size issues. By the universe axiom, there is a 2-category of categories containing **Set** as a member and admitting an inclusion viewing a set as a locally discrete category. Similarly, there is a 3-category of 2-categories containing the 2-category of categories as a member and admitting an inclusion viewing a category as a locally discrete 2-category. If it is important to distinguish between, for example, a 2-category of **Set**-small categories and a 2-category of categories containing **Set** as a member, we write **Cat** for the former and **CAT** for the latter. We adopt a similar convention for 3-categories of 2-categories.

Example 2.7. Any split fibration $F\colon \mathscr{F} \to \mathscr{C}$ is a split discrete 2-fibration. Dually, any split opfibration is a split discrete 2-opfibration. □

Example 2.8. Let $\mathbf{DFib}$ denote the 2-category of discrete fibrations $F\colon \mathscr{F} \to \mathscr{C}$, whose morphisms $F \to G$ are pairs of functors (H, K) making commutative squares

$$\begin{array}{ccc} \mathscr{F} & \xrightarrow{H} & \mathscr{G} \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle G} \\ \mathscr{C} & \xrightarrow[K]{} & \mathscr{D} \end{array}$$

and whose 2-cells $(H, K) \Rightarrow (L, M)$ are pairs of transformations (α, β) with $\alpha\colon H \Rightarrow L$ and $\beta\colon K \Rightarrow M$ satisfying the condition $G * \alpha = \beta * F$. The projection 2-functor

$$\mathrm{cod}\colon \mathbf{DFib} \to \mathbf{Cat} \tag{2.1}$$

taking a discrete fibration to its codomain and extended suitably to morphisms and 2-cells is a discrete 2-fibration. Identify the fiber of cod over $\mathscr{C} \in \mathbf{Cat}$ with the category $\mathbf{DFib}(\mathscr{C})$ of discrete fibrations over $\mathscr{C}$. □

Construction 2.9 (2-Category of Cylinders)**.** The cylinder construction [3, §8.2] associated to a 2-category is used in two examples below and gives the action for the monadicity results below in Section 4.2. Given a 2-category $\mathfrak{C}$, the 2-category of cylinders $\mathbf{Cyl}(\mathfrak{C})$ has as objects morphisms $f\colon A \to B$ of $\mathfrak{C}$, as arrows $f \to g$ 2-cells

$$\begin{array}{ccc} A & \xrightarrow{h} & C \\ {\scriptstyle f}\downarrow & \overset{\alpha}{\Rightarrow} & \downarrow{\scriptstyle g} \\ B & \xrightarrow[k]{} & D \end{array}$$

of $\mathfrak{C}$, and finally as 2-cells pairs of 2-cells (γ, δ) for which there is an equality of composite 2-cells as in the diagram

$$\begin{array}{ccc} A & \xrightarrow{m} & C \\ \| & \overset{\gamma}{\Rightarrow} & \| \\ A & \xrightarrow[h]{} & C \\ {\scriptstyle f}\downarrow & \overset{\alpha}{\Rightarrow} & \downarrow{\scriptstyle g} \\ B & \xrightarrow[k]{} & D \end{array} \quad = \quad \begin{array}{ccc} A & \xrightarrow{m} & C \\ {\scriptstyle f}\downarrow & \overset{\beta}{\Rightarrow} & \downarrow{\scriptstyle g} \\ B & \xrightarrow[n]{} & D \\ \| & \overset{\delta}{\Rightarrow} & \| \\ B & \xrightarrow[k]{} & D \end{array}$$

making a cylinder. In the above display think of α as the front face and β as the back face. Following Bénabou's conventions, the domain of the arrow α is f and the codomain is g. The arrows h and k are the source and target respectively. Denote *source* and *target* 2-functors by $\mathrm{src}\colon \mathbf{Cyl}(\mathfrak{C}) \to \mathfrak{C}$ and $\mathrm{tgt}\colon \mathbf{Cyl}(\mathfrak{C}) \to \mathfrak{C}$, respectively. The source 2-functor takes an object (a morphism of $\mathfrak{B}$) to its domain, takes a morphism (a square as above) to the morphism h, and a 2-cell pair (γ, δ) to γ. Target is defined analogously. There are 3 ways of composing cylinders. All of these are strictly associative since $\mathfrak{B}$ is a 2-category. Domain-to-codomain and front-to-back compositions give the 2-category structure on $\mathbf{Cyl}(\mathfrak{B})$. As a 2-category, domain-to-codomain is external composition and front-to-back is internal. Notice that this makes $\mathbf{Cyl}(\mathfrak{B})$ much like the ordinary arrow 2-category of $\mathfrak{B}$ but with arbitrary squares with a cell instead of just the commutative ones. Source-to-target composition, that is, *superposition* in the language of [3], is reserved for the action on 2-fibrations discussed in Section 4.2. What will be termed *opcylinders* have a form similar to that above, but the domain and codomains are switched. That is, a morphism $f \to g$ in the 2-category $\mathbf{Cyl}^{op}(\mathfrak{B})$ of opcylinders is a cell in the opposite direction

$$\begin{array}{ccc} A & \xrightarrow{h} & C \\ {\scriptstyle f}\downarrow & \overset{\alpha}{\Leftarrow} & \downarrow{\scriptstyle g} \\ B & \xrightarrow[k]{} & D. \end{array}$$

Opcylinders point from the front face α to the back face β and like ordinary cylinders the component 2-cells preserve the direction. By contrast, *cocylinders*, a concept not used in the paper, would reverse the direction of the component 2-cells. This 2-category of cylinders is a special case of our *lax comma 2-category* associated to any pair of 2-functors with common codomain (Construction 2.13) in the special case where both 2-functors are the identity. □

Example 2.10. For any 2-category $\mathfrak{C}$, the **lax slice** over an object $C \in \mathfrak{C}_0$ consists of arrows $f\colon B \to C$ as its objects, with morphisms $f \to g$ those pairs consisting of $h\colon B \to D$ and a 2-cell $\alpha\colon f \Rightarrow gh$, and finally with 2-cells those $\theta\colon h \Rightarrow k$ satisfying the compatibility condition from the example of $\mathbf{Cyl}(\mathfrak{C})$ but with $B = D$, $h = k$ and $\delta = 1$. The lax slice will be denoted by '$\mathfrak{C}/C$'. Of course, there are various candidates for 2-categorical slices. That is, the morphisms could alternatively have the 2-cell α be an isomorphism or an equality. The latter would perhaps be called the *strict slice* 2-category. Whereas the lax slice will occur in a number of examples, the strict slice will be used throughout §4 in considering monadicity. The two notions will not be used in the same context, so the same notation '$\mathfrak{C}/C$' will stand for each and the meaning will be explicitly stated whenever it arises. The domain projection $d_0\colon \mathfrak{C}/C \to \mathfrak{C}$ from the lax slice of a 2-category $\mathfrak{C}/C$ back to $\mathfrak{C}$ is a discrete 2-fibration. Notice that each $\mathfrak{C}/C$ is the fiber of the 2-functor tgt: $\mathbf{Cyl}(\mathfrak{C}) \to \mathfrak{C}$. This means, of course, that d_0 is a fragment of the source 2-functor. Discrete 2-fibrations isomorphic to those of the form d_0 are said to be **representable**. □

Example 2.11. Let $\mathfrak{K}$ denote a 2-category and $t\colon B \to B$ a monad in $\mathfrak{K}$ as in [26, §3.1]. Define a 2-category $t\mathbf{Alg}$ of t-algebras in $\mathfrak{K}$. The objects are t-algebras (s, ν) where $s\colon A \to B$ and $\nu\colon ts \Rightarrow s$ satisfying (3.2) of the reference. A morphism $(s, \nu) \to (r, \lambda)$ is a pair (g, σ) where $g\colon A \to C$ is a $\mathfrak{K}$-morphism and $\sigma\colon s \Rightarrow rg$ is a 2-cells satisfying the equation

$$(\lambda * g)(t * \sigma) = \sigma\nu,$$

basically the appropriate adaptation of (3.3) of the reference allowing the domain of the t-algebras to vary. The equation says that σ is a morphism of the t-algebras s and gr. A 2-cell $(g, \sigma) \Rightarrow (h, \tau)$ is one $\alpha\colon g \Rightarrow h$ such that $(r * \alpha)\sigma = \tau$ holds. Notice that by fixing the domain object in the t-algebra $s\colon A \to B$, this defines a category $t\mathbf{Alg}(A)$ of algebras with domain A. Since algebra structure is preserved by precomposition with any morphism and homomorphisms are preserved by precomposition with arbitrary 2-cells, these categories yield a functor $t\mathbf{Alg}(-)\colon \mathfrak{K}^{op} \to \mathbf{Cat}$. The forgetful 2-functor

$$\Pi\colon t\mathbf{Alg} \to \mathfrak{K}$$

is a discrete 2-fibration in the sense of Definition 2.4. This is because, as observed in the reference, pulling back by a morphism or by a 2-cell preserves t-algebra structure, but changes the domain or the target. □

Example 2.12 (Families)**.** Let $\mathfrak{B}$ denote a (small) 2-category. Note that 2-functors $\mathscr{C}^{op} \to \mathfrak{B}$ from a 1-category $\mathscr{C}$ amount to 1-functors $\mathscr{C}^{op} \to |\mathfrak{B}|$ since $\mathscr{C}$ has no nontrivial 2-cells. Take $\mathbf{fam}(\mathfrak{B})$ to denote the 2-category whose objects are pairs $(\mathscr{C}, F)$ where $F\colon \mathscr{C}^{op} \to |\mathfrak{B}|$ is a functor. The morphisms are of the same form as those in $\mathbf{Fam}(\mathfrak{B})$ from Example 3.20 below. The 2-cells are also essentially the same, but with the difference that the modification in the definition must be an identity. The projection

$$\Pi\colon \mathbf{fam}(\mathfrak{B}) \to \mathbf{Cat}$$

is then a discrete 2-fibration as in Definition 2.4. There is an inclusion $\mathbf{fam}(\mathfrak{B}) \to \mathbf{Fam}(\mathfrak{B})$ commuting with the projections to $\mathfrak{B}$. Since $\Pi\colon \mathbf{fam}(\mathfrak{B}) \to \mathbf{Cat}$ is *a fortiori* a 2-fibration, this might be seen as another main example of an inclusion of a sub-2-fibration insofar as the concept is of interest. Covariant families are discrete 2-cofibrations in the language of Remark 2.5. □

2.3 Universality Here will be given a description of the universal property of the elements construction associated to a category-valued 2-functor on a 2-category. In particular it fits into a lax comma square, generalizing the comma squares discussed in Section 1.2. The following construction gives the general form of the apex object of these lax comma squares. This should be seen as a generalization of the cylinder construction which is the lax comma square for the cospan formed by identity morphisms. It is close to [16, §I,2.5].

Construction 2.13 (Lax Comma Category)**.** The **lax comma category** of a 2-functor $F\colon \mathfrak{A} \to \mathfrak{B}$ over another $G\colon \mathfrak{C} \to \mathfrak{B}$, denoted by $F /\!\!/ G$, has as objects triples (A, f, C) with $f\colon FA \to GC$ an arrow of $\mathfrak{B}$; and with morphisms $(A, f, C) \to (B, g, D)$ those triples (h, k, α) making a cell in $\mathfrak{B}$ of the form

$$\begin{array}{ccc} FA & \xrightarrow{Fh} & FB \\ {\scriptstyle f}\downarrow & \overset{\alpha}{\Rightarrow} & \downarrow{\scriptstyle g} \\ GC & \xrightarrow[Gk]{} & GD \end{array}$$

and whose 2-cells $(h, k, \alpha) \Rightarrow (m, n, \beta)$ are pairs (γ, δ) satisfying the 2-cell cylinder equality

$$\begin{array}{ccc} FA & \xrightarrow{Fm} & FB \\ \| & \overset{\gamma}{\Rightarrow} & \| \\ FA & \xrightarrow[Fh]{} & FB \\ {\scriptstyle f}\downarrow & \overset{\alpha}{\Rightarrow} & \downarrow{\scriptstyle g} \\ GC & \xrightarrow[Gk]{} & GD \end{array} \quad = \quad \begin{array}{ccc} FA & \xrightarrow{Fm} & FB \\ {\scriptstyle f}\downarrow & \overset{\beta}{\Rightarrow} & \downarrow{\scriptstyle g} \\ GC & \xrightarrow[Gn]{} & GD \\ \| & \overset{\delta}{\Rightarrow} & \| \\ GC & \xrightarrow[Gk]{} & GD \end{array}$$

Take the target of the square α in the first display as the arrow k and the source as h. Ordinary composition of squares is by pasting those with matching domain and codomain; a further horizontal composition is given by pasting squares with matching source and target. Now, vertical composition of 2-cells is given by vertical composition of 2-cells in $\mathfrak{B}$ whereas horizontal composition is given by horizontal composition in $\mathfrak{B}$. A further composition of 2-cells is given by stacking cylinders. Source and target again define 2-functors $\mathrm{src}\colon F /\!\!/ G \to \mathfrak{A}$ and $\mathrm{tgt}\colon F /\!\!/ G \to \mathfrak{B}$. This construction is called a *2-comma* category in [16, §I,2.5], however, we are *not* using this terminology for the general construction. Rather by **2-comma category** we understand the special case where only commutative squares in $\mathfrak{B}$ are taken in the definition of the morphisms of $F /\!\!/ G$. This way we have terminology at hand to distinguish between the two constructions, the former emphasizing the arbitrary cells involved in the definition of morphism and the latter emphasizing the required strictness. The 2-comma will be denoted by F/G. Notice that with $F = G = 1_{\mathfrak{B}}$, the 2-category of cylinders $1 /\!\!/ 1 \cong \mathbf{Cyl}(\mathfrak{B})$ of Construction 2.9 is recovered. □

The notions of 2-functor, 2-natural transformation, and modification are well-known and together comprise the data for the 3-categorical structure giving **2Cat**. Perhaps less well-known is the idea of a lax natural transformation [16, §I,2.4], which for completeness is recalled here.

Definition 2.14. A **lax-natural transformation** $\alpha\colon F \Rightarrow G$ of 2-functors $F, G\colon \mathfrak{K} \rightrightarrows \mathfrak{L}$ consists of a family of arrows $\alpha_A\colon FA \to GA$ of $\mathfrak{L}$ indexed over the objects $A \in \mathfrak{K}_0$ together with, for each arrow f of $\mathfrak{K}$, a distinguished 2-cell

$$\begin{array}{ccc} FA & \xrightarrow{Ff} & FB \\ {\scriptstyle\alpha_A}\big\downarrow & \overset{\alpha_f}{\Rightarrow} & \big\downarrow{\scriptstyle\alpha_B} \\ GA & \xrightarrow[Gf]{} & GB \end{array}$$

satisfying the following two compatibility conditions.

1. For any composable arrows f and g of $\mathfrak{K}$, there is an equality of 2-cells

$$\begin{array}{ccccc} FA & \xrightarrow{Ff} & FB & \xrightarrow{Fg} & FC \\ {\scriptstyle\alpha_A}\big\downarrow & \overset{\alpha_f}{\Rightarrow} & \big\downarrow & \overset{\alpha_g}{\Rightarrow} & \big\downarrow{\scriptstyle\alpha_C} \\ GA & \xrightarrow[Gf]{} & GB & \xrightarrow[Gg]{} & GC \end{array} \quad = \quad \begin{array}{ccc} FA & \xrightarrow{F(gf)} & FB \\ {\scriptstyle\alpha_A}\big\downarrow & \overset{\alpha_{gf}}{\Rightarrow} & \big\downarrow{\scriptstyle\alpha_B} \\ GA & \xrightarrow[G(gf)]{} & GB \end{array}$$

2. For any 2-cell $\theta\colon f \Rightarrow g$ of $\mathfrak{K}$, there is an equality of 2-cells as depicted in the diagram

$$\begin{array}{ccc} FA & \xrightarrow{Fg} & FB \\ \big\| & \overset{F\theta}{\Rightarrow} & \big\| \\ FA & \longrightarrow & FB \\ {\scriptstyle\alpha_A}\big\downarrow & \overset{\alpha_f}{\Rightarrow} & \big\downarrow{\scriptstyle\alpha_B} \\ GA & \xrightarrow[Gf]{} & GB \end{array} \quad = \quad \begin{array}{ccc} FA & \xrightarrow{Fg} & FB \\ {\scriptstyle\alpha_A}\big\downarrow & \overset{\alpha_g}{\Rightarrow} & \big\downarrow{\scriptstyle\alpha_B} \\ GA & \longrightarrow & GB \\ \big\| & \overset{G\theta}{\Rightarrow} & \big\| \\ GA & \xrightarrow[Gf]{} & GB \end{array}$$

A lax-natural transformation is **pseudo natural** if the cells α_f are invertible. If they are identities, the transformation is **2-natural**. An **oplax transformation** $\alpha\colon F \Rightarrow G$ is defined in the same way with the difference that the coherence cells α_f point in the opposite direction. □

Example 2.15. Between the source and target 2-functors

$$\mathrm{src}, \mathrm{tgt}\colon \mathbf{Cyl}(\mathfrak{B}) \rightrightarrows \mathfrak{B}$$

from the 2-category of squares from Construction 2.9, there is a lax natural transformation $\beta\colon \mathrm{src} \Rightarrow \mathrm{tgt}$ given on a component $f\colon B \to C$ by $\beta_f := f$. If $\alpha\colon f \to g$ is an arrow of $\mathbf{Cyl}(\mathfrak{B})$ with source h and target k as in the first display of Construction 2.9, then take the coherence cell in $\mathfrak{B}$ to be α itself. Then the axioms for lax naturality are satisfied by the definitions of composition for 1- and 2-cells in $\mathbf{Cyl}(\mathfrak{B})$. Of course this is a special case of a more general situation for the lax comma category of a 2-functor F over another G. That is, $F /\!\!/ G$ and its projections $\mathrm{src}\colon F /\!\!/ G \to \mathfrak{A}$ and $\mathrm{tgt}\colon F /\!\!/ G \to \mathfrak{C}$ admit a lax natural transformation $\beta\colon F \circ \mathrm{src} \Rightarrow G \circ \mathrm{tgt}$ defined on components in an analogous way. □

Lemma 2.16. *The lax natural transformation $\beta\colon F \circ \mathrm{src} \Rightarrow G \circ \mathrm{tgt}$ of Example 2.15 is 1-dimensionally universal amid lax natural transformations in the sense that for any lax natural transformation*

$$\alpha\colon FH \Rightarrow GK$$

there is a unique 2-functor depicted as the dashed arrow in the diagram

$$\begin{array}{ccccc} \mathfrak{D} & & \xrightarrow{K} & & \mathfrak{C} \\ & \overset{U}{\searrow} & & & \\ & & F /\!/ G & \xrightarrow{\text{tgt}} & \mathfrak{C} \\ \downarrow H & & \text{src}\downarrow & \overset{\beta}{\Rightarrow} & \downarrow G \\ & & \mathfrak{A} & \xrightarrow[F]{} & \mathfrak{B} \end{array}$$

making two commutative triangles and satisfying $\beta * U = \alpha$.

Proof. This is a routine check. □

Remark 2.17. In fact $F /\!/ G$ is furthermore 2- and 3-dimensionally universal in a sense recalled in the next subsection. For now, we return to generalities on lax natural transformations. □

Now, the universality of the elements construction can be described.

Proposition 2.18. *For any 2-functor* $F\colon \mathfrak{B}^{op} \to \mathbf{Cat}$, *the opposite of the 2-category of elements as above presents the lax comma object* $\mathbf{Elt}(F^{op}) \cong */F$ *with a universal lax natural transformation*

$$\begin{array}{ccc} \mathbf{Elt}(F)^{op} & \xrightarrow{\Pi^{op}} & \mathfrak{B}^{op} \\ \downarrow & \overset{\phi}{\Rightarrow} & \downarrow F \\ \mathbf{1} & \xrightarrow[*]{} & \mathbf{Cat} \end{array}$$

where $*\colon \mathbf{1} \to \mathbf{Cat}$ *denotes the map sending the unique element of* $\mathbf{1}$ *to the terminal category.*

Proof. Straightforward check from the constructions. □

3. The Representation Theorem

The bulk of the work of [7, §2] consists in showing that the 2-category of elements construction above extends to a 3-functor $\mathbf{Elt}(-)\colon \mathbf{2Fib}(\mathfrak{B}) \to [\mathfrak{B}^{coop}, \mathbf{2Cat}]$ with suitable pseudo-inverse making an equivalence of 3-categories

$$[\mathfrak{B}^{coop}, \mathbf{2Cat}] \simeq \mathbf{2Fib}(\mathfrak{B}).$$

See specifically [7, Theorem 2.2.11] and its proof. This section presents the first main result of the paper in Section 3.2, namely, Theorem 3.7 showing that discrete 2-fibrations correspond via the category of elements construction to category-valued 2-functors indexed by the base 2-category making an equivalence

$$[\mathfrak{B}^{coop}, \mathbf{Cat}] \simeq \mathbf{D2Fib}(\mathfrak{B}).$$

This fulfills the first desiderata discussed in Remark 1.2. The candidate for the pseudo-inverse is developed first in Section 3.1 and subsequently shown to be the correct construction via lax pullback squares. 2-Fibrations are then discussed in Section 3.3 where it is shown in Theorem 3.23 that the equivalence restricts that for 2-fibrations in the sense that there is a commutative diagram of equivalences

$$\begin{array}{ccc} \mathbf{2Fib}(\mathfrak{B}) & \xrightarrow{\simeq} & [\mathfrak{B}^{op}, \mathbf{2Cat}] \\ \uparrow & & \uparrow \\ \mathbf{D2Fib}(\mathfrak{B}) & \xrightarrow[\simeq]{} & [\mathfrak{B}^{op}, \mathbf{Cat}] \end{array}$$

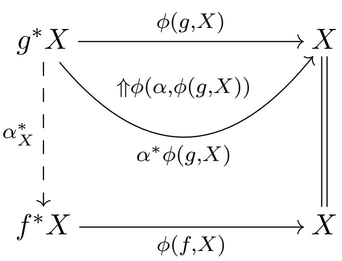
which fulfills the desiderata of Remark 1.5.

3.1 The Pseudo-Inverse The following development presents an adaptation of the pseudo-inverse presented in [7, §2.2.3] for the discrete case. It is of interest to see how the assumption that P is locally a discrete fibration simplifies a number of the computations. Throughout let $P\colon \mathfrak{E} \to \mathfrak{B}$ denote a discrete 2-fibration. Recall that $\mathfrak{E}_B$ denote the fiber of P above an object $B \in \mathfrak{B}$ consisting of the objects, arrows and 2-cells of $\mathfrak{E}$ above B and the various identities associated to it.

Lemma 3.1. *Each fiber $\mathfrak{E}_B$ is an ordinary category.*

Proof. If $\theta\colon u \Rightarrow v$ is a 2-cell between arrows $u, v\colon X \rightrightarrows Y$ of $\mathfrak{E}_B$, then $P\theta = 1_{1_B}$ holds by definition so that since P is locally a discrete opfibration θ must be 1_u. □

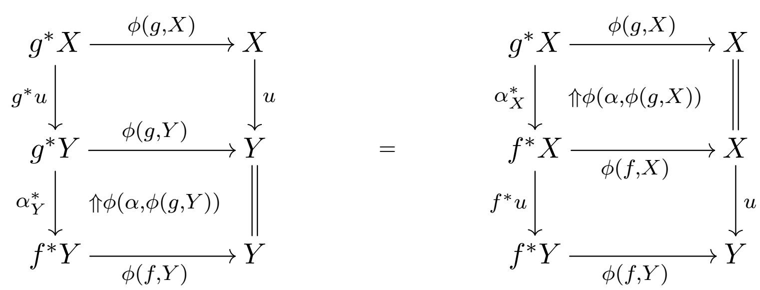
Construction 3.2 (Pseudo-Inverse)**.** Here define correspondences $F_P\colon \mathfrak{B}^{coop} \to \mathbf{Cat}$ amounting to a 2-functor. The assignments are essentially those of [7, §2.2.3] with the difference that the discrete fibration assumption makes many of the proofs easier. Take on objects $F_PB := \mathfrak{E}_B$ namely, the fiber category of P over the object $B \in \mathfrak{B}$. A morphism $f\colon B \to C$ of $\mathfrak{B}$ defines a transition functor $f^*\colon \mathfrak{E}_C \to \mathfrak{E}_B$ in the following way. Take f^*X for $X \in EC$ to be the domain of the chosen cartesian morphism over f, namely, the arrow $\phi(f, X)\colon f^*X \to X$ specified by the splitting ϕ. The arrow and 2-cell assignments are then given by the 1- and 2-dimensional lifting properties of such $\phi(f, X)$. So defined, f^* is a functor by uniqueness of these lifts. Finally, given a 2-cell $\alpha\colon f \Rightarrow g$ in $\mathfrak{B}$, there is an associated transition 2-cell $\alpha^*\colon g^* \Rightarrow f^*$ in the opposite direction. The component on an object $X \in EC$ is the dashed arrow in



The cell $\phi(\alpha, \phi(g, X))$ exists because P is locally a discrete fibration. Since its domain is over f, the dashed arrow making a commutative triangle exists by the 1-cell lifting property of $\phi(f, X)$. Notice that this means the equation

$$\phi(f, X)\alpha^*_X = \alpha^*\phi(g, X) \tag{3.1}$$

holds. These components are suitably natural in X by preservation properties of the splitting and uniqueness of lifts, as seen in the next Lemma. That is, given a morphism $u\colon X \to Y$ the equality



follows by the fact that the whiskered cells on each side are both lifts of α with target $u\phi(g, X)$. This means that the sources of these two cells in particular must be equal. Since $\phi(f, Y)$ is cartesian, the equation $\alpha^*_Y g^* u = f^*(u)\alpha^*_X$ must hold, as required. This is the hard part in the constructions of [7] but is straightforward here by the uniqueness properties of the discrete fibrations. $\square$

Proposition 3.3. *The assignments for* $F_P \colon \mathfrak{B}^{coop} \to \mathbf{Cat}$ *make it a 2-functor.*

Proof. That F_P is functorial on 1-cells is straightforward. Preservation of vertical composition of 2-cells $\alpha \colon f \Rightarrow g$ and $\beta \colon g \Rightarrow h$ between 1-cells $B \to C$ would follow from an equality of 2-cells

$$\begin{array}{ccc} h^*X & \xrightarrow{\phi(h,X)} & X \\ {\scriptstyle \beta^*_X}\downarrow & \Uparrow & \| \\ g^*X & \longrightarrow & X \\ {\scriptstyle \alpha^*_X}\downarrow & \Uparrow & \| \\ f^*X & \xrightarrow[\phi(f,X)]{} & X \end{array} \quad = \quad \begin{array}{ccc} h^*X & \xrightarrow{\phi(h,X)} & X \\ {\scriptstyle (\beta\alpha)^*_X}\downarrow & \Uparrow & \| \\ f^*X & \xrightarrow[\phi(f,X)]{} & X \end{array}$$

since $\phi(f, X)$ is cartesian. But the 2-cells on either side are in fact equal since they are both over the vertical composite $\beta\alpha$ with the same target $\phi(h, X)$.

Horizontal composition of 2-cells is also respected. Given 2-cells $\alpha \colon f \Rightarrow g$ and $\beta \colon h \Rightarrow k$ with $f, g \colon A \rightrightarrows B$ and $h, k \colon B \rightrightarrows C$, it needs to be seen that $\beta^* * \alpha^* = (\beta * \alpha)^*$ holds. Checking on a component at $X \in \mathfrak{E}_C$, this would follow from an equality of 2-cells as in

$$\begin{array}{ccccc} g^*k^*X & \xrightarrow{\phi(g,k^*X)} & k^*X & & \\ {\scriptstyle \alpha^*_{k^*X}}\downarrow & \Uparrow & \| & & \\ f^*k^*X & \xrightarrow[\phi(f,k^*X)]{} & k^*X & \xrightarrow{\phi(k,X)} & X \\ {\scriptstyle f^*\beta^*_X}\downarrow & & {\scriptstyle \beta^*_X}\downarrow & \Uparrow & \| \\ f^*h^*X & \xrightarrow[\phi(f,h^*X)]{} & h^*X & \xrightarrow[\phi(h,X)]{} & X \end{array} \quad = \quad \begin{array}{ccc} (hg)^*X & \xrightarrow{\phi(hg,X)} & X \\ {\scriptstyle (\beta*\alpha)^*_X}\downarrow & \Uparrow & \| \\ (hf)^*X & \xrightarrow[\phi(hf,X)]{} & X \end{array}$$

by the splitting equation $\phi(h, X)\phi(f, h^*X) = \phi(hf, X)$. But the composites on either side are over $\beta * \alpha$ with the same target by the same splitting equation. So, the cells must be equal since locally P is a discrete fibration. That units are respected is straightforward from the splitting assumptions and left to the reader. $\square$

3.2 Representation Theorem

The pseudo-inverse from the previous development is what is required to prove the main theorem, namely, Theorem 3.7 below, showing that discrete 2-fibrations over a base $\mathfrak{B}$ correspond to contravariant category-valued 2-functors on the base. That the correspondence $\mathbf{Elt}(-)$ is essentially surjective up to isomorphism can be seen using lax comma squares of Construction 2.13.

Construction 3.4. Let $P \colon \mathfrak{E} \to \mathfrak{B}$ denote a discrete 2-fibration. Construct a lax natural transformation

$$\begin{array}{ccc} \mathfrak{E}^{coop} & \xrightarrow{P^{coop}} & \mathfrak{B}^{coop} \\ \downarrow & \overset{\lambda}{\Rightarrow} & \downarrow{\scriptstyle F_P} \\ \mathbf{1} & \xrightarrow[*]{} & \mathbf{Cat} \end{array}$$

in the following way. The component corresponding to $X \in \mathfrak{E}_0$ is given by $X\colon \mathbf{1} \to \mathfrak{E}_{PX}$. Given $u\colon X \to Y$ of $\mathfrak{E}$, the corresponding lax naturality square is in fact a triangle as a left in the display

$$\begin{array}{ccc} \mathbf{1} & \xrightarrow{X} & \mathfrak{E}_{PX} \\ \Vert & \Downarrow \tilde{u} & \uparrow{\scriptstyle P(u)^*} \\ \mathbf{1} & \xrightarrow[Y]{} & \mathfrak{E}_{PY} \end{array} \qquad\qquad \begin{array}{ccc} X & \xrightarrow{u} & Y \\ {\scriptstyle \tilde{u}}\downarrow & & \Vert \\ P(u)^*Y & \xrightarrow[\phi(Pu,Y)]{} & Y \end{array}$$

where $\tilde{u}$ is the unique lift of 1 in $\mathfrak{E}_{PX}$ appearing on the right. That the data constitutes a lax natural transformation follows by construction of F_P and the splitting equations for P. ☐

Proposition 3.5. *For any discrete 2-fibration $P\colon \mathfrak{E} \to \mathfrak{B}$, the 2-functor $F_P\colon \mathfrak{B}^{coop} \to \mathbf{Cat}$ fits into a lax comma square of the form*

$$\begin{array}{ccc} \mathfrak{E}^{coop} & \xrightarrow{P^{coop}} & \mathfrak{B}^{coop} \\ \downarrow & \overset{\lambda}{\Rightarrow} & \downarrow{\scriptstyle F_P} \\ \mathbf{1} & \xrightarrow[*]{} & \mathbf{Cat} \end{array}$$

with λ as in Construction 3.4 above.

Proof. One could verify the three aspects of the universal property of the lax comma object. Alternatively, from the explicit description of $*/F_P$ it is straightforward to see that it is isomorphic as a 2-category to $\mathfrak{E}^{coop}$ over $\mathfrak{B}^{coop}$ via projections. The slight subtleties come in at the level of morphisms and 2-cells, but the assignments are given by the cartesian lifting properties enjoyed by the distinguished morphisms and 2-cells of $\mathfrak{E}$, with uniqueness ensuring that the assignments are functorial and bijections. ☐

Corollary 3.6. *For any discrete 2-fibration $P\colon \mathfrak{E} \to \mathfrak{B}$, the canonical map $\mathfrak{E} \to \mathbf{Elt}(F_P)$ is an isomorphism over $\mathfrak{B}$. In other words, an object assignment*

$$\mathbf{Elt}(-)\colon [\mathfrak{B}^{coop}, \mathbf{Cat}] \to \mathbf{D2Fib}(\mathfrak{B})$$

taking a functor to its elements construction is essentially surjective to within isomorphism.

Proof. Since $\mathbf{Elt}(F_P)^{coop}$ and $\mathfrak{E}^{coop}$ both present the lax comma object, they are canonically isomorphic via the map between them induced by the universal property of $*/F_P$. This shows that the domain 2-category of every discrete 2-fibration occurs, up to isomorphism, as the category of elements of some contravariant category-valued 2-functor, meaning that $\mathbf{Elt}(-)$ is essentially surjective up to isomorphism. ☐

Theorem 3.7 (Representation Theorem for Discrete 2-Fibrations)**.** *For any 2-category $\mathfrak{B}$, the assignment $\mathbf{Elt}(-)$ extends to a 2-functor making an equivalence of 2-categories*

$$[\mathfrak{B}^{coop}, \mathbf{Cat}] \simeq \mathbf{D2Fib}(\mathfrak{B})$$

between contravariant category-valued 2-functors on $\mathfrak{B}$ and discrete 2-fibrations over $\mathfrak{B}$.

Proof. The result is proved if it can be shown that $\mathbf{Elt}(-)$ extends to a 2-functor that is locally an isomorphism on hom-categories. Given a 2-natural transformation $\alpha\colon F \Rightarrow G$ of 2-functors $F, G\colon \mathfrak{B}^{coop} \rightrightarrows \mathbf{Cat}$, define $\mathbf{Elt}(\alpha)\colon \mathbf{Elt}(F) \to \mathbf{Elt}(G)$ on 0-, 1-, and 2-cells of $\mathbf{Elt}(F)$ by

$$\left((B,X) \underset{(g,\bar{g})}{\overset{(f,\bar{f})}{\Downarrow\theta}} (C,Y) \right) \mapsto \left((B,\alpha_B X) \underset{(g,\alpha_B\bar{g})}{\overset{(f,\alpha_B\bar{f})}{\Downarrow\theta}} (C,\alpha_C Y) \right)$$

These are well-defined and functorial by the construction of the 2-categories of elements and the 2-naturality properties of α. Similarly, given a modification $m\colon \alpha \Rrightarrow \beta$ of 2-natural transformations $\alpha, \beta\colon F \Rightarrow G$, define a transformation $\mathbf{Elt}(m)\colon \mathbf{Elt}(\alpha) \Rightarrow \mathbf{Elt}(\beta)$ by taking as component at (B, X) in $\mathbf{Elt}(F)$ the arrow

$$(1, m_{B,X})\colon (B, \alpha_B X) \to (B, \beta_B X)$$

of $\mathbf{Elt}(G)$. This is 2-natural and vertical over $\mathfrak{B}$ by construction and the modification property of m and that each m_B is natural. These assignments on arrows and 2-cells are thus well-defined and 2-functorial by construction. Moreover they are the same as in §2.2.11 of [7] where it is shown that they are indeed bijections. The same proofs work in the present context. □

Remark 3.8. This theorem fulfills Remark 1.2. In light of Remark 2.5, this is really a theorem template for correspondences between (co/op/coop) fibrations over $\mathfrak{B}$ and 2-functors valued in $\mathbf{Cat}$ that are variously co/contra-variant on arrows or cells. For example, what have been called *cofibrations* here correspond to $\mathbf{Cat}$-valued 2-functors that are contravariant on 1-cells only. □

As a result of Theorem 3.7, some light can be thrown on the previous examples.

Example 3.9. The canonical representable functor $\mathfrak{B}(-, X)\colon \mathfrak{B}^{op} \to \mathbf{CAT}$ corresponds to the domain discrete cofibration from the slice category $d_0\colon \mathfrak{B}/X \to \mathfrak{B}$ of Example 2.10. □

Example 3.10. The 2-functor

$$P\colon \mathbf{Cat}^{coop} \to \mathbf{CAT} \qquad \mathscr{C} \mapsto [\mathscr{C}^{op}, \mathbf{Set}]$$

corresponds to the codomain fibration $\mathrm{cod}\colon \mathbf{DFib} \to \mathbf{Cat}$ of Example 2.8. On the other hand, the 2-functor $\mathscr{C} \mapsto [\mathscr{C}, \mathbf{Set}]$ corresponds to $\mathrm{cod}\colon \mathbf{DOpf} \to \mathbf{Cat}$. □

Example 3.11. A 2-functor $\mathscr{C}^{op} \to \mathfrak{B}$ from any 1-category is really a functor $\mathscr{C}^{op} \to |\mathfrak{B}|$ since $\mathscr{C}$ has no nontrivial 2-cells. Thus, consider the 2-functor taking a category to contravariant functors into $\mathfrak{B}$, that is,

$$[-, |\mathfrak{B}|]\colon \mathbf{Cat}^{coop} \to \mathbf{Cat} \qquad \mathscr{C} \mapsto [\mathscr{C}^{op}, |\mathfrak{B}|]$$

and extended suitably to functors and transformations. The image of any such category under $[-, |\mathfrak{B}|]$ is of course a strict 1-category. The corresponding 2-category of elements projection is essentially the family fibration $\mathbf{fam}(\mathfrak{B}) \to \mathbf{Cat}$ from Example 2.12. Likewise covariant families correspond to the family discrete cofibration. □

Example 3.12. Let t denote a 2-monad in a 2-category $\mathfrak{K}$. The 2-functor

$$t\mathbf{Alg}(-)\colon \mathfrak{K}^{op} \to \mathbf{Cat} \qquad A \mapsto t\mathbf{Alg}(A)$$

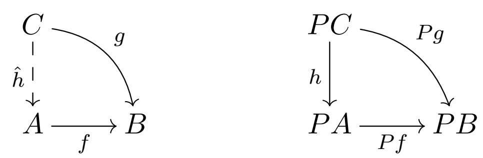
taking $A \in \mathfrak{K}_0$ to the category of t-algebras with domain A as in §3 of [26] corresponds under the restricted equivalence of Theorem 3.23 to the discrete cofibration $\Pi\colon t\mathbf{Alg} \to \mathfrak{K}$ of Example 2.11. □

3.3 2-Fibrations

Here is recalled the precise definition of a 2-fibration. It will be seen that every discrete 2-fibration is one. The subsection also fulfills Desiderata 2 from Remark 1.5 concerning the restricted elements equivalence in Theorem 3.23. Stating the definition a 2-fibration involves first a categorification of the definition of *cartesian morphism*, from [19, §2] and [7, §2.2.1], as follows.

Definition 3.13 (2-Cartesian Morphism)**.** Let $P\colon \mathfrak{E} \to \mathfrak{B}$ be a 2-functor. An arrow $f\colon A \to B$ of $\mathfrak{E}$ is **2-cartesian** if it satisfies the following two conditions.

1. Whenever $g\colon C \to B$ is an arrow of $\mathfrak{E}$ for which there is a morphism $h\colon PC \to PA$ making a commutative triangle in $\mathfrak{B}$ as on the right



   it follows that there is a unique $\hat{h}\colon C \to A$ in $\mathfrak{E}$ with $P\hat{h} = h$ making a commutative triangle in $\mathfrak{E}$ as on the left above.
2. Whenever $\theta\colon g \Rightarrow k$ is a 2-cell of $\mathfrak{E}$ for which there is a 2-cell $\gamma\colon h \Rightarrow l$ of $\mathfrak{B}$ such that $P\theta = Pf * \gamma$ holds, there is a unique lift 2-cell $\hat{\gamma}\colon \hat{h} \Rightarrow \hat{l}$ in $\mathfrak{E}$ with $P\hat{\gamma} = \gamma$ such that $f * \hat{\gamma} = \theta$.

A 2-functor $P\colon \mathfrak{E} \to \mathfrak{B}$ **has enough 2-cartesian arrows** if for each arrow $g\colon B \to PE$ of $\mathfrak{B}$ there is a 2-cartesian arrow $f\colon A \to E$ of $\mathfrak{E}$ such that $Pf = g$. □

Definition 3.14 (2-Fibration)**.** A 2-functor $P\colon \mathfrak{E} \to \mathfrak{B}$ is a **2-fibration** if

1. for each arrow $f\colon B \to PE$ of $\mathfrak{B}$ there is a cartesian arrow $g\colon A \to E$ of $\mathfrak{E}$ with $Pg = f$;
2. locally P is a fibration in that each $P_{A,B}\colon \mathfrak{E}(A, B) \to \mathfrak{B}(PA, PB)$ is a fibration;
3. and finally cartesian 2-cells are closed under horizontal composition.

Such 2-fibrations will always assumed to be split in the sense of Definition 3.16 below. Over a fixed base $\mathfrak{B}$ these are objects of a 3-category $\mathbf{2Fib}(\mathfrak{B})$ whose arrows are splitting-preserving 2-functors over $\mathfrak{B}$, together with vertical 2-natural transformations, and vertical modifications between them. □

Remark 3.15. The immediately forgoing definition is that of [7, §2.1.6] which adapts the original [19, Definition 2.3] (which appeared also in [2]) by asking for a slightly stronger closure property in the third clause. That is, Hermida's definition asks that cartesian cells are closed only under precomposition with any 1-cell whereas Buckley's definition amounts to closure under both pre- and postcomposition with any 1-cell. As explained in [7, Remark 2.1.8 and Remark 2.1.9] these ultimately are inequivalent definitions. In particular the extra assumption is needed for complete construction of the pseudo-inverse to the elements construction (a partial pseudo-inverse appears in [2, Theorem 5.1]) and ultimately to give the desired equivalence in the representation theorem for 2-fibrations. See the discussion [7, Remark 2.2.12]. □

3.3.1 Spliting Equations Let $P\colon \mathfrak{E} \to \mathfrak{B}$ denote a 2-fibration as in Definition 3.14. A **cleavage** specifies for each $f\colon B \to C$ of the base and each $X \in \mathfrak{E}_C$ a chosen cartesian arrow

$$\phi(f, X)\colon f^*X \to X$$

over f; and for each 2-cell $\alpha\colon f \Rightarrow Pg$ of the base a chosen cartesian 2-cell

$$\phi(\alpha, g)\colon \alpha^*g \Rightarrow g$$

over α, each of $\mathfrak{E}$. The splitting equations assert that these choices are functorial in a precise way. These are adaptations of [7, §2.1.10] suited for our conventions.

For composable arrows $f\colon B \to C$ and $g\colon C \to D$ in $\mathfrak{B}$ and an object $X \in \mathfrak{E}_D$, the splitting equation for 1-cells is the usual one, namely,

$$\phi(gf, X) = \phi(g, X)\phi(f, g^*X). \tag{3.2}$$

For internally composable 2-cells $\alpha\colon f \Rightarrow g$ and $\beta\colon g \Rightarrow Ph$, the splitting equation is

$$\phi(\beta\alpha, f) = \phi(\beta, h)\phi(\alpha, \beta^*h). \tag{3.3}$$

For externally composable 2-cells $\alpha\colon f \Rightarrow Ph$ and $\beta\colon g \Rightarrow Pk$, the splitting equation is

$$\phi(\beta * \alpha, kh) = \phi(\beta, k) * \phi(\alpha, h). \tag{3.4}$$

Finally, there are the two splitting equations for identities, namely,

$$\phi(1_P X, X) = 1_X \tag{3.5}$$

$$\phi(1_{1_f}, f) = 1_f \tag{3.6}$$

for any object $X \in \mathfrak{E}$ and any arrow $f \in \mathfrak{E}$.

Definition 3.16. A 2-fibration $P\colon \mathfrak{E} \to \mathfrak{B}$ is **split** if it is cloven with cleavage $\phi(-,-)$ satisfying equations Equation (3.2) through Equation (3.6) immediately above. A 2-functor over $\mathfrak{B}$ is **splitting-preserving** if it preserves the global and local splittings strictly. Let **2Fib**$(\mathfrak{B})$ denote the 3-category of split 2-fibrations over $\mathfrak{B}$, splitting-preserving 2-functors over $\mathfrak{B}$, transformations with vertical components and vertical modifications. □

The concept of a 2-fibration is a generalization of that of a discrete 2-fibration, just as the concept of a fibration generalizes that of a discrete fibration.

Proposition 3.17. *Every discrete 2-fibration as above is a 2-fibration as in Definition 3.14. The 2-category* **D2Fib**$(\mathfrak{B})$ *is identifiable as a (locally discrete) sub-3-category of* **2Fib**$(\mathfrak{B})$.

Proof. That there is a required cartesian 1-cell follows since $|E|\colon |\mathfrak{E}| \to |\mathfrak{B}|$ is a split fibration; but in particular the 2-cell lifting condition follows because locally E is a discrete opfibration. That the external composition of cartesian 2-cells is again cartesian also follows from the fact that E is locally a discrete fibration (in particular the uniqueness aspect of the definition). □

Example 3.18 (Source 2-Fibration)**.** The source 2-functor src: **Cyl**$(\mathfrak{C}) \to \mathfrak{C}$ from Construction 2.9 is a 2-fibration in the present sense of Definition 3.14. This is a lax 2-categorical analogue of the usual domain fibration dom: $\mathscr{C}^{\mathbf{2}} \to \mathscr{C}$ for an ordinary category $\mathscr{C}$ whose fibers are the coslice categories $X/\mathscr{C}$. For the fibers of src above are the lax coslice 2-categories $X/\mathfrak{C}$. □

Example 3.19 (Target 2-Fibration)**.** Again consider $\mathbf{Cyl}(\mathfrak{B})$ from Construction 2.9, but this time also the target 2-functor $\text{tgt}\colon \mathbf{Cyl}(\mathfrak{B}) \to \mathfrak{B}$. Suppose that $\mathfrak{B}$ has (strict) comma squares (see [37, §1] for example). The target 2-functor is then a 2-fibration as in Definition 3.14. The presence of comma squares suffice for constructing both the cartesian 1-cells and the cartesian 2-cells locally. This 2-functor tgt is the analogue of the codomain fibration $\text{cod}\colon \mathscr{C}^{\mathbf{2}} \to \mathscr{C}$ whenever $\mathscr{C}$ is a category with pullbacks. For the fibers of tgt are the lax slices $\mathfrak{B}/X$. □

Example 3.20 (Category-Indexed Families)**.** Let $\mathbf{Fam}(\mathfrak{B})$ denote the **2-category of families** in a 2-category $\mathfrak{B}$. That is, the objects are pseudo-functors $F\colon \mathscr{C}^{op} \to \mathfrak{B}$ from small 1-categories $\mathscr{C}$. Arrows $(\mathscr{C}, F) \to (\mathscr{D}, G)$ are pairs (H, α) where $H\colon \mathscr{C} \to \mathscr{D}$ is a functor and α is a pseudo-natural transformation $\alpha\colon F \Rightarrow GH$. Finally 2-cells are just appropriate pairs (σ, m) of a pseudo-natural transformation σ and a modification m as in [7, §2.3.1] adapted for our co- rather than the contravariant families of the reference. The projection $\Pi\colon \mathbf{Fam}(\mathfrak{B}) \to \mathbf{Cat}$ is then a 2-fibration in the sense of Definition 3.14. □

Example 3.21. This is based on [7, 2.3.12]. Start with a 2-category $\mathfrak{K}$. Let $\mathbf{Mnd}(\mathfrak{K})$ denote the 2-category of monads on $\mathfrak{K}$ and $\mathbf{Alg}_{lax}(\mathfrak{K})$ the 2-category of pairs $(S, (A, m))$ with S a 2-monad on $\mathfrak{K}$ and (A, m) an S-algebra; and with morphisms pairs consisting of a 2-monad morphism and an lax morphisms of algebras; and appropriate 2-cells. The forgetful 2-functor $\Pi\colon \mathbf{Alg}_{lax}(\mathfrak{K}) \to \mathbf{Mnd}(\mathfrak{K})$ is a 2-fibration in the sense of Definition 3.14. Note that $\Pi\colon \mathbf{Alg}_{oplax}(\mathfrak{K}) \to \mathbf{Mnd}(\mathfrak{K})$ would be a cofibration in the language of Remark 2.5. □

The next construction provides more examples and sheds light on both of those above. Higher 2-categories of elements originated in [5] and appeared later in [16, §1,2.5] and [7, §2.2.1]. The appropriate version for 2-category-valued functors is the following.

Construction 3.22 (2-Category of Elements)**.** For any 3-functor $F\colon \mathfrak{B}^{coop} \to \mathbf{2Cat}$ on a 2-category $\mathfrak{B}$, the **2-category of elements** of E is the 2-category whose

1. objects are pairs (B, X) with $B \in \mathfrak{B}_0$ and $X \in FB$;
2. arrows are pairs $(f, u)\colon (B, X) \to (C, Y)$ with $f\colon B \to C$ in $\mathfrak{B}$ and $u\colon X \to f^*Y$ in the fiber FB;
3. and whose 2-cells : $(f, u) \Rightarrow (g, v)$ are those pairs (α, σ) where $\alpha\colon f \Rightarrow g$ is in $\mathfrak{B}$ and σ is a 2-cell
$$\begin{array}{ccc} X & \xrightarrow{\bar{f}} & f^*Y \\ \Vert & \Downarrow\sigma & \uparrow{\scriptstyle \alpha^*_Y} \\ X & \xrightarrow[\bar{g}]{} & g^*Y \end{array}$$
of the 2-category FB.

Denote this 2-category by $\mathbf{Elt}(E)$. There is an evident projection 2-functor $\Pi\colon \mathbf{Elt}(E) \to \mathfrak{B}$. □

The final result of the subsection makes good on the desiderata of Remark 1.5 from the introduction, showing that the representation equivalence for discrete 2-fibration restricts that of 2-fibrations.

Theorem 3.23 (Restricted Equivalence)**.** *For any 2-category $\mathfrak{B}$, the Representation Equivalence* *for discrete 2-fibrations restricts that for 2-fibration in the sense that*

$$\begin{array}{ccc} \mathbf{2Fib}(\mathfrak{B}) & \xrightarrow{\simeq} & [\mathfrak{B}^{op}, \mathbf{2Cat}] \\ \uparrow & & \uparrow \\ \mathbf{D2Fib}(\mathfrak{B}) & \xrightarrow[\simeq]{} & [\mathfrak{B}^{op}, \mathbf{Cat}] \end{array}$$

commutes where the top row is viewed as an equivalence of locally discrete 3-categories.

Proof. On the one hand, every discrete 2-fibration is a 2-fibration (Proposition 3.17 above) and the elements construction for a 2-functor $F\colon \mathfrak{B}^{coop} \to \mathbf{Cat}$ is the same whether F is viewed as a 2-functor (Construction 2.1) or as a degenerate 3-functor (Construction 3.22) via the inclusion $\mathbf{Cat} \to \mathbf{2Cat}$. On the other hand, the pseudo-inverse of Construction 3.2 is a specialization of that in the reference. □

3.4 Representable Notions In this subsection, we shall show that the notions of 2-fibration and discrete 2-fibrations are not merely the naive representable ones in the sense that they are not fibrations internal to **2Cat** in the sense of the accepted definition. First recall the following from [30, Definition 3.1.1]. We will relate this definition to those of for example [37] in the discussion below.

Definition 3.24. An arrow $p\colon E \to B$ in a 2-category $\mathfrak{K}$ is a **fibration**, or for emphasis is an **internal fibration**, if for each object $X \in \mathfrak{K}$ the functor *post-composition with* p

$$p \circ -\colon \mathfrak{K}(X, E) \to \mathfrak{K}(X, B)$$

is an ordinary fibration in **Cat** and if for each arrow $f\colon X \to Y$ of $\mathfrak{K}$ the commutative square

$$\begin{array}{ccc} \mathfrak{K}(Y, E) & \xrightarrow{-\circ f} & \mathfrak{K}(X, E) \\ {\scriptstyle p\circ -}\downarrow & & \downarrow{\scriptstyle p\circ -} \\ \mathfrak{K}(Y, B) & \xrightarrow[-\circ f]{} & \mathfrak{K}(X, B) \end{array}$$

is a morphism of fibrations in that the topmost functor preserves cartesian morphisms. A morphism p satisfying just the first condition for all $X \in \mathfrak{K}$ is **representably a fibration**. □

Example 3.25. In [15, §3] it is proved that an ordinary fibration is an internal fibration in the 2-category **Cat**. See also [30, Example 3.1.4]. In particular, note that in this case being representably a fibration implies being an internal fibration by [15, Corollary 3.7] which shows that a transformation is cartesian for $P \circ -\colon \mathbf{Cat}(\mathscr{X}, \mathscr{E}) \to \mathbf{Cat}(\mathscr{X}, \mathscr{B})$ if and only if each component is cartesian for P. □

However, even though the internalized definition recovers the usual notion of fibration, not all internal fibrations in **2Cat** are 2-fibrations in the sense of Definition 3.14. First note that, in general, a 2-functor is representably a fibration if it has enough 2-cartesian arrows.

Lemma 3.26. *If $P\colon \mathfrak{E} \to \mathfrak{B}$ is a 2-functor, then if P has enough 2-cartesian arrows as in Definition 3.13, for any 2-category $\mathfrak{X}$, the functor*

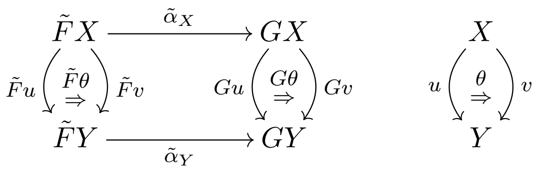
$$P \circ -\colon \mathbf{2Cat}(\mathfrak{X}, \mathfrak{E}) \to \mathbf{2Cat}(\mathfrak{X}, \mathfrak{B})$$

is an ordinary fibration in **Cat**.

Proof. Let $G\colon \mathfrak{X} \to \mathfrak{E}$ be any 2-functor and let $\alpha\colon F \Rightarrow PG$ denote any 2-natural transformation of 2-functors $F, PG\colon \mathfrak{X} \rightrightarrows \mathfrak{B}$. What needs to be shown is that there is a cartesian lift $\tilde{\alpha}\colon \tilde{F} \Rightarrow G$ over α. Letting $X \in \mathfrak{X}$ be any object, take as a definition of the component $\tilde{\alpha}_X$ the arrow

$$\tilde{\alpha}_X := \phi(\alpha_X, GX)\colon \alpha_X^*GX \to GX \qquad \tilde{F}X := \alpha_X^*GX.$$

that is, $\tilde{\alpha}_X$ is the chosen cartesian arrow above α_X. Likewise the domain of that chosen cartesian arrow is the object assignment for $\tilde{F}$. The arrow and 2-cell assignment for $\tilde{F}$ come about from the fact that each component of $\tilde{\alpha}$ is thus 2-cartesian. That is, for any cell $\theta\colon u \Rightarrow v$ of $\mathfrak{X}$, we have $\tilde{F}\theta$ making a cylinder as in



arising from the fact that in particular $\tilde{\alpha}_Y := \phi(\alpha_Y, GY)$ is 2-cartesian. In fact, *2-cartesian* does all the work here: that $\tilde{F}$ is a 2-functor; that $\tilde{\alpha}$ is 2-natural; and that the evidently constructed unique lift is well-defined all follow from this assumption. The details are tedious but routine. □

One might hope that something like the converse is true, namely, that if P is representably a fibration, then it is a 2-fibration or at least has enough 2-cartesian arrows. As with the proof of [15, Proposition 3.6], it seems reasonable simply to take $\mathfrak{X} = 1$ and deduce the result. However, such hopes are thwarted upon even a preliminary examination since this would seem to require either lax transformations or perhaps *modifications* to get all the 2-dimensional properties of 2-fibrations. One could be forgiven, then, for suspecting that the purely representable notion, interpreted in the naive setting **2Cat**, is not the correct characterization. That is, one may suspect that there are 2-functors that are at least representably fibrations, and perhaps even internal fibrations, that are not 2-fibrations. And indeed this is the case. For an example, recall [33, §1] that a *2-groupoid* is a 2-category in which all morphisms and cells are invertible. That such things exist is attested to also by [18, §2] which shows that a path 2-groupoid can be associated canonically to any Hausdorff topological space.

Example 3.27. Let $i\colon |\mathfrak{G}| \to \mathfrak{G}$ denote the inclusion of the underlying 1-category of any 2-groupoid $\mathfrak{G}$ back into itself. Of course $|\mathfrak{G}|$ is viewed as a 2-category with no non-identity 2-cells. One can show directly that i is representably a fibration. Alternatively, as a result of invertibility of arrows and cells, i satisfies the criteria of Lemma 3.26. But additionally i satisfies the second condition of Definition 3.24, namely, that the square

$$\begin{array}{ccc} \mathbf{2Cat}(\mathfrak{Y}, |\mathfrak{G}|) & \xrightarrow{-\circ F} & \mathbf{2Cat}(\mathfrak{X}, |\mathfrak{G}|) \\ {\scriptstyle i\circ -}\downarrow & & \downarrow{\scriptstyle i\circ -} \\ \mathbf{2Cat}(\mathfrak{Y}, \mathfrak{G}) & \xrightarrow[-\circ F]{} & \mathbf{2Cat}(\mathfrak{X}, \mathfrak{G}) \end{array}$$

is a morphism of fibrations for any 2-functor $F\colon \mathfrak{X} \to \mathfrak{Y}$. Therefore i is an internal fibration in **2Cat**. However, i is not a (discrete) 2-fibration since it is not a (discrete) fibration locally. □

This example shows as well that there is no hope that discrete 2-fibrations are discrete internal fibrations in **2Cat**. Likewise, discrete 2-fibrations are not merely internal discrete 2-fibrations in **2Cat**. First the definition, recalled from [40, §2.2] or [30, §3.2].

Definition 3.28. A morphism $p\colon E \to B$ in a 2-category $\mathfrak{K}$ is a **discrete fibration**, or for emphasis an **internal discrete fibration**, if it is representably so, that is, if for each object $X \in \mathfrak{K}$ the functor

$$p \circ -\colon \mathfrak{K}(X, E) \to \mathfrak{K}(X, B)$$

is an ordinary discrete fibration in **Cat**. □

Example 3.29. Ordinary discrete fibrations between small categories are precisely the discrete fibrations internal to the 2-category **Cat**. □

Lemma 3.30. *If $P\colon \mathfrak{E} \to \mathfrak{B}$ is a discrete fibration internal to* **2Cat***, then the underlying functor $|P|\colon |\mathfrak{E}| \to |\mathfrak{B}|$ is a discrete fibration. Consequently, discrete 2-fibrations are not discrete fibrations internal to* **2Cat**.

Proof. Let $\mathfrak{X} = 1$, the trivial 2-category with one object and only an identity morphism and identity cell. The commutativity of the diagram

$$\begin{array}{ccc} |\mathfrak{E}| & \xrightarrow{\cong} & \mathbf{2Cat}(1, \mathfrak{E}) \\ {\scriptstyle |P|}\downarrow & & \downarrow{\scriptstyle P\circ -} \\ |\mathfrak{B}| & \xrightarrow[\cong]{} & \mathbf{2Cat}(1, \mathfrak{B}) \end{array}$$

then shows that $|P|$ is a discrete fibration. Since according to definition Definition 2.4 a discrete 2-fibration at least is an honest fibration at the level of underlying 1-categories, and there are examples of such things, this shows that the representable notion of internal discrete fibration is too restrictive. □

To return to the discussion in the introduction and segue into the next section, we recall here that there are a number of equivalent *intrinsic characterizations* of internal fibrations and of discrete fibrations. In the purely 2-categorical setting, these again originate with [37, §2] which forms the basis for the bicategorical generalizations appearing for example in [38], [39]. Roughly, [37] works in the setting of a *representable 2-category* which is just to say a 2-category with not only suitably 2-categorical products and equalizers but also further weighted limits, namely, cotensors with the ordinal **2**. In this setting fibrations are defined as algebras for a certain lax-idempotent monad given by an action of a cotensor. This definition is then shown [37, Proposition 9] to be equivalent to an internalized *Chevalley criterion* based on the classical *left-adjoint-left-inverse* characterization of ordinary (op)fibrations from [15]. However, it is not our purpose to recount here these potential alternative intrinsic definitions in detail. For again [30, Proposition 3.1.3] shows that the present representable characterization of internal fibrations in Definition 3.24 above is in fact equivalent to the internal Chevalley criteria in [37], hence to the internal definitions in terms of algebras. Bottom lining the whole situation, then, the result of Example 3.27 is thus that the notion of a 2-fibration is *not* that of a fibration in a 2-category in the sense of [37] if by that one means the naive thing, namely, the 2-category **2Cat**. In particular, 2-fibrations are not algebras for the cotensor action as described in the reference. It was in fact the search for what type of action could be defined on a (discrete) 2-fibration that led to the results of the next section. The action on 2-fibrations is that of the cylinder 2-category Construction 2.9 which is shown to be a cotensor in a certain *lax 3-category* which is a notion introduced here specifically to describe this structure's universal property. A revised representability result appears in the concluding Section 4.4.

4. Monadicity

In this section, the other main result of the paper is given in Theorem 4.13, namely, that 2-fibrations are monadic over a slice of **2Cat**. The goal is really to prove that discrete 2-fibration are monadic, but the result for 2-fibrations comes at little extra cost. We shall develop a 2-monad T and consider an underlying monad $|T|$ for which there is a commutative square of equivalences

$$\begin{array}{ccc} \mathbf{D2Fib}(\mathfrak{B}) & \xrightarrow{\simeq} & |T|\mathbf{Alg} \\ \downarrow & & \downarrow \\ \mathbf{2Fib}(\mathfrak{B}) & \xrightarrow{\simeq} & T\mathbf{Alg}. \end{array}$$

The action of the (2-)monad is that of the 2-category of cylinders from Construction 2.9. These results should be seen as higher-dimensional versions of the well-known results for ordinary fibrations and discrete fibrations, reviewed in Section 4.1. The main theorems are proved in Section 4.2. The final subsection Section 4.3 is meant to introduce the setting to describe the universality of the 2-category of cylinders and prove our modified representability result.

4.1 (Discrete) Fibrations and Monadicity Let $\mathscr{C}$ denote an ordinary small category. Define a functor $T\colon \mathbf{Set}/\mathscr{C}_0 \to \mathbf{Set}/\mathscr{C}_0$ by taking an ordinary set function $f\colon X \to \mathscr{C}_0$ to the projection from the pullback composed with the domain arrow $d_0\colon \mathscr{C}_1 \to \mathscr{C}_0$ as in

$$\begin{array}{ccc} \mathscr{C}_1 \times_{\mathscr{C}_0} X & \xrightarrow{\pi_2} & X \\ {\scriptstyle d_1^* f}\downarrow & & \downarrow{\scriptstyle f} \\ \mathscr{C}_1 & \xrightarrow[d_1]{} & \mathscr{C}_0 \\ {\scriptstyle d_0}\downarrow & & \\ \mathscr{C}_0 & & \end{array}$$

Thus, in other words, define $T = d_0 \circ d_1^*(f)$. The arrow assignment is induced by the universal property of the pullback. So defined, T is an ordinary monad on $\mathbf{Set}/\mathscr{C}_0$. Now, on the one hand, if $F\colon \mathscr{F} \to \mathscr{C}$ is a discrete fibration, define an action $M\colon \mathscr{C}_1 \times_{\mathscr{C}_0} \mathscr{F}_0 \to \mathscr{F}_0$ by taking (f, x) with $d_1 f = fx$ to the domain of the unique arrow of $\mathscr{F}$ over f, denoted by f^*x. This is an action of $\mathscr{C}_1$ on $\mathscr{F}_0$ and makes F into a T-algebra. Universality of the constructions then yields a functor $\mathbf{DFib}(\mathscr{C}) \to T\mathbf{Alg}$ that is one-half of an equivalence.

Theorem 4.1. *There is an equivalence of categories*

$$\mathbf{DFib}(\mathscr{C}) \simeq T\mathbf{Alg}$$

for any small category $\mathscr{C}$.

Proof. On the other hand, any T-algebra $f\colon A \to \mathscr{C}_0$ yields a discrete fibration $F\colon \mathscr{F} \to \mathscr{C}$ by taking $\mathscr{F}_0 = A$ and $\mathscr{F}_1 = TA = \mathscr{C}_1 \times_{\mathscr{C}_0} A$. This extends uniquely to morphisms and gives the pseudo-inverse required for the equivalence. This is treated in more detail in [32, §V.7] and internally in any topos in [23, §2.3]. □

The goal is to give an analogous result for discrete 2-fibrations. To that end it will be helpful to recall the needed preliminaries on 2-monads and some results about ordinary fibrations. The theory of 2-monads goes back to Street's [36]. What is needed here for the most part is summarized in [26, §3]. Here the material is unpacked for the case $\mathfrak{K} = \mathbf{2Cat}$.

Definition 4.2. A **2-monad** on a 2-category $\mathfrak{K}$ is a 2-functor $T\colon \mathfrak{K} \to \mathfrak{K}$ with 2-natural transformations $\eta\colon 1 \Rightarrow T$ and $\mu\colon TT \Rightarrow T$ such that $\mu T\mu = \mu\mu T$ and $\mu T\eta = 1 = \mu\eta T$ all hold, as usual. An **algebra** for such a 2-monad is an object $A \in \mathfrak{K}_0$ with a structure map $a\colon TA \to A$ satisfying the usual equations, namely, $a\mu_A = aTa$ and $a\eta_A = 1$. A **morphism of algebras** (A, a) and (B, b) is a morphism of the 2-category $h\colon A \to B$ that preserves the unit and preserves the action. A **2-cell of morphisms of algebras** $h, k\colon A \rightrightarrows B$ is a 2-cell $\theta\colon h \Rightarrow k$ of the 2-category satisfying the compatibility condition $\theta * a = b * T\theta$. With these definitions $T\mathbf{Alg}$ denotes the 2-category of T-algebras, their morphisms and 2-cells. A 2-category is **monadic** over $\mathfrak{K}$ if it is equivalent to $T\mathbf{Alg}$ for some 2-monad T on $\mathfrak{K}$. □

Example 4.3. Consider the 2-monad in the sense of Definition 4.2 on $\mathbf{Cat}/\mathscr{C}$ given by sending a functor $H\colon \mathscr{X} \to \mathscr{C}$ to the pullback composed with d_0 as in

$$\begin{array}{ccc} \mathscr{C}^{\mathbf{2}} \times_{\mathscr{C}} \mathscr{X} & \xrightarrow{\pi_2} & \mathscr{X} \\ {\scriptstyle d_1^* H}\downarrow & & \downarrow{\scriptstyle H} \\ \mathscr{C}^{\mathbf{2}} & \xrightarrow[d_1]{} & \mathscr{C}_0 \\ {\scriptstyle d_0}\downarrow & & \\ \mathscr{C}_0 & & \end{array}$$

Denote this 2-monad by T. Split fibrations over $\mathscr{C}$ are precisely the normalized T-algebras as in Definition 4.2 for T as above in the sense that there is an equivalence of 2-categories

$$\mathbf{Fib}(\mathscr{C}) \simeq T\mathbf{Alg}.$$

If $F\colon \mathscr{F} \to \mathscr{C}$ is a split fibration, the action of $\mathscr{C}^{\mathbf{2}}$ on $\mathscr{F}$ is given by

$$M\colon \mathscr{C}^{\mathbf{2}} \times_{\mathscr{C}} \mathscr{F} \to \mathscr{F} \qquad (f, X) \mapsto f^*X$$

on objects and by the dashed arrow solution of the following lifting problem

$$\begin{array}{ccc} B & \xrightarrow{f} & C \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle k} \\ A & \xrightarrow[g]{} & D \end{array} \qquad \begin{array}{c} X \\ \downarrow{\scriptstyle u} \\ Y \end{array} \qquad \mapsto \qquad \begin{array}{ccc} f^*X & \xrightarrow{\phi(f,X)} & X \\ {\scriptstyle f^*u}\,\vdots & & \downarrow{\scriptstyle u} \\ g^*Y & \xrightarrow[\phi(f,Y)]{} & Y \end{array}$$

on arrows. Dually, split opfibrations over $\mathscr{C}$ are precisely the normalized 2-algebras for the 2-monad on $\mathbf{Cat}/\mathscr{C}$ given by pulling back along $d_0\colon \mathscr{C}^{\mathbf{2}} \to \mathscr{C}$ and then composing with $d_1\colon \mathscr{C}^{\mathbf{2}} \to \mathscr{C}$. The correspondence is discussed in [16, §I,3.5]. A detailed account is in [15]. This result led to the definition of fibrations in a 2-category as certain algebras in [37, §2]. □

Remark 4.4. In this sense, fibrations over $\mathscr{C}$ are algebras for an action of the cotensor $\mathscr{C}^{\mathbf{2}}$ in $\mathbf{Cat}$. Likewise, discrete fibrations over $\mathscr{C}$ are algebras for an action of $\mathscr{C}_1$, which is a fragment of $\mathscr{C}^{\mathbf{2}}$. In the next subsection on 2-fibrations, the role of $\mathscr{C}^{\mathbf{2}}$ is played by the opcylinders of Construction 2.9. Likewise discrete 2-fibrations are monadic for an action of a fragment of this construction. □

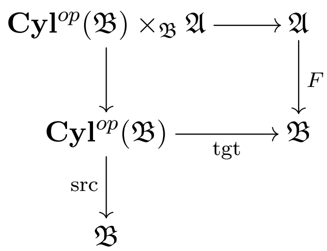
Remark 4.5 (Desiderata 3)**.** By the proof of Theorem 4.1, every $\mathscr{C}_1$-algebra determines a functor that is a discrete fibration by adding some extra structure. Thus, it determines a fibration, hence a $\mathscr{C}^{\mathbf{2}}$ algebra. In this way, view $\mathscr{C}_1$-algebras as including into $\mathscr{C}^{\mathbf{2}}$-algebras. These inclusions commute with the equivalences between fibrations and algebras. An analogous situation for (discrete) 2-fibrations is obtained in Theorem 4.13 below. □

4.2 Monadicity of (Discrete) 2-Fibrations Theorem 4.13 below presents the main result, namely, that 2-fibrations over a base 2-category $\mathfrak{B}$ are precisely the algebras for a 2-monad on the 2-slice of $\mathbf{2Cat}/\mathfrak{B}$. It will be seen that discrete 2-fibrations are also monadic for a restricted monad on $\mathbf{Cat}/|\mathfrak{B}|$. In each case, the monad will be given by an action of the 2-category of opcylinders in $\mathfrak{B}$ from Construction 2.9. This will be seen to be much like the cotensor with $\mathbf{2}$ in a lax 3-category in Section 4.3 below. Throughout opcylinders are used because this works with the convention that 2-fibrations are *doubly contravariant*, that is, correspond to functors dual on morphisms and 2-cells. Ordinary cylinders act on 2-cofibrations.

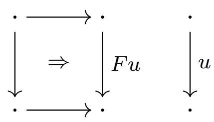
Construction 4.6. Define an endo-2-functor

$$T\colon \mathbf{2Cat}/\mathfrak{B} \to \mathbf{2Cat}/\mathfrak{B} \tag{4.1}$$

using the pull-push pattern above. That is, starting with a 2-functor $F\colon \mathfrak{A} \to \mathfrak{B}$, define TF to be the vertical composite along the left side of the diagram



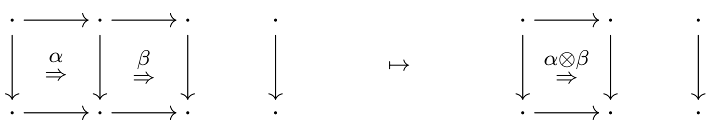
That is, take the 2-pullback of F along the target functor, and then pushforward along the source functor. Recall that arrows in the 2-category of opcylinders point from source to target. So, an arrow of the pullback above will be represented as



lining up the arrow u of $\mathfrak{A}$ with the target of the cell of $\mathfrak{B}$. Notice that this rotates the convention of Construction 2.9 by 90-degrees, putting the source and target as the vertically displayed arrows, rather than the horizontally displayed ones. This is a space-saving device and is more natural for the action of objects. In any case, this assignment extends in a natural way to an assignment on arrows and 2-cells using the universal property of the 2-pullback making a 2-functor T. There is a 2-natural transformation $\mu\colon T^2 \Rightarrow T$ given on a component $\mu_F\colon T^2F \to TF$

$$\mathbf{Cyl}^{op}(\mathfrak{B}) \times_{\mathfrak{B}} \mathbf{Cyl}^{op}(\mathfrak{B}) \times_{\mathfrak{B}} \mathfrak{A} \xrightarrow{-*-\times 1} \mathbf{Cyl}^{op}(\mathfrak{B}) \times_{\mathfrak{B}} \mathfrak{A}$$

by superposition of opsquares and opcylinders (denoted by '$\otimes$' diagrammatically) along shared source/target, crossed with identity or projection from $\mathfrak{A}$, as indicated by



and extended in the evident way to 2-cells. The 2-naturality arguments follow by unit and associativity for superposition of opcylinders. Similarly, there is a unit 2-natural transformation

$\eta \colon 1 \Rightarrow T$ given by inserting an identity square for horizontal composition in $\mathbf{Cyl}^{op}(\mathfrak{B})$. Likewise, there is a 2-monad

$$|T| \colon \mathbf{Cat}/|\mathfrak{B}| \to \mathbf{Cat}/|\mathfrak{B}| \tag{4.2}$$

on the ordinary 2-slice of $\mathbf{Cat}$ by $|\mathfrak{B}|$ defined by a similar pattern, just using the underlying 1-category of opcylinders. That is, for any $F \colon \mathscr{A} \to |\mathfrak{B}|$, take $|T|(F)$ to be the vertical composite on the left of

$$\begin{array}{ccc} |\mathbf{Cyl}^{op}(\mathfrak{B})| \times_{|\mathfrak{B}|} \mathscr{A} & \longrightarrow & \mathscr{A} \\ \downarrow & & \downarrow{\scriptstyle F} \\ |\mathbf{Cyl}^{op}(\mathfrak{B})| & \xrightarrow[|\mathrm{tgt}|]{} & |\mathfrak{B}| \\ {\scriptstyle |\mathrm{src}|}\downarrow & & \\ |\mathfrak{B}| & & \end{array}$$

with operations defined as above but ignoring the 2-cell assignments. The notation '$|T|$'reflects that fact that the evident diagram

$$\begin{array}{ccc} \mathbf{2Cat}/\mathfrak{B} & \xrightarrow{T} & \mathbf{2Cat}/\mathfrak{B} \\ {\scriptstyle |-|}\downarrow & & \downarrow{\scriptstyle |-|} \\ \mathbf{Cat}/|\mathfrak{B}| & \xrightarrow[|T|]{} & \mathbf{Cat}/|\mathfrak{B}| \end{array}$$

commutes. The way in which every $|T|$-algebra is a T-algebra is discussed in Lemma 4.11 below. □

Lemma 4.7. *The 2-functors T and $|T|$ of Construction 4.6 each define a 2-monad.*

Proof. The 2-monad axioms are exactly the associativity and unit laws for pasting of squares in $\mathbf{Cyl}^{op}(\mathfrak{B})$ since it suffices to check on components of μ and η. □

Construction 4.8. Let $P \colon \mathfrak{E} \to \mathfrak{B}$ denote a 2-fibration. Define an action 2-functor

$$M \colon \mathbf{Cyl}^{op}(\mathfrak{B}) \times_{\mathfrak{B}} \mathfrak{E} \to \mathfrak{E} \tag{4.3}$$

in the following way. For an object $(f \colon B \to C, X)$ of the purported domain, by construction of the pullback, $X \in \mathfrak{E}_C$ holds. Thus, take as definition of the action on objects $M(f, X) := f^*X$, that is, the domain of the chosen cartesian arrow $\phi(f, X)$ over f coming with the splitting. Now, a morphism of the domain of M is a pair (α, u). Under the action this is sent to the dashed arrow as at the right in the diagram

$$\begin{array}{ccc} B & \xrightarrow{f} & PX \\ {\scriptstyle h}\downarrow & \overset{\alpha}{\Rightarrow} & \downarrow{\scriptstyle Pu} \\ A & \xrightarrow[g]{} & PY \end{array} \quad \begin{array}{c} X \\ \downarrow{\scriptstyle u} \\ Y \end{array} \quad \mapsto \quad \begin{array}{ccc} f^*X & \xrightarrow{\phi(f,X)} & X \\ {\scriptstyle M(\alpha,u)}\downarrow & \overset{\phi(\alpha, u\phi(f,X))}{\Rightarrow} & \downarrow{\scriptstyle u} \\ g^*Y & \xrightarrow[\phi(g,Y)]{} & Y \end{array}$$

The dashed arrow exists because the target of the chosen 2-cell above α is over gh and $\phi(g, Y)$ is of course cartesian over g. Note that up to this point, with minor adaptations, these definitions would work to give assignments for an action

$$M \colon |\mathbf{Cyl}^{op}(\mathfrak{B})| \times_{|\mathfrak{B}|} |\mathfrak{E}| \to |\mathfrak{E}| \tag{4.4}$$

on a discrete 2-fibration $P\colon \mathfrak{E} \to \mathfrak{B}$. To continue with that for the general case Equation (4.3), the 2-cell assignment can be seen from penultimate display. That is, a 2-cell of the domain of M is given by a pair of 2-cells $h \Rightarrow h'$ and $u \Rightarrow u'$. Since the lift of α corresponding to u' is cartesian, there will be a unique lift of the composite 2-cell $gh \Rightarrow gh'$ between the sources of the lifts of α. The required 2-cell is then uniquely induced $M(\alpha, u) \Rightarrow M(\alpha', u')$ by the 2-cell lifting property of the 2-cartesian morphism $\phi(g, Y)$ as in Definition 3.13. By uniqueness of lifts M is a 2-functor. □

Lemma 4.9. *Any 2-fibration $P\colon \mathfrak{E} \to \mathfrak{B}$ is a T-algebra. If P is a discrete 2-fibration, then $|P|$ is a $|T|$-algebra.*

Proof. It suffices to see that the functor in Equation (4.3) satisfies the algebra laws summarized in Definition 4.2. But as in the proof of functoriality in the construction of M, this results from the uniqueness of lifted 2-cells owing to the fact that P is locally a discrete opfibration. □

The aim now is to prove that these object assignments extend to 2-functors with suitable pseudo-inverse making a commutative square of 2-equivalences

$$\begin{array}{ccc} \mathbf{D2Fib}(\mathfrak{B}) & \xrightarrow{\simeq} & |T|\mathbf{Alg} \\ \downarrow & & \downarrow \\ \mathbf{2Fib}(\mathfrak{B}) & \xrightarrow{\simeq} & T\mathbf{Alg} \end{array}$$

for any 2-category $\mathfrak{B}$ with T as in Construction 4.6. First show that every T-algebra is canonically a 2-fibration and that every $|T|$-algebra is a discrete 2-fibration. In fact the data in each case is almost exactly the same on each side, meaning that (discrete) 2-fibrations are in 1-1 correspondence with the appropriate algebras.

Lemma 4.10. *Any T-algebra $Q\colon \mathfrak{E} \to \mathfrak{B}$ with structure map $M\colon \mathbf{Cyl}^{op}(\mathfrak{B}) \times_{\mathfrak{B}} \mathfrak{E} \to \mathfrak{E}$ is a 2-fibration in a canonical way.*

Proof. Take $X \in \mathfrak{E}_0$ and a morphism $f\colon B \to QX$ of $\mathfrak{B}$. The chosen cartesian arrow above f is given by the image of the action $M((f,1),1_X)\colon M(f,X) \to X$ where '$(f,1)$' refers to the commutative square

$$\begin{array}{ccc} B & \xrightarrow{f} & C \\ {\scriptstyle f}\downarrow & & \| \\ C & = & C \end{array}$$

which is a morphism in $\mathbf{Cyl}^{op}(\mathfrak{B})$. Use as notation $f^*X = M(f,X)$ and $\phi(f,X) = M((f,1),1_X)$. The lifting property follows from commutativity conditions in $\mathbf{Cyl}^{op}(\mathfrak{B})$ and the fact that M is a 2-functor. This choice of cartesian arrow works for $Q\colon \mathscr{A} \to |\mathfrak{B}|$ with changes to notation. In the former case of an honest 2-functor $P\colon \mathfrak{E} \to \mathfrak{B}$, the 2-cell lifting property of $\phi(f,X)\colon f^*X \to X$ follows readily again using the definition of cells in $\mathbf{Cyl}^{op}(\mathfrak{B})$ and the fact that M is a 2-functor. That this provides a splitting for Q follows from the strict algebra equations.

It needs only to be seen that locally $Q\colon \mathfrak{E} \to \mathfrak{B}$ is a split fibration. Take an arrow $u\colon X \to Y$ of $\mathfrak{E}$ and a 2-cell $\alpha\colon f \Rightarrow Pu$ of $\mathfrak{B}$. Consider the 2-cell of $\mathbf{Cyl}^{op}(\mathfrak{B}) \times_{\mathfrak{B}} \mathfrak{E}$ as indicated by the equality

$$\begin{array}{ccc} PX & = & PX \\ {\scriptstyle f}\downarrow & \overset{\alpha}{\Rightarrow} & \left(\overset{1}{=}\right){\scriptstyle Pu} \\ PY & = & PY \end{array} \quad \begin{array}{c} X \\ \big){\scriptstyle u} \\ Y \end{array} \quad = \quad \begin{array}{ccc} PX & = & PX \\ {\scriptstyle f}\left(\overset{\alpha}{\Rightarrow}\right){\scriptstyle Pu} & & \downarrow{\scriptstyle Pu} \\ PY & = & PY \end{array} \quad \begin{array}{c} X \\ \downarrow{\scriptstyle u} \\ Y \end{array}$$

The image of this 2-cell under M in $\mathfrak{C}$ is the chosen cartesian 2-cell with codomain u above α. That it has the required lifting property follows from the definition of internal composition of 2-cells in $\mathbf{Cyl}^{op}(\mathfrak{B})$ and the fact that M is well-defined. That external composition preserves cartesian 2-cells follows from the definition. □

Lemma 4.11. *If $Q\colon \mathscr{A} \to |\mathfrak{B}|$ is a $|T|$-algebra with structure map $M\colon |\mathbf{Cyl}^{op}(\mathfrak{B})| \times_{|\mathfrak{B}|} \mathscr{A} \to \mathscr{A}$ then Q determines a discrete 2-fibration in a canonical way.*

Proof. The argument in the proof of Lemma 4.10 shows that $Q\colon \mathscr{A} \to |\mathfrak{B}|$ is a split fibration. Equip $\mathscr{A}$ with 2-cells resulting in a 2-category $\mathfrak{A}$ with $|\mathfrak{A}| = \mathscr{A}$ and a split discrete 2-fibration $Q\colon \mathfrak{A} \to \mathfrak{B}$. Form the pullback of categories

$$\begin{array}{ccc} |\mathbf{Cyl}^{op}(\mathfrak{B})| \times_{|\mathfrak{B}|} \mathscr{A} & \longrightarrow & \mathscr{A} \\ \downarrow & & \downarrow \\ |\mathbf{Cyl}^{op}(\mathfrak{B})| & \xrightarrow[\text{tgt}]{} & |\mathfrak{B}|. \end{array}$$

The 2-cells of the new 2-category $\mathfrak{A}$ are the horizontally globular arrows of the pullback category. That is, a 2-cell of $\mathfrak{A}$ is an arrow

$$\begin{array}{ccccc} \cdot & = & \cdot & & \cdot \\ \downarrow & \overset{\alpha}{\Rightarrow} & \downarrow {\scriptstyle Fu} & & \downarrow {\scriptstyle u} \\ \cdot & = & \cdot & & \cdot \end{array}$$

The source of the cell is $M(\alpha, u)$ in $\mathscr{A}$ and the target is u. The compositions and units of opcylinders make $\mathfrak{A}$ into a 2-category. The required 2-functor is Q on objects and arrows, and is given on 2-cells by restricting the projection $|\mathbf{Cyl}^{op}(\mathfrak{B})| \times_{|\mathfrak{B}|} \mathscr{A} \to |\mathbf{Cyl}^{op}(\mathfrak{B})|$. The resulting 2-functor $Q\colon \mathfrak{A} \to \mathfrak{B}$ is locally a discrete fibration by construction of the 2-cells in $\mathfrak{A}$. □

Remark 4.12. By this construction each $|T|$-algebra canonically determines a 2-fibration, hence a T-algebra. Notice that this is the analogue of the construction in the proof of Theorem 4.1 for this higher-dimensional structure, obtaining the fibration from underlying structure by freely adding missing higher cells. □

All in all, these lemmas show that given a 2-fibration P, there is an associated T-algebra (P, M) with the action as defined above; and conversely that for every T-algebra (P, M), the 2-functor P is a split 2-fibration. In other words, 2-fibrations are in 1-1 correspondence with T-algebras. Similarly for discrete 2-fibrations and $|T|$-algebras. Moreover, just as every discrete 2-fibration is a 2-fibration, every $|T|$-algebra canonically determines a T-algebra by adding some extra structure. These object assignments extend to well-defined 2-functors making a commutative square

$$\begin{array}{ccc} \mathbf{D2Fib}(\mathfrak{B}) & \longrightarrow & |T|\mathbf{Alg} \\ \downarrow & & \downarrow \\ \mathbf{2Fib}(\mathfrak{B}) & \longrightarrow & T\mathbf{Alg} \end{array}$$

which turn out to give equivalences of 2-categories in the horizontal rows. For any morphism of (discrete) 2-fibrations is splitting-preserving, which implies equivariance with respect to the algebra actions defined in the previous lemmas. Likewise, transformations between splitting-preserving functors with vertical components determine transformations of the corresponding equivariant morphisms satisfying the compatibility condition of Definition 4.2. In other words, the data for morphisms or 2-cells of fibrations is data for corresponding equivariant morphisms of algebras or 2-cells of such morphisms. These are 2-functorial since the data is identical and composition is defined in the same way on either side. Now, we have the main result, which fulfills the desiderata of Remark 4.5.

Theorem 4.13 (Discrete 2-Fibrations are 2-Monadic)**.** *There are 2-equivalences making a commutative square*

$$\begin{array}{ccc} \mathbf{D2Fib}(\mathfrak{B}) & \xrightarrow{\simeq} & |T|\mathbf{Alg} \\ \downarrow & & \downarrow \\ \mathbf{2Fib}(\mathfrak{B}) & \xrightarrow[\simeq]{} & T\mathbf{Alg} \end{array}$$

for any 2-category $\mathfrak{B}$ with T as above in Construction 4.6.

Proof. Conversely, any T- or $|T|$-equivariant morphism of algebras is a splitting-preserving morphism of the corresponding fibrations by the construction of the splittings in Lemma 4.10. Likewise the compatibility condition for 2-cells of algebra morphisms in Definition 4.2 implies that the components are vertical. □

Remark 4.14. Cofibrations in the language of Remark 2.5 are precisely the algebras for the analogous action of ordinary cylinders. □

4.3 Lax 3-Categories This subsection presents the setting needed to describe the way in which cylinders of Construction 2.9 arise as a universal construction. The setting is a *lax 3-category*, namely, a category enriched in 2-categories and lax functors. The main results of this subsection, namely, Theorem 4.26 describes this universality among lax natural transformations. The machinery of lax 3-categories is used because lax transformations do not organize into an ordinary 3-category even if we take 2-functors as the morphisms [35]. The template for the universality result is that of the ordinary 2-comma (cf. [37, §1, p.108]) summarized in the following result.

Proposition 4.15 (Universal Property of 2-Comma Category)**.** *Given 2-functors $F\colon \mathfrak{A} \to \mathfrak{B}$ and $G\colon \mathfrak{C} \to \mathfrak{B}$, the 2-comma category F/G of Construction 2.13 is 1-, 2- and 3-dimensionally universal in the following sense.*

1. *Given 2-functors $H\colon \mathfrak{D} \to \mathfrak{A}$ and $K\colon \mathfrak{D} \to \mathfrak{C}$ and any 2-cell $\theta\colon FH \Rightarrow GK$, there is a unique 2-functor $U\colon \mathfrak{D} \to F/G$ such that $\theta = \lambda * U$.*
2. *Given 2-natural transformations ξ and ζ satisfying the equality of 2-cells in the diagram*

$$\begin{array}{ccccc} \mathfrak{D} & \xrightarrow{U} & F/G & \xrightarrow{\text{tgt}} & \mathfrak{C} \\ {\scriptstyle V}\downarrow & \overset{\xi}{\Rightarrow} & \downarrow{\scriptstyle \text{src}} & \overset{\lambda}{\Rightarrow} & \downarrow{\scriptstyle G} \\ F/G & \xrightarrow[\text{src}]{} & \mathfrak{A} & \xrightarrow[F]{} & \mathfrak{B} \end{array} \quad = \quad \begin{array}{ccc} \mathfrak{D} & \xrightarrow{U} & F/G \\ {\scriptstyle V}\downarrow & \overset{\zeta}{\Rightarrow} & \downarrow{\scriptstyle \text{tgt}} \\ F/G & \xrightarrow{\text{tgt}} & \mathfrak{C} \\ {\scriptstyle \text{src}}\downarrow & \overset{\lambda}{\Rightarrow} & \downarrow{\scriptstyle G} \\ \mathfrak{A} & \xrightarrow[F]{} & \mathfrak{B} \end{array}$$

there is a unique 2-natural transformation $\omega\colon U \Rightarrow V$ such that the equations $\text{src} * \omega = \xi$ and $\text{tgt} * \omega = \zeta$ each hold.

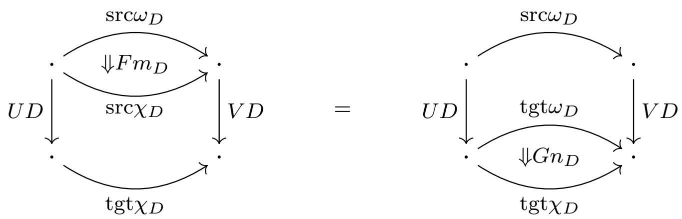
3. *Given 2-natural transformations* $\omega\colon U \Rightarrow V$ *and* $\chi\colon U \Rightarrow V$ *between* $U, V\colon \mathfrak{D} \rightrightarrows F/G$ *as above with modifications* $m\colon \mathrm{src} * \omega \Rrightarrow \mathrm{src} * \chi$ *and* $n\colon \mathrm{tgt} * \omega \Rrightarrow \mathrm{tgt} * \chi$ *satisfying*

$$(\lambda * V)Fm = Gn(\lambda * U) \tag{4.5}$$

there is a unique modification $l\colon \omega \Rrightarrow \chi$ *satisfying* $\mathrm{tgt} * l = n$ *and* $\mathrm{src} * l = m$.

These properties characterize F/G *up to isomorphism in* **2Cat**.

Proof. The proofs of the first two universality conditions are the same as for the ordinary 1-comma category in **Cat**, since the underlying 1-category of the 2-comma category is essentially the 1-comma category of the underlying functors. Thus, we have only to prove the third condition. But the compatibility condition Equation (4.5) just means that for each $D \in \mathfrak{D}_0$ there is an equality of composite 2-cells in $\mathfrak{B}$ of the form



But this is plainly the form of a 2-cell in F/G and thus gives the D-component of the purported modification l. Compatibility follows by naturality and the modification condition. □

Remark 4.16 (Non-Elementary Statement of Universal Property)**.** Recall from [24, §3.7] that the cotensor of an object C in a $\mathscr{V}$-category $\mathcal{B}$ by an object $X \in \mathscr{V}_0$ is an object $X \pitchfork C$ of $\mathcal{B}$ for which there is a $\mathscr{V}$-natural isomorphism with counit

$$\mathcal{B}(B, X \pitchfork C) \cong [X, \mathcal{B}(B, C)] \qquad X \to \mathcal{B}(X \pitchfork C, C) \tag{4.6}$$

for any $B \in \mathcal{B}_0$. Viewing **Cat** as **Set**-enriched, for any small category $\mathscr{C}$, the usual arrow category $\mathbf{Arr}(\mathscr{C}) = \mathbf{Cat}(\mathbf{2}, \mathscr{C})$ is the cotensor of $\mathscr{C}$ with the ordinal category **2**. Similarly, the usual arrow 2-category $\mathbf{2Cat}(\mathbf{2}, \mathfrak{C})$ is the cotensor of $\mathfrak{C}$ with **2** in **2Cat**. Now, the universal property of the 2-comma category says, essentially, that $1_{\mathfrak{B}}/1_{\mathfrak{B}}$, the comma category of $1_{\mathfrak{B}}$ with itself, is the cotensor of $\mathfrak{B}$ with **2** in the sense that there is an isomorphism

$$\mathbf{2Cat}(\mathfrak{A}, 1_{\mathfrak{B}}/1_{\mathfrak{B}}) \cong |\mathbf{Cat}|(\mathbf{2}, \mathbf{2Cat}(\mathfrak{A}, \mathfrak{B}))$$

induced by composition with the canonical 2-natural transformation β of Construction 2.13. This is probably easiest to see from the definitions using the fact that the hom-category on the right is isomorphic to the arrow 2-category $\mathbf{2Cat}(\mathfrak{A}, \mathfrak{B})^{\mathbf{2}}$ as presented in Construction 2.13. Note in particular that $1_{\mathfrak{B}}/1_{\mathfrak{B}}$ is isomorphic to the 2-arrow category $\mathfrak{B}^{\mathbf{2}}$. □

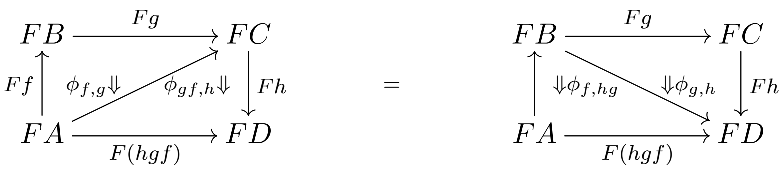
For now, $||\mathbf{2Cat}||$ denotes the ordinary category of 2-categories and 2-natural transformations. It is strict cartesian monoidal. A **3-category** is a $||\mathbf{2Cat}||$-enriched category. This means that a 3-category $\mathcal{A}$ is a set of objects $\mathcal{A}_0$ together with hom 2-categories $\mathcal{A}(A, B)$ for any $A, B \in \mathcal{A}_0$ together with appropriate composition and identity 2-functors satisfying the usual associativity and unit diagrams.

Example 4.17. 2-categories, 2-functors, 2-natural transformations, and their modifications comprise the 3-category, **2Cat**. □

Example 4.18. For any 2-category $\mathfrak{A}$, 2-functors $\mathfrak{A}^{op} \to \mathbf{2Cat}$, 2-natural transformations, modifications and *perturbations* between them [12] form a 3-category $[\mathfrak{A}^{op}, \mathbf{2Cat}]$. □

Definition 4.19 (Lax Functor)**.** A **(normalized) lax functor** between 2-categories $F\colon \mathfrak{A} \to \mathfrak{B}$ makes object, arrow, and 2-cells assignments $A \mapsto FA$, $f \mapsto Ff$, and $\alpha \mapsto F\alpha$ and comes equipped with coherence 2-cells $\phi_{f,g}\colon FgFf \Rightarrow F(gf)$ for any two composable arrows f and g, all satisfying the following conditions.

1. F strictly preserves domains, codomains, sources and targets, identity morphisms and 2-cells, and vertical composition of 2-cells.
2. For composable arrows $f\colon A \to B$, $g\colon B \to C$ and $h\colon C \to D$ of $\mathfrak{K}$, there is an equality of composite coherence 2-cells



3. For horizontally composable 2-cells $\alpha\colon f \Rightarrow g$ and $\beta\colon h \Rightarrow k$ with $f, g\colon A \rightrightarrows B$ and $h, k\colon B \rightrightarrows C$, there is an equality of composite 2-cells $\phi_{g,k}(F\beta * F\alpha) = F(\beta * \alpha)\phi_{f,h}$.

Let $|\mathbf{2Cat}_{lax}|$ denote the 1-category of 2-categories and lax functors between them. □

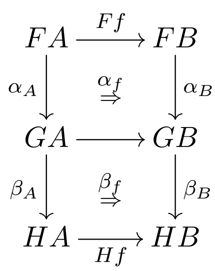
Remark 4.20. The notation '$|\mathbf{2Cat}_{lax}|$' is used with out double '|' because 2-categories with lax functors cannot be made into a 3-category with any reasonable notion of transformation. See [35]. The category $|\mathbf{2Cat}_{lax}|$ is cartesian monoidal with ordinary cartesian products of 2-categories giving the product. Thus, the following makes sense. □

Definition 4.21. A **lax 3-category** is a category enriched in $|\mathbf{2Cat}_{lax}|$. □

Although certainly every ordinary 3-category is an obvious (and trivial) example, note that due to laxity not every lax 3-category is a 2- or 3-category. The data for the main example **Lax**, consisting of 2-categories, 2-functors, lax transformations and modifications, is given in the following development. The hom 2-categories are considered in [25, §5] in connection with 2-categorical limits, so we adopt the same notation.

Lemma 4.22. *For any 2-categories $\mathfrak{A}$ and $\mathfrak{B}$, the 2-functors between them, with lax natural transformations and modifications, are the data of a 2-category, denoted here by '$\mathbf{Lax}(\mathfrak{A}, \mathfrak{B})$'.*

Proof. Composition of lax natural transformations is well-defined. Taking $\alpha\colon F \Rightarrow G$ and $\beta\colon G \Rightarrow H$, declare $(\beta\alpha)_A := \beta_A\alpha_A$, as expected, and take the 2-cell for lax naturality at an arrow $f\colon A \to B$ to be the juxtaposition of 2-cells



The lax naturality conditions are satisfied because they are satisfied by α and β. Internal composition of modifications is given by internal composition of 2-cells in $\mathfrak{B}$; similarly, external composition is given by that in $\mathfrak{B}$. The interchange law follows by interchange in $\mathfrak{B}$. □

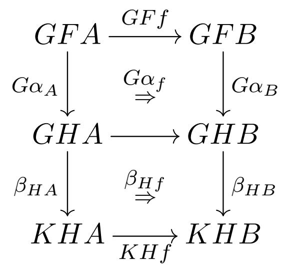
Construction 4.23 (Data for 2-Categories with Lax Transformations)**.** The hom-categories for a lax 3-category will be the 2-categories $\mathbf{Lax}(\mathfrak{A},\mathfrak{B})$. Required for enrichment are composition and identity morphisms in $|\mathbf{2Cat}_{lax}|$. First fix 2-categories $\mathfrak{A}$, $\mathfrak{B}$, and $\mathfrak{C}$. Construct what will be a lax functor

$$-\otimes -\colon \mathbf{Lax}(\mathfrak{B},\mathfrak{C})\times\mathbf{Lax}(\mathfrak{A},\mathfrak{B})\to\mathbf{Lax}(\mathfrak{A},\mathfrak{C}) \tag{4.7}$$

in the following way. On objects, i.e. composable pairs (G,F) of 2-functors, take $G\otimes F$ to be the ordinary composition GF. For horizontally composable lax natural transformations $\alpha\colon F\Rightarrow H$ and $\beta\colon G\Rightarrow K$ with $F,H\colon\mathfrak{A}\rightrightarrows\mathfrak{B}$ and $G,K\colon\mathfrak{B}\rightrightarrows\mathfrak{C}$, take $\beta\otimes\alpha$ to have components

$$(\beta\otimes\alpha)_A := \beta_{HA}G\alpha_A$$

indexed over $A\in\mathfrak{A}_0$. Given a morphism $f\colon A\to B$ of $\mathfrak{A}$, the 2-cell $(\beta\otimes\alpha)_f$ required for lax naturality is then the composite 2-cell



The conditions for lax naturality are satisfied since they are satisfied by the α_f and β_{Hf} over morphisms $f\colon A\to B$. Given further lax natural transformations $\gamma\colon F\Rightarrow H$ and $\delta\colon G\Rightarrow K$ and two modifications $m\colon\alpha\Rrightarrow\gamma$ and $n\colon\beta\Rrightarrow\delta$, define what will be the component of a modification $n\otimes m$ as the horizontal composite

$$(n\otimes m)_A := n_{HA} * Gm_A$$

over $A\in\mathfrak{A}_0$. That this is well-defined, that is, satisfies the modification condition, is just a result of the fact that both m and n are modifications and that G is functorial on 2-cells. □

Lemma 4.24. *The assignments of Construction 4.23 make*

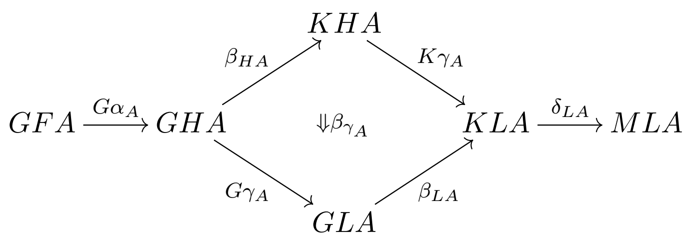
$$-\otimes -\colon \mathbf{Lax}(\mathfrak{B},\mathfrak{C})\times\mathbf{Lax}(\mathfrak{A},\mathfrak{B})\to\mathbf{Lax}(\mathfrak{A},\mathfrak{C}) \tag{4.8}$$

into a lax functor of 2-categories in the sense of Definition 4.19.

Proof. Preservation of domains, codomains, sources, targets and 1- and 2-cell identities are all straightforward to check. The compatibility cells, however, need to be exhibited and the two conditions of Definition 4.19 need to be checked. Given further lax natural transformations $\delta\colon K\Rightarrow M$ and $\gamma\colon H\Rightarrow L$, required is a compatibility cell, that is, a modification $(\delta\otimes\gamma)(\beta\otimes\alpha)\Rrightarrow\delta\beta\otimes\gamma\alpha$. Unraveling both sides at objects $A\in\mathfrak{A}_0$, we see that this amounts to giving 2-cells

$$\delta_{LA}K(\gamma_A)\beta_{HA}G(\alpha_A)\Rightarrow\delta_{LA}\beta_{LA}G\gamma_AG\alpha_A$$

satisfying the appropriate compatibility condition for a modification. But there is an evident choice by taking the cell to be the laxity coherence cell β_{γ_A} as in the diagram



The modification condition for a fixed arrow $f\colon A \to B$ of $\mathfrak{A}$ follows by the second lax naturality condition for the associated cell γ_f. That the associativity compatibility condition for lax functoriality on 1-cells is satisfied follows by the lax naturality of β and the definition of composition of lax natural transformations.

Preservation of vertical composition of 2-cells (that is, modifications) is straightforward using the fact that all the 2-functors involved strictly preserve vertical composition of 2-cells. However, the compatibility condition in Definition 4.19 for horizontal composition is less clear. Thus, suppose that further modifications $p\colon \tau \Rrightarrow \chi$ and $l\colon \sigma \Rrightarrow \rho$ are given between lax natural transformations $\tau, \chi\colon K \Rightarrow M$ and $\sigma, \rho\colon H \Rightarrow L$. One computes $(p * n) \otimes (l * m)$ on the one hand and $(p \otimes l) * (n \otimes m)$ on the other, and adding in the coherence cells as defined above, the compatibility condition will follow from an equality of the composite 2-cells

$$
\begin{array}{ccccc}
GHA & \xrightarrow{G\rho_A} & GLA & & \\
\Vert & \overset{Gl_A}{\Rightarrow} & \Vert & & \\
GHA & \longrightarrow & GLA & = & GLA \\
{\scriptstyle \beta_{HA}}\downarrow & \overset{\beta_{\sigma_A}}{\Rightarrow} & \downarrow & \overset{n_{LA}}{\Rightarrow} & \downarrow{\scriptstyle \delta_{LA}} \\
KHA & \xrightarrow[K\sigma_A]{} & KLA & = & KLA
\end{array}
\quad = \quad
\begin{array}{ccccc}
GHA & = & GHA & \xrightarrow{G\rho_A} & GLA \\
{\scriptstyle \beta_{HA}}\downarrow & \overset{n_{HA}}{\Rightarrow} & \downarrow & \overset{\delta_{\rho_A}}{\Rightarrow} & \downarrow{\scriptstyle \delta_{LA}} \\
KHA & = & KHA & \longrightarrow & KLA \\
 & & \Vert & \overset{KL_A}{\Rightarrow} & \Vert \\
 & & KHA & \xrightarrow[K\sigma_A]{} & KLA
\end{array}
$$

That this equality does in fact hold is now easy to establish, first using the modification condition for n at σ_A and then by the using the second lax naturality condition for δ at the 2-cell l_A. □

Theorem 4.25. *The data of Construction 4.23 makes 2-categories, 2-functors, lax natural transformations and modifications into a lax 3-category, denoted by* **Lax***.*

Proof. It remains to check the pentagonal associativity condition and the identity conditions. But these are now easy. For 2-categories $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ and $\mathfrak{D}$, the associativity condition asserts that there is an equality of lax functors

$$
\begin{array}{ccc}
(\mathbf{Lax}(\mathfrak{C},\mathfrak{D}) \times \mathbf{Lax}(\mathfrak{B},\mathfrak{C})) \times \mathbf{Lax}(\mathfrak{A},\mathfrak{B}) & \xrightarrow{\cong} & \mathbf{Lax}(\mathfrak{C},\mathfrak{D}) \times (\mathbf{Lax}(\mathfrak{B},\mathfrak{C}) \times \mathbf{Lax}(\mathfrak{A},\mathfrak{B})) \\
{\scriptstyle \otimes\times 1}\downarrow & & \downarrow{\scriptstyle 1\times\otimes} \\
\mathbf{Lax}(\mathfrak{B},\mathfrak{D}) \times \mathbf{Lax}(\mathfrak{A},\mathfrak{B}) & & \mathbf{Lax}(\mathfrak{C},\mathfrak{D}) \times \mathbf{Lax}(\mathfrak{A},\mathfrak{C}) \\
{\scriptstyle \otimes}\downarrow & & \downarrow{\scriptstyle \otimes} \\
\mathbf{Lax}(\mathfrak{A},\mathfrak{D}) & = & \mathbf{Lax}(\mathfrak{A},\mathfrak{D})
\end{array}
$$

But this follows readily. For composition of 2-functors is strictly associative. At the level of 1-cells, it is a direct computation using the definition of $\otimes$. Take 2-functors $F, K\colon \mathfrak{A} \rightrightarrows \mathfrak{B}$, $G, L\colon \mathfrak{B} \rightrightarrows \mathfrak{C}$ and $H, M\colon \mathfrak{C} \rightrightarrows \mathfrak{D}$ with lax natural transformations $\alpha\colon F \Rightarrow K$, $\beta\colon G \Rightarrow L$ and $\gamma\colon H \Rightarrow L$. On the one hand, computing around the counterclockwise direction of the diagram, we have

$$((\gamma \otimes \beta) \otimes \alpha)_A = (\gamma \otimes \beta)_{KA} HG\alpha_A = \gamma_{LKA} H(\beta_{KA}) H(G(\alpha_A)$$

and around the clockwise direction on the other hand

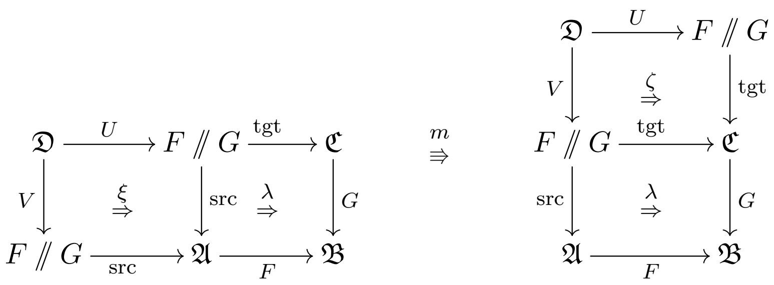
$$(\gamma \otimes (\beta \otimes \alpha))_A = \gamma_{LKA} H(\beta \otimes \alpha)_A = H(\beta_{KA}\alpha_A).$$

These results are evidently the same since H is a 2-functor. The computation at the 2-cell level is analogous. The identity conditions also follow by direct inspection. □

Now, the universal property of the lax comma $F /\!\!/ G$ in **Lax** from Construction 3.4 is a *laxification* of the universal property of the 2-comma category proved in Proposition 4.15. As already remarked, what is here called the *lax comma category* is called the *2-comma category* in [16, §I,2.5]. However the development there does not appear to describe completely the universality enjoyed by this construction. The complete statement is the following.

Theorem 4.26 (Universal Property of Lax Comma Category)**.** *Given 2-functors $F\colon \mathfrak{A} \to \mathfrak{B}$ and $G\colon \mathfrak{C} \to \mathfrak{B}$, the lax comma category $F /\!\!/ G$ of Construction 2.13 is 1-, 2- and 3-dimensionally universal in the following sense.*

1. *Given 2-functors $H\colon \mathfrak{D} \to \mathfrak{A}$ and $K\colon \mathfrak{D} \to \mathfrak{C}$ and any lax transformation $\theta\colon FH \Rightarrow GK$, there is a unique 2-functor $U\colon \mathfrak{D} \to F /\!\!/ G$ such that $\theta = \lambda * U$.*
2. *Given lax natural transformations ξ and ζ together with a modification m as in the diagram*



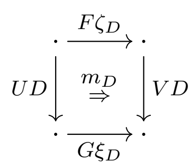
   there is a lax natural transformation $\omega\colon U \Rightarrow V$ such that the equations $\mathrm{tgt} * \omega = \xi$ *and* $\mathrm{src} * \omega = \zeta$ *each hold.*
3. *Given lax natural transformations $\omega\colon U \Rightarrow V$ and $\chi\colon U \Rightarrow V$ between $U, V\colon \mathfrak{D} \rightrightarrows F /\!\!/ G$ as above with modifications $m\colon \mathrm{src} * \omega \Rrightarrow \mathrm{src} * \chi$ and $n\colon \mathrm{tgt} * \omega \Rrightarrow \mathrm{tgt} * \chi$ satisfying*

$$(\lambda * V)Fm = Gn(\lambda * U) \tag{4.9}$$

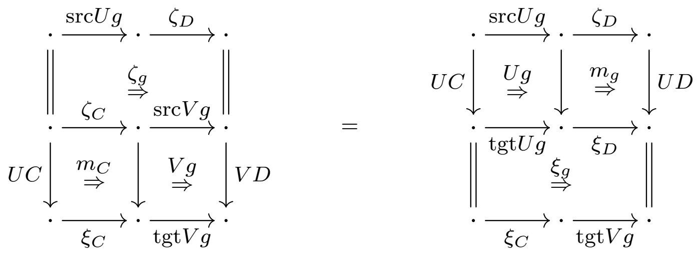
   there is a unique modification $l\colon \omega \Rrightarrow \chi$ satisfying $\mathrm{tgt} * l = n$ *and* $\mathrm{src} * l = m$.

These properties characterize $F /\!\!/ G$ up to isomorphism in **Lax**.

Proof. The 1-dimensional aspect of the universal property was proved in Lemma 2.16. The proof of the 3-dimensional aspect is the same as in the proof of Proposition 4.15 with suitable adaptations for lax naturality. Thus, we prove the second condition, that is, the 2-dimensional aspect of the universal property. Thus, given the data of ξ, ζ and m, we need to construction $\omega\colon U \Rightarrow V$. The component of the modification m at say $D \in \mathfrak{D}_0$ is a 2-cell



of $\mathfrak{B}$. Thus, define the component of ω at $D \in \mathfrak{D}_0$ to be the arrow (ζ_D, ξ_D, m_D) of $F /\!\!/ G$. Given an arrow $g\colon C \to D$ of $\mathfrak{D}$, there should be a lax naturality cell from (ζ_C, ξ_C, m_C) to (ζ_D, ξ_D, m_D), the arrows of which will be the 2-cells Ug and Vg. The actual 2-cell in $F /\!\!/ G$ is given by the lax naturality cells ζ_g and ξ_g. This does define the required lax naturality 2-cell in $F /\!\!/ G$ by the equality



which holds by the modification condition for m since the other cells on either side are the coherence cells for the composite lax natural transformations. The compatibility conditions for lax naturality as in Definition 2.14 are satisfied since the same conditions are satisfied by ξ and ζ. The lax natural transformation ω is by construction the unique one with the desired properties. □

The universality of cylinders is now a special case. The following is an analogue of the non-elementary statement of the universal property of the 2-comma object for $1_{\mathfrak{B}}$ in Remark 4.16.

Corollary 4.27. *The cylinder construction* $\mathbf{Cyl}(\mathfrak{B})$ *presents the cotensor with* $\mathbf{2}$ *in* $\mathbf{Lax}$ *in the sense that there is an isomorphism*

$$\mathbf{Lax}(\mathfrak{A}, \mathbf{Cyl}(\mathfrak{B})) \cong \mathbf{Lax}(\mathbf{2}, \mathbf{Lax}(\mathfrak{A}, \mathfrak{B}))$$

of 2-categories.

Proof. Use Theorem 4.26 in the case that $F = G = 1$. □

Remark 4.28. Notice that *mutatis mutandis* $\mathbf{Cyl}^{op}(\mathfrak{B})$ is thus a cotensor with $\mathbf{2}$ among 2-categories, 2-functors, oplax transformations and modifications. □

4.4 Representability Redux We return now to the problem posed in Section 3.4 which showed in particular that 2-fibrations are not internal fibrations in **2Cat**. However, the setting of the lax 3-category **Lax** allows a kind of partial recovery of a representable characterization. That is, we have the following result. Notice it is phrased for 2-*cofibrations* since this is what works with regard to the conventions concerning duals in this paper (Remark 2.5). A similar result works for 2-fibrations taking 2-categories, 2-functors, oplax transformations, and modifications as forming the ambient structure. Since we have been working with **Lax** throughout this section, for consistency we phrase the result in this context. Note first of all that postcomposition defines a 2-functor

$$P \circ - \colon \mathbf{Lax}(\mathfrak{X}, \mathfrak{E}) \to \mathbf{Lax}(\mathfrak{X}, \mathfrak{B})$$

for any given 2-functor $P \colon \mathfrak{X} \to \mathfrak{E}$. The issue raised in [35] is not that of whiskering with 2-functors but with lax functors.

Proposition 4.29. *A 2-functor* $P \colon \mathfrak{E} \to \mathfrak{B}$ *is a (split) 2-cofibration if, and only if, for each 2-category* $\mathfrak{X}$ *the 2-functor*

$$P \circ - \colon \mathbf{Lax}(\mathfrak{X}, \mathfrak{E}) \to \mathbf{Lax}(\mathfrak{X}, \mathfrak{B})$$

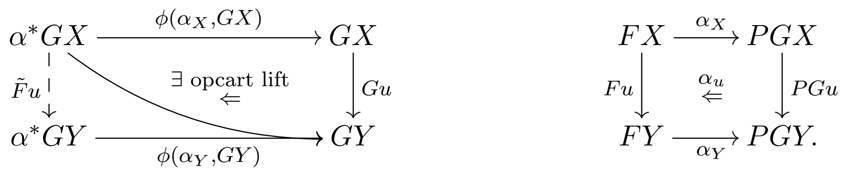
is one in **2Cat**.

Proof. The sufficiency of the condition is clear taking $\mathfrak{X} = 1$ since lax transformations between points $X, Y : 1 \rightrightarrows \mathfrak{E}$ amount to ordinary arrows $X \to Y$. Modifications are used here to get the 2-dimensional aspects of the various lifting properties.

On the other hand, assuming that P is a 2-cofibration, we need a candidate for a 2-cartesian lift $\tilde{\alpha} : \tilde{F} \Rightarrow G$ of a given lax natural transformation $\alpha : F \Rightarrow PG$ of 2-functors $F, PG : \mathfrak{X} \rightrightarrows \mathfrak{B}$. Letting $X \in \mathfrak{X}$ be any object, as in Lemma 3.26, take as the definition of $\tilde{\alpha}_X$ the arrow

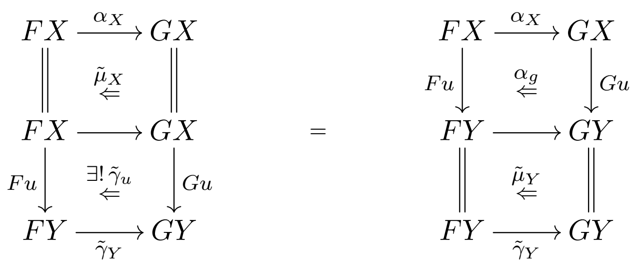
$$\tilde{\alpha}_X := \phi(\alpha_X, GX) : \alpha_X^* GX \to GX \qquad \tilde{F}X := \alpha_X^* GX.$$

That is, $\tilde{\alpha}_X$ is the chosen cartesian arrow above α_X. Again the domain of that chosen cartesian arrow is the object assignment for $\tilde{F}$. We need to see in this case that $\tilde{F}$ is a 2-functor and that the α_X are lax natural. These can be done simultaneously. Let $u : X \to Y$ denote any arrow of $\mathfrak{X}$. Take α_u to be the cell at the left:



following essentially the same procedure as Construction 4.8. That is, the cell exists because it is an opcartesian lift of the given laxity cell α_u in $\mathfrak{B}$. The dashed arrow then exists because $\phi(\alpha_Y, GY)$ in particular is (2-)cartesian. Notice that we need *op*cartesian here to make sure the lifted cell points in the right direction. In any case, this gives both the arrow assignment for $\tilde{F}$ and the candidate for the required laxity cells $\tilde{\alpha}_u$. The fact that $\phi(\alpha_Y, GY)$ is additionally 2-cartesian gives the 2-cell assignment for $\tilde{F}$. The cells $\tilde{\alpha}_u$ are genuinely lax natural by the splitting equations, closure properties of opcartesian 2-cells, and the construction of $\tilde{F}$. That the defined $\tilde{\alpha}$ has the necessary lifting property is a detailed but ultimately straightforward check.

It also needs to be seen that locally $P \circ -$ is a (split) opfibration. For this, fix 2-functors $F, G : \mathfrak{X} \rightrightarrows \mathfrak{E}$ and a lax transformation $\alpha : F \Rightarrow G$. Suppose further that we have a modification $\mu : P * \alpha \Rrightarrow \gamma$ for some lax transformation $\gamma : PF \Rightarrow PG$. Needed is an opcartesian lift $\tilde{\mu} : \alpha \Rrightarrow \tilde{\gamma}$ over μ where $\tilde{\gamma} : F \Rightarrow G$ is a lax natural transformation over γ. For each $X \in \mathfrak{X}$, there is a chosen opcartesian cell $\tilde{\mu}_X : \alpha_X \Rightarrow \mu_X^* \alpha_X$ over μ_X owing to the fact that locally P is a split opfibration. Take $\tilde{\gamma}_X := \mu_X^* \alpha_X$ and for $u : X \to Y$ define $\tilde{\gamma}_u$ to be the cell on the left



which exists as a lift of γ_u since the composite $Gu * \tilde{\mu}_X$ is opcartesian. Notice that if $\tilde{\gamma}$ is lax natural, then $\tilde{\mu}$ is a modification by construction since the equality above is precisely the modification condition. But that $\tilde{\gamma}$ is lax natural follows directly from the fact that the cells $\tilde{\gamma}_u$ are defined as unique lifts. Finally, that each component $\tilde{\mu}_X$ is opcartesian implies that the modification $\tilde{\mu}$ is itself opcartesian as well. □

By the proof of the previous proposition, the following is then immediate.

Corollary 4.30. *A 2-functor $P\colon \mathfrak{E} \to \mathfrak{B}$ is a discrete 2-cofibration if, and only if, for each 2-category $\mathfrak{X}$ the 2-functor*

$$P \circ -\colon \mathbf{Lax}(\mathfrak{X}, \mathfrak{E}) \to \mathbf{Lax}(\mathfrak{X}, \mathfrak{B})$$

is one in **2Cat**.

5. Prospectus

The paper closes here with some speculation about further avenues of inquiry.

5.1 Internalization Returning to the discussion of Section 1.3, there remains the question about the approach to be taken toward internalizing the notion of a discrete 2-fibration in some higher topos. Again the idea is to boost the internalization results for flat set-valued functors achieved by Diaconescu in [11] and [10] into the next highest dimension, giving an elementary account of flatness for something like 2- and pseudo-functors. Discrete fibrations were encoded in dimension 1 as algebras for an action of the cotensor of the base category with **2**. What has been discovered here, however, is that discrete 2-fibrations are algebras for the action of a structure $\mathbf{Cyl}(\mathfrak{B})$ that is not a cotensor is the expected venue, namely, **2Cat**, but in the more unusual 3-dimensional structure **Lax**.

The approach that was taken in [28] was to axiomatize the idea of a 2-category internal to a given 1-category $\mathscr{E}$. This included the definition of internal hom-categories making sense of how to discuss concepts defined for internal 2-categories locally. A discrete 2-fibration over a fixed base internal 2-category was then defined to be an internal 2-functor (1) whose underlying internal 1-functor is an algebra for the action of the internal arrow category of the underlying 1-category of the base; and (2) that is locally a discrete fibration in the sense of being an algebra for an internalization of the action as in the opening of Section 4.1. This approach worked to given an elementary version of the desired flatness results but was technically complicated in a way that fundamentally muddied what should have been a clear and elegant picture. The question, then, is whether the results of this paper give any insight into the possibility of a more straightforward elementary axiomatization of the setting in which the flatness results should be achieved.

It is not clear that this is the case. At least what one expects is that, just as the 1-dimensional flatness results were axiomatized in the internal category theory of a topos, the 2-dimensional flatness results should appear internally in some kind of 2-topos. As was shown in Section 3.4, the 2-fibration concepts introduced in [19] and [7] that were considered here are not representable ones in **2Cat**. The representable concepts appear in [16, §I,2.9], where a 2-fibration is defined using a Chevalley condition via the cotensor with **2** of the base. It is asserted there without proof that such 2-fibrations correspond in a strong way with 2-category-valued 2-functors on the base. The point is that the representable discrete 2-fibration concept would be the discretization of Gray's notion and again given by an action of the ordinary cotensor. This would perhaps be the correct notion of discrete fibration to internalize in a 2-topos as in [40], which is already a reasonable well-established categorification of the idea of an ordinary elementary topos. The issue, however, is again that the fibration concepts considered here are not the representable ones and the action of the cotensor object giving the algebra structure is not that of the cotensor in **2Cat**, but rather **Lax**, a much less studied 3-dimensional structure. It is neither clear how *2-toposy* **Lax** really is nor whether the fibration concepts studied here are the end of the story.

5.2 Double-Categorical Generalizations Discrete double fibrations are category objects in discrete fibrations [29]. It was suggested in Remark 2.6 that discrete 2-fibrations are not really a special case. This might be surprising seeing as double categories [13] embrace both 2- and bicategories as special cases by trivializing one direction of the double structure. A discrete double fibration is equivalently a double functor $P\colon \mathbb{E} \to \mathbb{B}$ such that each component P_1 and P_0 is a discrete fibration. So, interpreting the functor $P_0\colon \mathbb{E}_0 \to \mathbb{B}_0$ as the ordinary category part of the double functor, a discrete 2-fibration cannot be a special case since its ordinary category part is an honest fibration and not necessarily discrete in the fibers. Another way to look at it is that discrete double fibrations correspond to lax functors valued in the double category of sets with spans whereas discrete 2-fibrations are valued in honest categories.

Even moving to *double fibrations* [8], (discrete) 2-fibrations, as formulated here, are not straightforwardly a special case. Rather what is shown is that a 2-functor $P\colon \mathfrak{E} \to \mathfrak{B}$ is a 2-fibration if, and only if, the associated quintets construction double functor $\mathbb{Q}(P)\colon \mathbb{Q}(\mathfrak{E}) \to \mathbb{Q}(\mathfrak{B})$ is a double fibration [8, Proposition 2.39]. It seems rather that (discrete) double fibrations and (discrete) 2-fibrations are nearly orthogonal generalizations of ordinary (discrete) fibrations and that there is a missing common notion embracing both. This appears to be related to the notion of a *double 2-category*, that is, a category internal to 2- or bicategories [8, §3.2]. The common generalization would be fibration of such things, that is, an internal functor each of whose components is a fibration of 2-categories. A structure of this kind with no 1-part would be a 2-fibration in the present sense and one whose 2-category components are ordinary categories would be a double fibration. Orthogonal actions of the cylinder construction, exploiting the intercategory structure [14], should recover monadicity results for each of these structures as special cases of one for the over-structure.

5.3 Fibration Concepts An interesting pattern is suggested by the developments of the paper. The main result, Theorem 3.7, and all those results summarized in the introduction, generally speaking, take the following form: correspondences between a structured class of *geometric data* – that is, (discrete) fibrations – on the one hand and a structured class of representations of some gadget in a base structure of which the gadget is either (A) of the same status (i.e. presheaves representing a category in the category of sets) or (B) a member (i.e. pseudo-functors representing a category as parameterized categories). That is, on the one hand, there is a representing structure – perhaps some kind of n-category – denoted here by $\mathcal{K}$ and a higher $(n+1)$-structure, denoted by $\mathbf{Cat}(\mathcal{K})$ of which $\mathcal{K}$ is a member (think **Set** and **Cat** or **Cat** and **2Cat**). There is a represented object of $\mathbf{Cat}(\mathcal{K})$ – some n-category – denoted by $\mathbb{B}$. There is a hom-object in $\mathbf{Cat}(\mathcal{K})$ of the same overall structure as $\mathcal{K}$ of representations of $\mathbb{B}$, denoted by $[\mathbb{B}, \mathcal{K}]$ and a higher class of representations $[\mathbb{B}, \mathbf{Cat}(\mathcal{K})]$ of the same overall status and structure as $\mathbf{Cat}(\mathcal{K})$. There is an inclusion of representations $[\mathbb{B}, \mathcal{K}] \to [\mathbb{B}, \mathbf{Cat}(\mathcal{K})]$ where those of the source are thought of as the discrete representations relative to those of the target since $\mathcal{K} \to \mathbf{Cat}(\mathcal{K})$ is the inclusion into the ambient $(n+1)$-structure of the $(n+1)$-discrete structures (i.e. precisely the members of $\mathcal{K}$). A category of elements construction then establishes correspondences with geometric structures on the other side of n and $(n+1)$-equivalences making the whole following situation commute:

$$\begin{array}{ccc} \mathbf{Fib}(\mathbb{B}) & \xrightarrow{\simeq} & [\mathbb{B}, \mathbf{Cat}(\mathcal{K})] \\ \uparrow & & \uparrow \\ \mathbf{DFib}(\mathbb{B}) & \xrightarrow[\simeq]{} & [\mathbb{B}, \mathcal{K}] \end{array}$$

The question, then, is given examples of at least one of the representation hom-structures in $[\mathbb{B},\mathcal{K}] \to [\mathbb{B},\mathbf{Cat}(\mathcal{K})]$, what is the (discrete) fibration concept on the other side of some such equivalence? That is, what is the corresponding lifting property of some structure-preserving n-functor into the represented structure $\mathbb{B}$? Insofar as there are functor-category structures other than those considered so far, this is a potentially fruitful area of inquiry. For there are plenty of 2- and 3-dimensional representing structures such as bicategories of profunctors or of relations; 2-categories with lax functors or lax natural transformations; double categories of sets or of profunctors; higher n-categories – all of which have well-established associated functor categories and thus potentially corresponding fibration concepts waiting to be discovered. In fact this outline may be interesting even in considering lower-level representations of classical and well-known algebraic gadgets such as groups, rings, modules and their higher-dimensional analogues such as 2-groups and 2-rigs.

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References

[1] M. Artin, A. Grothendieck, and J.L. Verdier. *Théorie des Topos et Cohomologie Etale des Schémas (SGA 4) Tome 1*, volume 269 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1972.

[2] Igor Baković. Fibrations of bicategories, 2011. Preprint available at `https://www2.irb.hr/korisnici/ibakovic/groth2fib.pdf`.

[3] Jean Bénabou. Introduction to bicategories. In *Reports of the Midwest Category Seminar I*, volume 47 of *Lecture Notes in Mathematics*, pages 1–77. Springer, 1967.

[4] Jean Bénabou. Fibered categories and the foundations of naive category theory. *The Journal of Symbolic Logic*, 50(1):10–37, 1985.

[5] G.J. Bird. *Limits in 2-Categories of Locally-Presentable Categories*. PhD thesis, University of Sydney, 1984.

[6] Francis Borceux. *Handbook of Categorical Algebra 1: Basic Category Theory*, volume 50 of *Encyclopedia of Mathematics and Its Applications*. Cambridge University Press, Cambridge, 1994.

[7] Mitchell Buckley. Fibred 2-categories and bicategories. *Journal of Pure and Applied Algebra*, 218(6):1034–1074, 2014.

[8] G.S.H. Cruttwell, M.J. Lambert, D.A. Pronk, and M. Szyld. Double fibrations. *Theory and Applications of Categories*, 38(35):1326–1394, 2022.

[9] M.E. Descotte, E. J. Dubuc, and M. Szyld. Sigma colimits in 2-categories and flat pseudofunctors. *Advances in Mathematics*, 333:266–313, 2018.

[10] Radu Diaconescu. *Change of Base for Some Toposes.* PhD thesis, Dalhousie University, 1973.

[11] Radu Diaconescu. Change of base for toposes with generators. *Journal of Pure and Applied Algebra*, 6(3):191–218, 1975.

[12] R. Gordon, A. J. Power, and Ross Street. *Coherence for tricategories*, volume 117. American Mathematical Society, United States, September 1995.

[13] Marco Grandis and Robert Paré. Limits in double categories. *Cahiers de Topologie et Géométrie Différentielle Catégoriques*, 40(3):162–220, 1999.

[14] Marco Grandis and Robert Paré. Intercategories. *Theory and Applications of Categories*, 30(38):1215–1255, 2015.

[15] John W. Gray. Fibred and cofibred categories. In *Proceedings of the Conference on Categorical Algebra, La Jolla 1965*, pages 21–83. Springer-Verlag, 1966.

[16] John W. Gray. *Formal Category Theory: Adjointness for 2-Categories*, volume 391 of *Lecture Notes in Mathematics.* Springer-Verlag, Berlin, 1974.

[17] Alexander Grothendieck. Categories fibrees et descente. In *Revêtements Étales et Groupe Fondamental: Séminaire de Géométrie Algébrique du Bois Marie 1960-61 (SGA 1)*, volume 224 of *Lecture Notes in Mathematics*, pages 145–194. Springer, 1971.

[18] K.A. Hardie, K.H. Kamps, and R.W. Kieboom. A homotopy 2-groupoid of a hausdorff space. *Applied Categorical Structures*, 8:209–234, 2000.

[19] Claudio Hermida. Some properties of fib as a fibred 2-category. *Journal of Pure and Applied Algebra*, 134(1):83–109, 1999.

[20] Bart Jacobs. *Categorical Logic and Type Theory*, volume 141 of *Studies in Logic and the Foundations of Mathematics.* Elsevier, 1999.

[21] P.T. Johnstone. Fibrations and partial products in a 2-category. *Applied Categorical Structures*, 1:141–179, 1993.

[22] P.T. Johnstone. *Sketches of an Elephant: A Topos Theory Compendium, Volume 1*, volume 43 of *Oxford Logic Guides.* Clarendon Press, London, 2001.

[23] P.T. Johnstone. *Topos Theory.* Dover, 2014.

[24] G.M. Kelly. *Basic Concepts of Enriched Category Theory*, volume 64 of *London Math. Soc. Lecture Notes Series.* Cambridge University Press, Cambridge, 1982.

[25] G.M. Kelly. Elementary observations on 2-categorical limits. *Bulletin of the Austrailian Mathematical Soceity*, 39:301–317, 1989.

[26] G.M. Kelly and Ross Street. Review of the elements of 2-categories. In G. M. Kelly, editor, *Category Seminar: Proceedings Sydney Category Theory Seminar 1972 /1973*, volume 420 of *Lecture Notes in Mathematics*, pages 75–103. Springer, 1974.

[27] Stephen Lack. A 2-categories companion. In John C. Baez and J. Peter May, editors, *Towards Higher Categories*, volume 152 of *The IMA Volumes in Mathematics and its Applications*, pages 105–191. Springer, 2010.

[28] Michael Lambert. *An Elementary Account of Flat 2-Functors*. PhD thesis, Dalhousie University, 2019.

[29] Michael Lambert. Discrete double fibrations. *Theory and Applications of Categories*, 37(22):671–708, 2021.

[30] Fosco Loregian and Emily Riehl. Categorical notions of fibration. *Expositiones Mathematicae*, 38:496–514, 2020.

[31] Saunders Mac Lane. *Category Theory for the Working Mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, Berlin, 2nd edition, 1998.

[32] Saunders Mac Lane and Ieke Moerdijk. *Sheaves in Geometry and Logic: A First Introduction to Topos Theory*. Springer-Verlag, Berlin, 1992.

[33] Ieke Moerdijk and Jan-Alve Svennson. Algebraic classification of equivariant homotopy 2-types, I. *Journal of Pure and Applied Algebra*, 89(1-2):187–216, 1993.

[34] Michael Shulman. Framed bicategories and monoidal fibrations. *Theory and Applications of Categories*, 20(18):650–673, 2008.

[35] Mike Shulman. The problem with lax functors. https://golem.ph.utexas.edu/category/2009/12/the_problem_with_lax_functors.html, 2009. Accessed: 2019-12-03.

[36] Ross Street. The formal theory of monads. *Journal of Pure and Applied Algebra*, 2(2):149–168, 1972.

[37] Ross Street. Fibrations and Yoneda's lemma in a 2-category. In G. M. Kelly, editor, *Category Seminar: Proceedings Sydney Category Theory Seminar 1972 /1973*, volume 420 of *Lecture Notes in Mathematics*, pages 104–133. Springer, 1974.

[38] Ross Street. Fibrations in bicategories. *Cahiers de Topologie et Géométrie Différentielle Catégoriques*, 21(2):111–160, 1980.

[39] Ross Street. Conspectus of variable categories. *Journal of Pure and Applied Algebra*, 21(3):307–338, 1981.

[40] Mark Weber. Yoneda structures from 2-toposes. *Applied Categorical Structures*, 15:259–323, 2007.