Higher Structures 8(2):248–331, 2024.

HIGHER STRUCTURES

Koszul duality for simplicial restricted Lie algebras

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Abstract

Let $\mathsf{s}_0\mathsf{Lie}^r$ be the category of 0-reduced simplicial restricted Lie algebras over a fixed perfect field of positive characteristic p. We prove that there is a full subcategory $\mathrm{Ho}(\mathsf{s}_0\mathsf{Lie}^r_\xi)$ of the homotopy category $\mathrm{Ho}(\mathsf{s}_0\mathsf{Lie}^r)$ and an equivalence $\mathrm{Ho}(\mathsf{s}_0\mathsf{Lie}^r_\xi) \simeq \mathrm{Ho}(\mathsf{s}_1\mathsf{CoAlg}^{tr})$. Here $\mathsf{s}_1\mathsf{CoAlg}^{tr}$ is the category of 1-reduced simplicial truncated coalgebras; informally, a coaugmented cocommutative coalgebra C is truncated if $x^p = 0$ for any x from the augmentation ideal of the dual algebra C^*. Moreover, we provide a sufficient and necessary condition in terms of the homotopy groups $\pi_*(L_\bullet)$ for $L_\bullet \in \mathrm{Ho}(\mathsf{s}_0\mathsf{Lie}^r)$ to lie in the full subcategory $\mathrm{Ho}(\mathsf{s}_0\mathsf{Lie}^r_\xi)$.

As an application of the equivalence above, we construct and examine an analog of the unstable Adams spectral sequence of A. K. Bousfield and D. Kan in the category sLie^r. We use this spectral sequence to recompute the homotopy groups of a free simplicial restricted Lie algebra.

Communicated by: Muriel Livernet.
Received: 21st February, 2023. Accepted: August 27th, 2024.
MSC: 18M70; 17B56; 16T15; 18N50; 55S10.
Keywords: Koszul duality, restricted Lie algebras, truncated coalgebras, Λ-algebra.

1. Introduction

In [60, Theorem I], D. Quillen proved that there exists an equivalence of homotopy categories:

$$\mathrm{Ho}(\mathsf{DGL}_0) \simeq \mathrm{Ho}(\mathsf{DGC}_1), \tag{1.1}$$

where DGL_0 is the category of 0-connected differential graded Lie algebras over the rationals $\mathbf{Q}$, DGC_1 is the category of 1-connected differential graded cocommutative coalgebras over $\mathbf{Q}$, and the homotopy categories are taken with respect to quasi-isomorphisms.

Later, the equivalence (1.1) was generalized in a wide range of new contexts; an interested reader might check the following long but not exhaustive list of references: [51], [25], [22], [5],

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DOI: 10.21136/HS.2024.12

and [14]. We refer to phenomena like (1.1) as *(derived) Koszul duality.* We notice that each of these contexts of Koszul duality from the list above can be formally regarded as relating algebras over an operad with divided power coalgebras over a cooperad. The purpose of this paper is to study Koszul duality for restricted Lie algebras, which can be regarded as a divided power algebra over an operad, and therefore does not fit into the previously studied contexts.

Quillen's work provides a *Lie model* for the homotopy category $\mathrm{Ho}(\mathcal{S}_{\mathbf{Q}})$ of rational simply-connected topological spaces. The existence of a Lie model for the category $\mathrm{Ho}(\mathcal{S}^{\wedge}_p)$ of *p-adic homotopy types* is a more difficult problem and it still remains unresolved, see e.g. [42]. However, A. K. Bousfield and E. Curtis [9] showed that there is a certain relation between $\mathrm{Ho}(\mathcal{S}^{\wedge}_p)$ and the (homotopy) category $\mathrm{Ho}(\mathsf{sLie}^r)$ of simplicial restricted Lie algebras; in particular, they constructed the unstable Adams spectral sequence whose E_1-term is the homotopy groups of a free simplicial restricted Lie algebra and which converges to the homotopy groups of a space. Moreover, they showed that the E_1-term can be expressed in terms of an explicit differential graded algebra; namely, in terms of the lambda algebra Λ. The Koszul duality described in this paper will explain and conceptualize some of their calculations. Furthermore, the results of this paper are used in [39] to study the interaction of the unstable Adams spectral sequence and the Goodwillie tower.

1.1 Results. Throughout this paper, p is a fixed prime number and $\mathbf{F}$ is a fixed perfect field of characteristic p. Recall from [37, Definition V.4] that a *restricted Lie algebra* (L,ξ) over $\mathbf{F}$ is a Lie algebra L equipped with a (non-additive, in general) *p-operation* $\xi\colon L\to L$ (Definition 2.1.1). We write Lie^r for the category of restricted Lie algebras (over $\mathbf{F}$) and we denote by sLie^r the category of *simplicial objects* in Lie^r; i.e. sLie^r is the category of contravariant functors from the simplex category Δ to Lie^r. The category sLie^r will be the main object of this paper.

In [34] G. Hochschild defined the cohomology groups $H^*(L;\mathbf{F})$ for a restricted Lie algebra $L\in\mathsf{Lie}^r$, and later, his definition was extended to simplicial restricted Lie algebras by S. Priddy in [58]. More precisely, S. Priddy constructed a functor

$$\overline{W}U^r\colon \mathsf{sLie}^r\to \mathsf{sCoAlg}^{aug} \tag{1.1.1}$$

such that

$$H^*(L;\mathbf{F})\cong \mathrm{Hom}(\pi_*(\overline{W}U^r(L)),\mathbf{F}),\ L\in\mathsf{Lie}^r.$$

Here CoAlg^{aug} is the category of coaugmented cocommutative coalgebras over $\mathbf{F}$ and sCoAlg^{aug} is the category of simplicial objects in CoAlg^{aug}.

Let $C=(C,\eta\colon\mathbf{F}\to C)\in\mathsf{CoAlg}^{aug}$ be a *finite-dimensional* coaugmented cocommutative coalgebra and let $C^*=(C^*,\eta^*\colon C^*\to\mathbf{F})$ be its dual augmented algebra. We say that C is *truncated* if, for every $x\in\ker(\eta^*)$, we have $x^p=0$. An infinite-dimensional coalgebra $C\in\mathsf{CoAlg}^{aug}$ is called truncated if C is a union of finite-dimensional truncated sub-coalgebras. We write CoAlg^{tr} for the full subcategory of CoAlg^{aug} spanned by truncated ones. (In the main text, we will use a different but equivalent Definition 2.2.5 for truncated coalgebras.)

By [58, Proposition 5.10] and [45, Lemma 8.4], the essential image of the functor $\overline{W}U^r$ is contained in the full subcategory $\mathsf{sCoAlg}^{tr}\subset\mathsf{sCoAlg}^{aug}$ of simplicial truncated coalgebras. Moreover, for every $L_\bullet\in\mathsf{sLie}^r$, the simplicial coalgebra $\overline{W}U^r(L_\bullet)$ is *reduced*; i.e. the coalgebra $\overline{W}U^r(L_\bullet)_0$ of 0-simplices is isomorphic to $\mathbf{F}$.

We write $\mathsf{s}_0\mathsf{CoAlg}^{tr}$ for the category of reduced simplicial truncated coalgebras and we say that a map $f\colon C_\bullet\to D_\bullet$ in $\mathsf{s}_0\mathsf{CoAlg}^{tr}$ is a *weak equivalence* if f is a weak equivalence of underlying simplicial vector spaces, i.e. the induced map $f_*\colon\pi_*(C_\bullet)\to\pi_*(D_\bullet)$ is an isomorphism.

Theorem A (Theorem 3.1.32)**.** *The functor* $\overline{W}U^r\colon \mathsf{sLie}^r \to \mathsf{s_0CoAlg}^{tr}$ *has a left adjoint*

$$PG\colon \mathsf{s_0CoAlg}^{tr} \to \mathsf{sLie}^r$$

such that the unit map

$$\eta\colon C_\bullet \to \overline{W}U^r \circ PG(C_\bullet) \tag{1.1.2}$$

is a weak equivalence for any reduced simplicial truncated coalgebra $C_\bullet \in \mathsf{s_0CoAlg}^{tr}$*.*

Our proof of Theorem A has two main ingredients. The first one is the classical observation made in [49] that the category Hopf^{pr} of primitively generated Hopf algebras is simultaneously equivalent to Lie^r and to the category $\mathsf{Grp}(\mathsf{CoAlg}^{tr})$ of *group objects* in CoAlg^{tr}. The second ingredient was inspired by [66]. Namely, we adapt to our setting Stevenson's proof of the Kan theorem [38, Theorem 7.1] (see also [29, Corollary V.6.4]), which states that the homotopy category $\mathrm{Ho}(\mathsf{s_0Set})$ of reduced simplicial sets is equivalent to the homotopy category $\mathrm{Ho}(\mathsf{sGrp})$ of simplicial groups.

Similarly, we say that $f\colon L'_\bullet \to L_\bullet \in \mathsf{sLie}^r$ is a weak equivalence if $f_*\colon \pi_*(L'_\bullet) \to \pi_*(L_\bullet)$ is an isomorphism. We notice, however, that the dual of Theorem A is not fulfilled. Namely, according to Example 4.2.22, there is a simplicial restricted Lie algebra $L_\bullet \in \mathsf{sLie}^r$ such that the counit map

$$PG \circ \overline{W}U^r(L_\bullet) \to L_\bullet$$

is not a weak equivalence in sLie^r. Therefore we introduce the notion of an $\mathbf{F}$-*equivalence*: a map $f\colon L'_\bullet \to L_\bullet$ in sLie^r is an $\mathbf{F}$-equivalence if and only if $\overline{W}U^r(f)$ is a weak equivalence in $\mathsf{s_0CoAlg}^{tr}$ (Definition 3.2.21). By Corollary 3.2.15, any weak equivalence in sLie^r is an $\mathbf{F}$-equivalence.

Theorem B. *Let* $\mathcal{W}_{\mathsf{sLie}^r}$ *(resp.* $\mathcal{W}_{\mathbf{F}}$*) be the class of weak equivalences (resp.* $\mathbf{F}$*-equivalences) in* sLie^r*. Then there are model structures* $(\mathsf{sLie}^r, \mathcal{W}_{\mathsf{sLie}^r}, \mathcal{C}, \mathcal{F})$ *and* $(\mathsf{sLie}^r, \mathcal{W}_{\mathbf{F}}, \mathcal{C}_{\mathbf{F}}, \mathcal{F}_{\mathbf{F}})$ *on the category* sLie^r *such that*

1. $f \in \mathcal{F}$ *if and only if* f *is a fibration in* $\mathsf{sVect}_{\mathbf{F}}$ *(Remark 3.2.2);*
2. *the classes of cofibrations coincide,* $\mathcal{C}_{\mathbf{F}} = \mathcal{C}$*;*
3. *there is an inclusion* $\mathcal{F}_{\mathbf{F}} \subset \mathcal{F}$*;*
4. *both model structures are simplicial and combinatorial;*
5. *the model structure* $(\mathcal{W}_{\mathsf{sLie}^r}, \mathcal{C}, \mathcal{F})$ *is right proper and* $(\mathcal{W}_{\mathbf{F}}, \mathcal{C}_{\mathbf{F}}, \mathcal{F}_{\mathbf{F}})$ *is left proper.*

Theorem B is a combination of Theorems 3.2.3 and 3.2.24 from the main text. We will abuse notation and denote by sLie^r (resp. by sLie^r_ξ) the model category $(\mathsf{sLie}^r, \mathcal{W}_{\mathsf{sLie}^r}, \mathcal{C}, \mathcal{F})$ (resp. $(\mathsf{sLie}^r, \mathcal{W}_{\mathbf{F}}, \mathcal{C}_{\mathbf{F}}, \mathcal{F}_{\mathbf{F}})$) from Theorem B. We notice that the model category sLie^r_ξ is a (left) Bousfield localization of sLie^r ([33, Definition 3.3.1]), and so the homotopy category $\mathrm{Ho}(\mathsf{sLie}^r_\xi)$ is a full subcategory of $\mathrm{Ho}(\mathsf{sLie}^r)$. It follows from Theorem A that functors $\overline{W}U^r$ and PG induce an equivalence of homotopy categories:

$$\mathrm{Ho}(\mathsf{s_0CoAlg}^{tr}) \simeq \mathrm{Ho}(\mathsf{sLie}^r_\xi). \tag{1.1.3}$$

Moreover, there is a simplicial combinatorial model structure on $\mathsf{s_0CoAlg}^{tr}$ and the equivalence (1.1.3) can be enhanced to an equivalence between the underlying ∞-categories (see [40, Section A.2]) of simplicial model categories $\mathsf{s_0CoAlg}^{tr}$ and sLie^r_ξ. The next theorem follows from Propositions 3.2.29, 3.2.32, and Theorem 3.2.26.

Theorem C. *Let us denote by $s\mathcal{L}$ (resp. $s\mathcal{L}_\xi$, $s\mathcal{CA}_0$) the underlying ∞-category of the simplicial model category* sLie^r *(resp.* sLie^r_ξ, $\mathsf{s_0CoAlg}^{tr}$*). Then there is an equivalence of presentable ∞-categories:*

$$s\mathcal{CA}_0 \simeq s\mathcal{L}_\xi$$

and $s\mathcal{L}_\xi \subset s\mathcal{L}$ is a localization of the presentable ∞-category $s\mathcal{L}$.

Our next goal is to describe the full subcategory $s\mathcal{L}_\xi \subset s\mathcal{L}$ together with the $\mathbf{F}$-*completion* functor

$$L_\xi \colon s\mathcal{L} \to s\mathcal{L}_\xi \hookrightarrow s\mathcal{L},$$

see Definition 5.3.1. Since any object in sLie^r is fibrant, this problem is equivalent to identifying fibrant objects and fibrant replacements in the model category sLie^r_ξ. It seems to be difficult in general, so we restrict ourselves to the case of *connected* simplicial restricted Lie algebras. Here we say that a simplicial restricted Lie algebra $L_\bullet \in \mathsf{sLie}^r$ is connected if $\pi_0(L_\bullet) = 0$.

Let $L_\bullet \in \mathsf{sLie}^r$ be a simplicial restricted Lie algebra. The p-operation $\xi \colon L_\bullet \to L_\bullet$ is a map of simplicial sets, and so it induces a map of homotopy groups:

$$\xi_* \colon \pi_n(L_\bullet) \to \pi_n(L_\bullet),\ n \geq 0$$

which is additive for $n \geq 1$. Moreover, since the p-operation ξ is semi-linear, the map ξ_* is semi-linear as well, i.e. $\xi_*(ax) = a^p\xi_*(x)$, $a \in \mathbf{F}$, $x \in \pi_*(L_\bullet)$. In this way, all homotopy groups $\pi_n(L_\bullet), n \geq 1$ are naturally left modules over the ring of twisted polynomials $\mathbf{F}\{\xi\}$, see Definition 2.1.3.

The ring $\mathbf{F}\{\xi\}$ is non-commutative (if $\mathbf{F} \neq \mathbf{F}_p$), however it still shares a lot of common properties with the polynomial ring $\mathbf{F}[t]$, see Section 5.1. In particular, one can still define the *ξ-adic completion*

$$\widehat{M} = \lim_r M/\xi^r(M)$$

of a left $\mathbf{F}\{\xi\}$-module $M \in \mathsf{Mod}_{\mathbf{F}\{\xi\}}$, see Definition 5.1.8. The ξ-adic completion is not an exact functor, and so we introduce in Section 5.2 its *left derived functors* L_0 and L_1. The functor L_0 is equipped with a natural transformation

$$\phi_M \colon M \to L_0(M),\ M \in \mathsf{Mod}_{\mathbf{F}\{\xi\}}; \tag{1.1.4}$$

and we say that a left $\mathbf{F}\{\xi\}$-module M is *derived ξ-adic complete* if $L_1(M) = 0$ and the map ϕ_M is an isomorphism.

We say that $L_\bullet \in \mathsf{sLie}^r$ is $\mathbf{F}$-*complete* if $L_\bullet$ is a fibrant object of sLie^r_ξ, see Section 5.3.

Theorem D (Corollary 5.3.12)**.** *Let $L_\bullet \in \mathsf{sLie}^r$ be a connected simplicial restricted Lie algebra, $\pi_0(L_\bullet) = 0$. Then $L_\bullet$ is $\mathbf{F}$-complete if and only if all homotopy groups $\pi_n(L_\bullet), n \geq 1$ are derived ξ-adic complete left modules over the ring $\mathbf{F}\{\xi\}$.*

Let $s\mathcal{CA}_1$ be the ∞-category of 1-connected simplicial truncated coalgebras, i.e. $C_\bullet \in s\mathcal{CA}_1$ if and only if $\pi_0(C_\bullet) \cong \mathbf{F}$ and $\pi_1(C_\bullet) = 0$. Combining Theorem C with Theorem D yields the following corollary.

Corollary. *There is an equivalence of ∞-categories*

$$s\mathcal{CA}_1 \simeq s\mathcal{L}_{\xi,0},$$

where $s\mathcal{L}_{\xi,0}$ is the full subcategory of $s\mathcal{L}$ spanned by connected simplicial restricted Lie algebras whose homotopy groups are derived ξ-complete.

We note that it seems plausible to derive the last corollary from the results of [13] on the deformation theory of simplicial commutative **F**-algebras. The comparison between this paper and [13] will be presented elsewhere.

1.2 Applications. At the end of the paper, we provide several applications of Theorem C. Following [58], we define the reduced *cohomology groups* $\widetilde{H}^*(L_\bullet;\mathbf{F})$ of a simplicial restricted Lie algebra $L_\bullet \in \mathsf{sLie}^r$ by the formula

$$\widetilde{H}^q(L_\bullet;\mathbf{F}) = \mathrm{Hom}(\pi_q(\overline{W}U^r(L_\bullet);\mathbf{F}),\ q\geq 1 \ \text{ and } \ \widetilde{H}^0(L_\bullet;\mathbf{F})=0.$$

By the Eilenberg-Zilber theorem, we observe that $\widetilde{H}^*(L_\bullet;\mathbf{F})$ is a non-unital graded commutative algebra over $\mathbf{F}$. Moreover, by [58, Proposition 5.3] and [45, Theorem 8.5], there is an action of the Steenrod operations $\beta^\varepsilon P^a, a\geq 0, \varepsilon=0,1$ (resp. $Sq^a, a\geq 0$ if $p=2$) on the cohomology groups $\widetilde{H}^*(L_\bullet;\mathbf{F})$ such that P^0 (resp. Sq^0) acts by zero. These Steenrod operations are semi-linear and still satisfy the Adem relations, see Section 6.1. So, the cohomology groups $\widetilde{H}^*(L_\bullet;\mathbf{F})$ is also a left module (as an $\mathbf{F}_p$-vector space) over the *homogenized mod-p Steenrod algebra* $\mathcal{A}^h_p$, see Definition 6.1.5. We recall that the classical mod-p Steenrod algebra $\mathcal{A}_p$ has $P^0=1$ (resp. $Sq^0=1$), and so Adem relations in $\mathcal{A}_p$ have both quadratic and linear parts, while Adem relations in $\mathcal{A}^h_p$ have only the quadratic part (Remark 6.1.6).

We say that a positively graded (non-unital) commutative $\mathbf{F}$-algebra $A_*=\oplus_{q>0}A_q$ is an *unstable $\mathcal{A}^h_p$-algebra* (Definition 6.1.8) if A_* is equipped with a semi-linear action of the homogenized mod-p Steenrod algebra $\mathcal{A}^h_p$ such that the following non-stability relations are satisfied:

1. $\beta^\varepsilon P^a(x)=0$ if $2a+\varepsilon>|x|$ (resp. $Sq^a(x)=0$ if $a>|x|$ and $p=2$);
2. $P^a(x)=x^p$ if $2a=|x|$ (resp. $Sq^a(x)=x^2$ if $a=|x|$).

We observe that a cohomology ring $\widetilde{H}^*(L_\bullet;\mathbf{F}), L_\bullet\in\mathsf{sLie}^r$ is an unstable $\mathcal{A}^h_p$-algebra (Example 6.1.10). We use Theorem C together with [55, Proposition 6.2.1] in order to construct an analog of the Bousfield-Kan spectral sequence [12] in the setting of the category sLie^r.

Theorem E (Corollary 6.2.7)**.** *Let $L_\bullet$ be an $\mathbf{F}$-complete simplicial restricted Lie algebra such that its cohomology groups $\widetilde{H}^*(L_\bullet;\mathbf{F})$ are degreewise finite-dimensional. Then there is a completely convergent spectral sequence*

$$E^2_{s,t}=\mathrm{Ext}^s_{\mathcal{U}^h}(\widetilde{H}^*(L_\bullet;\mathbf{F}),\Sigma^{t+1}\mathbf{F})\Rightarrow \pi_{t-s}(L_\bullet),\quad d^r\colon E^r_{s,t}\to E^r_{s+r,t+r-1}.$$

Here $\mathrm{Ext}^*_{\mathcal{U}^h}$ are non-abelian Ext-groups in the category of unstable $\mathcal{A}^h_p$-algebras, see Definition 6.2.1.

Finally, we use Theorem E to tie together two classical computations. Let $L^r(V_\bullet)\in\mathsf{sLie}^r$ be a free simplicial restricted Lie algebra (Example 2.1.7) generated by a simplicial vector space $V_\bullet\in\mathsf{sVect}_\mathbf{F}$. The homotopy groups

$$\pi_*(L^r(V_\bullet))$$

were computed in [9, Theorem 8.5] and [69, Proposition 13.2] in terms of the algebra Λ of [10] and $\pi_*(V_\bullet)$. At the same time, by [56, Section 7], the algebra Λ is anti-isomorphic to the Koszul dual algebra $\mathcal{K}^*_p$ of $\mathcal{A}^h_p$, see Section 6.3. In Corollaries 6.4.9 and 6.4.15, we use the Curtis theorem [15], Theorem E, and the paper [56] to redo the computations of A. K. Bousfield, E. Curtis, and R. Wellington. Our approach is not easier than theirs, but perhaps, it is more fundamental and flexible for plausible generalizations. In particular, we derive the following theorem from Corollaries 6.4.9 and 6.4.15.

Theorem F. *Let $V_\bullet \in \mathsf{sVect}_{\mathbf{F}}$ be a simplicial vector space such that $\pi_*(V_\bullet)$ is one-dimensional. Then the spectral sequence of Theorem E*

$$E^2_{s,t} = \mathrm{Ext}^s_{\mathcal{U}^h}(\widetilde{H}^*(L^r(V_\bullet); \mathbf{F}), \Sigma^{t+1}\mathbf{F}) \Rightarrow \pi_{t-s}(L_\xi L^r(V_\bullet)) \tag{1.2.1}$$

degenerates at the second page. Here $L_\xi L^r(V_\bullet)$ is the $\mathbf{F}$-completion of the free simplicial restricted Lie algebra $L^r(V_\bullet)$.

The assumption that $\pi_*(L^r(V_\bullet))$ is one-dimensional is essential here; if it is not fulfilled, then it seems likely that this spectral sequence is highly non-trivial, see Remark 6.4.16.

1.3 Organization. In Section 2 we recall crucial facts about the categories Lie^r and CoAlg^{tr} needed to prove Theorem A. First, in Section 2.1 we recall that there is an equivalence

$$P : \mathsf{Hopf}^{pr} \rightleftarrows \mathsf{Lie}^r : U^r.$$

between the category of restricted Lie algebras Lie^r and the category Hopf^{pr} of primitively generated Hopf algebras. Here U^r is the universal enveloping algebra functor and P is the functor of primitive elements. After that, in Section 2.2 we define truncated coalgebras and prove several basic facts about them, and in Section 2.3 we show that the category Hopf^{pr} is also equivalent to the category $\mathsf{Grp}(\mathsf{CoAlg}^{tr})$ of *group objects* in CoAlg^{tr} (Corollary 2.3.6). Finally, in Proposition 2.3.8, we prove that the forgetful functor from Hopf^{pr} to CoAlg^{tr} has a left adjoint H and the algebra $H(C), C \in \mathsf{CoAlg}^{tr}$ is free associative if we forget about the comultiplication in $H(C)$.

In Section 3.1 we adapt the argument from [66] to our context and prove Theorem A as Theorem 3.1.32.

In Section 3.2 we construct simplicial combinatorial model structures on the categories sLie^r and $\mathsf{s}_0\mathsf{CoAlg}^{tr}$, see Theorems 3.2.3 and 3.2.10 respectively. We also introduce the notion of an $\mathbf{F}$-equivalence (Definition 3.2.21) and construct the model category sLie^r_ξ in Theorem 3.2.24. These results imply Theorem B. At the end of Section 3.2, we prove Propositions 3.2.29, 3.2.32, and Theorem 3.2.26, which together imply Theorem C.

In Section 4 we provide technical tools needed for the proof of Theorem D. In Section 4.4 we introduce the notion of *principal fibrations* in the category sLie^r (Definition 4.4.5). Then we show that a connected simplicial restricted Lie algebra has a *Postnikov tower* (Corollary 4.3.6) and each stage in this tower is weakly equivalent to a principal fibration (Corollary 4.4.13). Finally, in Section 4.5 we construct an analog of the *Serre spectral sequence* for principal fibrations (Corollary 4.5.12) in the category sLie^r.

In Section 5.1 we prove a few basic facts about the ring of twisted polynomials $\mathbf{F}\{\xi\}$ and we define the ξ-adic completion in Definition 5.1.8. In Section 5.2 we define the left derived functors for the ξ-adic completion. Finally, we prove Theorem D in Section 5.3 as Corollary 5.3.12. Our proof is based on the classical proof (given e.g. in [43, Theorem 11.1.1]) that a simply-connected space X is p-complete if and only if its homotopy groups $\pi_n(X), n \geq 2$ are derived p-complete.

In Section 6 we illustrate possible applications of the previous results. In Section 6.1 we recall properties of the Steenrod operations and we show that the cohomology ring $\widetilde{H}^*(L_\bullet; \mathbf{F}), L_\bullet \in \mathsf{sLie}^r$ is a left $\mathcal{A}^h_p$-module over the homogenized mod-p Steenrod algebra $\mathcal{A}^h_p$, see Example 6.1.10. Moreover, we introduce the category $\mathcal{U}^h$ of unstable $\mathcal{A}^h_p$-algebra (Definition 6.1.8), the category $\mathcal{M}^h$ of *unstable $\mathcal{A}^h_p$-modules* (Definition 6.1.13), and the category $\mathcal{M}^h_0$ of *strongly unstable $\mathcal{A}^h_p$-modules* (Definition 6.1.14). Both categories $\mathcal{M}^h$ and $\mathcal{M}^h_0$ are abelian and closely related to $\mathcal{U}^h$, see Remark 6.1.15. In Section 6.2 we prove Theorem E as Corollary 6.2.7.

In Section 6.3 we recall the definition of the lambda algebra Λ of [10]. (We point out that in this work we use the convention for Λ from [69, Definition 7.1] but not from the original paper.) Then, we compute unstable abelian Ext-groups $\mathrm{Ext}^s_{\mathcal{M}^h}(W, \Sigma^t\mathbf{F})$ and $\mathrm{Ext}^s_{\mathcal{M}^h_0}(W, \Sigma^t\mathbf{F})$ in terms of the algebra Λ for a trivial $\mathcal{A}^h_p$-module $W \in \mathsf{Vect}^{gr}_{\mathbf{F}}$ (Corollary 6.3.17).

In Section 6.4 we apply the spectral sequence of Theorem E to a free simplicial restricted Lie algebra $L_\bullet = L^r(V_\bullet)$ generated by a simplicial vector space $V_\bullet \in \mathsf{sVect}_{\mathbf{F}}$. Finally, we derive Theorem F from the Curtis theorem [15] and previous computations.

1.4 Notation. Here we describe some notation that will be used throughout the paper. As was said, p is a fixed prime number and $\mathbf{F}$ is a fixed perfect field of characteristic p. We denote by $\mathsf{Vect}_{\mathbf{F}}$ the category of vector spaces over $\mathbf{F}$; $\mathsf{Vect}^{gr}_{\mathbf{F}}$ (resp. $\mathsf{Vect}^{>0}_{\mathbf{F}}$) is the category of non-negatively (resp. positively) graded vector spaces over $\mathbf{F}$. We usually denote by $V_* = \oplus_{q\geq 0} V_q$ an object of $\mathsf{Vect}^{gr}_{\mathbf{F}}$.

Throughout most of the paper, except Section 4.5, we write ΣV_* for the shift of $V_* \in \mathsf{Vect}^{gr}_{\mathbf{F}}$, i.e. $(\Sigma V_*)_q = V_{q-1}$, $q \geq 0$. Moreover, $\Sigma^t V_* = \Sigma(\Sigma^{t-1}V_*)$, $t \geq 0$; we extend this notation to all integers in an usual way. In Section 4.5, for better readability of formulas, we denote the shift $\Sigma^t V_*$ by $V_*[t]$.

We denote by CoAlg the category of cocommutative coalgebras over $\mathbf{F}$ and we denote by Alg the category of associative algebras over $\mathbf{F}$. If it is not said otherwise, all algebras are unital and all coalgebras are counital. We denote by CoAlg^{aug} the category of coaugmented cocommutative coalgebras; an object of CoAlg^{aug} is a pair $(C, \eta\colon \mathbf{F} \to C)$, where $C \in \mathsf{CoAlg}$ and η is a map of coalgebras.

In this paper, all Hopf algebras are cocommutative, but not necessary commutative; we denote by Hopf the category of (cocommutative) Hopf algebras over $\mathbf{F}$. We notice that Hopf is equivalent to the category $\mathsf{Grp}(\mathsf{CoAlg})$ of group objects in CoAlg, since the direct product of cocommutative coalgebras C and D in the category CoAlg is the tensor product $C \otimes_{\mathbf{F}} D$, see e.g. [67, Theorem 6.4.5].

We define Lie algebras so that they satisfy the alternating condition $[x, x] = 0$, which always implies the antisymmetry $[x, y] = -[y, x]$, but is equivalent to it only over a field of characteristic $p \neq 2$. We write Lie for the category of Lie algebras over the field $\mathbf{F}$.

We denote by Δ the simplex category of finite nonempty linearly ordered sets. We denote by $[n], n \geq 0$ the object of Δ with $(n+1)$ elements. Let C be a category, then sC (resp. cC) is the category of simplicial (resp. cosimplicial) objects in C, i.e. sC (resp. cC) is the category of contravariant (resp. covariant) functors from Δ to C:

$$\mathsf{sC} = \mathrm{Fun}(\Delta^{op}, \mathsf{C}), \text{ and } \mathsf{cC} = \mathrm{Fun}(\Delta, \mathsf{C}).$$

We usually denote by $X_\bullet$ (resp. $X^\bullet$) an object of sC (resp. cC), where $X_n = X_\bullet([n])$ (resp. $X^n = X^\bullet([n])$) for $n \geq 0$. If the category C has a terminal object $* \in C$, then $\mathsf{s}_0\mathsf{C} \subset \mathsf{sC}$ is the full subcategory of reduced simplicial objects, $X_\bullet \in \mathsf{s}_0\mathsf{C}$ if and only if $X_0 \cong *$. This notation can be nested, e.g. csC is the category of cosimplicial simplicial objects in C.

Moreover, if the category C is complete and cocomplete, then the category sC is enriched, tensored, and cotensored over sSet. Therefore there is a canonical notion of homotopy between maps in sC, see [62, Definition 4, Section II.1]. In this case, we define (strong) deformation retracts in sC in a usual way. If sC is a simplicial model category, then deformation retracts are weak equivalences.

Let us denote by $\mathsf{Ch}_{\geq 0}$ (resp. $\mathsf{Ch}^{\geq 0}$) the category of connective chain (resp. cochain) complexes over $\mathbf{F}$. An object of $\mathsf{Ch}_{\geq 0}$ (resp. $\mathsf{Ch}^{\geq 0}$) is a pair $C_\bullet = (C_*, d)$ (resp. $C^\bullet = (C^*, d)$), where $C_\bullet \in \mathsf{Vect}^{gr}_{\mathbf{F}}$ and d is a differential

$$d\colon C_{n+1} \to C_n \ \ (\text{resp. } d\colon C^n \to C^{n+1}),\ n \geq 0$$

such that $d^2 = 0$. We denote by $H_*(C_\bullet)$ (resp. $H^*(C^\bullet)$) the homology (resp. cohomology) groups of $C_\bullet$ (resp. $C^\bullet$).

Let $V_\bullet \in \mathsf{sVect}_{\mathbf{F}}$ be a simplicial vector space. We denote by $\pi_*(V_\bullet)$ the homotopy groups of the underlying simplicial set. The normalized chain complex functor

$$N\colon \mathsf{sVect}_{\mathbf{F}} \to \mathsf{Ch}_{\geq 0}$$

is given by

$$(NV_\bullet)_n = \bigcap_{i=1}^{n} \ker(d_i),\ \ d = d_0\colon (NV_\bullet)_n \to (NV_\bullet)_{n-1}.$$

By the Dold-Kan correspondence, N is an equivalence and we denote by $\Gamma\colon \mathsf{Ch}_{\geq 0} \to \mathsf{sVect}_{\mathbf{F}}$ its inverse. Moreover, we recall that there is an isomorphism

$$\pi_*(V_\bullet) \cong H_*(NV_\bullet),\ V_\bullet \in \mathsf{sVect}_{\mathbf{F}}.$$

We write $F \dashv G$ or

$$F : \mathsf{C} \rightleftarrows \mathsf{D} : G$$

if the functor F is the left adjoint to the functor G. We write oblv (abbrv. to obliviate) for various forgetful functors. We note that an underlying vector space is not always a result of applying oblv. For example, suppose $(C, \eta) \in \mathsf{CoAlg}^{aug}$ is a coaugmented coalgebra, then the underlying vector space of (C, η) is C, but $\mathsf{oblv}(C, \eta)$ is $\operatorname{coker}(\eta)$.

2. Algebraic background

In this section we provide algebraic background for restricted Lie algebras, primitively generated Hopf algebras, and truncated coalgebras. At the end of Section 2.1, we recall that the category Lie^r of restricted Lie algebras is equivalent to the category Hopf^{pr} of primitively generated Hopf algebras. In Section 2.2, for a cocommutative coalgebra C, we define the *Verschiebung* operator $V\colon C \to C$ (Definition 2.2.4); C is called *truncated* if V is trivial (Definition 2.2.5). The main results of Section 2.2 are Propositions 2.2.14 and 2.2.15, where we describe a cofree truncated coalgebra $\mathrm{Sym}^{tr}(W)$ and show that the category CoAlg^{tr} of truncated coalgebras is locally presentable. In Section 2.3 we derive from [49, Proposition 4.20] that the category Hopf^{pr} is equivalent to the category $\mathsf{Grp}(\mathsf{CoAlg}^{tr})$ of group objects in CoAlg^{tr} (Proposition 2.3.3). Finally, in Proposition 2.3.8, we describe a *free Hopf algebra* $H(C)$ generated by $C \in \mathsf{CoAlg}^{tr}$.

2.1 Restricted Lie algebras. Let L be a Lie algebra and let $x \in L$. We denote by $\mathrm{ad}(x)\colon L \to L$ the map given by $y \mapsto \mathrm{ad}(x)(y) = [y, x]$.

Definition 2.1.1. Let L be a Lie algebra over $\mathbf{F}$. A *p-operation* on L is a map $\xi\colon L \to L$ such that

- $\xi(ax) = a^p \xi(x)$, $a \in \mathbf{F}, x \in L$;

- $\mathrm{ad}(\xi(x)) = \mathrm{ad}(x)^{\circ p} \colon L \to L$;
- $\xi(x+y) = \xi(x) + \xi(y) + \sum_{i=1}^{p-1} \dfrac{s_i(x,y)}{i}$, for all $x, y \in L$, where $s_i(x,y)$ is the coefficient of t^{i-1} in the formal expression $\mathrm{ad}(tx+y)^{\circ(p-1)}(x)$.

Definition 2.1.2 ([37, Definition V.4])**.** A *restricted Lie algebra* (L, ξ) is a Lie algebra L (over $\mathbf{F}$) equipped with a p-operation $\xi \colon L \to L$. A linear map

$$f \colon (L', \xi_{L'}) \to (L, \xi_L)$$

is a homomorphism of restricted Lie algebras if

$$[f(x), f(y)] = f([x,y]),\ x, y \in L',$$

and $\xi_L(f(x)) = f(\xi_{L'}(x))$, $x \in L'$. We will denote by Lie^r the category of restricted Lie algebras.

Recall that a Lie algebra L is called *abelian* if L is equipped with zero bracket. First, we will describe abelian restricted Lie algebras.

Definition 2.1.3. The *twisted polynomial ring* $\mathbf{F}\{\xi\}$ is defined as the set of polynomials in the variable ξ and coefficients in $\mathbf{F}$. It is endowed with a ring structure with the usual addition and with a non-commutative multiplication that can be summarized with the relation:

$$\xi a = a^p \xi, \quad a \in \mathbf{F}.$$

We denote by $\mathsf{Mod}_{\mathbf{F}\{\xi\}}$ (resp. $\mathsf{Mod}^{\mathbf{F}\{\xi\}}$) the abelian category of *left* (resp. *right*) $\mathbf{F}\{\xi\}$-modules.

The full subcategory of abelian restricted Lie algebras is equivalent to $\mathsf{Mod}_{\mathbf{F}\{\xi\}}$ because, if L is an abelian restricted Lie algebra, then the p-operation $\xi \colon L \to L$ is additive. We denote by $\mathsf{triv}_\xi(M)$ a unique abelian restricted Lie algebra with the underlying left $\mathbf{F}\{\xi\}$-module equal to $M \in \mathsf{Mod}_{\mathbf{F}\{\xi\}}$. Finally, we say that an abelian restricted Lie algebra is *p-abelian* if the p-operation $\xi \colon L \to L$ is trivial, i.e. $\xi = 0$.

We proceed with more examples of restricted Lie algebras.

Example 2.1.4. Given an associative $\mathbf{F}$-algebra A, we write A° for the restricted Lie algebra whose underlying vector space is A equipped with the bracket $[x,y] = xy - yx$ and the p-operation $\xi(x) = x^p$.

Definition 2.1.5. Let $(C, \eta \colon \mathbf{F} \to C)$ be a coaugmented cocommutative coalgebra over $\mathbf{F}$ with comultiplication $\Delta \colon C \to C \otimes C$. Recall that an element $x \in C$ is called *primitive* if

$$\Delta(x) = 1 \otimes x + x \otimes 1.$$

Here $1 = \eta(1) \in C$. We denote by $P(C)$ the set of primitive elements in C.

Example 2.1.6. Let H be a cocommutative Hopf algebra over $\mathbf{F}$. Then the set of primitive elements $P(H)$ is a restricted Lie subalgebra in H°, see Example 2.1.4.

Example 2.1.7. Let V be a vector space over $\mathbf{F}$ and let $T(V)$ be the tensor algebra generated by V. It is well-known that $T(V)$ has a unique structure of a cocommutative Hopf algebra such that generators $v \in V$ are primitive elements. Therefore $P(T(V))$ is a restricted Lie algebra, which we will denote by $L^r(V)$.

We say that a restricted Lie algebra L is *free* if L is isomorphic to $L^r(V)$ for some $V \in \mathsf{Vect}_{\mathbf{F}}$. This terminology is justified because the functor

$$L^r \colon \mathsf{Vect}_{\mathbf{F}} \to \mathsf{Lie}^r$$

is the left adjoint to the forgetful functor

$$\mathsf{oblv} \colon \mathsf{Lie}^r \to \mathsf{Vect}_{\mathbf{F}}.$$

Proposition 2.1.8. *The underlying vector space of a free restricted Lie algebra $L^r(V), V \in \mathsf{Vect}_{\mathbf{F}}$ splits as follows*

$$\mathsf{oblv} \circ L^r(V) \cong \bigoplus_{n\geq 1} L^r_n(V) = \bigoplus_{n\geq 1} (\mathbf{Lie}_n \otimes V^{\otimes n})^{\Sigma_n}.$$

Here $\mathbf{Lie}_n \in \mathsf{Vect}_{\mathbf{F}}$ is the n-th space of the Lie operad.

Proof. [23, Theorem 1.2.5]. □

The next proposition is standard, cf. [49, Chapter 6].

Proposition 2.1.9. *The category Lie^r is monadic over $\mathsf{Vect}_{\mathbf{F}}$ via the adjunction $L^r \dashv \mathsf{oblv}$. The category Lie^r is complete and cocomplete and the forgetful functor oblv creates limits and sifted colimits. Moreover, Lie^r is locally presentable.* □

For instance, the direct product $L_1 \times L_2$ is the direct sum $L_1 \oplus L_2$ as a vector space, with $[l_1, l_2] = 0, l_1 \in L_1, l_2 \in L_2$ and p-operation acting componentwise. We fix the following observation for later purposes (Proposition 4.4.3).

Proposition 2.1.10. *Let*

$$\begin{array}{ccc} L & \longrightarrow & L_1 \\ \downarrow & & \downarrow \\ L_2 & \longrightarrow & L_{12} \end{array}$$

be a pushout square in Lie^r and let $M \in \mathsf{Lie}^r$ be a restricted Lie algebra. Then the commutative diagram

$$\begin{array}{ccc} M \times L & \longrightarrow & M \times L_1 \\ \downarrow & & \downarrow \\ M \times L_2 & \longrightarrow & M \times L_{12} \end{array}$$

is again a pushout square in Lie^r.

Proof. We show that the natural map

$$(M \times L_1) \coprod_{M\times L} (M \times L_2) \to M \times (L_1 \coprod_{L} L_2) = M \times L_{12}$$

is an isomorphism. Let

$$f \colon M \coprod_{M} M \to (M \times L_1) \coprod_{M\times L} (M \times L_2)$$

be the map of restricted Lie algebras induced by the maps from the zero Lie algebra to L, L_1, and L_2, and let

$$g \colon M \to M \times L_{12},\ g(m) = (m, 0),\ m \in M$$

be the canonical embedding. Consider the following commutative diagram

$$\begin{array}{ccccc} M \coprod_M M & \overset{f}{\underset{0}{\rightrightarrows}} & (M \times L_1) \coprod_{M\times L} (M \times L_2) & \longrightarrow & L_1 \coprod_L L_2 \\ \downarrow \cong & & \downarrow & & \downarrow \cong \\ M & \overset{g}{\underset{0}{\rightrightarrows}} & M \times L_{12} & \longrightarrow & L_{12}. \end{array}$$

Since $g(M)$ is a (restricted) Lie ideal in $M \times L_{12}$, the bottom row is a coequalizer diagram both in Lie^r and in $\mathsf{Vect}_{\mathbf{F}}$. Furthermore, the top row is also a coequalizer diagram in Lie^r because pushouts commute with coequalizers. Since the outer vertical arrows are isomorphisms and f is a split monomorphism, it suffices to show that the top row is also a coequalizer diagram in $\mathsf{Vect}_{\mathbf{F}}$, i.e. the image $\mathrm{im}(f)$ is a (restricted) Lie ideal in the pushout $(M\times L_1)\coprod_{M\times L}(M\times L_2)$. However, this pushout is generated as a restricted Lie algebra by $\mathrm{im}(f)$, L_1, and L_2. Finally, we observe that $[m,l]=0$ for any $m \in \mathrm{im}(f)$ and $l \in L_i$, $i=1,2$, which imply the proposition. □

Definition 2.1.11. The *universal enveloping algebra* of $(L,\xi) \in \mathsf{Lie}^r$ is the quotient algebra

$$U^r(L) = T(L)/\langle x\otimes y - y\otimes x - [x,y],\ \ z^p - \xi(z),\ \ x,y,z \in L\rangle.$$

This construction is natural and the functor $U^r\colon \mathsf{Lie}^r \to \mathsf{Alg}$ is the left adjoint to the functor $(-)^\circ\colon\ \mathsf{Alg} \to \mathsf{Lie}^r$ of Example 2.1.4. In particular, $U^r L^r(V) \cong T(V)$.

We also note that U^r preserves colimits and takes direct products to the tensor product of algebras. In particular, $U^r(L)$ is an augmented cocommutative Hopf algebra via the diagonal map $\Delta\colon L \to L \times L$.

We write Hopf for the category of cocommutative Hopf algebras. By the previous paragraph, we have the functor

$$U^r\colon \mathsf{Lie}^r \to \mathsf{Hopf},$$

and by Example 2.1.6, we have the functor

$$P\colon \mathsf{Hopf} \to \mathsf{Lie}^r,\ H \mapsto P(H)$$

in the opposite direction.

Definition 2.1.12. We say that a Hopf algebra $H \in \mathsf{Hopf}$ is *primitively generated* if the subset of primitive elements $P(H) \subset H$ generates H, i.e. the natural map $T(P(H)) \to H$ is surjective. We denote by Hopf^{pr} the full subcategory of Hopf spanned by primitively generated Hopf algebras.

Since $U^r(L)$ is a quotient of a tensor algebra by a Hopf ideal, the Hopf algebra $U^r(L)$ is primitively generated for all $L \in \mathsf{Lie}^r$.

Theorem 2.1.13. *The functors $U^r\colon \mathsf{Lie}^r \to \mathsf{Hopf}^{pr}$ and $P\colon \mathsf{Hopf}^{pr} \to \mathsf{Lie}^r$ are inverse to each other. In other words, they provide the following equivalence of categories*

$$P : \mathsf{Hopf}^{pr} \rightleftarrows \mathsf{Lie}^r : U^r.$$

Proof. [49, Theorem 6.11]. □

2.2 Truncated coalgebras.

Recall that $\mathrm{char}(\mathbf{F}) = p$.

Definition 2.2.1. Let $W \in \mathsf{Vect}_{\mathbf{F}}$ be a vector space over $\mathbf{F}$. We define the (-1)*-th Frobenius twist* $W^{(-1)} \in \mathsf{Vect}_{\mathbf{F}}$ of W as follows. As an abelian group, $W^{(-1)} = W$ and we endow it with a new $\mathbf{F}$-action

$$- \cdot -\colon \mathbf{F} \times W^{(-1)} \to W^{(-1)}$$

given by $a \cdot w = a^p w$, $a \in \mathbf{F}$, $w \in W^{(-1)} = W$. Since the field $\mathbf{F}$ is perfect, there also exists the inverse operation; namely, we define the *Frobenius twist* of W as a unique $\mathbf{F}$-vector space $W^{(1)}$ such that $(W^{(1)})^{(-1)} = W$.

For any vector space $W \in \mathsf{Vect}_{\mathbf{F}}$ we have a natural $\mathbf{F}$-linear map

$$W^{(1)} \to \mathrm{Sym}^p(W) = (W^{\otimes p})_{\Sigma_p} \tag{2.2.2}$$

sending w to $w^{\otimes p}$. If W is finite-dimensional, we have the dual map

$$\Gamma^p(W) = (W^{\otimes p})^{\Sigma_p} \to W^{(1)}, \tag{2.2.3}$$

and we extend it to all vector spaces to taking filtered colimits.

Definition 2.2.4. Let C be a cocommutative coalgebra over $\mathbf{F}$ with comultiplication $\Delta\colon C \to C \otimes C$ and counit $\varepsilon\colon C \to \mathbf{F}$. We define the *Verschiebung* operator $V\colon C \to C^{(1)}$ as follows:

$$V\colon C \xrightarrow{\Delta^{p-1}} (C^{\otimes p})^{\Sigma_p} \xrightarrow{(2.2.3)} C^{(1)}.$$

We note that V is a $\mathbf{F}_p$-linear coalgebra homomorphism.

Definition 2.2.5. A cocommutative coalgebra C is called *truncated* if

$$\ker(V) = \ker(\varepsilon),$$

where $\varepsilon\colon C \to \mathbf{F}$ is the counit. We write CoAlg^{tr} for the full subcategory in the category of cocommutative coalgebras CoAlg spanned by *non-zero* truncated ones.

Remark 2.2.6. Let C be a finite-dimensional coalgebra, and let C^* be its dual algebra. Then C is truncated if and only if for any $x \in C^*$ the p-th power x^p is some scalar multiple of the unit $1 \in C^*$.

Example 2.2.7. Let W be a vector space over $\mathbf{F}$. We define the *trivial* coalgebra $\mathsf{triv}(W)$ as follows. The underlying vector space of $\mathsf{triv}(W)$ is $W \oplus \mathbf{F}$, where the second summand is spanned by the element $1 \in \mathbf{F}$. We define the comultiplication $\Delta\colon \mathsf{triv}(W) \to \mathsf{triv}(W) \otimes \mathsf{triv}(W)$ and the counit $\varepsilon\colon \mathsf{triv}(W) \to \mathbf{F}$ as follows

$$\Delta(1) = 1, \quad \Delta(w) = 1 \otimes w + w \otimes 1, w \in W;$$

$$\varepsilon(w) = 0, w \in W, \quad e(1) = 1.$$

This is easy to see that the coalgebra $\mathsf{triv}(W)$ is truncated.

Definition 2.2.8 (Chapter 8, [67])**.** A non-zero coalgebra $C \in \mathsf{CoAlg}$ is called *simple* if C has no non-zero proper subcoalgebras and $C \in \mathsf{CoAlg}$ is called *pointed* if all simple subcoalgebras of C are 1-dimensional.

Proposition 2.2.9.

1. *Any simple truncated coalgebra is* 1*-dimensional. In particular, any truncated coalgebra is pointed.*
2. *Any non-zero truncated coalgebra has a* unique *simple subcoalgebra.*

Proof. If C is simple, then C is finite-dimensional, and the dual algebra C^* is a finite field extension of $\mathbf{F}$, see [67, Lemma 8.0.1]. In particular, the Frobenius map

$$(-)^p\colon C^* \to C^*$$

is injective. However, since the coalgebra C is truncated, the image of this map is $\mathbf{F} \subseteq C^*$ by Remark 2.2.6. Therefore $C^* \cong \mathbf{F}$. This proves the first part.

For the second part, we show that $C \in \mathsf{CoAlg}^{tr}$ has at most one grouplike element. Recall that $0 \neq x \in C$ is called *grouplike* if

$$\Delta(x) = x \otimes x;$$

and the set $G(C) \subset C$ of grouplike elements in C one-to-one corresponds to simple subcoalgebras of C, see [67, Lemma 8.0.1]. If $x \in C$ is grouplike, then $V(x) = x$ and $\varepsilon(x) = 1$. Therefore, if $x, y \in C$ are grouplike, then $x - y \in \ker(\varepsilon)$. Since C is truncated, this implies $V(x - y) = 0$. So, we obtain

$$x - y = V(x) - V(y) = V(x - y) = 0.$$

Since any non-zero coalgebra contains a simple subcoalgebra, the proposition follows. □

In particular, there is an exactly one coalgebra map $\mathbf{F} \to C$ for any $C \in \mathsf{CoAlg}^{tr}$; that is any non-zero truncated coalgebra is *canonically* coaugmented. Therefore we will consider the category CoAlg^{tr} as a full subcategory of coaugmented cocommutative coalgebras CoAlg^{aug}.

Proposition 2.2.10. *The fully faithful embedding*

$$\mathsf{CoAlg}^{tr} \subset \mathsf{CoAlg}^{aug}$$

has both left and right adjoints. In particular, the category CoAlg^{tr} *has all limits, which can be computed in* CoAlg^{aug}*. Similarly, for all colimits in* CoAlg^{tr}*.*

Proof. The left adjoint $l\colon \mathsf{CoAlg}^{aug} \to \mathsf{CoAlg}^{tr}$ can be given as the coequalizer of the following diagram:

$$C^{(-1)} \underset{}{\overset{V^{(-1)}}{\rightrightarrows}} C \longrightarrow l(C),$$

where $C \in \mathsf{CoAlg}^{aug}$, and the lower arrow is the composition the counit map and the coaugmentation map. Similarly, the right adjoint $r\colon \mathsf{CoAlg}^{aug} \to \mathsf{CoAlg}^{tr}$ can be defined as the equalizer of the same diagram:

$$r(C) \longrightarrow C \overset{V}{\rightrightarrows} C^{(1)}.$$

□

Definition 2.2.11. A non-counital coalgebra $(C, \Delta\colon C \to C \otimes C)$ is called *conilpotent* if for any $x \in C$ there exists n such that

$$\Delta^{n-1}(x) = 0,$$

where $\Delta^{n-1}\colon C \to C^{\otimes n}$ is the $(n-1)$-fold composition of the comultiplication. A coaugmented coalgebra $(C, \eta\colon \mathbf{F} \to C)$ is called conilpotent if the non-counital coalgebra $\operatorname{coker}(\eta)$ is conilpotent.

Lemma 2.2.12. *A finite-dimensional coaugmented coalgebra (C, η) is conilpotent if and only if the dual augmented algebra (C^*, η^*) has the* nilpotent *augmentation ideal $I = \ker(\eta^*) \subset C^*$, i.e. $I^n = 0$ for some $n \in \mathbf{N}$.* □

Proposition 2.2.13. *Any truncated coalgebra C is conilpotent (as a coaugmented coalgebra).*

Proof. Let $x \in C$, we have to show that $\Delta^{n-1}(x) = 0 \in (C/\mathbf{F})^{\otimes n}$ for some $n \in \mathbf{N}$. We can assume that C is generated by x. Then by [67, Theorem 2.2.1], the coalgebra C is finite-dimensional. Therefore, by Lemma 2.2.12, it suffices to show that the augmentation ideal I in the dual algebra C^* is nilpotent. Since C is truncated, $x^p = 0$ for any $x \in I$. Then $I^{pm} = 0$, where $m = \dim(I)$. □

Let W be a vector space over $\mathbf{F}$. Then the symmetric algebra $\mathrm{Sym}(W)$ is a (cocommutative) Hopf algebra, which is truncated as a coalgebra. Since $w^p \in \mathrm{Sym}(W)$ is a primitive element for any $w \in W$, the ideal

$$I = (w^p \mid w \in W) \subset \mathrm{Sym}(W)$$

generated by all p-th powers is a Hopf ideal. Therefore $\mathrm{Sym}^{tr}(W) = \mathrm{Sym}(W)/I$ is a Hopf algebra, which is truncated as a coalgebra.

Proposition 2.2.14. *The functor*

$$\mathrm{Sym}^{tr} \colon \mathsf{Vect}_{\mathbf{F}} \to \mathsf{CoAlg}^{tr}$$

is the right adjoint to the forgetful functor

$$\mathsf{oblv} \colon \mathsf{CoAlg}^{tr} \hookrightarrow \mathsf{CoAlg}^{aug} \to \mathsf{Vect}_{\mathbf{F}}$$

given by $(C, \eta \colon \mathbf{F} \to C) \mapsto \mathrm{coker}(\eta)$.

Proof. By Propositions 2.2.10 and 2.2.13, the right adjoint $R \colon \mathsf{Vect}_{\mathbf{F}} \to \mathsf{CoAlg}^{tr}$ for the functor oblv is the equalizer of the diagram

$$R(W) \longrightarrow \Gamma(W) \overset{V}{\underset{}{\rightrightarrows}} \Gamma(W)^{(1)},$$

where $W \in \mathsf{Vect}_{\mathbf{F}}$ and

$$\Gamma(W) = \bigoplus_{i=0}^{\infty} (W^{\otimes i})^{\Sigma_i}$$

is a cofree conilpotent cocommutative coalgebra. We recall that $\Gamma(W)$ is a commutative Hopf algebra also known as the divided power algebra generated by W. Explicitly, $\Gamma(W)$ is generated by elements $\gamma_n(w)$, $w \in W$, $n \geq 0$ subjects to relations:

- $\gamma_0(w) = 1$
- $\gamma_n(aw) = a^n \gamma_n(w)$, $w \in W$, $a \in \mathbf{F}$;
- $\gamma_n(w + w') = \sum_{i=0}^{n} \gamma_i(w)\gamma_{n-i}(w')$, $w, w' \in W$;
- $\gamma_m(w)\gamma_n(w) = \dfrac{(m+n)!}{m!n!}\gamma_{m+n}(w)$, $w \in W$;
- $\Delta(\gamma_n(w)) = \sum_{i=0}^{n} \gamma_i(w) \otimes \gamma_{n-i}(w)$, $w \in W$.

In particular, $V(\gamma_{pn}(w)) = \gamma_n(w)$ and $R(W)$ is the sub-Hopf algebra generated by the elements $\gamma_0(w), \ldots, \gamma_{p-1}(w), w \in W$. The latter is isomorphic to $\mathrm{Sym}^{tr}(W)$. □

Proposition 2.2.15. *The category* CoAlg^{tr} *is comonadic over* $\mathsf{Vect}_{\mathbf{F}}$ *via the adjunction* $\mathsf{oblv} \dashv \mathrm{Sym}^{tr}$. *The category* CoAlg^{tr} *is complete and cocomplete and the forgetful functor* oblv *creates colimits. Moreover,* CoAlg^{tr} *is locally presentable.* □

Example 2.2.16. If $C, D \in \mathsf{CoAlg}^{tr}$, then their Cartesian product $C \times D$ is the tensor product $C \otimes_{\mathbf{F}} D$ equipped with the obvious comultiplication. The coCartesian coproduct $C \sqcup D$ is the wedge sum of augmented vector spaces $(C \oplus D)/\mathbf{F}$ equipped with the obvious comultiplication.

2.3 Primitively generated Hopf algebras.

Let L be a restricted Lie algebra, then the universal enveloping algebra $U^r(L)$ is naturally equipped with the increasing *Lie filtration* $F_\bullet$, which is inductively defined as follows:

- $F_s U^r(L) = 0$ if $s < 0$;
- $F_0 U^r(L) = \mathbf{F}$;
- $F_{s+1} U^r(L) = F_s U^r(L) + L \cdot F_s U^r(L) \subset U^r(L)$.

We denote by $E_0 U^r(L)$ the associated graded Hopf algebra. Since $E_0 U^r(L)$ is commutative, there is a natural map

$$\mathrm{Sym}^{tr}(L) \to E_0 U^r(L). \tag{2.3.1}$$

Theorem 2.3.2 (Poincaré-Birkhoff-Witt)**.** *The homomorphism* (2.3.1) *is an isomorphism of Hopf algebras.*

Proof. [49, Proposition 6.12]. □

Proposition 2.3.3. *A Hopf algebra H is primitively generated if and only if H is truncated as a cocommutative coalgebra.*

Proof. Let H be a primitively generated Hopf algebra, and let $I \subset H$ be the augmentation ideal. We observe that $V(I) = 0$ because I is generated by the set of primitive elements $P(H) \subset I$, the Verschiebung operator $V \colon H \to H^{(1)}$ is a Hopf algebra homomorphism, and $V(P(H)) = 0$. Therefore H is a truncated coalgebra.

Suppose now that H is a truncated coalgebra. It suffices to show that the natural morphism

$$U^r(P(H)) \to H \tag{2.3.4}$$

is surjective. For a coaugmented cocommutative coalgebra C, one can consider a natural (increasing) *conilpotent* filtration on C:

$$F_{(n)}C = \mathrm{eq}(C \overset{\Delta^n}{\underset{}{\rightrightarrows}} C^{\otimes n+1}).$$

A coalgebra C is conilpotent if and only if the conilpotent filtration $F_{(\bullet)}C$ is exhaustive. Moreover, for a Hopf algebra H the filtration $F_{(\bullet)}H$ is both multiplicative and comultiplicative. Hence the associated graded Hopf algebra $E_{(\bullet)}H$ is a connected graded Hopf algebra, $P(E_{(\bullet)}H) = P(H)$.

By Proposition 2.2.13, both Hopf algebras $U^r P(H)$ and H are conilpotent, therefore the map (2.3.4) is surjective if and only if the map

$$E_{(\bullet)}U^r P(H) \to E_{(\bullet)}H$$

is surjective; or equivalently, if and only if the induced map on the module of indecomposable elements

$$Q(E_{(\bullet)}U^r P(H)) \to Q(E_{(\bullet)}H) \tag{2.3.5}$$

is surjective.

Since H is truncated as a coalgebra, the Hopf algebra $E_{(\bullet)}H$ is truncated as well. The linear dual of [49, Proposition 4.20] implies that the algebras $E_{(\bullet)}H$, $E_{(\bullet)}U^rP(H)$ are primitively generated. Finally, the commutative diagram

$$\begin{array}{ccc} P(E_{(\bullet)}U^rP(H)) & = & P(E_{(\bullet)}H) \\ \downarrow & & \downarrow \\ Q(E_{(\bullet)}U^rP(H)) & \longrightarrow & Q(E_{(\bullet)}H) \end{array}$$

implies that the map (2.3.5) is surjective. □

Let C be a 1-category with finite products. Recall that a *group object* of C is an object $X \in \mathsf{C}$ equipped with a multiplication map $m\colon X \times X \to X$, a unit map $e\colon * \to X$, and with an inverse map $i\colon X \to X$, such that m is associative and unital. We denote by $\mathsf{Grp}(\mathsf{C})$ the category of group objects in C.

Corollary 2.3.6. *The category* $\mathsf{Grp}(\mathsf{CoAlg}^{tr})$ *is equivalent to the category* Hopf^{pr} *of primitively generated Hopf algebras.* □

Finally, we will describe free group objects in $\mathsf{Grp}(\mathsf{CoAlg}^{tr})$.

Definition 2.3.7. Let $C, D \in \mathsf{CoAlg}^{tr}$ be truncated coalgebras. We define the *smash product* $C \wedge D \in \mathsf{CoAlg}^{tr}$ as the following coequalizer

$$C \wedge D = \mathrm{coeq}(C \sqcup D \overset{\mathrm{can}}{\rightrightarrows} C \times D),$$

where the upper arrow is the canonical map from coproduct to product and the lower arrow is the composition the counit map and the coaugmentation map. In particular, $\mathsf{oblv}(C \wedge D) \cong \mathsf{oblv}(C) \otimes \mathsf{oblv}(D)$

Proposition 2.3.8. *The forgetful functor* $\mathsf{Hopf}^{pr} \to \mathsf{CoAlg}^{tr}$ *has a left adjoint*

$$H\colon \mathsf{CoAlg}^{tr} \to \mathsf{Hopf}^{pr}.$$

Moreover, as an algebra $H(C), C \in \mathsf{CoAlg}^{tr}$ *is the tensor algebra* $T(\mathsf{oblv}(C))$ *generated by the quotient vector space* $\mathsf{oblv}(C) = C/\mathbf{F}$.

Proof. The tensor algebra $T(\mathsf{oblv}(C))$ has a unique bialgebra structure such that the map $C \to T(\mathsf{oblv}(C))$ is a coalgebra homomorphism. We denote the resulting bialgebra by $H(C)$. More precisely, as a coaugmented coalgebra $H(C)$ is isomorphic to

$$\coprod_{n=0}^{\infty} C^{\wedge n} = \coprod_{n=0}^{\infty} C^{\otimes n}/\sim .$$

In particular, $H(C)$ is truncated, and so $H(C)$ is the free bialgebra generated by C. Therefore it is enough to show that $H(C)$ admits an antipode (which is unique if it exists).

By Proposition 2.2.9, the bialgebra $H(C)$ is pointed and has a unique grouplike element. By [67, Proposition 9.2.5], this implies that $H(C)$ is a Hopf algebra. □

3. Koszul duality

In this section we state and prove our main results summarized in Theorems A, B, and C from the introduction.

Section 3.1 is devoted to the proof of Theorem A. By [58], the functor $\overline{W}U^r$ is the composite of two functors

$$\mathsf{sLie}^r \xrightarrow{U^r} \mathsf{sHopf}^{pr} \xrightarrow{\overline{W}} \mathsf{s_0CoAlg}^{tr},$$

and by [49, Theorem 6.11], the leftmost arrow here is an equivalence. Inspired by [66], we present the functor $\overline{W}$ as the composite of simpler functors:

$$\overline{W}\colon \mathsf{sHopf}^{pr} \xrightarrow{N} \mathsf{s_0sCoAlg}^{tr} \xhookrightarrow{\iota} \mathsf{ssCoAlg}^{tr} \xrightarrow{T} \mathsf{sCoAlg}^{tr},$$

see Definition 3.1.22 and Remark 3.1.24. Here N is the *nerve functor* (3.1.3), ι is the embedding (3.1.21) of the full subcategory of *horizontally reduced* bisimplicial coalgebras into all, and T is the *Artin-Mazur codiagonal* (3.1.19). Therefore the left adjoint G to $\overline{W}$ is the composite

$$G\colon \mathsf{sCoAlg}^{tr} \xrightarrow{\mathrm{Dec}} \mathsf{ssCoAlg}^{tr} \xrightarrow{R} \mathsf{s_0sCoAlg}^{tr} \xrightarrow{GZ} \mathsf{sHopf}^{pr},$$

where Dec is the *total décalage* (3.1.12), R is given in (3.1.29), and GZ is the functor from Proposition 3.1.4.

By [53], the unit map $C_\bullet \to T\mathrm{Dec}(C_\bullet)$ is a weak equivalence for any $C_\bullet \in \mathsf{sCoAlg}^{tr}$ (Theorem 3.1.20). We examine the adjunction $GZ \dashv N$ in Lemma 3.1.10 and Corollary 3.1.18, and we show that the unit map for this adjunction is a weak equivalence for certain simplicial coalgebras. These observations and formal properties of functors Dec and T suffice to prove the main theorem of this section, Theorem 3.1.32.

The main results of Section 3.2 are Theorems 3.2.3, 3.2.10, and 3.2.24, where we construct simplicial model structures on the categories sLie^r and $\mathsf{s_0CoAlg}^{tr}$ equipped with various notions of weak equivalences. The model structure of Theorem 3.2.3 (resp. of Theorem 3.2.10) is a right (resp. a left) transferred model structure from $\mathsf{sVect}_{\mathbf{F}}$; for the proof, we apply [33, Theorem 11.3.2] (resp. [32, Theorem 2.2.1]).

In Definition 3.2.21, we introduce the notion of an $\mathbf{F}$*-equivalence*. Following [58], we demonstrate in Corollary 3.2.15 and Proposition 3.2.18 that the class of $\mathbf{F}$-equivalences is well-behaved. Using [40, Proposition A.2.6.15] we construct a simplicial model category sLie^r_ξ as a Bousfield localization of sLie^r. At the end of the section, we prove Theorem 3.2.26 and Proposition 3.2.32, which constitute together Theorem C from the introduction.

3.1 Kan loop group functor.

Definition 3.1.1. Let $\mathsf{sCoAlg}^{tr} = \mathrm{Fun}(\Delta^{op}, \mathsf{CoAlg}^{tr})$ denote the category of *simplicial objects* in the category CoAlg^{tr} of non-zero truncated coalgebras over the field $\mathbf{F}$. A simplicial coalgebra $C_\bullet \in \mathsf{sCoAlg}^{tr}$ is *reduced* if $C_0 \cong \mathbf{F}$. Let $\mathsf{s_0CoAlg}^{tr}$ denote the full subcategory of sCoAlg^{tr} spanned by reduced objects.

Definition 3.1.2. Let $H \in \mathsf{Hopf}^{pr}$ be a primitively generated Hopf algebra and let $\varepsilon\colon H \to \mathbf{F}$ be the counit. The *nerve* of H is a reduced simplicial coalgebra $N_\bullet H$ defined as follows:

$$N_0 H = \mathbf{F}, \;\; N_q H = H^{\otimes q}, \;\; q > 0;$$

with face and degeneracy maps given by

$$d_0(h_1 \otimes \ldots \otimes h_q) = \varepsilon(h_1)h_2 \otimes \ldots \otimes h_q, \quad q > 0;$$

$$d_i(h_1 \otimes \ldots \otimes h_q) = h_1 \otimes \ldots \otimes h_{i-1} \otimes h_i h_{i+1} \otimes \ldots \otimes h_q, \quad 1 \leq i \leq q-1, q > 0;$$

$$d_q(h_1 \otimes \ldots \otimes h_q) = \varepsilon(h_q)h_1 \otimes \ldots \otimes h_{q-1}, \quad q > 0;$$

$$s_0(h_1 \otimes \ldots \otimes h_q) = 1 \otimes h_1 \otimes \ldots \otimes h_q, \quad q \geq 0;$$

$$s_{i+1}(h_1 \otimes \ldots \otimes h_q) = h_1 \otimes \ldots \otimes h_i \otimes 1 \otimes h_{i+1} \otimes \ldots \otimes h_q, \quad i, q \geq 0.$$

By Proposition 2.3.3, all coalgebras $N_q H, q \geq 0$ are truncated, and so we obtain the *nerve functor*

$$N\colon \mathsf{Hopf}^{pr} \to \mathsf{s}_0\mathsf{CoAlg}^{tr}. \tag{3.1.3}$$

As usual, we observe that the simplicial set $N_\bullet H$ is 2-coskeletal, see e.g. [35, Proposition 2.2.3].

Proposition 3.1.4. *The nerve functor N has a left adjoint*

$$GZ\colon \mathsf{s}_0\mathsf{CoAlg}^{tr} \to \mathsf{Hopf}^{pr}.$$

Namely, $GZ(C_\bullet), C_\bullet \in \mathsf{s}_0\mathsf{CoAlg}^{tr}$ is the quotient of free Hopf algebra $H(C_1)$ of Proposition 2.3.8 by the two-sided Hopf ideal $I \subset H(C_1)$ generated by elements

$$r(c) = d_1(c) - \sum_i d_0(c_i^{(1)}) d_2(c_i^{(2)}),$$

where $c \in C_2$, and $\Delta(c) = \sum_i c_i^{(1)} \otimes c_i^{(2)}$.

Proof. The straightforward computation shows that the set $\mathrm{Hom}_{\mathsf{s}_0\mathsf{CoAlg}^{tr}}(C_\bullet, N_\bullet H)$ of simplicial coalgebra maps is a subset of $\mathrm{Hom}_{\mathsf{CoAlg}^{tr}}(C_1, H)$, and a map $f \in \mathrm{Hom}_{\mathsf{CoAlg}^{tr}}(C_1, H)$ belongs to $\mathrm{Hom}_{\mathsf{s}_0\mathsf{CoAlg}^{tr}}(C_\bullet, N_\bullet H)$ if and only if

$$f(d_1(c)) = \sum_i f(d_0(c_i^{(1)})) f(d_2(c_i^{(2)}))$$

for each $c \in C_2$, $\Delta(c) = \sum_i c_i^{(1)} \otimes c_i^{(2)}$. Now, the proposition follows by Proposition 2.3.8, which guarantees the existence of a free Hopf algebra. □

Remark 3.1.5. Note that $r(c_1 + c_2) = r(c_1) + r(c_2), c_1, c_2 \in C_2$, and $r(c) = 0$ if $c \in C_2$ is degenerate, i.e. $c = s_i(c'), c' \in C_1, i = 0, 1$.

Lemma 3.1.6. *The module of indecomposable elements $Q(GZ(C_\bullet))$ for the Hopf algebra $GZ(C_\bullet)$, $C_\bullet \in \mathsf{sCoAlg}^{tr}$ is naturally isomorphic to $\pi_1(C_\bullet)$.*

Remark 3.1.7. For the category Grp of groups the nerve construction is right adjoint to the fundamental group functor. Likewise, we regard GZ as a fundamental group construction.

Example 3.1.8. Let $C \in \mathsf{CoAlg}^{tr}$ be a truncated coalgebra. We will abbreviate $\mathsf{oblv}(C) \in \mathsf{Vect}_{\mathbf{F}}$ by $\widetilde{C}$, $C = \widetilde{C} \oplus \mathbf{F}$. Recall from [29, Section III.5] that the *Kan suspension* $\Sigma_\bullet C$ of C is a reduced simplicial coalgebra defined as follows:

$$\Sigma_0 C = \mathbf{F}, \quad \Sigma_q C = \coprod_{i=1}^{q} C \cong \mathbf{F} \oplus \bigoplus_{i=1}^{q} \widetilde{C}, \quad q > 0;$$

with face and degeneracy maps constant on the first summand and given by the following formulas on the second summand:

$$d_0(c_1,\dots,c_q) = d_q(c_1,\dots,c_q) = 0, \quad q>0;$$

$$d_i(c_1,\dots,c_q) = (c_1,\dots,c_{i-1},c_i+c_{i+1},\dots,c_q), \quad 1\leq i\leq q-1, q>0;$$

$$s_0(c_1,\dots,c_q) = (0,c_1,\dots,c_q), \quad q\geq 0;$$

$$s_{i+1}(c_1,\dots,c_q) = (c_1,\dots,c_i,0,c_{i+1},\dots,c_q), \quad i,q\geq 0.$$

Notice that the homotopy groups $\pi_*(\Sigma_\bullet C)$ of the underlying simplicial vector space are equal to:

$$\pi_*(\Sigma_\bullet C) = \begin{cases} \mathbf{F} & \text{if } * = 0, \\ \widetilde{C} & \text{if } * = 1, \\ 0 & \text{if } * > 1. \end{cases}$$

By Remark 3.1.5, we obtain that $GZ(\Sigma_\bullet C)\cong H(C)$, the free Hopf algebra generated by C.

Definition 3.1.9. A map $f\colon C_\bullet \to D_\bullet$ of simplicial coalgebras $C_\bullet, D_\bullet \in \mathsf{sCoAlg}$ is a *weak equivalence* if f is a weak equivalence of the underlying simplicial vector spaces, i.e. $\pi_*(f)$ is an isomorphism.

Lemma 3.1.10. *Let $C\in \mathsf{CoAlg}^{tr}$, then the natural map*

$$\Sigma_\bullet C \to N_\bullet GZ(\Sigma_\bullet C)$$

is a weak equivalence of simplicial coalgebras.

Proof. First, note that the underlying simplicial vector space of $N_\bullet H, H\in \mathsf{Hopf}^{pr}$ only depends on the underlying augmented algebra structure of H. Second, by Example 3.1.8, the Hopf algebra $GZ(\Sigma_\bullet C)$ is the free Hopf algebra $H(C)$, and by Proposition 2.3.8, there is an algebra isomorphism $H(C)\cong T(\mathsf{oblv}(C))$. Finally, for the free associative algebra $T(\mathsf{oblv}(C))$, we have

$$\pi_i(N_\bullet T(\mathsf{oblv}(C))) = 0, \ i>1,$$

and a natural isomorphism $\pi_1(N_\bullet T(\mathsf{oblv}(C)))\cong \mathsf{oblv}(C)$. □

Definition 3.1.11. Let $\mathsf{ssCoAlg}^{tr}$ denote the category of *bisimplicial* truncated coalgebras over $\mathbf{F}$, i.e. $\mathsf{ssCoAlg}^{tr} = \mathrm{Fun}(\Delta^{op}\times\Delta^{op}, \mathsf{CoAlg}^{tr})$. A bisimplicial map $f\colon C_{\bullet,\bullet}\to D_{\bullet,\bullet}$ is a *vertical* weak equivalence if the maps

$$f_{n,\bullet}\colon C_{n,\bullet}\to D_{n,\bullet}$$

are weak equivalences for each $n\geq 0$. Similarly, f is a *horizontal* weak equivalence if the maps

$$f_{\bullet,m}\colon C_{\bullet,m}\to D_{\bullet,m}$$

are weak equivalences for each $m\geq 0$. Finally, f is a *levelwise* weak equivalence if f is either a vertical or a horizontal weak equivalence.

Let $\sigma\colon \Delta\times\Delta\to\Delta$ denote the ordinal sum functor on the category Δ, described on objects via $\sigma([n],[m])=[n+1+m]$. The induced functor

$$\mathrm{Dec}=\sigma^*\colon \mathsf{sCoAlg}^{tr}\to \mathsf{ssCoAlg}^{tr} \tag{3.1.12}$$

is called *total décalage*, see [35, Chapitre VI.1.5]. In other words, the coalgebra of (n,m)-bisimplices of $\mathrm{Dec}(C_\bullet)$ is $\mathrm{Dec}(C_\bullet)_{n,m}=C_{n+1+m}$ and the degeneracy and face maps in $\mathrm{Dec}(C_\bullet)_{\bullet,\bullet}$ are recombinations of those for $C_\bullet\in\mathsf{sCoAlg}^{tr}$.

We denote by $\mathrm{Dec}_m\colon \mathsf{sCoAlg}^{tr}\to\mathsf{sCoAlg}^{tr}$ a functor given by

$$C_\bullet\mapsto \mathrm{Dec}(C)_{\bullet,m}.$$

Note that

$$\mathrm{Dec}_{m+1}(C_\bullet)=\mathrm{Dec}_0(\mathrm{Dec}_m(C_\bullet)). \tag{3.1.13}$$

Moreover, the simplicial coalgebra $\mathrm{Dec}_0(C_\bullet)$ is an augmented simplicial object via the map $d_0\colon \mathrm{Dec}_0(C_\bullet)\to C_0$, e.g. see [66, Section 2]. In particular, d_0 is a (strong) deformation retract in simplicial coalgebras.

Let $C_\bullet\in\mathsf{sCoAlg}^{tr}$ be a simplicial coalgebra and let $\mathrm{sk}_1C_\bullet\in\mathsf{sCoAlg}^{tr}$ be the 1-skeleton of $C_\bullet$. The natural map $\mathrm{sk}_1C_\bullet\to C_\bullet$ induces the map

$$\mathrm{Dec}_0(\mathrm{sk}_1C_\bullet)\to\mathrm{Dec}_0(C_\bullet)$$

such that $\mathrm{Dec}_0(\mathrm{sk}_1C_\bullet)_0=\mathrm{Dec}_0(C_\bullet)_0=C_1$ and both $\mathrm{Dec}_0(\mathrm{sk}_1C_\bullet)$ and $\mathrm{Dec}_0(C_\bullet)$ are deformation retracts of C_0.

Lemma 3.1.14. *The reduced simplicial coalgebra $\mathrm{Dec}_0(\mathrm{sk}_1C_\bullet)/C_1$ is isomorphic to the Kan suspension $\Sigma_\bullet(C_1/s_0C_0)$ of the quotient coalgebra C_1/s_0C_0.*

Proof. Straightforward computation. □

Lemma 3.1.15. *The map*

$$f\colon \mathrm{Dec}_0(\mathrm{sk}_1C_\bullet)/C_1\to\mathrm{Dec}_0(C_\bullet)/C_1$$

is a weak equivalence of reduced simplicial coalgebras.

Proof. We denote by $\widetilde{C}_\bullet\subset C_\bullet$ the kernel of the counit $\varepsilon\colon C_\bullet\to\mathbf{F}$. Then the underlying simplicial vector space of the quotient coalgebra $\mathrm{Dec}_0(C_\bullet)/C_1$ is a direct sum $(\mathrm{Dec}_0(\widetilde{C}_\bullet)/\widetilde{C}_1)\oplus\mathbf{F}$. We compute the homotopy groups $\pi_*(\mathrm{Dec}_0(\widetilde{C}_\bullet)/\widetilde{C}_1)$ via the long exact sequence:

$$\ldots\to\pi_{*+1}(\mathrm{Dec}_0(\widetilde{C}_\bullet)/\widetilde{C}_1)\to\pi_*(\widetilde{C}_1)\to\pi_*(\mathrm{Dec}_0(\widetilde{C}_\bullet))\to\pi_*(\mathrm{Dec}_0(\widetilde{C}_\bullet)/\widetilde{C}_1)\to\ldots$$

Since $\mathrm{Dec}_0(\widetilde{C}_\bullet)$ is a deformation retract of the constant simplicial vector space $\widetilde{C}_0$, we have $\pi_0(\mathrm{Dec}_0(\widetilde{C}_\bullet))=\widetilde{C}_0$, $\pi_0(\widetilde{C}_1)=\widetilde{C}_1$, and $\pi_*(\mathrm{Dec}_0(\widetilde{C}_\bullet))=\pi_*(\widetilde{C}_1)=0$ if $*>0$. Therefore,

$$\pi_1(\mathrm{Dec}_0(\widetilde{C}_\bullet)/\widetilde{C}_1)=\ker d_0=\operatorname{coker} s_0=\widetilde{C}_1/s_0\widetilde{C}_0$$

and $\pi_i(\mathrm{Dec}_0(\widetilde{C}_\bullet)/\widetilde{C}_1)=0$ if $i\neq 1$. This implies that f is a weak equivalence. □

Proposition 3.1.16. *The induced map*

$$GZ(f)\colon GZ(\mathrm{Dec}_0(\mathrm{sk}_1C_\bullet)/C_1)\to GZ(\mathrm{Dec}_0(C_\bullet)/C_1)$$

is an isomorphism of Hopf algebras.

Proof. By Lemma 3.1.6 and Lemma 3.1.15, the induced map

$$QGZ(f)\colon QGZ(\mathrm{Dec}_0(\mathrm{sk}_1 C_\bullet)/C_1) \to QGZ(\mathrm{Dec}_0(C_\bullet)/C_1)$$

of the modules of indecomposable elements is an isomorphism. Therefore $GZ(f)$ is surjective. We show that $GZ(f)$ is injective as well.

By Lemma 3.1.14 and Example 3.1.8, the left hand side $GZ(\mathrm{Dec}_0(\mathrm{sk}_1 C_\bullet)/C_1)$ is the free Hopf algebra $H(C_1/s_0C_0)$ generated by the quotient coalgebra C_1/s_0C_0. Whereas, the right hand side $GZ(\mathrm{Dec}_0(C_\bullet)/C_1)$ is the quotient of the free Hopf algebra $H(C_2/s_0C_1)$ subject to relations

$$r(c) = d_1(c) - \sum_i d_0(c_i^{(1)})d_2(c_i^{(2)}),$$

where $c \in C_3/s_0^2C_1$, and $\Delta(c) = \sum_i c_i^{(1)} \otimes c_i^{(2)}$. Finally, the Hopf algebra homomorphism $GZ(f)$ maps a generator $c \in C_1/s_0C_0$ to the generator $s_1(c) \in C_2/s_0C_1$. We will construct a left inverse

$$d\colon GZ(\mathrm{Dec}_0(C_\bullet)/C_1) \to GZ(\mathrm{Dec}_0(\mathrm{sk}_1 C_\bullet)/C_1)$$

to the map $GZ(f)$.

Note that a tensor algebra $T(V), V \in \mathsf{Vect}_{\mathbf{F}}$ has a canonical anti-automorphism $T(V) \xrightarrow{\cong} T(V)^{op}$ given on monomials by

$$v_1v_2 \cdot \ldots \cdot v_n \mapsto v_nv_{n-1} \cdot \ldots \cdot v_1.$$

Since as an algebra a free Hopf algebra $H(C), C \in \mathsf{CoAlg}^{tr}$ is a tensor algebra $T(\mathsf{oblv}(C))$, there exists the similar anti-automorphism of Hopf algebras:

$$H(C) \xrightarrow{\cong} H(C)^{op}. \tag{3.1.17}$$

Next, the face operators $d_0, d_1\colon C_2 \to C_1$ together with the map (3.1.17) produce maps

$$d_0\colon H(C_2) \to H(C_1) \cong H(C_1)^{op},$$

$$d_1\colon H(C_2) \to H(C_1)$$

of Hopf algebras. We define a map $d'\colon H(C_2) \to H(C_1)$ as the following composite:

$$\begin{aligned} d'\colon H(C_2) \xrightarrow{\Delta} H(C_2) \otimes H(C_2) &\xrightarrow{d_0 \otimes d_1} H(C_1)^{op} \otimes H(C_1) \\ &\xrightarrow{S \otimes \mathsf{id}} H(C_1) \otimes H(C_1) \\ &\xrightarrow{\nabla} H(C_1). \end{aligned}$$

Here $S\colon H(C_1)^{op} \to H(C_1)$ is the antipode map for the free Hopf algebra $H(C_2)$, and $\nabla\colon H(C_1)\otimes H(C_1) \to H(C_1)$ is the multiplication map.

A straightforward computation with simplicial relations shows that

$$d'(s_0(c)) = \varepsilon(c), \quad c \in C_1 \quad \text{and} \quad d'(r(c)) = 0, \quad c \in C_3.$$

Therefore the map $d'\colon H(C_2) \to H(C_1)$ factors through a unique map

$$d\colon H(C_2/s_0C_1)/I \to H(C_1/s_0C_0),$$

where $I \subset H(C_2/s_0C_1)$ is the two-sided ideal generated by relations $r(c) = 0, c \in C_3$. We show that d is a left inverse for the map $GZ(f) = s_1$, i.e

$$ds_1(x) = x,\ x \in H(C_1/s_0C_0).$$

We can assume that $x \in H(C_1/s_0C_0)$ is a generator, i.e $x \in C_1/s_0C_0$. Let $\Delta(x) = \sum_i x_i^{(1)} \otimes x_i^{(2)}$, then

$$ds_1(x) = \sum_i S(d_0s_1(x_i^{(1)}))d_1s_1(x_i^{(2)}) = \sum_i S(s_0d_1(x_i^{(1)}))(x_i^{(2)}) = \sum_i \varepsilon(x_i^{(1)})(x_i^{(2)}) = x. \qquad \square$$

Corollary 3.1.18. *Let $C_\bullet \in \mathsf{sCoAlg}^{tr}$, then the induced map*

$$\mathrm{Dec}_0(C_\bullet)/C_1 \to NGZ(\mathrm{Dec}_0(C_\bullet)/C_1)$$

is a weak equivalence.

Proof. In the commutative diagram

$$\begin{array}{ccc} \mathrm{Dec}_0(\mathrm{sk}_1 C_\bullet)/C_1 & \longrightarrow & NGZ(\mathrm{Dec}_0(\mathrm{sk}_1 C_\bullet)/C_1) \\ \downarrow & & \downarrow \\ \mathrm{Dec}_0(C_\bullet)/C_1 & \longrightarrow & NGZ(\mathrm{Dec}_0(C_\bullet)/C_1), \end{array}$$

the top arrow is a weak equivalence by Lemma 3.1.10, the left vertical arrow is a weak equivalence by Lemma 3.1.15, and the right vertical arrow is an isomorphism by Proposition 3.1.16. $\square$

Since the category CoAlg^{tr} is complete, the total décalage functor Dec has a right adjoint

$$T\colon \mathsf{ssCoAlg}^{tr} \to \mathsf{sCoAlg}^{tr}, \tag{3.1.19}$$

which is called the *Artin-Mazur codiagonal*, see [66].

Theorem 3.1.20. *The Artin-Mazur codiagonal functor T satisfies following properties:*

1. *the unit map $C_\bullet \to T\mathrm{Dec}(C_\bullet)$ is a weak equivalence for any $C_\bullet \in \mathsf{sCoAlg}^{tr}$.*
2. *there is a natural weak equivalence $d \to T$, where $d\colon \mathsf{ssCoAlg}^{tr} \to \mathsf{sCoAlg}^{tr}$ is the diagonal functor, i.e. $d(C_{\bullet,\bullet})_n = C_{n,n}$, $C_{\bullet,\bullet} \in \mathsf{ssCoAlg}^{tr}$.*
3. *T maps levelwise weak equivalences of bisimplicial coalgebras to weak equivalences of simplicial coalgebras.*

Proof. By [53, Corollary 2.3], the unit map $C_\bullet \to T\mathrm{Dec}(C_\bullet)$ is a deformation retract, and so it is a weak equivalence.

The natural weak equivalence $d \to T$ of simplicial coalgebras was constructed in the proof of [66, Theorem 1.1]. Finally, T preserves levelwise weak equivalence because d does so. $\square$

Let us denote by sHopf^{pr} the category of simplicial objects in the category Hopf^{pr} of primitively generated Hopf algebras. Furthermore, let

$$\mathsf{s}_0\mathsf{sCoAlg}^{tr} \subset \mathsf{ssCoAlg}^{tr} \tag{3.1.21}$$

be the full subcategory of bisimplicial coalgebras spanned by *horizontally* reduced ones, i.e. $C_{\bullet,\bullet} \in \mathsf{ssCoAlg}^{tr}$ lies in $\mathsf{s}_0\mathsf{sCoAlg}^{tr}$ if and only if $C_{0,m} = \mathbf{F}$ for all $m \geq 0$. Finally, we extend the nerve functor (3.1.3) on simplicial Hopf algebras as follows

$$N\colon \mathsf{sHopf}^{pr} \to \mathsf{s}_0\mathsf{sCoAlg}^{tr},$$
$$N(H_\bullet)_{n,m} = N_n(H_m).$$

Definition 3.1.22. The *twisted bar construction* $\overline{W}\colon \mathsf{sHopf}^{pr} \to \mathsf{sCoAlg}^{tr}$ is given by the following composite:

$$\overline{W}\colon \mathsf{sHopf}^{pr} \xrightarrow{N} \mathsf{s}_0\mathsf{sCoAlg}^{tr} \overset{\iota}{\hookrightarrow} \mathsf{ssCoAlg}^{tr} \xrightarrow{T} \mathsf{sCoAlg}^{tr}.$$

Let sAlg^{aug} be the category of simplicial (augmented) associative algebras over a field $\mathbf{F}$. Similar to Definition 3.1.22, we define the twisted bar construction for augmented associative algebras

$$\overline{W}\colon \mathsf{sAlg}^{aug} \to \mathsf{sVect}_{\mathbf{F}}. \tag{3.1.23}$$

Note that the following diagram

$$\begin{array}{ccc} \mathsf{sHopf}^{pr} & \xrightarrow{\overline{W}} & \mathsf{sCoAlg}^{tr} \\ \downarrow{\scriptstyle \mathsf{oblv}} & & \downarrow{\scriptstyle \mathsf{oblv}} \\ \mathsf{sAlg}^{aug} & \xrightarrow{\overline{W}} & \mathsf{sVect}_{\mathbf{F}} \end{array}$$

commutes.

Remark 3.1.24. By [20] or [66, Lemma 5.2], the definition of the twisted bar construction $\overline{W}$ given here coincides with the one given in [58, Section 2.3] (see also [50, p.12-05]).

Proposition 3.1.25. *The twisted bar construction $\overline{W}\colon \mathsf{sAlg}^{aug} \to \mathsf{sVect}_{\mathbf{F}}$ (resp. $\overline{W}\colon \mathsf{sHopf}^{pr} \to \mathsf{sCoAlg}^{tr}$) preserves weak equivalences.*

Proof. Let $f\colon A_\bullet \to A'_\bullet$ be a weak equivalence of simplicial augmented associative algebras. Then $N(f)$ is a vertical weak equivalence of bisimplicial vector spaces, and so is $\overline{W}(f) = TN(f)$ by Theorem 3.1.20 (see also [66, Theorem 1.1]). □

Definition 3.1.26. Let $Q\colon \mathsf{sAlg}^{aug} \to \mathsf{sVect}_{\mathbf{F}}$ denote the functor of *indecomposable elements*, i.e. $Q(A_\bullet) = I_\bullet/I_\bullet^2$, where $I_\bullet \subset A_\bullet$ is the augmentation ideal.

Proposition 3.1.27. *There is a natural transformation*

$$\eta_{A_\bullet}\colon \overline{W}(A_\bullet)/\mathbf{F} \to \Sigma_\bullet Q(A_\bullet), \quad A_\bullet \in \mathsf{sAlg}^{aug},$$

where $\mathbf{F} \subset \overline{W}(A_\bullet)$ is a constant simplicial vector space spanned by the unit elements in $A_\bullet$ and $\Sigma_\bullet Q(A_\bullet) \in \mathsf{sVect}_{\mathbf{F}}$ is the Kan suspension of the simplicial vector space $Q(A_\bullet)$. Moreover, $\eta_{A_\bullet}$ is a weak equivalence if $A_\bullet \in \mathsf{sAlg}^{aug}$ is a degreewise free associative algebra.

Proof. There is a natural map of vector spaces

$$A/\mathbf{F} \cong I \to I/I^2 = Q(A), \quad A \in \mathsf{sAlg}^{aug},$$

where $I \subset A$ is the augmentation ideal. This map induces the following map of simplicial vector spaces

$$\overline{W}_\bullet(A_\bullet)/\mathbf{F} \to \Sigma_\bullet Q(A_\bullet). \tag{3.1.28}$$

We show now that the map (3.1.28) is a weak equivalence if $A_t \cong T(V_t)$ is a tensor algebra, $V_t \in \mathsf{Vect}_{\mathbf{F}}$, $t \geq 0$. By part (2) of Theorem 3.1.20 and the spectral sequence of a bisimplicial set, we have a strongly convergent spectral sequence

$$E^1_{s,t} = \mathrm{Tor}^{A_t}_s(\mathbf{F}, \mathbf{F}) \Rightarrow \pi_{s+t}(dN_\bullet A_\bullet) \cong \pi_{s+t}(\overline{W}A_\bullet),$$

where $A_\bullet \in \mathsf{sAlg}^{aug}$, see e.g. [59] or [29, Section IV.2.2]. Since $A_t \cong T(V_t)$ is a free associative algebra, we have $E^1_{s,t} = 0$ for $s > 1$ and $E^1_{1,t} \cong V_t$, see e.g. [56, Example 2.2(1)] or [54, §1.2, 1.5]. This implies the proposition. □

Note that the fully faithful embedding $\iota\colon \mathsf{s_0sCoAlg}^{tr} \hookrightarrow \mathsf{ssCoAlg}^{tr}$ has a left adjoint

$$R\colon \mathsf{ssCoAlg}^{tr} \to \mathsf{s_0sCoAlg}^{tr}$$

given by

$$R(C_{\bullet,\bullet})_{n,m} = C_{n,m}/C_{0,m} \tag{3.1.29}$$

for all $C_{\bullet,\bullet} \in \mathsf{ssCoAlg}^{tr}$. Therefore the twisted bar construction $\overline{W}$ has a left adjoint

$$G\colon \mathsf{sCoAlg}^{tr} \to \mathsf{sHopf}^{pr}, \tag{3.1.30}$$

which is given by $G(C_\bullet)_m = GZ(\mathrm{Dec}_m(C_\bullet)/C_{m+1})$, $C_\bullet \in \mathsf{sCoAlg}^{tr}$.

Definition 3.1.31. Let $C_\bullet$ be a simplicial truncated coalgebra, then $G(C_\bullet) \in \mathsf{sHopf}^{pr}$ is called the *Hopf-Kan loop algebra* of $C_\bullet$.

Theorem 3.1.32. *Let $C_\bullet \in \mathsf{s_0CoAlg}^{tr}$ be a reduced simplicial truncated coalgebra. Then the unit map*

$$\eta\colon C_\bullet \to \overline{W}G(C_\bullet)$$

is a weak equivalence.

Proof. The units of the adjunctions $\mathrm{Dec} \dashv T$, $R \dashv \iota$, and $N \dashv GZ$ give a factorization of η

$$C_\bullet \to T\mathrm{Dec}C_\bullet \to T\iota R\mathrm{Dec}(C_\bullet) \to T\iota NGZR\mathrm{Dec}(C_\bullet)$$

in sCoAlg^{tr}. The map $C_\bullet \to T\mathrm{Dec}C_\bullet$ is a weak equivalence by Theorem 3.1.20. The maps $T\mathrm{Dec}(C_\bullet) \to T\iota R\mathrm{Dec}(C_\bullet)$ and $T\iota R\mathrm{Dec}(C_\bullet) \to T\iota NGZR\mathrm{Dec}(C_\bullet)$ are induced by the maps

$$\mathrm{Dec}(C_\bullet) \to \iota R\mathrm{Dec}(C_\bullet) \quad \text{and} \quad R\mathrm{Dec}(C_\bullet) \to NGZR\mathrm{Dec}(C_\bullet)$$

in $\mathsf{ssCoAlg}^{tr}$. We will show that both of these maps are levelwise weak equivalences.

The first map is a vertical weak equivalence, since

$$(R\mathrm{Dec}(C_\bullet))_{n,\bullet} = \mathrm{Dec}(C_\bullet)_{n,\bullet}/\mathrm{Dec}(C_\bullet)_{0,\bullet}$$

and $\mathrm{Dec}(C_\bullet)_{0,\bullet}$ is a deformation retract of $C_0 \cong \mathbf{F}$.

The second map is a horizontal weak equivalence due to Corollary 3.1.18 and the observation (3.1.13). Since T maps levelwise weak equivalences of bisimplicial coalgebras to weak equivalences, the theorem follows. □

Using Example 3.1.8 one can check the following.

Proposition 3.1.33. *The Hopf-Kan loop functor $G\colon \mathsf{sCoAlg}^{tr} \to \mathsf{sHopf}^{pr}$ is given as follows.*

1. *The Hopf algebra $G(C_\bullet)_m$ is the free Hopf algebra $H(C_{m+1}/s_0C_m)$ generated by the quotient coalgebra C_{m+1}/s_0C_m.*
2. *The face and degeneracy operators are defined on the generators of $G(C_\bullet)$ by the following formulas*

$$d_i[x] = [d_{i+1}x] \ \text{ if } i>0, \quad s_i[x] = [s_{i+1}x] \ \text{ if } i \geq 0,$$
$$d_0[x] = \sum_i S([d_0(x_i^{(1)})])[d_1(x_i^{(2)})];$$

where $x \in C_{m+1}$, $\Delta(x) = \sum_i x_i^{(1)} \otimes x_i^{(2)}$, $[y]$ is the class of $y \in C_{n+1}$ in C_{n+1}/s_0C_n, and

$$S\colon H(C_m/s_0C_{m-1})^{op} \to H(C_m/s_0C_{m-1})$$

is the antipode. □

3.2 Model structures. In this section we will construct model structures for categories of simplicial restricted Lie algebras sLie^r and reduced simplicial truncated coalgebras $\mathsf{s_0CoAlg}^{tr}$. We refer the reader to [62], [33], and [40, Appendices A.2-3] for most of definitions in this section.

Definition 3.2.1 (J. H. Smith)**.** A model category C is *combinatorial* if C is cofibrantly generated ([33, Definition 11.1.2]) and C is locally presentable.

We recall generating sets for simplicial model categories sSet and $\mathsf{sVect}_{\mathbf{F}}$. Define the following sets of morphisms in sSet

$$I_\Delta = \{\partial\Delta^n \hookrightarrow \Delta^n \mid n \geq 0\}, \quad J_\Delta = \{\Lambda^n_i \hookrightarrow \Delta^n \mid n > 0,\ 0 \leq i \leq n\},$$

where $\Lambda^n_i \subset \Delta^n$ (resp. $\partial\Delta^n \subset \Delta^n$) is the i-th horn (resp. the boundary) of Δ^n. Then the model category sSet is cofibrantly generated by I_Δ, J_Δ. Similarly, define

$$I_{\mathsf{Vect}} = \{\mathbf{F}(\partial\Delta^n) \hookrightarrow \mathbf{F}(\Delta^n) \mid n \geq 0\},$$

$$J_{\mathsf{Vect}} = \{\mathbf{F}(\Lambda^n_i) \hookrightarrow \mathbf{F}(\Delta^n) \mid n > 0,\ 0 \leq i \leq n\},$$

where $\mathbf{F}(X_\bullet), X_\bullet \in \mathsf{sSet}$ is the simplicial vector space spanned by $X_\bullet$. Then the model category sVect is cofibrantly generated by $I_{\mathsf{Vect}}, J_{\mathsf{Vect}}$.

Remark 3.2.2. By [62, Proposition 1, pII.3.8], a map $f\colon U_\bullet \to W_\bullet \in \mathsf{sVect}_{\mathbf{F}}$ of simplicial vector spaces is a *fibration* in the model structure above if and only if

1. the induced map $U_\bullet \to W_\bullet \times_{\pi_0(W_\bullet)} \pi_0(U_\bullet)$ is degreewise surjective, where $\pi_0(U_\bullet)$ and $\pi_0(W_\bullet)$ are constant simplicial vector spaces;
2. or equivalently, the induced map $N(f)\colon N(U_\bullet) \to N(W_\bullet)$ of normalized chain complexes (see Section 1.4) is surjective in positive degrees.

Theorem 3.2.3. *There exists a simplicial combinatorial right proper model structure on the category* sLie^r *such that a map* $f\colon L'_\bullet \to L_\bullet$ *is*

- *a* weak equivalence *if and only if* $\pi_*(f)$ *is an isomorphism;*
- *a* fibration *if and only if* $\mathsf{oblv}(f)\colon \mathsf{oblv}L'_\bullet \to \mathsf{oblv}L_\bullet$ *is a fibration in* $\mathsf{sVect}_{\mathbf{F}}$ *(see Remark 3.2.2);*
- *a* cofibration *if and only if* f *has the left lifting property with respect to all acyclic fibrations.*

Moreover,

$$I_{\mathsf{Lie}} = L^r(I_{\mathsf{Vect}}) = \{L^r(u) \mid u \in I_{\mathsf{Vect}}\}$$

is a set of generating cofibrations and

$$J_{\mathsf{Lie}} = L^r(J_{\mathsf{Vect}}) = \{L^r(v) \mid v \in J_{\mathsf{Vect}}\}$$

is a set of generating trivial cofibrations for the model category sLie^r.

Proof. Since the category Lie^r is complete and cocomplete, the category sLie^r of simplicial objects has the canonical simplicial enrichment. By [62, Theorem 4, pII.4.1], the category sLie^r is a simplicial model category equipped with notions of weak equivalences, fibrations and cofibrations as defined above. Furthermore, by an implicit argument in ibid. the category sLie^r is cofibrantly generated; more explicitly, one can use [33, Theorem 11.3.2]. Indeed, the sets $I_{\mathsf{Lie}}, J_{\mathsf{Lie}}$ clearly permit the small object argument ([33, Definition 10.5.15]) and the forgetful functor oblv takes relative J_{Lie}-cell complexes to weak equivalences because any map in J_{Vect} is a homotopy equivalence.

Since any object in sLie^r is fibrant, the model category sLie^r is *right proper* by [33, Corollary 13.1.3(2)]. Finally, the category sLie^r is locally presentable by Proposition 2.1.9. □

Remark 3.2.4. At the time of writing, we are not aware whether or not the model structure of Theorem 3.2.3 on sLie^r is *left proper*, cf. [63, Section 2.3].

We will discuss cofibrations in sLie^r; we encourage the reader to compare the next definitions with the definition of an almost-free morphism in the category of simplicial commutative algebras, e.g. given in [27, p. 23] or in [48, Definition 3.3'].

Definition 3.2.5. The *almost-simplex category* $\widetilde{\Delta}$ is the category of finite ordered sets $[n] = \{0, \dots, n\}, n \geq 0$ together with order-preserving maps which send 0 to 0 (cf. the first definition of Section 2 in [48]).

Let us denote by $\tilde{\mathsf{s}}\mathsf{C}$ the category of *almost-simplicial objects* in a category C, i.e. $\tilde{\mathsf{s}}\mathsf{C}$ is the category of contravariant functors from $\widetilde{\Delta}$ to C.

Remark 3.2.6. By an analog of the Dold-Kan correspondence, the category $\tilde{\mathsf{s}}\mathsf{Vect}_{\mathbf{F}}$ of almost-simplicial vector spaces is equivalent to the category of *graded* vector spaces, see [48, pp.607-608]. Thus, a map $f\colon U_\bullet \to W_\bullet$ in sVect is a monomorphism if and only if there is an almost-simplicial vector subspace $V_\bullet$ of $W_\bullet$ such that the natural map $U_n \oplus V_n \to W_n$ is an isomorphism for each $n \geq 0$.

The last paragraph inspires the next definition.

Definition 3.2.7. A morphism $f\colon L'_\bullet \to L_\bullet$ in sLie^r is called *almost-free* if there is an almost-simplicial vector subspace $V_\bullet$ of $L_\bullet$ such that the natural map of almost-simplicial restricted Lie algebras $L'_\bullet \sqcup L^r(V_\bullet) \to L_\bullet$ is an isomorphism in $\tilde{\mathsf{s}}\mathsf{Lie}^r$.

Finally, we say that a simplicial restricted Lie algebra $L_\bullet \in \mathsf{sLie}^r$ is *almost-free* if the morphism $0 \to L_\bullet$ is almost-free. We notice that an almost-free simplicial restricted Lie algebra is "free" in the sense of [58, Section 3.2], but not vice versa.

The following proposition can be proved exactly as the similar result in [47, Theorem 3.4] (see also the correction [48]).

Proposition 3.2.8. *Any almost-free morphism $f\colon L'_\bullet \to L_\bullet$ is a cofibration.* □

Remark 3.2.9. Similar to the case of simplicial commutative algebras, one can show that for any map $f\colon L'_\bullet \to L_\bullet$ there is a functorial factorization

$$f\colon L'_\bullet \xrightarrow{i} Q_\bullet(f) \xrightarrow{p} L_\bullet,$$

where i is almost-free and p is an acyclic fibration, cf. [27, Theorem 1.3 and Proposition 1.4]. In particular, any cofibration in sLie^r is a retract of an almost-free morphism.

Next, we show that the category $\mathsf{s}_0\mathsf{CoAlg}^{tr}$ also can be equipped with a model structure.

Theorem 3.2.10. *There exists a simplicial combinatorial left proper model structure on the category $\mathsf{s}_0\mathsf{CoAlg}^{tr}$ such that $f\colon C_\bullet \to D_\bullet$ is*

- *a* weak equivalence *if and only if $\pi_*(f)$ is an isomorphism;*
- *a* cofibration *if and only if $f\colon C_\bullet \to D_\bullet$ is degreewise injective;*
- *a* fibration *if and only if f has the right lifting property with respect to all acyclic cofibrations.*

Remark 3.2.11. P. Goerss in [28, Section 3] showed that the category sCoAlg of simplicial coalgebras over a field $\mathbf{F}$ endowed with the same notions of weak equivalences and cofibrations is a simplicial model category. It seems likely that one can straightforwardly adapt his argument for the category $\mathsf{s_0CoAlg}^{tr}$ as well. Nevertheless, we will prove Theorem 3.2.10 by applying more general technique.

Proof. Since the category CoAlg^{tr} of truncated coalgebras is complete and cocomplete, the category $\mathsf{s_0CoAlg}^{tr}$ of reduced simplicial objects has the canonical simplicial enrichment. Recall that finite coproducts in CoAlg^{tr} are wedge sums (see Example 2.2.16), therefore a coproduct of reduced simplicial truncated coalgebras (computed in sCoAlg^{tr}) is still reduced. Moreover, for any $C_\bullet \in \mathsf{s_0CoAlg}^{tr}$, we have a factorization

$$C_\bullet \sqcup C_\bullet = C_\bullet \times \partial\Delta^1 \overset{j}{\hookrightarrow} C_\bullet \times \Delta^1 \overset{p}{\to} C_\bullet \tag{3.2.12}$$

of the fold map $\nabla\colon C_\bullet \sqcup C_\bullet \to C_\bullet$ such that j is a cofibration and p is a weak equivalence.

We now apply Theorem 2.2.1 from [32] to the adjoint pair

$$\mathsf{oblv} : \mathsf{s_0CoAlg}^{tr} \rightleftarrows \mathsf{s_0Vect}_{\mathbf{F}} : \mathrm{Sym}^{tr}$$

in order to obtain the desired model structure on $\mathsf{s_0CoAlg}^{tr}$. Indeed, the model category $\mathsf{s_0Vect}_{\mathbf{F}}$ is cofibrantly generated; the category $\mathsf{s_0CoAlg}^{tr}$ is locally presentable by Proposition 2.2.15; $\mathsf{oblv}(C_\bullet), C_\bullet \in \mathsf{s_0CoAlg}^{tr}$ is a cofibrant object in $\mathsf{s_0Vect}_{\mathbf{F}}$ because any object in $\mathsf{s_0Vect}_{\mathbf{F}}$ is cofibrant; and finally, the factorization (3.2.12) fulfills the third condition of [32, Theorem 2.2.1]. The obtained model structure on $\mathsf{s_0CoAlg}^{tr}$ is cofibrantly generated by sets due to [6, Theorem 2.2.3].

It is clear that the axiom SM7b from [62, pII.2.3] holds for the constructed model structure on $\mathsf{s_0CoAlg}^{tr}$, and so $\mathsf{s_0CoAlg}^{tr}$ is a simplicial model category. Finally, $\mathsf{s_0CoAlg}^{tr}$ is left proper because any object in $\mathsf{s_0CoAlg}^{tr}$ is cofibrant. □

Remark 3.2.13. Let κ be an infinite regular cardinal greater than the cardinality of $\mathbf{F}$, and let I_{CoAlg} be the set of isomorphism classes of inclusions

$$C_\bullet \hookrightarrow D_\bullet,\ C_\bullet, D_\bullet \in \mathsf{s_0CoAlg}^{tr}$$

such that the cardinality of a basis in $D_\bullet$ is at most κ. Similarly, let $J_{\mathsf{CoAlg}} \subset I_{\mathsf{CoAlg}}$ be the set of isomorphism classes of inclusions as above which are weak equivalences. Then one can show that I_{CoAlg} (resp. J_{CoAlg}) is a set of generating (resp. trivial) cofibrations for the model structure of Theorem 3.2.10 on the category $\mathsf{s_0CoAlg}^{tr}$. We are not aware if it is possible to choose more practical generating sets.

The next proposition was proven by S. Priddy in [58, Proposition 2.8]. Here we repeat the argument for the reader's convenience.

Proposition 3.2.14. *The functor $U^r\colon \mathsf{sLie}^r \to \mathsf{sHopf}^{pr}$ preserves weak equivalences.*

Proof. Let $f\colon L'_\bullet \to L_\bullet$ be a weak equivalence. Filter both $U^r(L'_\bullet)$ and $U^r(L_\bullet)$ by their Lie filtrations, see Section 2.3. Since $U^r(f)$ preserves Lie filtrations, it induces a map of associated spectral sequences:

$$\begin{array}{ccc} \pi_*E_0U^r(L'_\bullet) & \xrightarrow{E_0U^r(f)} & \pi_*E_0U^r(L_\bullet) \\ \Downarrow & & \Downarrow \\ \pi_*U^r(L'_\bullet) & \xrightarrow{\pi_*U^r(f)} & \pi_*U^r(L_\bullet). \end{array}$$

According to the Poincaré-Birkhoff-Witt theorem (Theorem 2.3.2), we have isomorphisms $E_0U^r(L'_\bullet) \cong \mathrm{Sym}^{tr}(L'_\bullet)$ and $E_0U^r(L_\bullet) \cong \mathrm{Sym}^{tr}(L_\bullet)$, and so the induced map $E_0U^r(f)$ is an isomorphism as well. The Lie filtrations on $U^r(L'_\bullet)$ and $U^r(L_\bullet)$ are complete and bounded below, and so both spectral sequences converge strongly. Therefore $E^\infty U^r(f)$ and $\pi_*U^r(f)$ are also isomorphisms. □

Corollary 3.2.15. *The functor $\overline{W}U^r\colon \mathsf{sLie}^r \to \mathsf{s_0CoAlg}^{tr}$ preserves weak equivalences.*

Proof. See Propositions 3.2.14 and 3.1.25. □

Next, following [58, Proposition 3.5], we calculate the composite

$$\mathsf{oblv} \circ \overline{W}U^r\colon \mathsf{sLie}^r \to \mathsf{sVect}_{\mathbf{F}}.$$

For that we consider a functor

$$\mathsf{triv}\colon \mathsf{Vect}_{\mathbf{F}} \to \mathsf{Lie}^r$$

which maps a vector space V to the restricted p-abelian Lie algebra $\mathsf{triv}(V)$ with the underlying vector spaces equal to V equipped with identically zero Lie bracket and p-operation.

We observe that the functor triv has a left adjoint

$$\mathsf{Ab}_\xi\colon \mathsf{Lie}^r \to \mathsf{Vect}_{\mathbf{F}} \tag{3.2.16}$$

given by $L \mapsto L/([L,L]+\xi(L))$, where $[L,L] \subset L$ is the (restricted) Lie ideal generated by all elements of the form $[x,y]$, $x,y \in L$.

We extend the adjoint pair $\mathsf{Ab}_\xi \dashv \mathsf{triv}$ degreewise to the adjunction

$$\mathsf{Ab}_\xi : \mathsf{sLie}^r \rightleftarrows \mathsf{sVect}_{\mathbf{F}} : \mathsf{triv} \tag{3.2.17}$$

between categories of simplicial objects. Note that the adjunction (3.2.17) is a Quillen adjunction because the composite $\mathsf{oblv} \circ \mathsf{triv} = \mathsf{id}$, and so the functor triv preserves weak equivalences and fibrations.

Proposition 3.2.18. *There is a natural transformation*

$$\eta_{L_\bullet}\colon \mathsf{oblv} \circ \overline{W}U^r(L_\bullet) \to \Sigma_\bullet \mathsf{Ab}_\xi(L_\bullet), \quad L_\bullet \in \mathsf{sLie}^r.$$

Moreover, $\eta_{L_\bullet}$ is a weak equivalence if $L_\bullet \in \mathsf{sLie}^r$ is almost-free.

Here $\Sigma_\bullet \mathsf{Ab}_\xi(L_\bullet) \in \mathsf{sVect}_{\mathbf{F}}$ is the Kan suspension of the simplicial vector space $\mathsf{Ab}_\xi(L_\bullet)$, see [29, Section III.5].

Proof. Note that there is a natural isomorphism

$$QU^r(L) \cong \mathsf{Ab}_\xi(L), \quad L \in \mathsf{Lie}^r,$$

where $U^r(L) \in \mathsf{Hopf}^{pr}$ is the universal enveloping algebra and $QU^r(L)$ is the module of indecomposable elements, see Definition 3.1.26. Finally, Proposition 3.1.27 implies the assertion. □

Example 3.2.19. Let $V_\bullet \in \mathsf{sVect}_{\mathbf{F}}$ be a simplicial vector space, let $\Sigma_\bullet V$ be the Kan suspension of $V_\bullet$ (see [29, Section III.5]), and let $\mathsf{triv}(\Sigma_\bullet V)$ be a trivial simplicial coalgebra (Example 2.2.7). Using Proposition 3.1.33 one can show that $PG(\mathsf{triv}\Sigma_\bullet V) \cong L^r(V_\bullet)$. Moreover, the adjoint map

$$\mathsf{triv}(\Sigma_\bullet V) \to \overline{W}U^r(L^r(V_\bullet)) \tag{3.2.20}$$

is a weak equivalence by Theorem 3.1.32.

Definition 3.2.21. A map $f\colon L'_\bullet \to L_\bullet$ in sLie^r is an $\mathbf{F}$-*equivalence* if and only if $\overline{W}U^r(f)$ is a weak equivalence in $\mathsf{s}_0\mathsf{CoAlg}^{tr}$.

Example 3.2.22. By Corollary 3.2.15, any weak equivalence in sLie^r is an $\mathbf{F}$-equivalence. Furthermore, by Theorem 3.1.32, if $f\colon C_\bullet \to D_\bullet$ is a weak equivalence in $\mathsf{s}_0\mathsf{CoAlg}^{tr}$, then $PG(f)$ is an $\mathbf{F}$-equivalence.

Lemma 3.2.23. *Let*

$$\begin{array}{ccc} X_\bullet & \xrightarrow{f} & X'_\bullet \\ \big\downarrow{\scriptstyle g} & & \big\downarrow{\scriptstyle g'} \\ Y_\bullet & \longrightarrow & Y'_\bullet \end{array}$$

be a pushout square in the category sAlg^{aug} *of simplicial augmented associative algebras over the field* $\mathbf{F}$*. Suppose that* $\overline{W}(g)$ *is a weak equivalence of simplicial vector spaces and* f *is an almost-free morphism. Then* $\overline{W}(g')$ *is also a weak equivalence.*

Proof. Since the model structure on the category sAlg^{aug} is left proper (see [63, Section 2.3 and Example 2.7]) and the functor $\overline{W}$ preserves weak equivalences (see Proposition 3.1.25), we can assume that $X_\bullet, Y_\bullet$ are almost-free objects in sAlg^{aug} and $g\colon X_\bullet \to Y_\bullet$ is an almost-free morphism. Then both $X'_\bullet, Y'_\bullet$ are almost-free, and so by Proposition 3.1.27, it suffices to show that

$$\Sigma_\bullet Q(g')\colon \Sigma_\bullet Q(X'_\bullet) \to \Sigma_\bullet Q(Y'_\bullet)$$

is a weak equivalence in $\mathsf{sVect}_\mathbf{F}$. However, by the assumption, the morphism $Q(g)$ is a weak equivalence and the functor $Q\colon \mathsf{sAlg}^{aug} \to \mathsf{sVect}_\mathbf{F}$ preserves cofibrations and colimits. Since the model category $\mathsf{sVect}_\mathbf{F}$ is left proper, the lemma follows. □

Theorem 3.2.24. *There exists a simplicial combinatorial left proper model structure on the category* sLie^r *such that a map* $f\colon L'_\bullet \to L_\bullet$ *is*

- *a* weak equivalence *if and only if* f *is an* $\mathbf{F}$*-equivalence;*
- *a* cofibration *if and only if* f *is a cofibration in the model structure of Theorem 3.2.3.*
- *a* fibration *if and only if* f *has the right lifting property with respect to all* $\mathbf{F}$*-acyclic cofibrations.*

Proof. We construct the desired model structure by using Proposition A.2.6.15 from [40]. Let I_{Lie} be the generating set of cofibrations from Theorem 3.2.3 and let $\mathcal{W}_\mathbf{F}$ be the class of morphisms in sLie^r spanned by $\mathbf{F}$-equivalences. It is enough to prove that

1. the class $\mathcal{W}_\mathbf{F}$ is perfect, see [40, Definition A.2.6.12]. We recall that a class of morphisms $\mathcal{W}$ is called *perfect* if all isomorphisms are in $\mathcal{W}$, $\mathcal{W}$ has the two-out-of-three property, $\mathcal{W}$ is closed under filtered colimits, and there is a (small) subset $\mathcal{W}_0 \subset \mathcal{W}$ which generates $\mathcal{W}$ by filtered colimits.
2. a pushout of $g \in \mathcal{W}_\mathbf{F}$ along an almost-free morphisms is again in $\mathcal{W}_\mathbf{F}$;
3. if $g\colon L'_\bullet \to L_\bullet$ is a morphism in sLie^r which has the right lifting property with respect to all morphisms in I_{Lie}, then g is in $\mathcal{W}_\mathbf{F}$.

By the definition, the class $\mathcal{W}_\mathbf{F}$ is the preimage $(\overline{W}U^r)^{-1}(\mathcal{W}_{\mathsf{CoAlg}})$, where $\mathcal{W}_{\mathsf{CoAlg}}$ is the class of weak equivalences in $\mathsf{s}_0\mathsf{CoAlg}^{tr}$. We claim that the class $\mathcal{W}_{\mathsf{CoAlg}}$ is perfect. Indeed, any isomorphism is in $\mathcal{W}_{\mathsf{CoAlg}}$; $\mathcal{W}_{\mathsf{CoAlg}}$ has the two-out-of-three-property; $\mathcal{W}_{\mathsf{CoAlg}}$ is closed under filtered colimits; and finally, $\mathcal{W}_{\mathsf{CoAlg}}$ is an accessible subcategory in the arrow category $\mathrm{Ar}(\mathsf{s}_0\mathsf{CoAlg}^{tr})$ by

Theorem 3.2.10 and [40, Corollary A.2.6.8]. Since the functor $\overline{W}U^r$ preserves filtered colimits, the class $\mathcal{W}_{\mathbf{F}}$ is also perfect by [40, Corollary A.2.6.14].

For the second part, let

$$\begin{array}{ccc} X_\bullet & \xrightarrow{f} & X'_\bullet \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g'} \\ Y_\bullet & \longrightarrow & Y'_\bullet \end{array}$$

be a pushout square in sLie^r such that g is an $\mathbf{F}$-equivalence and f is almost-free. We will show that g' is again an $\mathbf{F}$-equivalence. Since the functor $U^r\colon \mathsf{Lie}^r \to \mathsf{Alg}$ is left adjoint, the diagram

$$\begin{array}{ccc} U^r(X_\bullet) & \xrightarrow{U^r(f)} & U^r(X'_\bullet) \\ \downarrow{\scriptstyle U^r(g)} & & \downarrow{\scriptstyle U^r(g')} \\ U^r(Y_\bullet) & \longrightarrow & U^r(Y'_\bullet) \end{array}$$

is a pushout square in sAlg, and so is in sAlg^{aug}. Moreover, $U^r(f)$ is an almost-free morphism of augmented associative algebras and $\overline{W}(U^r(g))$ is a weak equivalence of simplicial vector spaces. The claim follows now by Lemma 3.2.23.

Finally, if $g\colon L'_\bullet \to L_\bullet$ is a morphism in sLie^r which has the right lifting property with respect to all morphisms in I_{Lie}, then g is a weak equivalence by Theorem 3.2.3. By Corollary 3.2.15, any weak equivalence is in $\mathcal{W}_{\mathbf{F}}$. □

Using the notation from the proof of Theorem 3.2.24 we write $\mathcal{W}_{\mathsf{CoAlg}}$ for the class of weak equivalences in the category $\mathsf{s_0CoAlg}^{tr}$ and we write $\mathcal{W}_{\mathbf{F}}$ for the class of $\mathbf{F}$-equivalences in sLie^r. Moreover, we use $\mathcal{W}_{\mathsf{Lie}}$ for the class of usual weak equivalences in sLie^r from Theorem 3.2.3.

Definition 3.2.25. We denote by $s\mathcal{CA}_0 = \mathsf{s_0CoAlg}^{tr}[\mathcal{W}_{\mathsf{CoAlg}}^{-1}]$ the ∞-category obtained from the (ordinary) category $\mathsf{s_0CoAlg}^{tr}$ by inverting the class $\mathcal{W}_{\mathsf{CoAlg}}$, see [41, Definition 1.3.4.1]. Similarly, $s\mathcal{L} = \mathsf{sLie}^r[\mathcal{W}_{\mathsf{Lie}}^{-1}]$ is the ∞-category obtained from sLie^r by inverting $\mathcal{W}_{\mathsf{Lie}}$, and $s\mathcal{L}_\xi = \mathsf{sLie}^r[\mathcal{W}_{\mathbf{F}}^{-1}]$ is the ∞-category obtained from sLie^r by inverting $\mathcal{W}_{\mathbf{F}}$.

Theorem 3.2.26. *The adjoint pair $PG \dashv \overline{W}U^r$ induces an equivalence of ∞-categories*

$$PG : s\mathcal{CA}_0 \simeq s\mathcal{L}_\xi : \overline{W}U^r. \tag{3.2.27}$$

Proof. By Definition 3.2.21, we have $\overline{W}U^r(\mathcal{W}_{\mathbf{F}}) = \mathcal{W}_{\mathsf{CoAlg}}$, and by Theorem 3.1.32, we have $PG(\mathcal{W}_{\mathsf{CoAlg}}) \subset \mathcal{W}_{\mathbf{F}}$. Therefore PG and $\overline{W}U^r$ induce functors

$$PG\colon s\mathcal{CA}_0 \to s\mathcal{L}_\xi, \quad \overline{W}U^r\colon s\mathcal{L}_\xi \to s\mathcal{CA}_0$$

between localized ∞-categories. Similarly, natural transformations

$$\mathsf{id}_{\mathsf{s_0CoAlg}^{tr}} \to \overline{W}U^r \circ PG, \quad PG \circ \overline{W}U^r \to \mathsf{id}_{\mathsf{sLie}^r}$$

induces natural transformations

$$\mathsf{id}_{s\mathcal{CA}_0} \to \overline{W}U^r \circ PG, \quad PG \circ \overline{W}U^r \to \mathsf{id}_{s\mathcal{L}_\xi}$$

for functors between obtained ∞-categories. By Theorem 3.1.32, these natural transformations are natural equivalences. This implies the theorem. □

Remark 3.2.28. We are not aware whether or not the adjoint pair

$$PG : \mathsf{s_0CoAlg}^{tr} \rightleftarrows \mathsf{sLie}^r : \overline{W}U^r.$$

is a Quillen adjunction for any of two model structures on sLie^r.

Let C be a simplicial model category. Then the (simplicial) nerve $N(\mathsf{C}^\circ)$ forms an ∞-category, where $\mathsf{C}^\circ \subset \mathsf{C}$ is the full simplicial subcategory of fibrant-cofibrant objects. The ∞-category $N(\mathsf{C}^\circ)$ is called the *underlying ∞-category* of C, see [40, Section A.2].

Proposition 3.2.29. *The ∞-categories $s\mathcal{L}$, $s\mathcal{L}_\xi$, and $s\mathcal{CA}_0$ are presentable. In particular, $s\mathcal{L}$, $s\mathcal{L}_\xi$, and $s\mathcal{CA}_0$ are complete and cocomplete.*

Proof. Let us denote by sLie^r_ξ the category sLie^r equipped with the model structure of Theorem 3.2.24. By [41, Theorem 1.3.4.20], there are equivalences of ∞-categories:

$$s\mathcal{L} = \mathsf{sLie}^r[\mathcal{W}^{-1}_{\mathsf{Lie}}] \simeq N((\mathsf{sLie}^r)^o), \quad s\mathcal{L}_\xi = \mathsf{sLie}^r[\mathcal{W}^{-1}_{\mathbf{F}}] \simeq N((\mathsf{sLie}^r_\xi)^o),$$

and

$$s\mathcal{CA}_0 = \mathsf{s_0CoAlg}^{tr}[\mathcal{W}^{-1}_{\mathsf{CoAlg}}] \simeq N((\mathsf{s_0CoAlg}^{tr})^o).$$

By [40, Proposition A.3.7.6] and Theorems 3.2.3, 3.2.10, 3.2.24, we get the proposition. □

Recall that sLie^r_ξ is the model category from Theorem 3.2.24. Note that the identity functor $\mathsf{id}_{\mathsf{sLie}^r}$ produces a Bousfield localization

$$\mathsf{sLie}^r \rightleftarrows \mathsf{sLie}^r_\xi, \tag{3.2.30}$$

which by [40, Proposition 5.2.4.6] induces the adjoint pair

$$L_\xi : s\mathcal{L} \rightleftarrows s\mathcal{L}_\xi : \iota_\xi \tag{3.2.31}$$

between underlying ∞-categories.

Proposition 3.2.32. *The functor ι_ξ is fully faithful. In particular, the full subcategory $s\mathcal{L}_\xi \subset s\mathcal{L}$ is a localization.*

Proof. See [40, Appendix A.3.7]. □

4. Homotopy theory of simplicial restricted Lie algebras

This section is a technical heart of the paper. In our proof of Theorem D, we mimic the classical proof (given e.g in [43, Theorem 11.1.1]) that a simply-connected space X is p-complete if and only if its homotopy groups $\pi_n(X), n \geq 2$ are derived p-complete. In order to transfer this proof to our context, we need to show that the category sLie^r shares a lot of common properties and features with the category of (pointed and connected) spaces; in this section we carefully check all required properties. We state our results in a form minimal enough for the proof of Theorem D, although many of them can be easily generalized.

In Section 4.1 we prove that the homotopy excision theorem (Theorem 4.1.3) holds in the category sLie^r of simplicial restricted Lie algebras.

Inspired by Proposition 3.2.18, we define *homology* $H_*(L_\bullet; M)$ (Definition 4.2.4) and *cohomology groups* $H^*(L_\bullet; M)$ (Definition 4.2.23) of $L_\bullet \in \mathsf{sLie}^r$ with coefficients in any module M

over the ring $\mathbf{F}\{\xi\}$. We reformulate Proposition 3.2.18 as follows: $\pi_*(\overline{W}U^r(L_\bullet))$ is the homology groups $H_*(L_\bullet;\mathbf{F})$ with coefficients in the trivial module $\mathbf{F}$. Following [21], we prove the Hurewicz theorem (Theorem 4.2.10), which combined with the homotopy excision theorem gives the relative Hurewicz theorem (Corollary 4.2.17). Finally, in Proposition 4.2.21, we compute the homology groups $H_*(\mathsf{triv}_\xi(M);\mathbf{F})$ of an abelian restricted Lie algebra $\mathsf{triv}_\xi(M)$ provided M is a torsion-free $\mathbf{F}\{\xi\}$-module.

In Section 4.3 we show that any $L_\bullet\in\mathsf{sLie}^r$ has a *Postnikov tower* and we prove in Proposition 4.3.4 that this Postnikov tower has *k-invariants* provided $\pi_0(L_\bullet)=0$. In Section 4.4 we study group objects and *principal fibrations* in sLie^r. We define principal fibrations in Definition 4.4.5 and we show that they are a homotopy invariant of the base in Lemma 4.4.6. Moreover, in Theorem 4.4.10, we construct a *classifying object* $BM_\bullet$ (Definition 4.4.8) for principal $M_\bullet$-fibrations. Finally, we observe in Corollary 4.4.13 that each stage in the Postnikov tower of $L_\bullet$, $\pi_0(L_\bullet)=0$ is weakly equivalent to a principal fibration.

In Section 4.5 we adapt classical approaches of [64, Chapter 9.4] and [68, Chapter 15] to obtain an analog of the *Serre spectral sequence* for principal fibrations in the category sLie^r (Theorem 4.5.11). At the time of writing, we are not aware how to generalize our analog of the Serre spectral sequence to arbitrary fibrations, see Remark 4.5.13.

4.1 Homotopy excision theorem.

We fix some notation.

Definition 4.1.1. A simplicial restricted Lie algebra $L_\bullet\in\mathsf{sLie}^r$ is *n-connected* if $\pi_i(L_\bullet)=0$ for all $i\leq n$. A simplicial restricted Lie algebra $L_\bullet$ is *connected* if it is 0-connected, i.e. $\pi_0(L_\bullet)=0$.

Definition 4.1.2. A morphism $f\colon L'_\bullet\to L_\bullet$ in sLie^r is *n-connected* if the induced map on homotopy groups

$$\pi_i(f)\colon \pi_i(L'_\bullet)\to\pi_i(L_\bullet)$$

is an isomorphism for $i<n$ and a surjection for $i=n$.

Let $\mathrm{fib}(f)\in\mathsf{sLie}^r$ denote the *homotopy fiber* of a morphism $f\colon L'_\bullet\to L_\bullet$. Then f is n-connected if and only if $\mathrm{fib}(f)$ is $(n-1)$-connected and $\pi_0(f)$ is a surjection. Similarly, we write $\mathrm{cofib}(f)\in\mathsf{sLie}^r$ for the *homotopy cofiber* of the morphism f.

Theorem 4.1.3 (Homotopy excision theorem)**.** *Let $f\colon L'_\bullet\to L_\bullet$ be a n-connected morphism in sLie^r, $L'_\bullet$ is connected, and $n\geq 0$. Then the natural map (in the homotopy category* $\mathrm{Ho}(\mathsf{sLie}^r)$*)*

$$\mathrm{fib}(f)\to\Omega\,\mathrm{cofib}(f) \tag{4.1.4}$$

is $(n+1)$-connected.

Remark 4.1.5. Since the underlying ∞-category $s\mathcal{L}$ of sLie^r is not an ∞-topos, we can not apply (at least directly) the generalized Blakers-Massey theorem, see e.g. [2]. Instead, we will prove the homotopy excision theorem using model-theoretic approach, and only in the particular case as above.

Definition 4.1.6. A map $f\colon L'_\bullet\to L_\bullet$ in sLie^r is *n-reduced* if the maps $f_i\colon L'_i\to L_i$ are isomorphisms for $i\leq n$.

Note that an almost-free n-reduced map in sLie^r is n-connected. The opposite is true up to weak equivalences and the proof is standard.

Lemma 4.1.7. *Let $f\colon L'_\bullet \to L_\bullet$ be a n-connected morphism in sLie^r, $n\geq 0$. Then there exists a commutative diagram*

$$\begin{array}{ccc} \widetilde{L'}_\bullet & \xrightarrow{\tilde{f}} & \widetilde{L}_\bullet \\ \downarrow & & \downarrow \\ L'_\bullet & \xrightarrow{f} & L_\bullet \end{array}$$

such that the vertical arrows are weak equivalences in sLie^r and the map $\tilde{f}$ is almost-free and n-reduced. □

Let $f\colon L'_\bullet \to L_\bullet$ be an almost-free n-reduced map. Notice that f is an inclusion of the underlying simplicial vector spaces. Let us denote by $L_\bullet//L'_\bullet$ the quotient simplicial vector space

$$L_\bullet//L'_\bullet = \mathsf{oblv}(L_\bullet)/\mathsf{oblv}(L'_\bullet).$$

Similarly, we denote by $L_\bullet/L'_\bullet$ the underlying simplicial vector space of the quotient Lie algebra, i.e. $L_\bullet/L'_\bullet$ is the underlying simplicial vector of the coequalizer

$$\mathrm{coeq}(L'_\bullet \underset{0}{\overset{f}{\rightrightarrows}} L_\bullet).$$

There are equivalences

$$\mathsf{oblv}(\mathrm{fib}(f)) \simeq \Sigma^{-1}L_\bullet//L'_\bullet, \quad \mathsf{oblv}(\mathrm{cofib}(f)) \simeq L_\bullet/L'_\bullet,$$

and the map (4.1.4) is equivalent to the desuspension of the canonical map

$$L_\bullet//L'_\bullet \to L_\bullet/L'_\bullet \tag{4.1.8}$$

in $\mathsf{sVect}_{\mathbf{F}}$. Thus, Theorem 4.1.3 is equivalent to the following statement

(∗) Let $f\colon L'_\bullet \to L_\bullet$ be an almost-free n-reduced map in sLie^r, $L'_0=0$, and $n\geq 0$, then the map (4.1.8) is $(n+2)$-connected.

We first show (∗) for free maps.

Lemma 4.1.9. *Let $i\colon U_\bullet \to W_\bullet$ be n-reduced inclusion of simplicial vector spaces, and let*

$$f = L^r(i)\colon L^r(U_\bullet) \to L^r(W_\bullet)$$

be the induced map between free simplicial restricted Lie algebras. Then (∗) holds for f provided $U_0=0$ and $n\geq 0$.

Proof. Set $V_\bullet = W_\bullet/U_\bullet$ the quotient simplicial vector space, $V_i=0$ if $i\leq n$. By Proposition 2.1.8, there is a splitting

$$\mathsf{oblv}\circ L^r(X_\bullet) \cong \bigoplus_{m\geq 1} L^r_m(X_\bullet) = \bigoplus_{m\geq 1} (\mathbf{Lie}_m \otimes X_\bullet^{\otimes m})^{\Sigma_m},$$

where $X_\bullet \in \mathsf{sVect}_{\mathbf{F}}$. Therefore the map (4.1.8) is the direct sum of surjective maps

$$p_m\colon L^r_m(W_\bullet)/L^r_m(U_\bullet) \to L^r_m(V_\bullet), \quad m\geq 1, \tag{4.1.10}$$

and it suffices to show that each map p_m is $(n+2)$-connected.

The simplicial vector space $L^r_m(W_\bullet)$ has an increasing filtration

$$0 = F_{-1}L^r_m(W_\bullet) \subset F_0L^r_m(W_\bullet) \subset F_1L^r_m(W_\bullet) \subset \ldots \subset F_mL^r_m(W_\bullet) = L^r_m(W_\bullet)$$

such that the quotient vector space $F_j/F_{j-1} \in \mathsf{sVect}_{\mathbf{F}}$ is isomorphic to

$$F_jL^r_m(W_\bullet)/F_{j-1}L^r_m(W_\bullet) \cong (\mathbf{Lie}_m \otimes U_\bullet^{\otimes(m-j)} \otimes V_\bullet^{\otimes j})^{\Sigma_{m-j}\times\Sigma_j},$$

where $0 \leq j \leq m$. Since $U_0 = 0$ and $V_i = 0$ for $i \leq n$, we obtain that

$$\pi_i(F_jL^r_m(W_\bullet)/F_{j-1}L^r_m(W_\bullet)) = 0$$

for $i \leq n+1$ and $0 < j < m$. Therefore,

$$\pi_i(F_{m-1}L^r_m(W_\bullet)/F_0L^r_m(W_\bullet)) = 0$$

if $i \leq n+1$, and so each map p_m is $(n+2)$-connected, $m \geq 1$. □

Next, we will resolve any almost-free n-reduced map in sLie^r by free maps. Let $L \in \mathsf{Lie}^r$ be a restricted Lie algebra. Recall that the *bar construction* $B_\bullet(L)$ of L is the simplicial restricted Lie algebra defined as follows

$$B_s(L) = (L^r \circ \mathsf{oblv})^{\circ(s+1)}(L), \quad s \geq 0,$$

where the face operators are induced by the counit map $L^r \circ \mathsf{oblv} \to \mathsf{id}$, and the degeneracy operators are induced by the unit map $\mathsf{id} \to \mathsf{oblv} \circ L^r$. Notice that there is a canonical map

$$B_\bullet(L) \to L \in \mathsf{sLie}^r$$

to the constant simplicial restricted Lie algebra L, and this map is a weak equivalence. We are now ready to prove Theorem 4.1.3.

Proof of Theorem 4.1.3. Let $f\colon L'_\bullet \to L_\bullet$ be an almost-free n-reduced map in sLie^r, $L'_0 = 0$, $n \geq 0$. Consider the induced map of bisimplicial objects

$$B_\bullet(f)\colon B_\bullet(L'_\bullet) \to B_\bullet(L_\bullet).$$

Note that each $B_s(f), s \geq 0$ is the map of free simplicial restricted Lie algebras induced by the n-reduced inclusion of simplicial vector spaces

$$\mathsf{oblv} \circ (L^r \circ \mathsf{oblv})^{\circ s}(L'_\bullet) \to \mathsf{oblv} \circ (L^r \circ \mathsf{oblv})^{\circ s}(L_\bullet).$$

Thus, by Lemma 4.1.9, the map

$$B_s(L_\bullet)//B_s(L'_\bullet) \to B_s(L_\bullet)/B_s(L'_\bullet)$$

is $(n+2)$-connected for each $s \geq 0$. Moreover,

$$\pi_i(B_s(L_\bullet)//B_s(L'_\bullet)) = 0$$

for $s \geq 0$ and $i \leq n$.

There are (chains of) weak equivalences

$$L_\bullet//L'_\bullet \simeq d(B_\bullet(L_\bullet)//B_\bullet(L'_\bullet)),$$

$$L_\bullet/L'_\bullet \simeq d(B_\bullet(L_\bullet)/B_\bullet(L'_\bullet)),$$

where $d\colon \mathsf{ssVect}_{\mathbf{F}} \to \mathsf{sVect}_{\mathbf{F}}$, $d(V_{\bullet,\bullet})_s = V_{s,s}$, $V_{\bullet,\bullet} \in \mathsf{ssVect}_{\mathbf{F}}$ is the diagonal simplicial vector space. Thus, there is a map of strongly convergent spectral sequences

$$\begin{array}{ccc} E^1_{s,t} = \pi_t(B_s(L_\bullet)//B_s(L'_\bullet)) & \Longrightarrow & \pi_{s+t}(L_\bullet//L'_\bullet) \\ \downarrow & & \downarrow \\ \widetilde{E}^1_{s,t} = \pi_t(B_s(L_\bullet)/B_s(L'_\bullet)) & \Longrightarrow & \pi_{s+t}(L_\bullet/L'_\bullet), \end{array}$$

where the differentials act as follows $d^r\colon E^r_{s,t} \to E^r_{s-r,t+r-1}$, $\tilde{d}^r\colon \widetilde{E}^r_{s,t} \to \widetilde{E}^r_{s-r,t+r-1}$.

Note that $E^1_{s,t} = \widetilde{E}^1_{s,t} = 0$ if $t \leq n$, $E^1_{s,n+1} \to \widetilde{E}^1_{s,n+1}$ is an isomorphism for all $s \geq 0$, and $E^1_{s,n+2} \twoheadrightarrow \widetilde{E}^1_{s,n+2}$ is a surjection for all $s \geq 0$. Therefore the map $E^2_{s,n+1} \to \widetilde{E}^2_{s,n+1}$ is an isomorphism for all $s \geq 0$, and $E^2_{0,n+2} \twoheadrightarrow \widetilde{E}^2_{0,n+2}$ is a surjection. Hence, $E^\infty_{0,n+1} \cong \widetilde{E}^\infty_{0,n+1}$, $E^\infty_{1,n+1} \cong \widetilde{E}^\infty_{1,n+1}$, and $E^\infty_{0,n+2} \twoheadrightarrow \widetilde{E}^\infty_{0,n+2}$ is a surjection. This implies the theorem. □

4.2 Homology and cohomology. Recall that $\mathbf{F}\{\xi\}$ is the twisted polynomial ring (Definition 2.1.3), $\mathsf{Mod}_{\mathbf{F}\{\xi\}}$ is the abelian category of left $\mathbf{F}\{\xi\}$-modules, and

$$\mathsf{triv}_\xi\colon \mathsf{Mod}_{\mathbf{F}\{\xi\}} \to \mathsf{Lie}^r$$

is the functor which maps a left module M to the restricted Lie algebra $\mathsf{triv}_\xi(M)$ with the underlying vector spaces equal to M, p-operation given by ξ, and with identically zero bracket.

We observe that the functor triv_ξ has a left adjoint

$$\mathsf{Ab}\colon \mathsf{Lie}^r \to \mathsf{Mod}_{\mathbf{F}\{\xi\}}$$

given by $L \mapsto L/[L,L]$, where $[L,L] \subset L$ is the (restricted) Lie ideal generated by all elements of the form $[x,y]$, $x,y \in L$.

We extend the adjoint pair $\mathsf{Ab} \dashv \mathsf{triv}_\xi$ degreewise to the adjunction

$$\mathsf{Ab} : \mathsf{sLie}^r \rightleftarrows \mathsf{sMod}_{\mathbf{F}\{\xi\}} : \mathsf{triv}_\xi \tag{4.2.1}$$

between categories of simplicial objects. Note that the adjunction (4.2.1) is a Quillen adjunction because the composite

$$\mathsf{oblv} \circ \mathsf{triv}_\xi\colon \mathsf{sMod}_{\mathbf{F}\{\xi\}} \to \mathsf{sVect}_{\mathbf{F}}$$

maps a left $\mathbf{F}\{\xi\}$-module to the underlying vector space, and so the functor triv_ξ preserves weak equivalences and fibrations.

Remark 4.2.2. Note that the simplicial left $\mathbf{F}\{\xi\}$-module $\mathsf{Ab}(L_\bullet)$ is degreewise projective (and even free) provided a simplicial restricted Lie algebra $L_\bullet \in \mathsf{sLie}^r$ is almost-free.

The pair (4.2.1) induces the adjoint pair of derived functors

$$\mathbb{L}\mathsf{Ab} : s\mathcal{L} \rightleftarrows D_{\geq 0}(\mathsf{Mod}_{\mathbf{F}\{\xi\}}) : \mathsf{triv}_\xi \tag{4.2.3}$$

between underlying ∞-categories.

Recall that we denote by $\mathsf{Mod}^{\mathbf{F}\{\xi\}}$ the abelian category of *right* $\mathbf{F}\{\xi\}$-modules.

Definition 4.2.4. Let $M \in \mathsf{Mod}^{\mathbf{F}\{\xi\}}$ be a right $\mathbf{F}\{\xi\}$-module. We define the *chain complex* $\widetilde{C}_*(L_\bullet; M) \in D(\mathsf{Vect}_{\mathbf{F}})$ of $L_\bullet \in \mathsf{sLie}^r$ with coefficients in M by the formula

$$\widetilde{C}_*(L_\bullet; M) = \Sigma M \otimes_{\mathbf{F}\{\xi\}} \mathbb{L}\mathsf{Ab}(L_\bullet).$$

Here $- \otimes_{\mathbf{F}\{\xi\}} -$ is the derived tensor product

$$- \otimes_{\mathbf{F}\{\xi\}} -\colon D(\mathsf{Mod}^{\mathbf{F}\{\xi\}}) \times D(\mathsf{Mod}_{\mathbf{F}\{\xi\}}) \to D(\mathsf{Vect}_{\mathbf{F}}).$$

Furthermore, we define the *s-th homology group* $\widetilde{H}_s(L_\bullet; M)$ of $L_\bullet \in \mathsf{sLie}^r$ with coefficients in M by the next rule

$$\widetilde{H}_s(L_\bullet; M) = \pi_s(\widetilde{C}_*(L_\bullet; M)), \quad s \geq 0.$$

Consider the field $\mathbf{F}$ as a (diagonal) $\mathbf{F}\{\xi\}$-bimodule with ξ acting by zero, then we have $\mathbf{F} \otimes_{\mathbf{F}\{\xi\}} \mathsf{Ab}(L) = \mathsf{Ab}_\xi(L)$, see formula (3.2.16). Thus, Proposition 3.2.18 together with the Eilenberg-Zilber theorem imply that the homology groups

$$\widetilde{H}_*(L_\bullet; \mathbf{F}) = \bigoplus_{n>0} \widetilde{H}_n(L_\bullet; \mathbf{F}), \quad L_\bullet \in \mathsf{sLie}^r$$

form naturally a graded non-unital cocommutative coalgebra. Let us denote by $H_*(L_\bullet; \mathbf{F})$ the graded *coaugmented* coalgebra associated with $\widetilde{H}_*(L_\bullet; \mathbf{F})$, i.e.

$$H_*(L_\bullet; \mathbf{F}) = \mathbf{F} \oplus \widetilde{H}_*(L_\bullet; \mathbf{F}),$$

where the first summand is in degree 0.

Corollary 4.2.5 (Künneth formula)**.** *There is a natural isomorphism*

$$H_*(L_\bullet \times L'_\bullet; \mathbf{F}) \cong H_*(L_\bullet; \mathbf{F}) \otimes H_*(L'_\bullet; \mathbf{F}),$$

where $L_\bullet, L'_\bullet \in \mathsf{sLie}^r$. □

Remark 4.2.6. We are not aware if the Künneth formula is true for homology groups with any other coefficients.

Note that the functor $\mathsf{Ab}\colon \mathsf{sLie}^r \to \mathsf{sMod}_{\mathbf{F}\{\xi\}}$ comes with the natural transformation

$$\mathsf{id} \to \mathsf{Ab} \tag{4.2.7}$$

given by the quotient map $L_\bullet \to L_\bullet/[L_\bullet, L_\bullet] = \mathsf{Ab}(L_\bullet)$. This natural transformation induces the *Hurewicz homomorphism*

$$h\colon \pi_s(L_\bullet) \to \widetilde{H}_{s+1}(L_\bullet; \mathbf{F}\{\xi\}), \quad L_\bullet \in \mathsf{sLie}^r, \quad s \geq 0. \tag{4.2.8}$$

We notice that both sides of (4.2.8) are naturally endowed with an action of ξ. Indeed, the homology groups $\widetilde{H}_s(L_\bullet; \mathbf{F}\{\xi\}), s \geq 1$ are left $\mathbf{F}\{\xi\}$-modules by Definition 4.2.4; and the p-operation $\xi\colon L_\bullet \to L_\bullet$, $L_\bullet \in \mathsf{sLie}^r$ is a map of *simplicial sets*, so it induces a (in general, non-linear) map

$$\xi_*\colon \pi_*(L_\bullet) \to \pi_*(L_\bullet). \tag{4.2.9}$$

We point out here that the Hurewicz homomorphism (4.2.8) is compatible with these ξ-actions on both sides. Finally, we notice that $\pi_0(L_\bullet), L_\bullet \in \mathsf{sLie}^r$ is itself a restricted Lie algebra.

Theorem 4.2.10 (Hurewicz theorem)**.** *Let $L_\bullet \in \mathsf{sLie}^r$ be a simplicial restricted Lie algebra. Then the Hurewicz homomorphism $h\colon \pi_0(L_\bullet) \to \widetilde{H}_1(L_\bullet; \mathbf{F}\{\xi\})$ induces an isomorphism*

$$\mathsf{Ab}(\pi_0(L_\bullet)) \cong \widetilde{H}_1(L_\bullet; \mathbf{F}\{\xi\}).$$

If $\pi_i(L_\bullet) = 0$ for $0 \leq i \leq n$, then

$$h\colon \pi_{n+1}(L_\bullet) \xrightarrow{\cong} \widetilde{H}_{n+2}(L_\bullet; \mathbf{F}\{\xi\})$$

is an isomorphism, and

$$h\colon \pi_{n+2}(L_\bullet) \twoheadrightarrow \widetilde{H}_{n+3}(L_\bullet; \mathbf{F}\{\xi\})$$

is a surjection.

Proof. The first statement is clear, since the functor $\mathsf{Ab}\colon \mathsf{Lie}^r \to \mathsf{Mod}_{\mathbf{F}\{\xi\}}$ is a left adjoint, and so it commutes with colimits.

For the second part, we use Lemma 4.1.7, so we can assume that $L_\bullet \in \mathsf{sLie}^r$ is an almost-free object $L_\bullet = L^r(V_\bullet)$, $V_\bullet \in \tilde{\mathsf{s}}\mathsf{Vect}_{\mathbf{F}}$ and $V_i = 0$ for $0 \leq i \leq n$.

We show that $\pi_{n+1}[L_\bullet, L_\bullet] = 0$, where $[L_\bullet, L_\bullet] \subset L_\bullet$ is the Lie ideal generated by all elements of the form $[x, y]$, $x, y \in L_\bullet$. By the previous paragraph, it suffices to construct an element $\{x, y\} \in [L_{n+2}, L_{n+2}]$, $x, y \in L_{n+1}$ such that

$$\partial\{x, y\} = \sum_{i=0}^{n+3} (-1)^i d_i\{x, y\} = [x, y].$$

We set $\{x, y\} = [s_1 y, s_0 x - s_1 x]$, then the straightforward computation with the simplicial relations shows that $\partial\{x, y\} = [x, y]$.

Finally, there is a short exact sequence of simplicial vector spaces

$$0 \to \mathsf{oblv}([L_\bullet, L_\bullet]) \to \mathsf{oblv} L_\bullet \to \mathsf{Ab}(L_\bullet) \to 0$$

which induces the long exact sequence of homotopy groups

$$\ldots \to \pi_{n+2}(L_\bullet) \to \pi_{n+2}\mathsf{Ab}(L_\bullet) \to \pi_{n+1}[L_\bullet, L_\bullet] \to \pi_{n+1}(L_\bullet) \to \pi_{n+1}\mathsf{Ab}(L_\bullet) \to 0.$$

Since $\pi_{n+1}[L_\bullet, L_\bullet] = 0$, the theorem follows. □

Remark 4.2.11. Our proof is almost identical to the proof of the Hurewicz theorem in the category of simplicial Lie algebras (non-necessary equipped with a p-operation), see [21, Theorem 8].

We say that $(L_\bullet, A_\bullet)$ is a *pair* in sLie^r if $A_\bullet$ is a simplicial restricted Lie subalgebra of $L_\bullet$. A map of pairs

$$f\colon (L'_\bullet, A'_\bullet) \to (L_\bullet, A_\bullet)$$

is a map $f\colon L'_\bullet \to L_\bullet$ in sLie^r such that $f(A'_\bullet) \subset A_\bullet$. Notice that we do not ask $A_\bullet$ be a Lie ideal in $L_\bullet$ in opposite to [45, §8].

Definition 4.2.12. Let $(L_\bullet, A_\bullet)$ be a pair sLie^r. We define the *s-th relative homotopy group* $\pi_s(L_\bullet, A_\bullet)$ by the formula

$$\pi_s(L_\bullet, A_\bullet) = \pi_{s-1}(\mathrm{fib}(\iota)), \quad s \geq 1,$$

where $\mathrm{fib}(\iota) \in \mathsf{sLie}^r$ is the homotopy fiber of $\iota\colon A_\bullet \hookrightarrow L_\bullet$. Similarly, we define the *s-th relative homology group* $H_s(L_\bullet, A_\bullet; M)$, $M \in \mathsf{Mod}_{\mathbf{F}\{\xi\}}$ as follows

$$H_s(L_\bullet, A_\bullet; M) = \widetilde{H}_s(\mathrm{cofib}(\iota); M), \quad s \geq 0,$$

where $\mathrm{cofib}(\iota) \in \mathsf{sLie}^r$ is the homotopy cofiber of ι.

Let $(L_\bullet, A_\bullet)$ be a pair in sLie^r, then there are long exact sequences of homotopy and homology groups:

$$\ldots \to \pi_s(A_\bullet) \to \pi_s(L_\bullet) \to \pi_s(L_\bullet, A_\bullet) \xrightarrow{\delta} \pi_{s-1}(A_\bullet) \to \ldots \tag{4.2.13}$$

$$\begin{aligned} \ldots \to \widetilde{H}_s(A_\bullet; M) \to \widetilde{H}_s(L_\bullet; M) \to H_s(L_\bullet, A_\bullet; M) \\ \xrightarrow{\delta} \widetilde{H}_{s-1}(A_\bullet; M) \to \widetilde{H}_{s-1}(L_\bullet; M) \to H_{s-1}(L_\bullet, A_\bullet; M) \to \ldots \end{aligned} \tag{4.2.14}$$

Furthermore, there are similar long exact sequences for *triples* in sLie^r. Namely, if $(L_\bullet, A_\bullet, B_\bullet)$ is a triple in sLie^r, that is $B_\bullet \subset A_\bullet \subset L_\bullet$, then there are long sequences

$$\ldots \to \pi_s(A_\bullet, B_\bullet) \to \pi_s(L_\bullet, B_\bullet) \to \pi_s(L_\bullet, A_\bullet) \xrightarrow{\delta} \pi_{s-1}(A_\bullet, B_\bullet) \to \ldots \tag{4.2.15}$$

$$\begin{aligned} \ldots \to H_s(A_\bullet, B_\bullet; M) \to H_s(L_\bullet, B_\bullet; M) \to H_s(L_\bullet, A_\bullet; M) \\ \xrightarrow{\delta} H_{s-1}(A_\bullet, B_\bullet; M) \to H_{s-1}(L_\bullet, B_\bullet; M) \to H_{s-1}(L_\bullet, A_\bullet; M) \to \ldots \end{aligned} \tag{4.2.16}$$

Note that the natural map $\mathrm{fib}(\iota) \to \Omega\,\mathrm{cofib}(\iota)$ induces the *relative Hurewicz homomorphism*:

$$h\colon \pi_s(L_\bullet, A_\bullet) \to H_{s+1}(L_\bullet, A_\bullet; \mathbf{F}\{\xi\}),$$

compatible with the exact sequences above. We say that a pair $(L_\bullet, A_\bullet) \in \mathsf{sLie}^r$ is *n-connected* if $\pi_i(L_\bullet, A_\bullet) = 0$ for each $i \le n$. The homotopy excision theorem 4.1.3 immediately implies the following corollary.

Corollary 4.2.17 (Relative Hurewicz theorem)**.** *Suppose that $(L_\bullet, A_\bullet)$ is a n-connected pair in sLie^r and $\pi_0(A_\bullet) = 0$, $n \ge 1$. Then the relative Hurewicz homomorphism*

$$\pi_{n+1}(L_\bullet, A_\bullet) \to H_{n+2}(L_\bullet, A_\bullet; \mathbf{F}\{\xi\})$$

is an isomorphism, and

$$\pi_{n+2}(L_\bullet, A_\bullet) \to H_{n+3}(L_\bullet, A_\bullet; \mathbf{F}\{\xi\})$$

is a surjection. □

As usual, the relative Hurewicz theorem implies the homological Whitehead theorem.

Corollary 4.2.18 (Homological Whitehead theorem)**.** *Let $f\colon L'_\bullet \to L_\bullet$ be a map between connected simplicial restricted Lie algebras. Then f is a weak equivalence if and only if the induced map*

$$f_*\colon \widetilde{H}_i(L'_\bullet; \mathbf{F}\{\xi\}) \to \widetilde{H}_i(L_\bullet; \mathbf{F}\{\xi\})$$

is an isomorphism for all $i \ge 1$. □

Let $M \in \mathsf{Mod}_{\mathbf{F}\{\xi\}}$ be a left $\mathbf{F}\{\xi\}$-module and consider the abelian Lie algebra $\mathsf{triv}_\xi(M) \in \mathsf{Lie}^r \subset \mathsf{sLie}^r$. We will compute $H_*(\mathsf{triv}_\xi(M); \mathbf{F})$ provided M is torsion-free. First, by the Hurewicz theorem, we have

$$H_1(\mathsf{triv}_\xi(M); \mathbf{F}) \cong \mathbf{F} \otimes_{\mathbf{F}\{\xi\}} M = M/\xi(M),\ M \in \mathsf{Mod}_{\mathbf{F}\{\xi\}} \tag{4.2.19}$$

Next, using the addition operation $M \times M \to M$, $(m_1, m_2) \mapsto m_1 + m_2$ we observe that $\mathsf{triv}_\xi(M)$ is canonically a commutative group object of sLie^r. Therefore, by the Künneth formula, $H_*(\mathsf{triv}_\xi(M); \mathbf{F})$ is a graded commutative and cocommutative Hopf algebra. Using the isomorphism (4.2.19) we get a natural map of graded Hopf algebras

$$\gamma\colon \mathrm{Sym}^*(M/\xi(M)) \to H_*(\mathsf{triv}_\xi(M); \mathbf{F}), \tag{4.2.20}$$

where $\mathrm{Sym}^*(M/\xi(M))$ is the free graded commutative algebra generated by the vector space $M/\xi(M)$.

Recall that a left $\mathbf{F}\{\xi\}$-module M is called *torsion-free* if for any non-zero $a \in \mathbf{F}\{\xi\}$ the equation $ax = 0, x \in M$ implies $x = 0$.

Proposition 4.2.21. *Let $M \in \mathsf{Mod}_{\mathbf{F}\{\xi\}}$ be a torsion-free left $\mathbf{F}\{\xi\}$-module. Then the map γ naturally factorizes via an isomorphism*

$$\gamma'\colon \Lambda^*(M/\xi(M)) \xrightarrow{\cong} H_*(\mathsf{triv}_\xi(M); \mathbf{F}),$$

where $\Lambda^*(M/\xi(M))$ is the exterior algebra generated by $M/\xi(M)$.

Proof. First, assume that $M \cong \mathbf{F}\{\xi\}$ is a free $\mathbf{F}\{\xi\}$-module of rank 1. Then the abelian Lie algebra $\mathsf{triv}_\xi(M)$ is canonically isomorphic to the free restricted Lie algebra $L^r(M/\xi(M))$. Therefore,

$$H_i(\mathsf{triv}_\xi(M), \mathbf{F}) \cong H_i(L^r(M/\xi(M)); \mathbf{F}) \cong \begin{cases} \mathbf{F} & \text{if } i = 0, \\ M/\xi(M) & \text{if } i = 1, \\ 0, & \text{otherwise;} \end{cases}$$

and the proposition holds for free modules of rank 1.

Next, using an induction on the rank r and the Künneth formula we obtain the desired isomorphism for all free $\mathbf{F}\{\xi\}$-modules of finite rank, i.e. for $M \cong \mathbf{F}\{\xi\}^{\oplus r}$. By Corollary 5.1.5, any torsion-free $\mathbf{F}\{\xi\}$-module is a filtered colimit of finitely generated free modules, which implies the proposition. □

Example 4.2.22. Let $M = \mathbf{F}\{\xi^{\pm}\} = \mathbf{F}\{\xi\}[1/\xi]$ be the ring of twisted Laurent polynomials, see Section 5.2. By Proposition 4.2.21, we have

$$\widetilde{H}_*(\mathsf{triv}_\xi(M); \mathbf{F}) \cong 0, \quad \text{but} \quad \widetilde{H}_1(\mathsf{triv}_\xi(M); \mathbf{F}\{\xi\}) \cong M \neq 0.$$

Therefore,

$$\widetilde{H}_*(\mathsf{triv}_\xi(\Sigma^n M); \mathbf{F}) \cong 0, \quad \text{but} \quad \widetilde{H}_{n+1}(\mathsf{triv}_\xi(\Sigma^n M); \mathbf{F}\{\xi\}) \cong M \neq 0$$

for all $n \geq 0$. Here $\Sigma^n M \in D_{\geq 0}(\mathsf{Mod}_{\mathbf{F}\xi})$ is the shift of M.

In a similar way, one can also define the cohomology groups of $L_\bullet \in \mathsf{sLie}^r$.

Definition 4.2.23. Let $M \in \mathsf{Mod}_{\mathbf{F}\{\xi\}}$ be a *left* $\mathbf{F}\{\xi\}$-module. We define the *cochain complex* $\widetilde{C}^*(L_\bullet; M) \in D(\mathsf{Vect}_{\mathbf{F}})$ of $L_\bullet \in \mathsf{sLie}^r$ with coefficients in M as follows

$$\widetilde{C}^*(L_\bullet; M) = \mathbb{R}\mathrm{Hom}_{\mathbf{F}\{\xi\}}(\Sigma \mathbb{L}\mathsf{Ab}(L_\bullet), M).$$

Here $\mathbb{R}\mathrm{Hom}_{\mathbf{F}\{\xi\}}(-,-)$ is the derived Hom-functor

$$\mathbb{R}\mathrm{Hom}_{\mathbf{F}\{\xi\}}(-,-)\colon D(\mathsf{Mod}_{\mathbf{F}\{\xi\}}) \times D(\mathsf{Mod}_{\mathbf{F}\{\xi\}}) \to D(\mathsf{Vect}_{\mathbf{F}}).$$

Furthermore, we define the *s-th cohomology group* $\widetilde{H}^s(L_\bullet; M)$ of $L_\bullet \in \mathsf{sLie}^r$ with coefficients in M by the rule

$$\widetilde{H}^s(L_\bullet; M) = \pi_{-s}(\widetilde{C}^*(L_\bullet; M)), \quad s \geq 0.$$

Remark 4.2.24. Similar to the case of singular cohomology of spaces, we have the *universal coefficient formula*. Namely, there is a natural exact sequence

$$0 \to \mathrm{Ext}^1_{\mathbf{F}\{\xi\}}(\widetilde{H}_{s-1}(L_\bullet;\mathbf{F}\{\xi\}), M) \to \widetilde{H}^s(L_\bullet;M) \to \mathrm{Hom}_{\mathbf{F}\{\xi\}}(\widetilde{H}_s(L_\bullet;\mathbf{F}\{\xi\}), M) \to 0$$

for any $L_\bullet \in \mathsf{sLie}^r$, $M \in \mathsf{Mod}_{\mathbf{F}\{\xi\}}$, and $s \geq 1$. Moreover, this exact sequence splits, but the splitting may not be natural.

Remark 4.2.25. The adjoint pair (4.2.3) implies that the homotopy functor

$$\widetilde{H}^s(-;M)\colon \mathrm{Ho}(\mathsf{sLie}^r)^{op} \to \mathsf{Vect}_{\mathbf{F}}$$

is representable by the abelian Lie algebra $\mathsf{triv}_\xi \Sigma^{s-1} M \in \mathsf{sLie}^r$, cf. [58, Proposition 3.4]. Here $\mathrm{Ho}(\mathsf{sLie}^r)$ is the homotopy category of the model category sLie^r, see Theorem 3.2.3.

4.3 Postnikov system. Let us denote by $\Delta_{\leq(n+1)} \subset \Delta$ the full subcategory in the simplex category Δ spanned by $[i]$, $i \leq n+1$. If C is a category, we write $\mathsf{s}_{\leq(n+1)}\mathsf{C}$ for the category of contravariant functors from $\Delta_{\leq(n+1)}$ to C. Assume that the category C is complete, then the restriction functor

$$\mathrm{tr}^*_{(n+1)}\colon \mathsf{sC} \to \mathsf{s}_{\leq(n+1)}\mathsf{C}$$

has a right adjoint

$$\mathrm{tr}_{(n+1)*}\colon \mathsf{s}_{\leq(n+1)}\mathsf{C} \to \mathsf{sC}.$$

We write

$$\mathrm{cosk}_{n+1}\colon \mathsf{sC} \to \mathsf{sC}$$

for the composite $\mathrm{tr}_{(n+1)*} \circ \mathrm{tr}^*_{(n+1)}$ and $\alpha^n\colon \mathsf{id} \to \mathrm{cosk}_{n+1}$ for the unit map.

Assume that $\mathsf{C} = \mathsf{Vect}_{\mathbf{F}}$ is the category of vector spaces and let $V_\bullet \in \mathsf{sVect}_{\mathbf{F}}$. Then $\pi_i(\mathrm{cosk}_{n+1}V_\bullet) = 0$ for $i > n$, and the induced map $\pi_i(\alpha^n)$ is an isomorphism for $i \leq n$, see [44, Section II.8].

Since the functor $\mathsf{oblv}\colon \mathsf{sLie}^r \to \mathsf{sVect}_{\mathbf{F}}$ is a right adjoint, it preserves limits, and so we have a natural isomorphism:

$$\mathsf{oblv}(\mathrm{cosk}_{n+1}(L_\bullet)) \cong \mathrm{cosk}_{n+1}(\mathsf{oblv}(L_\bullet)),\ L_\bullet \in \mathsf{sLie}^r,\ n \geq 0.$$

Therefore the natural map

$$\alpha^n\colon L_\bullet \to \mathrm{cosk}_{n+1} L_\bullet \tag{4.3.1}$$

again induces an isomorphism on π_i for $i \leq n$, and $\pi_i \mathrm{cosk}_{n+1} L_\bullet = 0$ for $i > n$.

Here we slightly change the notation: for the rest of the paper, we will write $L_\bullet^{\leq n}$ for $\mathrm{cosk}_{n+1}L_\bullet$, $L_\bullet \in \mathsf{sLie}^r$, $n \geq 0$. Finally, we note that $\alpha^n\colon L_\bullet \to L_\bullet^{\leq n}$ is a fibration in sLie^r and we write $L_\bullet^{>n}$ for its fiber.

Definition 4.3.2. Let M be a left $\mathbf{F}\{\xi\}$-module and $n \geq 0$. A simplicial restricted Lie algebra $L_\bullet \in \mathsf{sLie}^r$ is an *Eilenberg–MacLane Lie algebra* of type $K(M,n)$ if it has the n-th homotopy group $\pi_n(L_\bullet)$ isomorphic to M (as a left $\mathbf{F}\{\xi\}$-module) and all other homotopy groups are trivial.

Example 4.3.3. An abelian Lie algebra $\mathsf{triv}_\xi \Sigma^n M \in \mathsf{sLie}^r, M \in \mathsf{Mod}_{\mathbf{F}\{\xi\}}$ is the Eilenberg-MacLane Lie algebra of type $K(M,n)$.

The Hurewicz theorem together with Remarks 4.2.24 and 4.2.25 immediately implies that there is a unique (up to a weak equivalence) Eilenberg-MacLane Lie algebra in sLie^r of a given type $K(M,n)$. Therefore we will abuse notation and call any such Lie algebra by $K(M,n)$.

Proposition 4.3.4. *Let $f\colon L'_\bullet \to L_\bullet$ be a map in sLie^r such that $\pi_0(L'_\bullet) = \pi_0(L_\bullet) = 0$ and the homotopy fiber $\mathrm{fib}(f)$ is an Eilenberg-MacLane Lie algebra $K(M,n)$, $M \in \mathsf{Mod}_{\mathbf{F}\{\xi\}}$, $n \geq 0$. Then there exists a map*

$$k\colon L_\bullet \to K(M,n+1)$$

such that the sequence

$$L'_\bullet \xrightarrow{f} L_\bullet \xrightarrow{k} K(M,n+1) \tag{4.3.5}$$

is a fiber sequence in sLie^r.

Proof. Note that the map $f\colon L'_\bullet \to L_\bullet$ is n-connected. Therefore, by the homotopy excision theorem 4.1.3, we have

$$\pi_i(\mathrm{cofib}(f)) = 0, \quad i \leq n,$$

$$\pi_{n+1}(\mathrm{cofib}(f)) \cong \pi_n(\mathrm{fib}(f)) = M,$$

where $\mathrm{cofib}(f)$ is the cofiber of the map f. Hence $\mathrm{cofib}(f)^{\leq(n+1)} \in \mathsf{sLie}^r$ is an Eilenberg-MacLane Lie algebra of type $K(M,n+1)$, and we define the desired map k as the composite

$$L_\bullet \to \mathrm{cofib}(f) \to \mathrm{cofib}(f)^{\leq(n+1)} \simeq K(M,n+1).$$

The straightforward diagram chase shows that the sequence (4.3.5) is a fiber sequence. □

We summarize the results of this section in the next corollary.

Corollary 4.3.6 (Postnikov tower). *Let $L_\bullet \in \mathsf{sLie}^r$ be a simplicial restricted Lie algebra. Then there are a natural tower of fibrations*

$$\ldots \xrightarrow{\beta^{n+1}} L_\bullet^{\leq(n+1)} \xrightarrow{\beta^n} L_\bullet^{\leq n} \xrightarrow{\beta^{n-1}} \ldots$$

and compatible maps $\alpha^n\colon L_\bullet \to L_\bullet^{\leq n}$ such that

1. *$\pi_i(L_\bullet^{\leq n}) = 0$ if $i > n$;*
2. *the induced map $\pi_i(\alpha^n)$ is an isomorphism for $i \leq n$;*
3. *$L_\bullet \simeq \mathrm{holim}_n\, L_\bullet^{\leq n}$.*

Moreover, if $\pi_0(L_\bullet) = 0$, then there exist k-invariants, i.e. there are maps

$$k^n\colon L_\bullet^{\leq n} \to K(\pi_{n+1}(L_\bullet), n+2), \quad n \geq 0$$

such that for each $n \geq 0$ the sequence

$$L_\bullet^{\leq n+1} \xrightarrow{\beta^n} L_\bullet^{\leq n} \xrightarrow{k^n} K(\pi_{n+1}(L_\bullet), n+2)$$

is a fiber sequence in sLie^r. □

4.4 Principal fibrations. In this section we will sketch the theory of principal fibrations in the category sLie^r of simplicial restricted Lie algebras. The corresponding theory in the category sSet of simplicial sets is well-known, and we will follow along its lines. We will use [29, Sections V.2-V.3] as our main reference.

Since the category $\mathsf{Mod}_{\mathbf{F}\{\xi\}}$ of left $\mathbf{F}\{\xi\}$-modules is abelian, we have the natural equivalence

$$\mathsf{Grp}(\mathsf{Mod}_{\mathbf{F}\{\xi\}}) \xrightarrow{\simeq} \mathsf{Mod}_{\mathbf{F}\{\xi\}},$$

where the left hand side is the category of group objects in $\mathsf{Mod}_{\mathbf{F}\{\xi\}}$. Moreover, it is not hard to see that the functor triv_ξ induces the equivalence

$$\mathsf{triv}_\xi \colon \mathsf{Grp}(\mathsf{Mod}_{\mathbf{F}\{\xi\}}) \xrightarrow{\simeq} \mathsf{Grp}(\mathsf{Lie}^r).$$

Indeed, if $\mu\colon L\times L\to L$ is a group multiplication in Lie^r, then μ coincides with the usual vector addition, and the latter is a Lie algebra homomorphism if and only if the Lie bracket is trivial. Similarly, we have equivalences for the categories of simplicial objects:

$$\mathsf{Grp}(\mathsf{sMod}_{\mathbf{F}\{\xi\}}) \xrightarrow{\simeq} \mathsf{sMod}_{\mathbf{F}\{\xi\}},$$

$$\mathsf{triv}_\xi \colon \mathsf{Grp}(\mathsf{sMod}_{\mathbf{F}\{\xi\}}) \xrightarrow{\simeq} \mathsf{Grp}(\mathsf{sLie}^r).$$

Let $M_\bullet \in \mathsf{sMod}_{\mathbf{F}\{\xi\}}$ be a simplicial left $\mathbf{F}\{\xi\}$-module. We say that $M_\bullet$ *acts* on a simplicial restricted Lie algebra $L_\bullet \in \mathsf{sLie}^r$ if there exists a morphism in sLie^r

$$\mu\colon \mathsf{triv}_\xi M_\bullet \times L_\bullet \to L_\bullet$$

which satisfies the associative and unit axioms. Note that

$$\bar{\mu}\colon \mathsf{triv}_\xi M_\bullet \to L_\bullet,\ \bar{\mu}(-) = \mu(-,0)$$

is a map of simplicial restricted Lie algebras, $\bar{\mu}(M_\bullet)$ is a Lie ideal in $L_\bullet$, and $\mu(m,l) = \bar{\mu}(m)+l$.

Let $M_\bullet$ act on $L_\bullet \in \mathsf{sLie}^r$. We denote by $L_\bullet/M_\bullet$ the *group action quotient*, i.e.

$$L_\bullet/M_\bullet = \mathrm{coeq}(\mathsf{triv}_\xi M_\bullet \times L_\bullet \underset{pr_2}{\overset{\mu}{\rightrightarrows}} L_\bullet).$$

We point out that this notation is consistent with the notation for a *categorical* quotient associated with a single morphism used before. Indeed, we have an isomorphism

$$\mathrm{coeq}(\mathsf{triv}_\xi M_\bullet \times L_\bullet \underset{pr_2}{\overset{\mu}{\rightrightarrows}} L_\bullet) \cong \mathrm{coeq}(\mathsf{triv}_\xi M_\bullet \underset{0}{\overset{\bar{\mu}}{\rightrightarrows}} L_\bullet).$$

Finally, we say that $M\in \mathsf{Mod}_{\mathbf{F}\{\xi\}}$ acts *freely* on $L\in \mathsf{Lie}^r$ if there is a M-equivariant isomorphism

$$L\cong \mathsf{triv}_\xi(M)\times X$$

in Lie^r, where M acts on the right hand side via $\mathsf{triv}_\xi(M)$. Note that the action is free if and only if $\bar{\mu}\colon \mathsf{triv}_\xi M\to L$ is a *split* monomorphism in Lie^r and $\bar{\mu}(\mathsf{triv}_\xi M)$ is a *Lie ideal* in L.

Let $\mathsf{sLie}^r_{M_\bullet}$ be the category of simplicial restricted Lie algebras with $M_\bullet$-action. Note that $\mathsf{sLie}^r_{M_\bullet}$ is a simplicial category such that the forgetful functor

$$\mathsf{oblv}\colon \mathsf{sLie}^r_{M_\bullet} \to \mathsf{sLie}^r$$

is simplicial. Moreover, oblv has a left adjoint

$$\mathsf{triv}_\xi M_\bullet \times (-)\colon \mathsf{sLie}^r \to \mathsf{sLie}^r_{M_\bullet},$$

which is also simplicial.

Lemma 4.4.1. *The adjoint pair*

$$\mathsf{triv}_\xi M_\bullet \times (-) : \mathsf{sLie}^r \rightleftarrows \mathsf{sLie}^r_{M_\bullet} : \mathsf{oblv}$$

is monadic and the forgetful functor oblv *preserves pushout squares.*

Proof. The adjoint pair is monadic by the definition of the category $\mathsf{sLie}^r_{M_\bullet}$. For the second part, it suffices to show that the endofunctor

$$\mathsf{triv}_\xi M_\bullet \times (-) \colon \mathsf{sLie}^r \to \mathsf{sLie}^r$$

preserves pushout squares, see [7, Proposition 4.3.2]. Finally, the assertion follows by Proposition 2.1.10. □

Since the model category sLie^r is cofibrantly generated (Theorem 3.2.3), we can transfer this model structure to $\mathsf{sLie}^r_{M_\bullet}$. Namely, we obtain the following theorem.

Theorem 4.4.2. *There exists a simplicial combinatorial model structure on* $\mathsf{sLie}^r_{M_\bullet}$ *such that* $f\colon L'_\bullet \to L_\bullet$ *is*

- *a* weak equivalence *if and only if* f *is a weak equivalence in* sLie^r*;*
- *a* fibration *if and only if* f *is a fibration in* sLie^r *(i.e.* $\mathsf{oblv}(f)$ *is a fibration in* $\mathsf{sVect}_{\mathbf{F}}$*, see Remark 3.2.2);*
- *a* cofibration *if and only if* f *has the left lifting property with respect to all acyclic fibrations.*

Proof. We again use Theorem 11.3.2 from [33]. Recall from Theorem 3.2.3 that I_{Lie} is the set of generating cofibrations and J_{Lie} is the set of generating trivial cofibrations for the model structure on sLie^r. Define the following sets of morphisms in $\mathsf{sLie}^r_{M_\bullet}$

$$I_{M_\bullet} = \{\mathsf{triv}_\xi(M_\bullet) \times u \mid u \in I_{\mathsf{Lie}}\}, \quad J_{M_\bullet} = \{\mathsf{triv}_\xi(M_\bullet) \times v \mid v \in J_{\mathsf{Lie}}\}.$$

It suffices to show that $I_{M_\bullet}$, $J_{M_\bullet}$ permit the small object argument (see [33, Definition 10.5.15]) and the functor

$$\mathsf{oblv}\colon \mathsf{sLie}^r_{M_\bullet} \to \mathsf{sLie}^r$$

takes $J_{M_\bullet}$-cell complexes to weak equivalences in sLie^r. The first part is clear because oblv preserves filtered colimits; and the second part is clear because oblv preserves pushouts (see Lemma 4.4.1) and any map in $\mathsf{oblv}(J_{M_\bullet})$ is a homotopy equivalence. □

Similar to [29, Corollary V.2.10], we obtain a description of cofibrant objects in $\mathsf{sLie}^r_{M_\bullet}$.

Proposition 4.4.3. *A simplicial restricted Lie* $M_\bullet$*-algebra* $L_\bullet \in \mathsf{sLie}^r_{M_\bullet}$ *is cofibrant if and only if*

1. M_n *acts freely on* L_n *for each* $n \geq 0$*;*
2. *the quotient* $L_\bullet/M_\bullet$ *is a cofibrant object in* sLie^r*.*

We first prove an analog of [29, Lemma 2.5].

Lemma 4.4.4. *Let* $L_\bullet \in \mathsf{sLie}^r_{M_\bullet}$ *be a simplicial restricted Lie* $M_\bullet$*-algebra such that* M_i *acts freely on* L_i *for each* $i \geq 0$ *and the quotient* $L_\bullet/M_\bullet = L^r(\mathbf{F}(\Delta^n))$*, where* $\mathbf{F}(\Delta^n)$ *is the simplicial vector* *space spanned by* $\Delta^n \in \mathsf{sSet}$. *Then there is an isomorphism* $\psi\colon \mathsf{triv}_\xi(M_\bullet) \times L^r(\mathbf{F}(\Delta^n)) \xrightarrow{\cong} L_\bullet$ *in* $\mathsf{sLie}^r_{M_\bullet}$ *such that the diagram*

$$\begin{array}{ccc} \mathsf{triv}_\xi(M_\bullet) \times L^r(\mathbf{F}(\Delta^n)) & \xrightarrow{\psi} & L_\bullet \\ & {\scriptstyle pr_2}\searrow \quad \swarrow{\scriptstyle q} & \\ & L^r(\mathbf{F}(\Delta^n)), & \end{array}$$

commutes.

Proof. By various adjunctions, there is a section $z\colon L^r(\mathbf{F}(\Delta^n)) \to L_\bullet$ of the quotient map $q\colon L_\bullet \to L^r(\mathbf{F}(\Delta^n))$. Then, we define ψ as the following composite

$$\psi\colon \mathsf{triv}_\xi(M_\bullet) \times L^r(\mathbf{F}(\Delta^n)) \xrightarrow{\mathsf{id}\times z} \mathsf{triv}_\xi(M_\bullet) \times L_\bullet \xrightarrow{\mu} L_\bullet,$$

where $\mu\colon \mathsf{triv}_\xi(M_\bullet) \times L_\bullet \to L_\bullet$ is the action map. Since M_i acts freely on L_i for each $i \geq 0$, the map ψ is an isomorphism. □

Proof of Proposition 4.4.3. Let $L_\bullet$ be a cofibrant object in $\mathsf{sLie}^r_{M_\bullet}$, then $L_\bullet$ is a retract of an $I_{M_\bullet}$-cell complex, see [33, Corollary 11.2.2]. By the cell induction, we obtain that M_n acts freely on L_n for each $n \geq 0$. Furthermore, since the quotient functor

$$(-)/M_\bullet\colon \mathsf{sLie}^r_{M_\bullet} \to \mathsf{sLie}^r$$

preserves colimits and $(I_{M_\bullet})/M_\bullet = I_{\mathsf{Lie}}$, we get that the quotient $L_\bullet/M_\bullet$ is a retract of an I_{Lie}-cell complex, and so $L_\bullet/M_\bullet$ is a cofibrant object in sLie^r.

Suppose now that the quotient $X_\bullet = L_\bullet/M_\bullet \in \mathsf{sLie}^r$ is a cofibrant object. Then $X_\bullet$ is a retract of an I_{Lie}-cell complex $Y_\bullet$; that is $Y_\bullet = \operatorname{colim}_n Y_\bullet^{(n)}$, $Y_\bullet^{(-1)} = 0$, and for each $n \geq 0$, there is a pushout diagram

$$\begin{array}{ccc} \coprod_\alpha A_\alpha & \longrightarrow & Y_\bullet^{(n-1)} \\ \downarrow{\scriptstyle \sqcup f_\alpha} & & \downarrow \\ \coprod_\alpha B_\alpha & \longrightarrow & Y_\bullet^{(n)} \end{array}$$

such that all $f_\alpha\colon A_\alpha \to B_\alpha$ belong to the generating set I_{Lie}. Define $L_\bullet^{(n)}$ as a pullback of the quotient map $q\colon L_\bullet \to L_\bullet/M_\bullet$ along $Y_\bullet^{(n)} \hookrightarrow Y_\bullet \to X_\bullet$. Then $L_\bullet$ is a retract of $L'_\bullet = \operatorname{colim}_n L_\bullet^{(n)}$. By applying inductively Lemma 4.4.4 together with Proposition 2.1.10, we obtain that $L'_\bullet$ is an $I_{M_\bullet}$-cell complex, and so $L_\bullet$ is a cofibrant object in $\mathsf{sLie}^r_{M_\bullet}$. □

Definition 4.4.5. Let $M_\bullet \in \mathsf{sMod}_{\mathbf{F}\{\xi\}}$ be a simplicial left $\mathbf{F}\{\xi\}$-module. A *principal fibration* (or principal $M_\bullet$-fibration) is a fibration $\pi\colon E_\bullet \to B_\bullet$ in $\mathsf{sLie}^r_{M_\bullet}$ so that

1. the base $B_\bullet$ is cofibrant and has trivial $M_\bullet$ action;
2. $E_\bullet$ is a cofibrant object in $\mathsf{sLie}^r_{M_\bullet}$;
3. the induced map from the quotient $E_\bullet/M_\bullet \to B_\bullet$ is an isomorphism.

As usual, we say that two principal fibrations $\pi\colon E_\bullet \to B_\bullet$ and $\pi'\colon E'_\bullet \to B_\bullet$ are *isomorphic* if there is an isomorphism $f\colon E_\bullet \to E'_\bullet$ in $\mathsf{sLie}^r_{M_\bullet}$ such that $\pi = \pi' \circ f$. Furthermore, we point out that any map $f\colon E_\bullet \to E'_\bullet$ of principal fibrations over $B_\bullet$ such that $f/M_\bullet = \mathsf{id}$ is an isomorphism of principal fibrations. We will write $PF_{M_\bullet}(B_\bullet)$ for the set of isomorphism classes of principal $M_\bullet$-fibrations over $B_\bullet \in \mathsf{sLie}^r$.

We fix some notation. Let $\pi\colon E_\bullet \to B_\bullet$ be a principal $M_\bullet$-fibration and let $f\colon B'_\bullet \to B_\bullet$ be any map in sLie^r between cofibrant objects. Then, by Proposition 4.4.3, the pullback

$$\pi'\colon E_\bullet \times_{B_\bullet} B'_\bullet \to B'_\bullet$$

is again a principal fibration and we write $E_\bullet|_f$ for its total space $E_\bullet \times_{B_\bullet} B'_\bullet$. We denote by $\bar{f}\colon E_\bullet|_f \to E_\bullet$ the map between total spaces induced by f.

Moreover, if

$$f\colon (B'_\bullet, A'_\bullet) \to (B_\bullet, A_\bullet)$$

is a map of pairs and $E_\bullet$ is a principal fibration over $B_\bullet$, then we denote by $E_\bullet|_{\partial f}$ the restriction of $E_\bullet|_f$ to $A'_\bullet$. This notation is slightly awkward and non-standard, but nevertheless it will be convenient for us in the next section.

Note that $PF_{M_\bullet}(-)$ is a contravariant functor on the category of cofibrant objects in sLie^r and we will show that this is a homotopy one.

Lemma 4.4.6. *Suppose that $f_1, f_2\colon B'_\bullet \to B_\bullet$ are two homotopic maps in sLie^r between cofibrant objects. Then $PF_{M_\bullet}(f_1) = PF_{M_\bullet}(f_2)$.*

Proof. Compare with [29, Lemma V.3.4]. It is enough to consider the universal example: given a principal fibration $\pi\colon E_\bullet \to B_\bullet \times \Delta^1$, the restrictions $E_\bullet|_0 \to B_\bullet$ and $E_\bullet|_1 \to B_\bullet$ over the vertices of Δ^1 are isomorphic. For this consider the lifting problem in $\mathsf{sLie}^r_{M_\bullet}$

$$\begin{array}{ccc} E_\bullet|_0 & \longrightarrow & E_\bullet \\ {\scriptstyle d^0}\big\downarrow & \nearrow & \big\downarrow{\scriptstyle \pi} \\ E_\bullet|_0 \times \Delta^1 & \longrightarrow & B_\bullet \times \Delta^1. \end{array}$$

Since $E_\bullet|_0$ is cofibrant in $\mathsf{sLie}^r_{M_\bullet}$, the left vertical arrow

$$E_\bullet|_0 = E_\bullet|_0 \times \Delta^0 \xrightarrow{\mathsf{id}\times d^0} E_\bullet|_0 \times \Delta^1$$

is a trivial cofibration. Therefore a lifting exists and defines an isomorphism of principal fibrations $E_\bullet|_0 \times \Delta^1 \cong E_\bullet$ over $B_\bullet$. The pullback of the last isomorphism along d^1 gives the desired isomorphism $E_\bullet|_0 \cong E_\bullet|_1$. □

As a corollary, we obtain the following statement, cf. [29, Lemma V.3.5].

Lemma 4.4.7. *Suppose that $B_\bullet$ is a contractible cofibrant object in sLie^r. Then any principal fibration $E_\bullet$ over $B_\bullet$ is trivializable, i.e. $E_\bullet \cong \mathsf{triv}_\xi M_\bullet \times B_\bullet$.* □

We now define the classifying object for principal fibrations. Recall that $M_\bullet \in \mathsf{sMod}_{\mathbf{F}\{\xi\}}$ is a simplicial left $\mathbf{F}\{\xi\}$-module.

Definition 4.4.8. Let $EM_\bullet$ be any cofibrant object in $\mathsf{sLie}^r_{M_\bullet}$ such that the map $EM_\bullet \to 0$ is a weak equivalence. Let $BM_\bullet = EM_\bullet/M_\bullet$ and $\pi_{M_\bullet}\colon EM_\bullet \to BM_\bullet$ be the resulting principal fibration.

Example 4.4.9. Note that the Kan cone $\mathsf{triv}_\xi C_\bullet M_\bullet \in \mathsf{sLie}^r$ ([29, Section III.5]) is contractible and it has a left $M_\bullet$-action such that M_n acts freely on

$$\mathsf{triv}_\xi C_n M_\bullet = \mathsf{triv}_\xi(M_n \times M_{n-1} \times M_{n-2} \times \ldots \times M_0).$$

However, $\mathsf{triv}_\xi C_\bullet M_\bullet$ is not $EM_\bullet$ because the Kan suspension

$$\mathsf{triv}_\xi \Sigma_\bullet M_\bullet = \mathsf{triv}_\xi C_\bullet M_\bullet / M_\bullet$$

is not a cofibrant object in sLie^r. This can be fixed as follows. Let $f\colon B_\bullet \to \mathsf{triv}_\xi \Sigma_\bullet M_\bullet$ be a cofibrant replacement. Then we observe that the pullback

$$E_\bullet = B_\bullet \times_{\mathsf{triv}_\xi \Sigma_\bullet M_\bullet} \mathsf{triv}_\xi (C_\bullet M_\bullet)$$

in $\mathsf{sLie}^r_{M_\bullet}$ is contractible, M_n acts freely on E_n for each $n \geq 0$, and the quotient $E_\bullet / M_\bullet = B_\bullet$ is cofibrant. In this way, we constructed a principal fibration from Definition 4.4.8.

Theorem 4.4.10. *For all cofibrant objects $B_\bullet \in \mathsf{sLie}^r$, the map*

$$\theta\colon [B_\bullet, BM_\bullet] \to PF_{M_\bullet}(B_\bullet)$$

sending a class $[f] \in [B_\bullet, BM_\bullet]$ to the pullback of $\pi_{M_\bullet}\colon EM_\bullet \to BM_\bullet$ along f is a bijection.

Proof. Note that θ is well-defined by Lemma 4.4.6. We will show that θ is a bijection by constructing an inverse. If $\pi\colon E_\bullet \to B_\bullet$ is a principal fibration, there is a lifting in the diagram in $\mathsf{sLie}^r_{M_\bullet}$

$$\begin{array}{ccc} 0 & \longrightarrow & EM_\bullet \\ \big\downarrow & \nearrow & \big\downarrow \\ E_\bullet & \longrightarrow & 0 \end{array} \tag{4.4.11}$$

because $E_\bullet$ is cofibrant and $EM_\bullet$ is fibrant. Let $f\colon B_\bullet \to BM_\bullet$ be the quotient map. We define

$$\Psi\colon PF_{M_\bullet}(B_\bullet) \to [B_\bullet, BM_\bullet]$$

by sending $\pi\colon E_\bullet \to B_\bullet$ to the class of f. The last map is well-defined because a lifting in the diagram (4.4.11) is unique up to an equivariant homotopy.

Note that if $EM_\bullet|_f$ is the pullback of $\pi_{M_\bullet}$ along $f = \Psi(\pi)$, then there is a commutative diagram

$$\begin{array}{ccc} E_\bullet & \longrightarrow & EM_\bullet|_f \\ & {\scriptstyle\pi}\searrow \quad \swarrow & \\ & B_\bullet. & \end{array}$$

Here the horizontal arrow is a map of principal fibrations over the same base, and so this is an isomorphism. Therefore, $\theta \circ \Psi = \mathsf{id}$. On the other hand, given a representative $g\colon B_\bullet \to BM_\bullet$ of a homotopy class in $[B_\bullet, BM_\bullet]$, the map $\bar{g}$ in the pullback diagram

$$\begin{array}{ccc} \theta(g) = EM_\bullet|_g & \xrightarrow{\bar{g}} & EM_\bullet \\ \big\downarrow & & \big\downarrow{\scriptstyle \pi_{M_\bullet}} \\ B_\bullet & \xrightarrow{g} & BM_\bullet \end{array}$$

gives a lifting in the diagram (4.4.11), so $\Psi \circ \theta = \mathsf{id}$. □

Corollary 4.4.12. *Let $M_\bullet \in \mathsf{sMod}_{\mathbf{F}\{\xi\}}$ be a simplicial left $\mathbf{F}\{\xi\}$-module. Then the classifying Lie algebra $BM_\bullet \in \mathsf{sLie}^r$ is unique up to homotopy. Moreover, $\pi_n(BM_\bullet) = \pi_{n-1}(M_\bullet)$.*

Proof. By Example 4.4.9, the classifying Lie algebra $BM_\bullet$ exists, and by Theorem 4.4.10, $BM_\bullet$ represents a contravariant functor

$$PF_{M_\bullet}\colon \mathrm{Ho}(\mathsf{sLie}^r)^{op} \to \mathsf{Set},$$

where $\mathrm{Ho}(\mathsf{sLie}^r)$ is the homotopy category of sLie^r. Therefore, by the Yoneda lemma, $BM_\bullet$ is unique up to a homotopy. The last statement follows from the long exact sequence of homotopy groups applying to the fibration $\pi_{M_\bullet}\colon EM_\bullet \to BM_\bullet$. □

Corollary 4.4.12 and Proposition 4.3.4 imply together the following statement.

Corollary 4.4.13. *Let $\pi\colon E_\bullet \to B_\bullet$ be a fibration in sLie^r such that the total space $E_\bullet$ and the base $B_\bullet$ are connected, and the fiber $\mathrm{fib}(\pi)$ is an Eilenberg-MacLane Lie algebra $K(M,n)$, $M \in \mathsf{Mod}_{\mathbf{F}\{\xi\}}$, $n \geq 0$. Then there exist a principal fibration $\pi'\colon E'_\bullet \to B'_\bullet$ and a commutative diagram*

$$\begin{array}{ccc} E'_\bullet & \xrightarrow{\bar{g}} & E_\bullet \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ B'_\bullet & \xrightarrow{g} & B_\bullet \end{array}$$

such that both maps g and $\bar{g}$ are weak equivalences. □

4.5 Serre spectral sequence.

Let $V_* \in \mathsf{Vect}^{gr}_{\mathbf{F}}$ be a graded vector space. Throughout this section we write $V_*[t] = \Sigma^t V_*$ for the shift of V_* by $t \in \mathbf{Z}$.

Let $\pi\colon E_\bullet \to B_\bullet$ be a fibration in the category sLie^r over a reduced base $B_\bullet$, $B_0 = 0$ with the fiber $F_\bullet = \pi^{-1}(0)$. Note that the base Lie algebra $B_\bullet$ is equipped with the increasing complete *skeletal filtration*:

$$B^{(s)}_\bullet = \begin{cases} 0 & \text{if } s = 0, \\ \mathrm{sk}_{s-1} B_\bullet & \text{if } s > 0; \end{cases}$$

and we define the increasing filtration $E^{(s)}_\bullet$ on the total space $E_\bullet$ by taking the preimages, i.e. $E^{(s)}_\bullet = \pi^{-1}(B^{(s)}_\bullet)$ if $s \geq 0$, and $E^{(-1)}_\bullet = 0$. By applying (non-reduced) homology to the filtered simplicial restricted Lie algebra $E^{(s)}_\bullet$, we obtain the following result.

Theorem 4.5.1. *Let $\pi\colon E_\bullet \to B_\bullet$ be a fibration in sLie^r. For homology with any coefficient module $M \in \mathsf{Mod}^{\mathbf{F}\{\xi\}}$ there is a convergent E^1-spectral sequence with*

$$E^1_{s,t} \cong H_{s+t}(E^{(s)}_\bullet, E^{(s-1)}_\bullet; M),$$

where the differential d_1 is the boundary operator of the triple $(E^{(s)}_\bullet, E^{(s-1)}_\bullet, E^{(s-2)}_\bullet)$, and E^∞ is the bigraded module associated to the filtration of $H_*(E_\bullet; M)$ defined by

$$F_s H_*(E_\bullet; M) = \mathrm{im}\left(H_*(E^{(s)}_\bullet; M) \to H_*(E_\bullet; M)\right). \qquad \square$$

Example 4.5.2. Let $\pi = \mathsf{id}\colon B_\bullet \to B_\bullet$ be the identity map, and suppose that the base Lie algebra $B_\bullet = L^r(V_\bullet)$ is almost-free, $V_\bullet \in \tilde{\mathsf{s}}\mathsf{Vect}_{\mathbf{F}}$. Then the quotient Lie algebra $B^{(s)}_\bullet / B^{(s-1)}_\bullet$, $s > 0$ is a free simplicial restricted Lie algebra $L^r(W_\bullet)$ generated by

$$W_\bullet = \Gamma(NV_{s-1}[s-1]),$$

where $NV_{s-1} \subseteq V_{s-1}$ is the vector subspace of normalized chains in $V_\bullet$, $NV_{s-1}[s-1]$ is the chain complex having NV_{s-1} concentrated in degree $s-1$, and

$$\Gamma\colon \mathsf{Ch}_{\geq 0} \to \mathsf{sVect}_{\mathbf{F}}$$

is the inverse to the normalized chain complex functor N, see Section 1.4. Therefore, $E^1_{s,t} = 0$ if $t \neq 0$ in the spectral sequence of Theorem 4.5.1, and so it degenerates at the second page. Moreover, the complex

$$C_s(B_\bullet; M) = \left(H_s(B^{(s)}, B^{(s-1)}; M), d_1\right),$$

where d_1 is the boundary operator δ of the triple $(B^{(s)}_\bullet, B^{(s-1)}_\bullet, B^{(s-2)}_\bullet)$ (see the sequence (4.2.16)) computes the homology groups $H_s(B_\bullet; M)$.

Next, we will compute the second page $E^2_{s,t}$ of the spectral sequence in Theorem 4.5.1 provided the fibration $\pi\colon E_\bullet \to B_\bullet$ is a principal fibration (Section 4.4) and the base $B_\bullet$ is reduced. We will follow the classical approach of [64, Chapter 9.2] and [68, Chapter 15].

We begin with defining analogs of a disk and a sphere in the category sLie^r. Let $W_* \in \mathsf{Vect}^{gr}_{\mathbf{F}}$ be a graded vector space concentrated in only one degree $s \geq 1$; that is $W_s = W$ and $W_* = 0$ if $* \neq s$. Consider a chain complex $C(W_*)_\bullet$ given as follows

$$C(W_*)_k = \begin{cases} W & \text{if } k = s, s-1, \\ 0, & \text{otherwise,} \end{cases}$$

endowed with the only one non-trivial differential $\partial_s = \mathsf{id}\colon C(W_*)_s \to C(W_*)_{s-1}$. Note that W_* can be considered itself as a chain complex; then $C(W_*)$ is the cone of the identity morphism $\mathsf{id}\colon W_*[-1] \to W_*[-1]$. We write $D(W_*) \in \mathsf{sLie}^r$ (resp. $\partial D(W_*) \in \mathsf{sLie}^r$) for the free simplicial restricted Lie algebra $L^r(\Gamma C(W_*))$ (resp. $L^r(\Gamma W_*[-1])$).

Remark 4.5.3. We list several properties of $D(W_*)$ and $\partial D(W_*)$, $W_* \in \mathsf{Vect}^{gr}_{\mathbf{F}}$. Both $D(W_*)$ and $\partial D(W_*)$ are cofibrant objects in sLie^r. The unique map $0 \to D(W_*)$ is a weak equivalence, and the simplicial restricted Lie algebra $D(W_*)$ is contractible. The canonical inclusion $\partial D(W_*) \subset D(W_*)$ is a cofibration and the quotient Lie algebra $D(W_*)/\partial D(W_*)$ is isomorphic to $\partial D(W_*[1])$. Since any $L_\bullet \in \mathsf{sLie}^r$ is a fibrant object, one has

$$\pi_s(L_\bullet) \cong [\partial D(\mathbf{F}[s+1]), L_\bullet],\ s \geq 0,$$

where $[\partial D(\mathbf{F}[s+1]), L_\bullet]$ is the set of homotopy classes of maps in sLie^r. Finally, for a pair $(L_\bullet, A_\bullet) \in \mathsf{sLie}^r$, one has

$$\pi_s(L_\bullet, A_\bullet) \cong [(D(\mathbf{F}[s]), \partial D(\mathbf{F}[s])), (L_\bullet, A_\bullet)],\ s \geq 1,$$

where the left hand side is the relative homotopy group (Definition 4.2.12) and the right hand side is the set of homotopy classes of maps of pairs in sLie^r.

Let $\pi\colon E_\bullet \to B_\bullet$ be a principal fibration with the fiber $M_\bullet \in \mathsf{sMod}_{\mathbf{F}\{\xi\}}$ and with the almost-free base $B_\bullet = L^r(V_\bullet)$, $V_\bullet \in \tilde{s}\mathsf{Vect}_{\mathbf{F}}$. Suppose

$$\alpha\colon (D(\mathbf{F}[s-1]), \partial D(\mathbf{F}[s-1])) \to (B^{(s)}_\bullet, B^{(s-1)}_\bullet),\ s \geq 2$$

is a map of pairs. The simplicial restricted Lie algebra $D(\mathbf{F}[s-1])$ is contractible, so by Lemma 4.4.7, there is an isomorphism of principal fibrations

$$\theta_\alpha\colon \mathsf{triv}_\xi M_\bullet \times D(\mathbf{F}[s-1]) \xrightarrow{\cong} E_\bullet|_\alpha.$$

Then we have the composite κ_α given by

$$\begin{aligned}\kappa_\alpha\colon H_t(\mathsf{triv}_\xi M_\bullet;\mathbf{F}) &\xrightarrow{\sigma} H_{t+s}(\mathsf{triv}_\xi M_\bullet \times D(\mathbf{F}[s-1]), \mathsf{triv}_\xi M_\bullet \times \partial D(\mathbf{F}[s-1]);\mathbf{F})\\ &\xrightarrow{\theta_{\alpha *}} H_{t+s}(E_\bullet|_\alpha, E_\bullet|_{\partial\alpha};\mathbf{F})\\ &\xrightarrow{\bar{\alpha}_*} H_{t+s}(E_\bullet^{(s)}, E_\bullet^{(s-1)};\mathbf{F}),\end{aligned}$$

where σ is given by $\sigma(x) = x\times\kappa_s$,

$$\kappa_s \in H_s(D(\mathbf{F}[s-1]), \partial D(\mathbf{F}[s-1]);\mathbf{F}) = \widetilde{H}_s(\partial D(\mathbf{F}[s]);\mathbf{F})$$

is the canonical generator; here, we use the Künneth isomorphism (Corollary 4.2.5). Since any two isomorphisms of a principal fibration over a contractible object in sLie^r are homotopic to each other, the map κ_α does not depend on the choice of the trivialization θ_α. Similarly, suppose

$$\alpha'\colon (D(\mathbf{F}[s-1]), \partial D(\mathbf{F}[s-1])) \to (B_\bullet^{(s)}, B_\bullet^{(s-1)})$$

is another map of pairs homotopic to α. By Lemma 4.4.6, there is an isomorphism $\theta\colon E_\bullet|_\alpha \cong E_\bullet|_{\alpha'}$ such that $\bar{\alpha}'\circ\theta = \bar{\alpha}$. Therefore, if

$$\theta_{\alpha'}\colon \mathsf{triv}_\xi M_\bullet \times D(\mathbf{F}[s-1]) \xrightarrow{\cong} E_\bullet|_{\alpha'}$$

is a trivialization, then there exists an isomorphism:

$$\phi\colon \mathsf{triv}_\xi M_\bullet \times D(\mathbf{F}[s-1]) \xrightarrow{\cong} \mathsf{triv}_\xi M_\bullet \times D(\mathbf{F}[s-1])$$

such that $\theta\circ\theta_\alpha = \theta_{\alpha'}\circ\phi$. Then we have

$$\bar{\alpha}_*\circ\theta_{\alpha *}\circ\sigma = \bar{\alpha}'_*\circ\theta_*\circ\theta_{\alpha *}\circ\sigma = \bar{\alpha}'_*\circ\theta_{\alpha' *}\circ\phi_*\circ\sigma.$$

Since ϕ is an isomorphism of a principal fibration over a contractible object, $\phi_* = \mathsf{id}$, and so $\kappa_\alpha = \kappa_{\alpha'}$. In this way, the following map is well-defined

$$\begin{aligned}\kappa\colon H_t(\mathsf{triv}_\xi M_\bullet;\mathbf{F})\times\pi_{s-1}(B_\bullet^{(s)}, B_\bullet^{(s-1)}) &\to H_{t+s}(E_\bullet^{(s)}, E_\bullet^{(s-1)};\mathbf{F}),\\ (x,[\alpha]) &\mapsto \kappa_\alpha(x).\end{aligned} \tag{4.5.4}$$

Lemma 4.5.5. *κ is bilinear.*

Proof. Compare the next proof with [68, Lemma 15.23]. Clearly, the mapping κ is linear in the first variable. The sum in

$$\pi_{s-1}(B_\bullet^{(s)}, B^{(s-1)\bullet}) = [(D(\mathbf{F}[s-1]), \partial D(\mathbf{F}[s-1])), (B_\bullet^{(s)}, B^{(s-1)\bullet})]$$

is defined by using the coproduct

$$\mu\colon D(\mathbf{F}[s-1]) \to D((\mathbf{F}\oplus\mathbf{F})[s-1]) \cong D(\mathbf{F}[s-1])\sqcup D(\mathbf{F}[s-1]),$$

induced by the diagonal $\mathbf{F}\to\mathbf{F}\oplus\mathbf{F}$. Namely, if $[\alpha],[\beta]\in\pi_{s-1}(B_\bullet^{(s)}, B^{(s-1)\bullet})$, then $[\alpha]+[\beta]$ is the class $[\nabla\circ(\alpha\sqcup\beta)\circ\mu]$, where $\nabla\colon B_\bullet^{(s)}\sqcup B_\bullet^{(s)}\to B_\bullet^{(s)}$ is the codiagonal. Note that, the pullback $E_\bullet|_{\nabla\circ(\alpha\sqcup\beta)}$ can be identified with the pushout (Proposition 2.1.10):

$$E_\bullet|_\alpha \sqcup_{\mathsf{triv}_\xi M_\bullet} E_\bullet|_\beta,$$

where $\mathsf{triv}_\xi M_\bullet \hookrightarrow E_\bullet|_\alpha$ and $\mathsf{triv}_\xi M_\bullet \hookrightarrow E_\bullet|_\beta$ are embeddings of the fibers. So, if

$$\theta_\alpha\colon \mathsf{triv}_\xi M_\bullet \times D(\mathbf{F}[s-1]) \xrightarrow{\cong} E_\bullet|_\alpha$$

and

$$\theta_\beta\colon \mathsf{triv}_\xi M_\bullet \times D(\mathbf{F}[s-1]) \xrightarrow{\cong} E_\bullet|_\beta$$

are trivializations, we obtain a trivialization

$$\theta_\alpha \sqcup \theta_\beta\colon \mathsf{triv}_\xi M_\bullet \times D((\mathbf{F}\oplus\mathbf{F})[s-1]) \to E_\bullet|_{\nabla\circ(\alpha\sqcup\beta)}$$

by "gluing" together θ_α and θ_β. The pullback of $\theta_\alpha \sqcup \theta_\beta$ along μ is an isomorphism

$$\theta_{\alpha+\beta}\colon \mathsf{triv}_\xi M_\bullet \times D(\mathbf{F}[s-1]) \xrightarrow{\cong} E_\bullet|_{\alpha+\beta}$$

such that $\bar{\mu}\circ\theta_{\alpha+\beta} = (\theta_\alpha\sqcup\theta_\beta)\circ(\mathsf{id}\times\mu)$.

Since

$$H_*(E_\bullet|_{\nabla\circ(\alpha\sqcup\beta)}, E_\bullet|_{\partial\nabla\circ(\alpha\sqcup\beta)};\mathbf{F}) \cong H_*(E_\bullet|_\alpha, E_\bullet|_{\partial\alpha};\mathbf{F}) \oplus H_*(E_\bullet|_\beta, E_\bullet|_{\partial\beta};\mathbf{F}),$$

we calculate

$$\begin{aligned}\kappa_{\alpha+\beta}(x) &= \overline{\nabla}_*\circ(\overline{\alpha\sqcup\beta})_*\circ\bar{\mu}_*\circ\theta_{(\alpha+\beta)*}\circ\sigma(x)\\ &= \overline{\nabla}_*\circ(\bar{\alpha}\sqcup\bar{\beta})_*\circ(\theta_\alpha\sqcup\theta_\beta)_*\circ(\mathsf{id}\times\mu)_*\circ\sigma(x)\\ &= \bar{\alpha}_*\circ\theta_{\alpha*}\circ\sigma(x)+\bar{\beta}_*\circ\theta_{\beta*}\circ\sigma(x) = \kappa_\alpha(x)+\kappa_\beta(x)\end{aligned}$$

for all $x\in H_*(\mathsf{triv}_\xi M_\bullet;\mathbf{F})$. □

Thus, κ induces a natural homomorphism

$$\kappa\colon H_t(\mathsf{triv}_\xi M_\bullet;\mathbf{F})\otimes\pi_{s-1}(B^{(s)}_\bullet, B^{(s-1)}_\bullet)\to H_{t+s}(E^{(s)}_\bullet, E^{(s-1)}_\bullet;\mathbf{F}).$$

Lemma 4.5.6. *The map κ commutes with the boundary operators of triples $(B^{(s)}_\bullet, B^{(s-1)}_\bullet, B^{(s-2)}_\bullet)$ and $(E^{(s)}_\bullet, E^{(s-1)}_\bullet, E^{(s-2)}_\bullet)$ in the sense that the following diagram commutes*

$$\begin{array}{ccc} H_t(\mathsf{triv}_\xi M_\bullet;\mathbf{F})\otimes\pi_{s-1}(B^{(s)}_\bullet, B^{(s-1)}_\bullet) & \xrightarrow{\kappa} & H_{s+t}(E^{(s)}_\bullet, E^{(s-1)}_\bullet;\mathbf{F})\\ \downarrow{\scriptstyle \mathsf{id}\otimes\delta} & & \downarrow{\scriptstyle\delta}\\ H_t(\mathsf{triv}_\xi M_\bullet;\mathbf{F})\otimes\pi_{s-2}(B^{(s-1)}_\bullet, B^{(s-2)}_\bullet) & \xrightarrow{\kappa} & H_{s+t-1}(E^{(s-1)}_\bullet, E^{(s-2)}_\bullet;\mathbf{F}).\end{array}$$

Proof. Compare with [68, Lemma 15.24]. Let $x\in H_t(\mathsf{triv}_\xi M_\bullet;\mathbf{F})$ and $[\alpha]\in\pi_{s-1}(B^{(s)}_\bullet, B^{(s-1)}_\bullet)$. Then

$$\begin{aligned}\delta\circ\kappa(x\otimes[\alpha]) &= j_*\circ\partial\circ\bar{\alpha}_*\circ\theta_{\alpha*}(x\times\kappa_s)\\ &= j_*\circ(\bar{\alpha}|E_\bullet|_{\partial\alpha})_*\circ\theta_{\partial\alpha*}(x\times\partial\kappa_s),\end{aligned}$$

where

$$\partial\colon H_{s+t}(E^{(s)}_\bullet, E^{(s-1)}_\bullet;\mathbf{F})\to\widetilde{H}_{s+t-1}(E^{(s-1)}_\bullet;\mathbf{F})$$

is the boundary homomorphism, $j\colon(E^{(s-1)}_\bullet,0)\to(E^{(s-1)}_\bullet, E^{(s-2)}_\bullet)$ is the evident map of pairs, and $\partial\kappa_s\in H_{s-1}(\partial D(\mathbf{F}[s-1]);\mathbf{F})$ is a generator.

Let $\rho\colon (D(\mathbf{F}[s-2]), \partial D(\mathbf{F}[s-2])) \to (\partial D(\mathbf{F}[s-1]), 0)$ be the standard projection; then $\delta[\alpha]$ is the class of the composite

$$(D(\mathbf{F}[s-2]), \partial D(\mathbf{F}[s-2])) \xrightarrow{\rho} (\partial D(\mathbf{F}[s-1]), 0) \xrightarrow{\alpha|_{\partial D(\mathbf{F}[s-1])}} (B_\bullet^{(s-1)}, 0) \xrightarrow{j} (B_\bullet^{(s-1)}, B_\bullet^{(s-2)}).$$

For convenience, we abbreviate this composite by γ. Then there is a trivialization

$$\theta_\gamma\colon \mathsf{triv}_\xi M_\bullet \times D(\mathbf{F}[s-2]) \xrightarrow{\cong} E_\bullet|_\gamma$$

such that $\bar{\rho} \circ \theta_\gamma = \theta_{\partial f} \circ (\mathsf{id} \times \rho)$. Finally, we have

$$\begin{aligned}
\kappa \circ (\mathsf{id} \otimes \delta)(x \otimes [\alpha]) &= \bar{\gamma}_* \circ \theta_{\gamma *}(x \times \kappa_{s-1}) \\
&= j_* \circ (\bar{\alpha}|E_\bullet|_{\partial\alpha})_* \circ \bar{\rho}_* \circ \theta_{\gamma *}(x \times \kappa_{s-1}) \\
&= j_* \circ (\bar{\alpha}|E_\bullet|_{\partial\alpha})_* \circ \theta_{\partial f *} \circ (\mathsf{id} \times \rho)_*(x \times \kappa_{s-1}) \\
&= j_* \circ (\bar{\alpha}|E_\bullet|_{\partial\alpha})_* \circ \theta_{\partial f *}(x \times \partial\kappa_s) \\
&= \delta \circ \kappa(x \otimes [\alpha]).
\end{aligned}$$

□

Lemma 4.5.7. *Let $\pi\colon E_\bullet \to B_\bullet$ be a principal fibration with the fiber $M_\bullet \in \mathsf{sMod}_{\mathbf{F}\{\xi\}}$. Suppose that $B_\bullet = D(W_*)$, $W_* \in \mathsf{Vect}^{gr}_{\mathbf{F}}$, $W_k = 0$ if $k \neq s-1$, and $s \geq 2$. Then*

1. *$\kappa(x, \xi_*[\alpha]) = 0$ for all $x \in H_t(\mathsf{triv}_\xi M_\bullet; \mathbf{F})$ and $[\alpha] \in \pi_{s-1}(B_\bullet^{(s)}, B_\bullet^{(s-1)})$;*
2. *The induced map*

$$\kappa\colon H_t(\mathsf{triv}_\xi M_\bullet; \mathbf{F}) \otimes \pi_{s-1}(B_\bullet^{(s)}, B_\bullet^{(s-1)})/\xi \to H_{t+s}(E_\bullet^{(s)}, E_\bullet^{(s-1)}; \mathbf{F})$$

is an isomorphism.

Proof. Since the base $B_\bullet$ is contractible, we can assume that π is a trivial principal fibration (Lemma 4.4.7). In this case, the diagram

$$\begin{array}{ccc}
H_t(\mathsf{triv}_\xi M_\bullet; \mathbf{F}) \otimes \pi_{s-1}(B_\bullet^{(s)}, B_\bullet^{(s-1)}) & \xrightarrow{\kappa} & H_{s+t}(E_\bullet^{(s)}, E_\bullet^{(s-1)}; \mathbf{F}) \\
\downarrow \cong & & \uparrow \cong \\
H_t(\mathsf{triv}_\xi M_\bullet; \mathbf{F}) \otimes H_s(B_\bullet^{(s)}, B_\bullet^{(s-1)}; \mathbf{F}\{\xi\}) & \longrightarrow & H_t(\mathsf{triv}_\xi M_\bullet; \mathbf{F}) \otimes H_s(B_\bullet^{(s)}, B_\bullet^{(s-1)}; \mathbf{F}).
\end{array}$$

commutes, where the left vertical arrow is the Hurewicz isomorphism and the right one is the Künneth isomorphism. Since

$$H_s(B_\bullet^{(s)}, B_\bullet^{(s-1)}; \mathbf{F}\{\xi\})/\xi \cong H_s(B_\bullet^{(s)}, B_\bullet^{(s-1)}; \mathbf{F}),$$

the lemma follows. □

Let $B_\bullet = L^r(V_\bullet)$ be an almost-free simplicial restricted Lie algebra, $V_\bullet \in \tilde{s}\mathsf{Vect}_{\mathbf{F}}$. Then the natural inclusion of normalized chains

$$NV_{s-1} \hookrightarrow V_{s-1} \hookrightarrow L^r(V_{s-1})$$

induce the following map of pairs

$$c_s\colon (D(NV_{s-1}), \partial D(NV_{s-1})) \to (B_\bullet^{(s)}, B_\bullet^{(s-1)}),$$

where we consider NV_{s-1} as a graded vector space concentrated in degree $s-1$. The next lemma follows immediately from the relative Hurewicz Theorem (Corollary 4.2.17).

Lemma 4.5.8. *For $B_\bullet = L^r(V_\bullet)$ as above, suppose that $B_\bullet$ is reduced. Then the induced map*

$$c_{s*}\colon \pi_{s-1}(D(NV_{s-1}), \partial D(NV_{s-1})) \to \pi_{s-1}(B_\bullet^{(s)}, B_\bullet^{(s-1)})$$

is an isomorphism. Moreover,

$$\pi_{s-1}(D(NV_{s-1}), \partial D(NV_{s-1})) \cong \mathbf{F}\{\xi\} \otimes_{\mathbf{F}} NV_{s-1}.$$

□

Proposition 4.5.9. *Let $\pi\colon E_\bullet \to B_\bullet$ be a principal fibration with the fiber $M_\bullet \in \mathsf{sMod}_{\mathbf{F}\{\xi\}}$ and the base $B_\bullet = L^r(V_\bullet)$, $V_\bullet \in \tilde{s}\mathsf{Vect}_{\mathbf{F}}$. Then the induced map*

$$\bar{c}_{s*}\colon H_*(E_\bullet|_{c_s}, E_\bullet|_{\partial c_s}; \mathbf{F}) \to H_*(E_\bullet^{(s)}, E_\bullet^{(s-1)}; \mathbf{F})$$

is an isomorphism for any $s \geq 0$.

Proof. Let $(X_\bullet, A_\bullet)$ be a pair in sLie^r. There is a strongly convergent functorial spectral sequence

$$E^1_{n,m} = H_n(L_m, A_m; \mathbf{F}) \Rightarrow H_{n+m}(L_\bullet, A_\bullet; \mathbf{F}),$$

where $H_*(L_m, A_m; \mathbf{F})$ are the homology groups of the *constant* simplicial restricted Lie algebra pair (L_m, A_m).

If $(L_\bullet, A_\bullet) = (E_\bullet^{(s)}, E_\bullet^{(s-1)})$, then by the definition of a principal fibration, we have

$$H_n(E_m^{(s)}, E_m^{(s-1)}; \mathbf{F}) \cong H_n(\mathsf{triv}_\xi M_m \times B_m^{(s)}, \mathsf{triv}_\xi M_m \times B_m^{(s-1)}; \mathbf{F}).$$

Since $B_\bullet$ is almost-free, the relative homology $H_i(B_m^{(s)}, B_m^{(s-1)}; \mathbf{F})$ are concentrated only in degree $i = 1$. Hence, by the Künneth isomorphism, we have

$$H_n(E_m^{(s)}, E_m^{(s-1)}; \mathbf{F}) \cong H_{n-1}(\mathsf{triv}_\xi M_m; \mathbf{F}) \otimes H_1(B_m^{(s)}, B_m^{(s-1)}; \mathbf{F}).$$

In a similar way, we obtain that $H_n(E_m|_{c_s}, E_m|_{\partial c_s}; \mathbf{F})$ is isomorphic to

$$H_{n-1}(\mathsf{triv}_\xi M_m; \mathbf{F}) \otimes H_1(D_m(NV_{s-1}), \partial D_m(NV_{s-1}); \mathbf{F}).$$

Since the map c_s induces the isomorphism

$$c_{s*}\colon H_1(D_m(NV_{s-1}), \partial D_m(NV_{s-1}); \mathbf{F}) \xrightarrow{\cong} H_1(B_m^{(s)}, B_m^{(s-1)}; \mathbf{F}),\ m, s \geq 0,$$

the proposition follows. □

Proposition 4.5.10. *Let $\pi\colon E_\bullet \to B_\bullet$ be a principal fibration with the fiber $M_\bullet \in \mathsf{sMod}_{\mathbf{F}\{\xi\}}$. Suppose that the base $B_\bullet = L^r(V_\bullet), V_\bullet \in \tilde{s}\mathsf{Vect}_{\mathbf{F}}$ is almost-free, and $V_0 = 0$. Then*

1. *$\kappa(x, \xi_*[\alpha]) = 0$ for all $x \in H_t(\mathsf{triv}_\xi M_\bullet; \mathbf{F})$ and $[\alpha] \in \pi_{s-1}(B_\bullet^{(s)}, B_\bullet^{(s-1)})$;*
2. *The induced map*

$$\kappa\colon H_t(\mathsf{triv}_\xi M_\bullet; \mathbf{F}) \otimes \pi_{s-1}(B_\bullet^{(s)}, B_\bullet^{(s-1)})/\xi \to H_{t+s}(E_\bullet^{(s)}, E_\bullet^{(s-1)}; \mathbf{F})$$

is an isomorphism for all $t, s \geq 0$.

Proof. In the commutative diagram

$$\begin{array}{ccc} H_t(\mathsf{triv}_\xi M_\bullet; \mathbf{F}) \otimes \pi_{s-1}(D(NV_{s-1}), \partial D(NV_{s-1})) & \xrightarrow{\kappa} & H_{s+t}(E_\bullet^{(s)}|_{c_s}, E_\bullet^{(s-1)}|_{\partial c_s}; \mathbf{F}) \\ \downarrow{\scriptstyle c_{s*}} & & \downarrow{\scriptstyle \bar{c}_{s*}} \\ H_t(\mathsf{triv}_\xi M_\bullet; \mathbf{F}) \otimes \pi_{s-1}(B_\bullet^{(s)}, B_\bullet^{(s-1)}) & \xrightarrow{\kappa} & H_{s+t}(E_\bullet^{(s)}, E_\bullet^{(s-1)}; \mathbf{F}) \end{array}$$

vertical arrows are isomorphisms by Lemma 4.5.8 and Proposition 4.5.9. Thus Lemma 4.5.7 implies the proposition. □

Theorem 4.5.11 (Serre spectral sequence). *Let $\pi\colon E_\bullet \to B_\bullet$ be a principal fibration with the fiber $M_\bullet \in \mathsf{sMod}_{\mathbf{F}\{\xi\}}$. Suppose that the base $B_\bullet = L^r(V_\bullet)$, $V_\bullet \in \tilde{s}\mathsf{Vect}_{\mathbf{F}}$ is almost-free, and $V_0 = 0$. Then there is a strongly convergent spectral sequence with*

$$E^2_{s,t} \cong H_t(\mathsf{triv}_\xi M_\bullet; \mathbf{F}) \otimes H_s(B_\bullet; \mathbf{F})$$

and E^∞ is the bigraded module associated to the filtration of $H_*(E_\bullet; \mathbf{F})$ defined by

$$F_s H_*(E_\bullet; \mathbf{F}) = \operatorname{im}\left(H_*(E^{(s)}_\bullet; \mathbf{F}) \to H_*(E_\bullet; \mathbf{F})\right).$$

Proof. In the spectral sequence of Theorem 4.5.1, we have

$$E^1_{s,t} \cong H_{s+t}(E^{(s)}_\bullet, E^{(s-1)}_\bullet; \mathbf{F});$$

the differential d_1 is the boundary operator δ of the triple $(E^{(s)}_\bullet, E^{(s-1)}_\bullet, E^{(s-2)}_\bullet)$, see the sequence (4.2.16). By Proposition 4.5.10 and Lemma 4.5.6, the map κ

$$\kappa\colon (H_t(\mathsf{triv}_\xi M_\bullet; \mathbf{F}) \otimes H_s(B^{(s)}_\bullet, B^{(s-1)}_\bullet; \mathbf{F}), \mathsf{id} \otimes \delta) \xrightarrow{\cong} (H_{s+t}(E^{(s)}_\bullet, E^{(s-1)}_\bullet; \mathbf{F}), \delta).$$

is an isomorphism of chain complexes. Therefore, by Example 4.5.2, we obtain that

$$E^2_{s,t} \cong H_t(\mathsf{triv}_\xi M_\bullet; \mathbf{F}) \otimes H_s(B_\bullet; \mathbf{F}). \qquad \square$$

By combining Theorem 4.5.11 with Corollary 4.4.13, we obtain the following statement.

Corollary 4.5.12. *Let $\pi\colon E_\bullet \to B_\bullet$ be a fibration in sLie^r such that the total space $E_\bullet$ and the base $B_\bullet$ are connected, and the fiber $\operatorname{fib}(\pi)$ is an Eilenberg-MacLane Lie algebra $K(M,n)$, $M \in \mathsf{Mod}_{\mathbf{F}\{\xi\}}$, $n \geq 0$. Then there is a functorial convergent spectral sequence*

$$E^2_{s,t} = H_t(K(M,n); \mathbf{F}) \otimes H_s(B_\bullet; \mathbf{F}) \Rightarrow H_{s+t}(E_\bullet; \mathbf{F}). \qquad \square$$

Remark 4.5.13. It seems likely that the spectral sequence of Corollary 4.5.12 exists for *any* fibration over a connected base. All steps in the proof can be generalized that much, except Proposition 4.5.9. At the time of writing, we do not know how to extend this proposition on any larger class of fibrations.

We also point out that Proposition 4.5.9 can be viewed as a consequence of Mather's second cube theorem in the category sLie^r, see [17, Definition 1.4] and [16]. At the time of writing, we are not aware if the cube theorem holds for sLie^r.

5. F-complete simplicial restricted Lie algebras

In this section we prove Theorem D from the introduction. In Section 5.1 we discuss basic properties of the ring $\mathbf{F}\{\xi\}$, define *ξ-adic completion* (Definition 5.1.8), and prove the Artin-Rees property for the ideal (ξ) in $\mathbf{F}\{\xi\}$ (Proposition 5.1.6). The latter implies the exactness property of the ξ-adic completion for finitely-generated modules, see Proposition 5.1.9. For non-finitely generated $\mathbf{F}\{\xi\}$-modules, the ξ-completion is not exact, and we define its *left derived functors* L_0 and L_1 in Section 5.2. We define *derived ξ-complete modules* in Definition 5.2.3 and show in Proposition 5.2.13 that derived ξ-complete modules form a weak Serre subcategory in $\mathsf{Mod}_{\mathbf{F}\{\xi\}}$. After that, we completely set up to prove Theorem D.

In Section 5.3 we define both $\mathbf{F}$-*complete* objects in sLie^r and the $\mathbf{F}$-*completion* functor L_ξ (Definition 5.3.1). We prove Theorem D by induction along the Postnikov tower. In Corollary 5.3.7, we show that Theorem D holds for Eilenberg-MacLane Lie algebras $K(M,n)$. After that, we heavily use the Serre spectral sequence (Corollary 4.5.12) to prove Theorem D as Corollary 5.3.12. Furthermore, in Corollary 5.3.11, we describe the homotopy groups of the $\mathbf{F}$-completion $L_\xi L_\bullet$ in terms of $\pi_*(L_\bullet)$ and derived functors L_0 and L_1.

5.1 ξ-complete modules. We begin with a few algebraic preliminaries. Recall from Definition 2.1.3 that $\mathbf{F}\{\xi\}$ is the ring of twisted polynomials. If $\mathbf{F} \neq \mathbf{F}_p$, then $\mathbf{F}\{\xi\}$ is a noncommutative ring, however it still shares a lot of common properties with the usual polynomial ring $\mathbf{F}[t]$. First of all, we note that the ring $\mathbf{F}\{\xi\}$ has no zero divisors.

We say that a twisted polynomial

$$f(\xi) = a_0 + a_1\xi + a_2\xi^2 + \ldots + a_n\xi^n \in \mathbf{F}\{\xi\}$$

has *degree* n if the leading coefficient $a_n \neq 0$. We will denote by $\deg(f)$ the degree of f. The next lemma shows that the function $\deg\colon \mathbf{F}\{\xi\} \setminus 0 \to \mathbf{N}$ can be used for left and right divisions with a remainder. The proof is straightforward and we leave it to the reader.

Lemma 5.1.1. *If f and g are in $\mathbf{F}\{\xi\}$ and g is nonzero, then there are $q, r \in \mathbf{F}\{\xi\}$ such that $f = qg + r$ and either $r = 0$ or $\deg(r) < \deg(g)$. Similarly, there are q' and r' such that $f = gq' + r'$ and either $r' = 0$ or $\deg(r') < \deg(g)$.* □

Corollary 5.1.2. *Any left (resp. right) ideal in $\mathbf{F}\{\xi\}$ is principal. In particular, the ring $\mathbf{F}\{\xi\}$ is left (resp. right) Noetherian.* □

Corollary 5.1.3. *Let $M \in \mathsf{Mod}_{\mathbf{F}\{\xi\}}$ (resp. $M \in \mathsf{Mod}^{\mathbf{F}\{\xi\}}$) be a free module. Then any submodule N of M is also free.* □

Corollary 5.1.4. *Let $M \in \mathsf{Mod}_{\mathbf{F}\{\xi\}}$ (resp. $M \in \mathsf{Mod}^{\mathbf{F}\{\xi\}}$) be any left (resp. right) module over $\mathbf{F}\{\xi\}$. Then there is a short exact sequence*

$$0 \to F_1 \to F_0 \to M \to 0,$$

where F_0, F_1 are free modules. □

Corollary 5.1.5. *Any torsion-free left (resp. right) $\mathbf{F}\{\xi\}$-module is a filtered colimit of finitely generated free submodules.* □

Consider now the *left* ideal $(\xi) \subset \mathbf{F}\{\xi\}$ generated by the element ξ. Since the field $\mathbf{F}$ is perfect, the ideal (ξ) is actually two-sided and coincides with the *right* ideal of $\mathbf{F}\{\xi\}$ generated by the same element ξ. Recall from [30, Section 13] that an ideal I in a ring R has the *left Artin-Rees property* if, for every left ideal $K \subset R$, there is a positive integer n such that $K \cap I^n = IK$.

Proposition 5.1.6. *The ideal $I = (\xi) \subset \mathbf{F}\{\xi\}$ has the left Artin-Rees property.*

Proof. Recall that the *Rees ring* $\mathcal{R}(\xi)$ is the subring of the polynomial ring $\mathbf{F}\{\xi\}[x]$ generated by $\mathbf{F}\{\xi\} + Ix$, that is,

$$\mathcal{R}(\xi) = \mathbf{F}\{\xi\} + Ix + I^2x^2 + \ldots I^jx^j + \ldots$$

Note that the ring $\mathcal{R}(\xi)$ is generated by $\mathbf{F}\{\xi\}$ together with the element $y = \xi x$. Therefore $\mathcal{R}(\xi)$ is a quotient of the twisted polynomial ring $\mathbf{F}\{\xi, y\}$, and so it is left Noetherian by [30, Theorem 1.14]. Finally, this implies that the ideal (ξ) has the left Artin-Rees property by [30, Lemma 13.2]. □

Let $M \in \mathsf{Mod}_{\mathbf{F}\{\xi\}}$ be a left module over the ring $\mathbf{F}\{\xi\}$ of twisted polynomials. Since the field $\mathbf{F}$ is perfect, the subset

$$\xi^r(M) = \{\xi^r \cdot m \mid m \in M\} \subset M, \quad r \geq 0$$

is a *submodule* of M. Similar to Definition 2.2.1, we define the *r-th Frobenius twist* $M^{(r)} \in \mathsf{Mod}_{\mathbf{F}\{\xi\}}$ of M as follows. As an abelian group, $M^{(r)} = M$ and we endow it with a new $\mathbf{F}\{\xi\}$-action

$$- \cdot -\colon \mathbf{F}\{\xi\} \times M^{(r)} \to M^{(r)}$$

given by formulas: $\xi \cdot m = \xi m$, $a \cdot m = \varphi^{-r}(a)m$, where $a \in \mathbf{F}$, $m \in M^{(r)} = M$, and $\varphi\colon \mathbf{F} \to \mathbf{F}$ is the Frobenius automorphism, $\varphi(a) = a^p$. Then a map

$$\xi^r\colon M^{(r)} \to M, \quad m \mapsto \xi^r m \tag{5.1.7}$$

is a homomorphism of $\mathbf{F}\{\xi\}$-modules and $\xi^r(M) = \mathrm{im}(\xi^r)$. We set $M_r = \ker(\xi^r)$; as an abelian group, M_r consists of all $m \in M$ such that $\xi^r m = 0$.

Definition 5.1.8. Define the *ξ-adic completion* (or *ξ-completion*) of a left $\mathbf{F}\{\xi\}$-module M to be

$$\widehat{M} = \lim_r M/\xi^r(M) \in \mathsf{Mod}_{\mathbf{F}\{\xi\}}.$$

A left $\mathbf{F}\{\xi\}$-module M is *ξ-adic complete* (or *ξ-complete*) if the natural map $M \to \widehat{M}$ is an isomorphism.

We write $\mathbf{F}\{\{\xi\}\}$ for the ξ-completion $\widehat{\mathbf{F}\{\xi\}}$. Since ideals $(\xi)^r = (\xi^r)$ are two-sided, $\mathbf{F}\{\{\xi\}\}$ is endowed with a (non-commutative) ring structure such that $\mathbf{F}\{\xi\} \to \mathbf{F}\{\{\xi\}\}$ is a ring homomorphism. We observe that the ξ-completion takes value in the category $\mathsf{Mod}_{\mathbf{F}\{\{\xi\}\}}$ of left $\mathbf{F}\{\{\xi\}\}$-modules. We note that $\mathbf{F}\{\{\xi\}\}$ is a *torsion-free* left $\mathbf{F}\{\xi\}$-module.

The Artin-Rees property of the ideal (ξ) implies the following exactness statement.

Proposition 5.1.9. *Suppose that*

$$0 \to M' \to M \to M'' \to 0$$

is an exact sequence of finitely generated left $\mathbf{F}\{\xi\}$-modules. Then the completed sequence

$$0 \to \widehat{M}' \to \widehat{M} \to \widehat{M}'' \to 0$$

is again exact.

Proof. By [30, Lemma 13.1(a)], the ξ-adic topology on M' is induced from the ξ-adic topology on M. This implies the proposition, cf. e.g. [4, Proposition 10.12]. □

5.2 Derived ξ-complete modules. Here we define left derived functors of the ξ-adic completion functor. By Corollary 5.1.4, for any left $\mathbf{F}\{\xi\}$-module M there is a free resolution

$$0 \to F_1 \to F_0 \to M \to 0.$$

We define the *left derived functors* of the ξ-completion by formulas

$$L_0(M) = \mathrm{coker}(\widehat{F}_1 \to \widehat{F}_0), \quad L_1(M) = \ker(\widehat{F}_1 \to \widehat{F}_0).$$

These groups are independent on the choice of the resolution and they are functorial in M, as one checks by comparing resolutions. The commutative square

$$\begin{array}{ccc} F_1 & \longrightarrow & F_0 \\ \downarrow & & \downarrow \\ \widehat{F}_1 & \longrightarrow & \widehat{F}_0 \end{array}$$

induces a natural map

$$\phi_M\colon M \to L_0(M). \tag{5.2.1}$$

The next lemma follows immediately from the results of the previous section.

Lemma 5.2.2. *The functors L_0 and L_1 take values in left $\mathbf{F}\{\{\xi\}\}$-modules. If M is either a finitely generated or a free left $\mathbf{F}\{\xi\}$-module, then $L_0M = \widehat{M}$, $L_1M = 0$, and $\phi_M\colon M \to L_0M$ coincides with the ξ-adic completion.* □

Definition 5.2.3. Let $M \in \mathsf{Mod}_{\mathbf{F}\{\xi\}}$ be a left $\mathbf{F}\{\xi\}$-module. We define the *derived ξ-adic completion* (or derived ξ-completion) of M to be the homomorphism $\phi_M\colon M \to L_0M$. A left $\mathbf{F}\{\xi\}$-module M is *derived ξ-adic complete* (or derived ξ-complete) if $\phi_M\colon M \to L_0M$ is an isomorphism and $L_1M = 0$.

Remark 5.2.4. As we will see in Proposition 5.2.11, if M is ξ-complete, then $L_1M = 0$.

Lemma 5.2.5. *For a short exact sequence of left $\mathbf{F}\{\xi\}$-modules*

$$0 \to M' \to M \to M'' \to 0,$$

there is a natural six term exact sequence of $\mathbf{F}\{\{\xi\}\}$-modules

$$0 \to L_1M' \to L_1M \to L_1M'' \to L_0M' \to L_0M \to L_0M'' \to 0.$$ □

Next, we will give an interpretation of the derived ξ-adic completion in terms of Hom and Ext-functors. We begin with the following observation. Let $M \in \mathsf{Mod}_{\mathbf{F}\{\xi\}}$ be a left $\mathbf{F}\{\xi\}$-module and let N be an $\mathbf{F}\{\xi\}$-bimodule. Then the set of maps $\mathrm{Hom}_{\mathbf{F}\{\xi\}}(N,M)$ is naturally a left $\mathbf{F}\{\xi\}$-module. Indeed, if $f \in \mathrm{Hom}_{\mathbf{F}\{\xi\}}(N,M)$ and $a \in \mathbf{F}\{\xi\}$, then we set $af\colon N \to M$, $n \mapsto f(na)$, $n \in N$. By the same argument, the Ext-group $\mathrm{Ext}_{\mathbf{F}\{\xi\}}(N,M)$ is a left $\mathbf{F}\{\xi\}$-module as well.

We denote $\mathbf{F}\{\xi^{\pm}\}$ the ring of *twisted Laurent polynomials*, i.e. $\mathbf{F}\{\xi^{\pm}\}$ is defined as the set of Laurent polynomials in the variable ξ and coefficients in $\mathbf{F}$. It is endowed with a ring structure with the usual addition and with a non-commutative multiplication that can be summarized with relations:

$$\xi a = \varphi(a)\xi = a^p\xi, \;\; \xi^{-1}a = \varphi^{-1}(a)\xi^{-1}, \;\; a \in \mathbf{F}.$$

Here $\varphi\colon \mathbf{F} \to \mathbf{F}$ is the Frobenius automorphism.

We denote by $\mathbf{F}\{\xi\}/\xi^{\infty}$ the quotient $\mathbf{F}\{\xi\}$-bimodule $\mathbf{F}\{\xi^{\pm}\}/\mathbf{F}\{\xi\}$. The bimodule $\mathbf{F}\{\xi\}/\xi^{\infty}$ is the union of its subbimodules

$$C_r \subset \mathbf{F}\{\xi\}/\xi^{\infty}, \;\; r \geq 1,$$

where C_r is generated by $\xi^{-r} \in \mathbf{F}\{\xi\}/\xi^{\infty}$. Notice that there is a bimodule isomorphism $C_r \cong \mathbf{F}\{\xi\}/\xi^r$.

Lemma 5.2.6. *Let $M \in \mathsf{Mod}_{\mathbf{F}\{\xi\}}$ be a left $\mathbf{F}\{\xi\}$-module. Then*

1. $\operatorname{Hom}_{\mathbf{F}\{\xi\}}(C_r, M) \cong M_r$, *where M_r is the kernel* (5.1.7).
2. $\operatorname{Ext}_{\mathbf{F}\{\xi\}}(C_r, M) \cong M/\xi^r(M) = \operatorname{coker}(\xi^r)$.
3. $\operatorname{Hom}_{\mathbf{F}\{\xi\}}(\mathbf{F}\{\xi\}/\xi^\infty, M) \cong \lim_r \operatorname{Hom}_{\mathbf{F}\{\xi\}}(C_r, M)$.

Moreover, there is a short exact sequence:

$$0 \to \lim\nolimits^1 \operatorname{Hom}_{\mathbf{F}\{\xi\}}(C_r, M) \to \operatorname{Ext}_{\mathbf{F}\{\xi\}}(\mathbf{F}\{\xi\}/\xi^\infty, M) \to \widehat{M} \to 0. \qquad \square$$

For convenience, we set

$$\mathbb{E}_\xi M = \operatorname{Ext}_{\mathbf{F}\{\xi\}}(\mathbf{F}\{\xi\}/\xi^\infty, M) \ \text{ and } \ \mathbb{H}_\xi M = \operatorname{Hom}_{\mathbf{F}\{\xi\}}(\mathbf{F}\{\xi\}/\xi^\infty, M),$$

where $M \in \mathsf{Mod}_{\mathbf{F}}\{\xi\}$. Then Lemma 5.2.6 implies that there is an exact sequences in $\mathsf{Mod}_{\mathbf{F}\{\xi\}}$:

$$0 \to \mathbb{H}_\xi M \to \prod_{r\geq 1} M_r \to \prod_{r\geq 1} M_r \to \mathbb{E}_\xi M \to \prod_{r\geq 1} M/\xi^r M \to \prod_{r\geq 1} M/\xi^r M \to 0. \tag{5.2.7}$$

Corollary 5.2.8. *Let $M \in \mathsf{Mod}_{\mathbf{F}\{\xi\}}$ be a left $\mathbf{F}\{\xi\}$-module. Then there are natural isomorphisms*

$$L_0 M \cong \mathbb{E}_\xi M, \quad L_1 M \cong \mathbb{H}_\xi M.$$

Proof. By Lemma 5.2.6, we have $\mathbb{H}_\xi F = 0$ and $\mathbb{E}_\xi F \cong \widehat{F}$ for a free $\mathbf{F}\{\xi\}$-module $F \in \mathsf{Mod}_{\mathbf{F}\{\xi\}}$. Let $0 \to F_1 \to F_0 \to M \to 0$ be a free resolution of M. Then, in the commutative diagram

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \mathbb{H}_\xi M & \longrightarrow & \mathbb{E}_\xi F_1 & \longrightarrow & \mathbb{E}_\xi F_0 & \longrightarrow & \mathbb{E}_\xi M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow\cong & & \downarrow\cong & & \downarrow & & \\
0 & \longrightarrow & L_1 M & \longrightarrow & \widehat{F}_1 & \longrightarrow & \widehat{F}_0 & \longrightarrow & L_0 M & \longrightarrow & 0,
\end{array}$$

both rows are exact sequence and two middle vertical arrows are isomorphisms. This implies the statement. $\square$

Remark 5.2.9. Under the isomorphism of Corollary 5.2.8, the derived completion map (5.2.1)

$$\phi_M \colon M \to L_0 M$$

coincides with the boundary homomorphism

$$\delta_M \colon M = \operatorname{Hom}_{\mathbf{F}\{\xi\}}(\mathbf{F}\{\xi\}, M) \to \operatorname{Ext}_{\mathbf{F}\{\xi\}}(\mathbf{F}\{\xi\}/\xi^\infty, M),$$

which is associated with the short exact sequence

$$0 \to \mathbf{F}\{\xi\} \to \mathbf{F}\{\xi^\pm\} \to \mathbf{F}\{\xi\}/\xi^\infty \to 0. \tag{5.2.10}$$

Proposition 5.2.11. *Let M be a left $\mathbf{F}\{\xi\}$-module and let N be any of $\widehat{M}$, $\mathbb{H}_\xi M$, and $\mathbb{E}_\xi M$. Then $L_1 N = 0$ and $\phi_N \colon N \to L_0 N$ is an isomorphism. In particular, if $\phi_M \colon M \to L_0 M$ is an isomorphism, then $L_1 M \cong L_1 L_0 M = 0$.*

Proof. Using the exact sequence (5.2.10) and Corollary 5.2.8 we obtain that $L_1 N = 0$ and ϕ_N is an isomorphism if and only if

$$\operatorname{Hom}_{\mathbf{F}\{\xi\}}(\mathbf{F}\{\xi^\pm\}, N) = 0 \ \text{ and } \ \operatorname{Ext}_{\mathbf{F}\{\xi\}}(\mathbf{F}\{\xi^\pm\}, N) = 0. \tag{5.2.12}$$

This condition certainly holds if $p^r N$ for some $r \geq 1$, so it holds for all M_r and $M/\xi^r M$. If (5.2.12) holds for modules $N_i \in \mathsf{Mod}_{\mathbf{F}\{\xi\}}$, then it holds for their product $\prod N_i$. It is also clear that if (5.2.12) holds for any two modules in a short exact sequence

$$0 \to M' \to M \to M'' \to 0,$$

then it holds for the third one as well. Now the proposition follows for $\widehat{M}$ by applying these observations to a short exact sequence

$$0 \to \widehat{M} \to \prod_{r\geq 1} M/\xi^r M \to \prod_{r\geq 1} M/\xi^r M \to 0.$$

In a similar way, the proposition follows for $L_0 M = \mathbb{E}_\xi M$ and $L_1 M = \mathbb{H}_\xi M$ as well by considering the exact sequence (5.2.7). □

We write $\mathsf{Mod}^{\mathrm{comp}}_{\mathbf{F}\{\xi\}}$ for the full subcategory of $\mathsf{Mod}_{\mathbf{F}\{\xi\}}$ spanned by derived ξ-complete left $\mathbf{F}\{\xi\}$-modules. We summarize the results of this section in the next proposition.

Proposition 5.2.13.

1. *Let $M \in \mathsf{Mod}_{\mathbf{F}\{\xi\}}$. Then $M \in \mathsf{Mod}^{\mathrm{comp}}_{\mathbf{F}\{\xi\}}$ if and only if* $\mathrm{Hom}(\mathbf{F}\{\xi^{\pm}\}, N)$ *and* $\mathrm{Ext}(\mathbf{F}\{\xi^{\pm}\}, N)$ *vanish together.*
2. *The subcategory $\mathsf{Mod}^{\mathrm{comp}}_{\mathbf{F}\{\xi\}} \subset \mathsf{Mod}_{\mathbf{F}\{\xi\}}$ is closed under taking any limits, cokernels, and extensions. In particular, $\mathsf{Mod}^{\mathrm{comp}}_{\mathbf{F}\{\xi\}}$ is a weak Serre subcategory in $\mathsf{Mod}_{\mathbf{F}\{\xi\}}$, see e.g. [65, Tag 02MO]* □

5.3 The F-completion of simplicial restricted Lie algebras. Recall that a map $f\colon L'_\bullet \to L_\bullet$ in sLie^r is an $\mathbf{F}$-equivalence if the induced map

$$f_*\colon \widetilde{H}_*(L'_\bullet;\mathbf{F}) \to \widetilde{H}_*(L_\bullet;\mathbf{F})$$

is an isomorphism. Note that this definition coincides with Definition 3.2.21 because of Proposition 3.2.18. We say that $Z_\bullet \in \mathsf{sLie}^r$ is $\mathbf{F}$-*complete* if

$$f^*\colon [L_\bullet, Z_\bullet] \to [L'_\bullet, Z_\bullet]$$

is a bijection for any $\mathbf{F}$-equivalence $f\colon L'_\bullet \to L_\bullet$ between cofibrant objects in sLie^r.

Definition 5.3.1. A map $\phi\colon L_\bullet \to L_\xi L_\bullet$ is called the $\mathbf{F}$-*completion* of $L_\bullet \in \mathsf{sLie}^r$ if ϕ is an $\mathbf{F}$-equivalence and $L_\xi L_\bullet$ is $\mathbf{F}$-complete. Note that a simplicial restricted Lie algebra $Z_\bullet \in \mathsf{sLie}^r$ is $\mathbf{F}$-complete if and only if $Z_\bullet$ is a fibrant object in the model structure of Theorem 3.2.24.

Remark 5.3.2. By Theorem 3.2.24, any $\mathbf{F}$-equivalence between $\mathbf{F}$-complete objects is a weak equivalence, the $\mathbf{F}$-completion exists, and it is unique up to (a chain of) weak equivalences.

In this section we will give an explicit construction of the $\mathbf{F}$-completion

$$\phi\colon L_\bullet \to L_\xi L_\bullet$$

provided $\pi_0(L_\bullet) = 0$. We begin with Eilenberg-MacLane Lie algebras.

Lemma 5.3.3. *The following Eilenberg-MacLane Lie algebras are* $\mathbf{F}$*-complete:*

1. *$K(V,n) \in \mathsf{sLie}^r$, $V \in \mathsf{Vect}_{\mathbf{F}}$ is a vector space over $\mathbf{F}$, $n \geq 0$;*

2. *$K(F/\xi^r, n) \in \mathsf{sLie}^r$, $F \in \mathsf{Mod}_{\mathbf{F}\{\xi\}}$ is a free left $\mathbf{F}\{\xi\}$-module, $r \geq 1$, $n \geq 0$;*
3. *$K(\widehat{F}, n) \in \mathsf{sLie}^r$, $\widehat{F}$ is the ξ-adic completion of a free $\mathbf{F}\{\xi\}$-module F, $n \geq 0$.*

Proof. By Proposition 4.2.25 and the universal coefficient theorem (Remark 4.2.24), we have

$$[-, K(V, n)] \cong \widetilde{H}^{n+1}(-; V) \cong \operatorname{Hom}(\widetilde{H}_{n+1}(-; \mathbf{F}), V),$$

which proves the first part.

We will prove the second part by induction on r. The base case, $r = 1$ is already done because F/ξ is a vector space over $\mathbf{F}$. Assume that $K(F/\xi^r, n)$ is $\mathbf{F}$-complete; we will show that $K(F/\xi^{r+1}, n)$ is $\mathbf{F}$-complete as well. There is a fiber sequence

$$K(F/\xi^{r+1}, n) \to K(F/\xi^r, n) \to K(\xi^r F/\xi^{r+1}F, n+1) \simeq K(F/\xi, n+1).$$

By the inductive assumption, $K(F/\xi^r, n)$ and $K(\xi^r F/\xi^{r+1}F, n+1)$ are $\mathbf{F}$-complete, and so is the fiber $K(F/\xi^{r+1}, n)$.

Finally, $K(\widehat{F}, n)$ is $\mathbf{F}$-complete since there is a fiber sequence

$$K(\widehat{F}, n) \to \prod_{r\geq 1} K(F/\xi^r, n) \to \prod_{r\geq 1} K(F/\xi^r, n). \qquad \square$$

Lemma 5.3.4. *Let F be a free left $\mathbf{F}\{\xi\}$-module. Then the ξ-adic completion map $\phi_F\colon F \to \widehat{F}$ induces an $\mathbf{F}$-equivalence*

$$\phi_F^n\colon K(F, n) \to K(\widehat{F}, n)$$

for any $n \geq 0$.

Proof. Note that $\widehat{F} \in \mathsf{Mod}_{\mathbf{F}\{\xi\}}$ is a torsion-free module. Therefore, by Proposition 4.2.21, there is a commutative diagram

$$\begin{array}{ccc} H_*(K(F,0); \mathbf{F}) & \xrightarrow{\phi^0_{F*}} & H_*(K(\widehat{F},0); \mathbf{F}) \\ \cong\uparrow & & \cong\uparrow \\ \Lambda^*(F/\xi) & \xrightarrow{\phi_F} & \Lambda^*(\widehat{F}/\xi), \end{array}$$

where $\Lambda^*(-)$ is the exterior algebra and both vertical arrows are isomorphisms. Since $F/\xi \cong \widehat{F}/\xi$, the lemma follows for $n = 0$. For $n > 0$, consider a map of fiber sequences

$$\begin{array}{ccccc} K(F,n) & \longrightarrow & 0 & \longrightarrow & K(F, n+1) \\ \downarrow\phi^n_F & & \downarrow & & \downarrow\phi^{n+1}_F \\ K(\widehat{F}, n) & \longrightarrow & 0 & \longrightarrow & K(\widehat{F}, n+1). \end{array}$$

By Corollary 4.5.12, it induces the map $f^r_{s,t}\colon E^r_{s,t} \to \widehat{E}^r_{s,t}$ of strongly convergent spectral sequences:

$$E^2_{s,t} = H_t(K(F,n); \mathbf{F}) \otimes H_s(K(F, n+1); \mathbf{F}) \Rightarrow H_{s,t}(0; \mathbf{F}),$$

$$\widehat{E}^2_{s,t} = H_t(K(\widehat{F},n); \mathbf{F}) \otimes H_s(K(\widehat{F}, n+1); \mathbf{F}) \Rightarrow H_{s,t}(0; \mathbf{F}),$$

such that $f^2_{0,*} = \phi^n_{F*}$, $f^2_{*,0} = \phi^{n+1}_{F*}$, and $f^\infty_{*,*}$ is an isomorphism. Then Zeeman's comparison theorem (see e.g. [46, Theorem 3.26]) implies that ϕ^n_{F*} is an isomorphism if and only if ϕ^{n+1}_{F*} is an isomorphism. Finally, the induction by n implies the lemma. $\square$

Let $M \in \mathsf{Mod}_{\mathbf{F}\{\xi\}}$ be any left $\mathbf{F}\{\xi\}$-module and let

$$0 \to F_1 \xrightarrow{f} F_0 \to M \to 0$$

be a free resolution of M. The map f induces the map $\hat{f}\colon \widehat{F}_1 \to \widehat{F}_0$ of ξ-adic completions, and so it induces the map

$$\hat{f}^{n+1}\colon K(\widehat{F}_1, n+1) \to K(\widehat{F}_0, n+1)$$

of Eilenberg-MacLane Lie algebras. We denote by $L_\xi K(M,n)$ the *homotopy fiber* of $\hat{f}^n$. There is a map of fiber sequences in sLie^r:

$$\begin{array}{ccccccc} K(F_0,n) & \longrightarrow & K(M,n) & \longrightarrow & K(F_1,n+1) & \xrightarrow{f^{n+1}} & K(F_0,n+1) \\ \downarrow{\scriptstyle \phi^n_{F_0}} & & \downarrow{\scriptstyle \phi^n_M} & & \downarrow{\scriptstyle \phi^{n+1}_{F_1}} & & \downarrow{\scriptstyle \phi^{n+1}_{F_0}} \\ K(\widehat{F}_0,n) & \longrightarrow & L_\xi K(M,n) & \longrightarrow & K(\widehat{F}_1,n+1) & \xrightarrow{\hat{f}^{n+1}} & K(\widehat{F}_0,n+1), \end{array} \qquad (5.3.5)$$

where ϕ^n_M is the induced map between homotopy fibers $K(M,n) \simeq \mathrm{fib}(f^{n+1})$ and $L_\xi K(M,n) = \mathrm{fib}(\hat{f}^{n+1})$.

Proposition 5.3.6. *ϕ^n_M is the $\mathbf{F}$-completion of $K(M,n) \in \mathsf{sLie}^r$.*

Proof. By Lemma 5.3.3, Eilenberg-MacLane Lie algebras $K(\widehat{F}_1, n+1)$ and $K(\widehat{F}_0, n+1)$ are $\mathbf{F}$-complete, and so is $L_\xi K(M,n)$ as a homotopy fiber between $\mathbf{F}$-complete objects in sLie^r.

By Lemma 5.3.4, the maps $\phi^n_{F_0}$ and $\phi^{n+1}_{F_1}$ are $\mathbf{F}$-equivalences. By applying the Serre spectral sequence of Corollary 4.5.12 to the leftmost map of fiber sequences in (5.3.5), we obtain that ϕ^n_M is an $\mathbf{F}$-equivalence as well. □

Corollary 5.3.7.

1. *There are natural isomorphisms*

$$\pi_n(L_\xi K(M,n)) \cong L_0 M, \quad \pi_{n+1}(L_\xi K(M,n)) \cong L_1 M,$$

and the induced map $\pi_n(\phi^n_M)$ is the derived ξ-completion map (5.2.1).

2. *An Eilenberg-MacLane Lie algebra $K(M,n), n \geq 0$ is $\mathbf{F}$-complete if and only if $M \in \mathsf{Mod}_{\mathbf{F}\{\xi\}}$ is a left derived ξ-adic complete $\mathbf{F}\{\xi\}$-module.*

Proof. The first part follows immediately from definitions. For the second part, suppose first that $K(M,n)$ is $\mathbf{F}$-complete. Then, by Proposition 5.3.6, the map ϕ^n_M is a weak equivalence. Therefore $M \cong L_0 M$ and $L_1 M = 0$.

In the opposite direction, suppose that M is derived ξ-complete. Then by the first part, ϕ^n_M is a weak equivalence, and so the Eilenberg-MacLane Lie algebra $K(M,n)$ is $\mathbf{F}$-complete by Proposition 5.3.6. □

Let $L_\bullet \in \mathsf{sLie}^r$ be a *connected* simplicial restricted Lie algebra. We will define inductively the $\mathbf{F}$-completions

$$\phi_n\colon L_\bullet^{\leq n} \to L_\xi(L_\bullet^{\leq n})$$

of the Postnikov truncations $L_\bullet^{\leq n}$ of $L_\bullet$, see Section 4.3.

If $n=0$, then $L_\bullet^{\leq 0}$ is contractible, and so we set $L_\xi L_\bullet^{\leq 0} = 0$. Clearly, the unique map

$$\phi_0\colon L_\bullet^{\leq 0} \to L_\xi L_\bullet^{\leq 0} = 0$$

is a weak equivalence.

Next, assume that the **F**-completion $\phi_n\colon L_\bullet^{\leq n} \to L_\xi L_\bullet^{\leq n}$ is defined, $n \geq 0$; we will construct an **F**-completion ϕ_{n+1}. Consider the fiber sequence

$$K(M, n+1) \to L_\bullet^{\leq(n+1)} \xrightarrow{\beta^n} L_\bullet^{\leq n} \xrightarrow{k^n} K(M, n+2)$$

from Corollary 4.3.6, where $M = \pi_{n+1}(L_\bullet)$. By the universal property of **F**-completions, there is a map

$$\hat{k}^n\colon L_\xi L_\bullet^{\leq n} \to L_\xi K(M, n+2)$$

such that the diagram

$$\begin{array}{ccc} L_\bullet^{\leq n} & \xrightarrow{k^n} & K(M, n+2) \\ \phi_n \downarrow & & \downarrow \phi_M^{n+2} \\ L_\xi L_\bullet^{\leq n} & \xrightarrow{\hat{k}^n} & L_\xi K(M, n+2) \end{array}$$

is homotopy commutative in sLie^r. We define $L_\xi L_\bullet^{\leq(n+1)}$ as the homotopy fiber $\mathrm{fib}(\hat{k}^n)$ of the map $\hat{k}^n$, $L_\xi L_\bullet^{\leq(n+1)} = \mathrm{fib}(\hat{k}^n)$; and then we define

$$\phi_{n+1}\colon L_\bullet^{\leq(n+1)} \to L_\xi L_\bullet^{\leq(n+1)}$$

as the induced map between homotopy fibers of k^n and $\hat{k}^n$.

Lemma 5.3.8. *ϕ_{n+1} is the **F**-completion of $L_\bullet^{\leq(n+1)}$.*

Proof. Clearly, $L_\xi L_\bullet^{\leq(n+1)}$ is **F**-complete as a homotopy fiber of a map between **F**-complete objects in sLie^r. Moreover, there is a map of fiber sequences

$$\begin{array}{ccccc} K(M, n+1) & \longrightarrow & L_\bullet^{\leq(n+1)} & \xrightarrow{\beta^n} & L_\bullet^{\leq n} \\ \downarrow \phi_M^{n+1} & & \downarrow \phi_{n+1} & & \downarrow \phi_n \\ L_\xi K(M, n+1) & \longrightarrow & L_\xi L_\bullet^{\leq(n+1)} & \xrightarrow{\hat{\beta}^n} & L_\xi L_\bullet^{\leq n} \end{array}$$

such that the maps ϕ_M^{n+1} and ϕ_n are **F**-equivalences. By applying the Serre spectral sequence (Corollary 4.5.12), we obtain that ϕ_{n+1} is an **F**-equivalence as well. □

Note that one can compute homotopy groups $\pi_*(L_\xi L_\bullet^{\leq n})$ of the **F**-completion $L_\xi L_\bullet^{\leq n}$ by using this inductive construction. Namely, we have the following statement.

Corollary 5.3.9. *Let $L_\bullet \in \mathsf{sLie}^r$ be a simplicial restricted Lie algebra, $\pi_0(L_\bullet) = 0$. Then we have*

$$\pi_i(L_\xi L_\bullet^{\leq n}) = \begin{cases} L_1\pi_n(L_\bullet^{\leq n}) = L_1\pi_n(L_\bullet) & \text{if } i = n+1, \\ 0 & \text{if } i > n+1. \end{cases}$$

Moreover, there is a short exact sequence

$$0 \to L_1\pi_{i-1}(L_\bullet) \to \pi_i(L_\xi L_\bullet^{\leq n}) \to L_0\pi_i(L_\bullet) \to 0$$

for each $i \leq n$. □

Finally, we define a map $\phi\colon L_\bullet \to L_\xi L_\bullet$ as follows

$$\phi = \operatorname*{holim}_n \phi_n\colon L_\bullet \to L_\xi L_\bullet = \operatorname*{holim}_n L_\xi(L_\bullet^{\leq n}),\ L_\bullet \in \mathsf{sLie}^r,\ \pi_0(L_\bullet) = 0.$$

Proposition 5.3.10. *ϕ is the $\mathbf{F}$-completion of $L_\bullet \in \mathsf{sLie}^r$, $\pi_0(L_\bullet)=0$.*

Proof. The simplicial restricted Lie algebra $L_\xi L_\bullet \in \mathsf{sLie}^r$ is $\mathbf{F}$-complete as a homotopy limit over a diagram of $\mathbf{F}$-complete objects in sLie^r. We will show that ϕ is an $\mathbf{F}$-equivalence. For each $i\geq 0$, there is an integer n_i such that maps

$$\alpha^n_*\colon \widetilde{H}_i(L_\bullet;\mathbf{F}) \to \widetilde{H}_i(L^{\leq n}_\bullet;\mathbf{F}),$$

$$\hat{\alpha}^n_*\colon \widetilde{H}_i(L_\xi L_\bullet;\mathbf{F}) \to \widetilde{H}_i(L_\xi L^{\leq n}_\bullet;\mathbf{F})$$

induced by the projections $\alpha^n\colon L_\bullet \to L^{\leq n}_\bullet$, $\hat{\alpha}^n\colon L_\xi L_\bullet \to L_\xi L^{\leq n}_\bullet$ are isomorphisms for all $n\geq n_i$. Indeed, by Corollary 5.3.9, the connectivity of α^n and $\hat{\alpha}^n$ increases with n; this fact and Lemma 4.1.7 imply the claim. Finally, the proposition follows by a commutative diagram

$$\begin{array}{ccc}
\widetilde{H}_i(L_\bullet;\mathbf{F}) & \xrightarrow{\phi_*} & \widetilde{H}_i(\operatorname{holim} L_\xi L^{\leq n}_\bullet;\mathbf{F}) \\
{\scriptstyle\alpha^n_*}\downarrow & & \downarrow{\scriptstyle\hat{\alpha}^n_*} \\
\widetilde{H}_i(L^{\leq n}_\bullet;\mathbf{F}) & \xrightarrow{\phi_{n*}} & \widetilde{H}_i(L_\xi L^{\leq n}_\bullet;\mathbf{F})
\end{array}$$

and Lemma 5.3.8. □

Corollary 5.3.11. *Let $L_\bullet \in \mathsf{sLie}^r$ be a simplicial restricted Lie algebra, $\pi_0(L_\bullet)=0$. Then we have a short exact sequence*

$$0\to L_1\pi_{i-1}(L_\bullet)\to \pi_i(L_\xi L_\bullet)\to L_0\pi_i(L_\bullet)\to 0$$

for each $i\geq 0$. □

Corollary 5.3.12. *A connected simplicial restricted Lie algebra $L_\bullet \in \mathsf{sLie}^r$ is $\mathbf{F}$-complete if and only if all homotopy groups $\pi_i(L_\bullet), i\geq 0$ are derived ξ-adic complete left modules over the ring $\mathbf{F}\{\xi\}$.*

Proof. First, suppose that $\pi_i(L_\bullet), i\geq 0$ are derived ξ-adic complete. Then

$$L_1\pi_i(L_\bullet)=0 \ \text{ and } \ L_0\pi_i(L_\bullet)\cong \pi_i(L_\bullet)$$

for each $i\geq 0$. By Corollary 5.3.11, this implies that $\pi_i(L_\xi L_\bullet)\cong \pi_i L_\bullet$, $i\geq 0$. Therefore the $\mathbf{F}$-completion map $\phi\colon L_\bullet \to L_\xi L_\bullet$ is a weak equivalence, and so $L_\bullet$ is $\mathbf{F}$-complete.

Next, suppose that $L_\bullet \in \mathsf{sLie}^r$ is $\mathbf{F}$-complete. Then $\phi\colon L_\bullet \to L_\xi L_\bullet$ is a weak equivalence, and so, by Corollary 5.3.11, there is a short exact sequence

$$0\to L_1\pi_{i-1}(L_\bullet)\to \pi_i(L_\bullet)\to L_0\pi_i(L_\bullet)\to 0,\ i\geq 0.$$

By Proposition 5.2.11, the outer terms in this exact sequence are derived ξ-complete $\mathbf{F}\{\xi\}$-modules. Therefore, by Proposition 5.2.13, the homotopy groups $\pi_i(L_\bullet)$, $i\geq 0$ are derived ξ-adic complete as well. □

6. Adams-type spectral sequence

In this section we illustrate our previous results by proving Theorems E and F from the introduction.

In Section 6.1 we recall basic properties of the Steenrod operations. In Definition 6.1.5, we introduce the *homogenized mod-p Steenrod algebra* $\mathcal{A}_p^h$, which is the associated graded algebra of the classical Steenrod algebra $\mathcal{A}_p$ with respect to a certain filtration, see Remark 6.1.6. In Definition 6.1.8, we define the category $\mathcal{U}^h$ of *unstable* $\mathcal{A}_p^h$*-algebras* and we show that the reduced cohomotopy groups $\widetilde{\pi}^*(C_\bullet), C_\bullet \in \mathsf{s_0CoAlg}^{tr}$ and the cohomology groups $\widetilde{H}^*(L_\bullet;\mathbf{F}), L_\bullet \in \mathsf{sLie}^r$ are unstable $\mathcal{A}_p^h$-algebras in Examples 6.1.9 and 6.1.10 respectively. We also define abelian categories $\mathcal{M}^h$ and $\mathcal{M}_0^h$ of *unstable* $\mathcal{A}_p^h$*-modules* and *strongly unstable* $\mathcal{A}_p^h$*-algebras* in Definitions 6.1.13 and 6.1.14 respectively. We describe free objects in each of these categories $\mathcal{U}_p^h$, $\mathcal{M}^h$, and $\mathcal{M}_0^h$ in Remark 6.1.19. Finally, we recall from [55] that the reduced cohomotopy groups $\widetilde{\pi}^*(\mathrm{Sym}^{tr}(V_\bullet))$ of a cofree simplicial truncated coalgebra $\mathrm{Sym}^{tr}(V_\bullet)$ is a free unstable $\mathcal{A}_p^h$-algebra (Theorem 6.1.21).

In Section 6.2 we prove Theorem E as Corollary 6.2.7. First, we recall the definition of non-abelian Ext-groups (Definition 6.2.1). Then we apply Theorem 6.1.11 to obtain the spectral sequence of Theorem 6.2.3, which computes homotopy groups of a derived mapping space

$$\mathrm{map}_{s\mathcal{C}\mathcal{A}_0}(C_\bullet, D_\bullet),\ C_\bullet, D_\bullet \in \mathsf{s_0CoAlg}^{tr}$$

from reduced cohomotopy groups $\widetilde{\pi}^*(C_\bullet)$ and $\widetilde{\pi}^*(D_\bullet)$. Finally, we use Theorem C to derive Theorem E from Theorem 6.2.3.

In Section 6.3 we recall the definition of the lambda algebra Λ of [10] and recall that the algebra Λ is anti-isomorphic to the Koszul dual algebra $\mathcal{K}_p^*$ of the algebra $\mathcal{A}_p^h$ (see (6.3.4)). Then, we construct *unstable* and *strongly unstable Koszul complexes* $K_\bullet(W)$ (see (6.3.14)) and $K_\bullet^0(W)$ (see (6.3.15)) for a trivial $\mathcal{A}_p^h$-module $W \in \mathsf{Vect}_{\mathbf{F}}^{gr}$ respectively. In Proposition 6.3.16, we show that these complexes are acyclic. Thus, we use them to compute unstable abelian Ext-groups $\mathrm{Ext}^s_{\mathcal{M}^h}(W, \Sigma^t\mathbf{F})$ and $\mathrm{Ext}^s_{\mathcal{M}_0^h}(W, \Sigma^t\mathbf{F})$ in terms of the algebra Λ in Corollary 6.3.17.

In the final section 6.4 we examine the spectral sequence (6.2.8) in the particular case $L_\bullet = L^r(V_\bullet)$ is a free restricted Lie algebra. First, in Proposition 6.4.4, we use the Curtis theorem [15], to compute the homotopy groups $\pi_*(L_\xi L^r(V_\bullet))$ of the $\mathbf{F}$-completion $L_\xi L^r(V_\bullet)$. Then, we observe from Corollary 6.3.17 that the second page of the spectral sequence (1.2.1) is computable provided $\pi_*(V_\bullet)$ is one-dimensional, see Corollaries 6.4.9 and 6.4.15. This will allow us to derive Theorem F. In Remarks 6.4.16 and 6.4.17, we discuss the spectral sequence (6.2.8) and the homotopy groups $\pi_*(L^r(V_\bullet))$ in the case $\dim \pi_*(V_\bullet) > 1$. We end the section with Remark 6.4.19 concerning the connection between the generators of the algebra Λ and the generators of the algebra $\mathcal{A}_p^h$.

In this section we heavily use Steenrod operations. We recall that the standard notation is different for p is odd and $p = 2$; we will enclose the case $p = 2$ in parentheses.

6.1 Steenrod operations.

Definition 6.1.1. Let $V_\bullet \in \mathsf{sVect}_{\mathbf{F}}$ be a simplicial vector space. For each $q \geq 0$, let $\pi^q(V_\bullet)$ denote the *q-th cohomotopy group* of $V_\bullet$, that is $\pi^q(V_\bullet) = \mathrm{Hom}(\pi_q(V_\bullet), \mathbf{F})$. Similarly, let $C_\bullet \in \mathsf{s_0CoAlg}^{tr}$ be a reduced simplicial truncated coalgebra. Set $\widetilde{\pi}^*(C_\bullet)$ to be the *reduced* cohomotopy groups of $C_\bullet$:

$$\widetilde{\pi}^*(C_\bullet) = \pi^*(\mathsf{oblv}(C_\bullet)) = \bigoplus_{q>0} \pi^q(C_\bullet).$$

Let $C_\bullet \in \mathsf{sCoAlg}$ be a simplicial coalgebra over $\mathbf{F}$. Then by the Eilenberg-Zilber theorem, we obtain that $\pi^*(C_\bullet)$ is a *graded* commutative algebra over $\mathbf{F}$. Furthermore, in [18] (see also [45]) A. Dold constructed functorial *Steenrod operations*

$$Sq^a\colon \pi^q(C_\bullet) \to \pi^{q+a}(C_\bullet), q, a \geq 0 \text{ if } p=2, \text{ and}$$

$$\beta^\varepsilon P^a\colon \pi^q(C_\bullet) \to \pi^{q+2a(p-1)+\varepsilon}(C_\bullet), q, a \geq 0,\ \varepsilon = 0,1 \text{ if } p>2.$$

These operations satisfy the following list of properties (where, by abuse of notation, $\beta^1 P^a = \beta P^a$ and $\beta^0 P^a = P^a$):

1. $\beta^\varepsilon P^a(\alpha x) = \alpha^p \beta^\varepsilon P^a(x)$ (resp. $Sq^a(\alpha x) = \alpha^2 Sq^a(x)$), $\alpha \in \mathbf{F}$, $x \in \pi^q(C_\bullet)$.
2. $\beta^\varepsilon P^a(x) = 0$ (resp. $Sq^a(x)=0$), if $2a+\varepsilon > q$ (resp. $a>q$) and $x \in \pi^q(C_\bullet)$.
3. $P^a(x) = x^p$ (resp. $Sq^a = x^2$) if $q=2a$ (resp. $q=a$).
4. Cartan formula:
$$P^a = \sum_{i=0}^{a} P^i \otimes P^{a-i} \text{ and } \beta P^a = \sum_{i=0}^{a} (\beta P^i \otimes P^{a-i} + P^i \otimes \beta P^{a-i})$$
(resp. $Sq^a = \sum_{i=0}^a Sq^i \otimes Sq^{a-i}$) on $\pi^*(C_\bullet \times C'_\bullet) \cong \pi^*(C_\bullet) \otimes \pi^*(C'_\bullet)$.
5. Stability: if $\sigma\colon \pi^q(C_\bullet) \xrightarrow{\cong} \pi^{q+1}(\Sigma_\bullet C_\bullet)$ is the suspension isomorphism, then $\sigma\beta^\varepsilon P^a = (-1)^\varepsilon \beta^\varepsilon P^a \sigma$ (resp. $\sigma Sq^a = Sq^a \sigma$).
6. Adem-Epstein relations. If p is odd, $a<pb$, and $\varepsilon=0,1$, then
$$\beta^\varepsilon P^a P^b = \sum_{j=0}^{a+b} (-1)^{a+j} \binom{(p-1)(b-j)-1}{a-pj} \beta^\varepsilon P^{a+b-j} P^j; \tag{6.1.2}$$
and if $a \leq pb$ and $\varepsilon=0,1$, then
$$\begin{aligned}\beta^\varepsilon P^a \beta P^b = &\sum_{j=0}^{a+b} (-1)^{a+j-1} \binom{(p-1)(b-j)-1}{a-pj-1} \beta^\varepsilon P^{a+b-j} \beta P^j \\ &+ (1-\varepsilon) \sum_{j=0}^{a+b} (-1)^{a+j} \binom{(p-1)(b-j)}{a-pj} \beta P^{a+b-j} P^j.\end{aligned} \tag{6.1.3}$$
Similarly, if $p=2$ and $a<2b$, then
$$Sq^a Sq^b = \sum_{j=0}^{a+b} \binom{b-j-1}{a-2j} Sq^{a+b-j} Sq^j. \tag{6.1.4}$$
7. the operation P^0 (resp. Sq^0) is induced by the Verschiebung operator (see Definition 2.2.4)
$$V\colon C_\bullet \to C_\bullet^{(1)}.$$

Recall that the *mod-p Steenrod algebra* is the $\mathbf{F}_p$-algebra $\mathcal{A}_p$ generated by Steenrod operations $\beta^\varepsilon P^a, a \geq 0, \varepsilon = 0,1$ (resp. $Sq^a, a\geq 0$) subject to Adem relations (6.1.2) and (6.1.3) (resp. the Adem relation (6.1.4)) and subject to the additional relation:

$$P^0 = 1 \text{ (resp. } Sq^0 = 1).$$

Definition 6.1.5. The *homogenized mod-p Steenrod algebra* $\mathcal{A}_p^h$ is the associative algebra over $\mathbf{F}_p$ generated by the elements $\beta^\varepsilon P^a, a \geq 0, \varepsilon = 0, 1$ (resp. $Sq^a, a \geq 0$ if $p = 2$) subject to Adem relations (6.1.2) and (6.1.3) (resp. the relation (6.1.4)) and subject to the additional relation

$$P^0 = 0 \text{ (resp. } Sq^0 = 0).$$

Remark 6.1.6. One can endow the Steenrod algebra $\mathcal{A}_p$ with an increasing multiplicative *weight* filtration $F_w\mathcal{A}_p$ by defining $F_0\mathcal{A}_p$ be spanned by the unit $1 \in \mathcal{A}_p$ and $F_1\mathcal{A}_p$ be spanned by the set of generators $\beta^\varepsilon P^a, a \geq 0, \varepsilon = 0, 1$ (resp. $Sq^a, a \geq 0$), see [56] and [54]. Then the associated graded algebra $\mathrm{gr}_F\mathcal{A}_p$ is isomorphic to $\mathcal{A}_p^h$. In particular, the algebra $\mathcal{A}_p^h$ is *bigraded*:

$$|\beta^\varepsilon P^a| = (2a(p-1) + \varepsilon, 1) \text{ and } |Sq^a| = (a, 1).$$

We refer to the first grading as *internal* and to the second one as *weight*.

Definition 6.1.7. A *left* $\mathcal{A}_p^h$*-module* $M_* = \bigoplus_{q>0} M_q$ is a positively graded vector space over $\mathbf{F}$ equipped with an $\mathbf{F}_p$-linear left action of the homogenized Steenrod algebra $\mathcal{A}_p^h$ such that

1. $\beta^\varepsilon P^a(M_q) \subset M_{q+2a(p-1)+\varepsilon}, a \geq 0, \varepsilon = 0, 1$ (resp. $Sq^a(M_q) \subset M_{q+a}, a \geq 0$);
2. $\beta^\varepsilon P^a(\alpha x) = \alpha^p \beta^\varepsilon P^a(x)$ (resp. $Sq^a(\alpha x) = \alpha^2 Sq^a(x)$), $\alpha \in \mathbf{F}$, $x \in M_q$.

We denote by $\mathsf{Mod}_{\mathcal{A}_p^h}$ the abelian category of left $\mathcal{A}_p^h$-modules.

Definition 6.1.8. An *unstable* $\mathcal{A}_p^h$*-algebra* is a left $\mathcal{A}_p^h$-module $A_* \in \mathsf{Mod}_{\mathcal{A}_p^h}$ which is a non-unital graded commutative algebra such that

1. $P^a(x) = x^p$ for all $x \in A_{2a}$ (resp. $Sq^a(x) = x^2$ for all $x \in A_a$.)
2. $\beta^\varepsilon P^a(x) = 0$ for all $x \in A_q$ and $2a + \varepsilon > q$ (resp. $Sq^a(x) = 0$ for all $x \in A_q$ and $a > q$.)

We denote by $\mathcal{U}^h$ the category of unstable $\mathcal{A}_p^h$-algebras.

Example 6.1.9. Let $C_\bullet \in \mathsf{s}_0\mathsf{CoAlg}^{tr}$ be a reduced simplicial truncated coalgebra. By the definition, the Verschiebung operator $V\colon C_\bullet \to C_\bullet^{(1)}$ factors through the constant coalgebra $\mathbf{F}_\bullet$. Therefore P^0 (resp. Sq^0) acts by zero on $\pi^i(C_\bullet)$ for all $i > 0$, and so the reduced non-unital algebra

$$\widetilde{\pi}^*(C_\bullet) = \pi^*(\mathsf{oblv}(C_\bullet)) = \bigoplus_{q>0} \pi^q(C_\bullet)$$

is an unstable $\mathcal{A}_p^h$-algebra.

Example 6.1.10. Let $L_\bullet \in \mathsf{sLie}^r$ be a restricted Lie algebra. Then $\overline{W}U^r(L_\bullet)$ is a reduced simplicial truncated coalgebra and

$$\widetilde{\pi}^*(\overline{W}U^r(L_\bullet)) \cong \widetilde{H}^*(L_\bullet, \mathbf{F})$$

by Proposition 3.2.18 and Definition 4.2.23. Therefore the cohomology groups $\widetilde{H}^*(L_\bullet, \mathbf{F})$ form an unstable $\mathcal{A}_p^h$-algebra, cf. [58, Section 5] and [45, Theorem 8.5].

Definition 6.1.11. We define the *suspension* $\Sigma A_* \in \mathcal{U}^h$ of an unstable $\mathcal{A}_p^h$-algebra A_* as follows

1. $(\Sigma A_*)_{q+1} = A_q$ for $q \geq 0$. If $x \in A_q$, then we write σx for the corresponding element in $(\Sigma A_*)_{q+1}$;
2. ΣA_* has zero multiplication;
3. $\beta^\varepsilon P^a(\sigma x) = (-1)^\varepsilon \sigma \beta^\varepsilon P^a(x)$ for all $x \in A_q$ (resp. $Sq^a(\sigma x) = \sigma Sq^a(x)$ for all $x \in A_q$).

Finally, $\Sigma^t A_* = \Sigma(\Sigma^{t-1} A_*)$.

Example 6.1.12. Let $C_\bullet \in \mathsf{s_0CoAlg}^{tr}$. In this section we will write $\Sigma C_\bullet \in \mathsf{s_0CoAlg}^{tr}$ for the Kan suspension $\Sigma_\bullet C_\bullet$ of $C_\bullet$, see [29, Section III.5]. Then we have

$$\widetilde{\pi}^*(\Sigma C_\bullet) \cong \Sigma\widetilde{\pi}^*(C_\bullet)$$

as unstable $\mathcal{A}^h_p$-algebras.

Definition 6.1.13. An $\mathcal{A}^h_p$-module M_* is called *unstable* if $\beta^\varepsilon P^a(x) = 0$ for all $x \in M_q$ and $2a+\varepsilon > q$ (resp. $Sq^a(x) = 0$ for all $x \in M_q$ and $a > q$). We denote by $\mathcal{M}^h$ the full abelian subcategory of $\mathsf{Mod}_{\mathcal{A}^h_p}$ spanned by unstable modules.

Definition 6.1.14. An $\mathcal{A}^h_p$-module M_* is called *strongly unstable* if $\beta^\varepsilon P^a(x) = 0$ for all $x \in M_q$ and $2a+\varepsilon \geq q$ (resp. $Sq^a(x) = 0$ for all $x \in M_q$ and $a \geq q$). We denote by $\mathcal{M}^h_0$ the full abelian subcategory of $\mathcal{M}^h$ spanned by strongly unstable modules.

Remark 6.1.15. Any unstable $\mathcal{A}^h_p$-algebra is an unstable $\mathcal{A}^h_p$-module. Moreover, any strongly unstable $\mathcal{A}^h_p$-module is a commutative group object in $\mathcal{U}^h$ and the category $\mathcal{M}^h_0$ is precisely the full subcategory of $\mathcal{U}^h$ spanned by those.

The next proposition is standard.

Proposition 6.1.16. *The categories $\mathcal{U}^h$, $\mathcal{M}^h$, and $\mathcal{M}^h_0$ are monadic over the category $\mathsf{Vect}^{\geq 0}_{\mathbf{F}}$ of positively graded vector spaces.* □

More precisely, the last proposition means that forgetful functors

$$\mathsf{oblv}_{\mathcal{U}^h} \colon \mathcal{U}^h \to \mathsf{Vect}^{\geq 0}_{\mathbf{F}},\ \ \mathsf{oblv}_{\mathcal{M}^h} \colon \mathcal{M}^h \to \mathsf{Vect}^{\geq 0}_{\mathbf{F}},\ \text{ and }\ \mathsf{oblv}_{\mathcal{M}^h_0} \colon \mathcal{M}^h_0 \to \mathsf{Vect}^{\geq 0}_{\mathbf{F}}$$

have respectively left adjoints

$$F_{\mathcal{U}^h} \colon \mathsf{Vect}^{\geq 0}_{\mathbf{F}} \to \mathcal{U}^h,\ \ F_{\mathcal{M}^h} \colon \mathsf{Vect}^{\geq 0}_{\mathbf{F}} \to \mathcal{M}^h,\ \text{ and }\ F_{\mathcal{M}^h_0} \colon \mathsf{Vect}^{\geq 0}_{\mathbf{F}} \to \mathcal{M}^h_0 \tag{6.1.17}$$

such that $\mathcal{U}^h$ is equivalent to the category $\mathbb{T}_{\mathcal{U}^h}\text{-}\mathsf{Alg}$ of algebras over the monad $\mathbb{T}_{\mathcal{U}^h} = \mathsf{oblv}_{\mathcal{U}^h} \circ F_{\mathcal{U}^h}$, $\mathcal{M}^h$ is equivalent to the category $\mathbb{T}_{\mathcal{M}^h}\text{-}\mathsf{Alg}$ of algebras over the monad $\mathbb{T}_{\mathcal{M}^h} = \mathsf{oblv}_{\mathcal{M}^h} \circ F_{\mathcal{M}^h}$, and $\mathcal{M}^h_0$ is equivalent to the category $\mathbb{T}_{\mathcal{M}^h}\text{-}\mathsf{Alg}$ of algebras over the monad $\mathbb{T}_{\mathcal{M}^h_0} = \mathsf{oblv}_{\mathcal{M}^h_0} \circ F_{\mathcal{M}^h_0}$

Remark 6.1.18. By Proposition 6.1.16, the forgetful functor $\mathsf{oblv} \colon \mathcal{U}^h \to \mathcal{M}^h$ has a left adjoint

$$\mathcal{F} \colon \mathcal{M}^h \to \mathcal{U}^h.$$

The functor $\mathcal{F}$ can given by the formula

$$\mathcal{F}(M_*) = \mathbf{F}[M_*]/(m^p - P^a(m) \mid m \in M_{2a}, a > 0)$$

(resp. $\mathbf{F}[M_*]/(m^2 - Sq^a(m) \mid m \in M_a, a > 0)$), where $\mathbf{F}[M_*]$ is the free (non-unital) graded commutative algebra generated by M_*.

Remark 6.1.19. Recall that a (possibly void) sequence $I = (i_1, \ldots, i_k)$ is called *admissible* if $i_j \geq pi_{j+1}, 1 \leq j \leq k-1$, see [55, Section 4]. The *excess* of I, denoted by $e(I)$, is defined by

$$e(I) = i_1 - (p-1)(i_2 + \ldots i_k), \quad e(\emptyset) = -1.$$

For a sequence I, we set $St^I = St^{i_1} \cdot \ldots \cdot St^{i_k} \in \mathcal{A}_p^h$ be a monomial in $\mathcal{A}_p^h$, where

$$St^i = \begin{cases} Sq^a & \text{if } p = 2 \text{ and } i = a, \\ P^a & \text{if } p > 2 \text{ and } i = 2a(p-1), \\ \beta P^a & \text{if } p > 2 \text{ and } i = 2a(p-1)+1, \\ 0 & \text{otherwise.} \end{cases}$$

Then the free unstable $\mathcal{A}_p^h$-module $F_{\mathcal{M}^h}(\iota_l)$ generated by a single element ι_l of degree l is the vector space $\mathbf{F}(St^I\iota_l)$ spanned by the elements $St^I\iota_l$, where I is any admissible sequence of positive integers with $e(I) \leq (p-1)l$; whereas, the free strongly unstable $\mathcal{A}_p^h$-module $F_{\mathcal{M}_0^h}(\iota_l)$ is the vector space spanned by all elements $St^I\iota_l$ with $e(I) < (p-1)l$.

Similarly, the free unstable $\mathcal{A}_p^h$-algebra $F_{\mathcal{A}^h}(\iota_l) \cong \mathbf{F}[St^I\iota_l]$ is the free graded commutative (non-unital) algebra generated by the same elements $St^I\iota_l$ with $e(I) < (p-1)l$.

Definition 6.1.20. A graded vector space $V_* = \bigoplus_{q\geq 0} V_q$ is called *of finite type* if $V_0 = 0$ and $\dim(V_q) < \infty$ for $q \geq 1$. A simplicial vector space $V_\bullet \in \mathsf{sVect}$ is called *of finite type* if $\pi_*(V_\bullet)$ is a graded vector space of finite type.

We write $\mathsf{Vect}_{\mathbf{F}}^{ft}$ for the category of graded vector spaces of finite type; and we write $\mathsf{sVect}_{\mathbf{F}}^{ft}$ for the category of simplicial vector spaces of finite type.

Theorem 6.1.21 (Priddy)**.** *Let $V_\bullet \in \mathsf{sVect}_{\mathbf{F}}^{ft}$ be a simplicial vector space of finite type. There is a natural isomorphism of unstable $\mathcal{A}_p^h$-algebras:*

$$\widetilde{\pi}^*(\mathrm{Sym}^{tr}(V_\bullet)) = \bigoplus_{q>0} \pi^q(\mathrm{Sym}^{tr}(V_\bullet)) \cong F_{\mathcal{A}}(\pi^*(V_\bullet)).$$

Here $\mathrm{Sym}^{tr}(V_\bullet)$ is the free simplicial truncated coalgebra generated by $V_\bullet$, see Proposition 2.2.14.

Proof. See [55, Proposition 6.2.1]. □

6.2 Bousfield-Kan spectral sequence. Let $\mathbb{T}\colon \mathsf{C} \to \mathsf{C}$ be a monad on a category C. The monad $\mathbb{T}$ induces the adjoint pair

$$F_{\mathbb{T}} : \mathsf{C} \rightleftarrows \mathbb{T}\text{-}\mathsf{Alg}(\mathsf{C}) : \mathsf{oblv}_{\mathbb{T}}$$

such that $\mathbb{T} = \mathsf{oblv}_{\mathbb{T}} \circ F_{\mathbb{T}}$. Given a $\mathbb{T}$-algebra $A \in \mathbb{T}\text{-}\mathsf{Alg}(\mathsf{C})$, we denote by $\mathbb{T}_\bullet(A) \in \mathbb{T}\text{-}\mathsf{sAlg}(\mathsf{C})$ the *bar-construction* $B_\bullet(F_{\mathbb{T}}, \mathbb{T}, A)$, i.e. $\mathbb{T}_\bullet(A)$ is an (almost-free) simplicial $\mathbb{T}$-algebra such that

$$\mathbb{T}_q(A) = F_{\mathbb{T}} \circ \mathbb{T}^{\circ q} \circ \mathsf{oblv}_{\mathbb{T}}(A), \ q \geq 0.$$

Similarly, let $\mathbb{R}\colon \mathsf{C} \to \mathsf{C}$ be a comonad which induces the adjoint pair $\mathsf{oblv}_{\mathbb{R}} \dashv C_{\mathbb{R}}$. Given a $\mathbb{R}$-coalgebra $C \in \mathbb{R}\text{-}\mathsf{CoAlg}(\mathsf{C})$, we denote by $\mathbb{R}^\bullet(C) \in \mathbb{R}\text{-}\mathsf{cCoAlg}(\mathsf{C})$ the *cobar-construction* $C^\bullet(C_{\mathbb{R}}, \mathbb{R}, C)$, i.e. $\mathbb{R}^\bullet(C)$ is an (almost-cofree) cosimplicial $\mathbb{R}$-coalgebra such that

$$\mathbb{R}^q(C) = C_{\mathbb{R}} \circ \mathbb{R}^{\circ q} \circ \mathsf{oblv}_{\mathbb{R}}(C), \ q \geq 0.$$

Suppose now the category C is $\mathbf{F}$-linear. Then

$$\mathrm{Hom}_{\mathbb{T}}(\mathbb{T}_\bullet(A), A'), \ A, A' \in \mathbb{T}\text{-}\mathsf{Alg}(\mathsf{C})$$

is a cosimplicial vector space and we define the Ext-group $\mathrm{Ext}^s_{\mathbb{T}}(A, A')$ as the s-th cohomotopy group of that, i.e.

$$\mathrm{Ext}^s_{\mathbb{T}}(A, A') = \pi^s \operatorname{Hom}_{\mathbb{T}}(\mathbb{T}_\bullet(A), A'),\ A, A' \in \mathbb{T}\text{-}\mathsf{Alg}(\mathsf{C}).$$

In other words, $\mathrm{Ext}^s_{\mathbb{T}}(A, A')$ is the *right non-abelian derived functor* of the Hom-functor

$$\operatorname{Hom}_{\mathbb{T}}(-, A')\colon (\mathbb{T}\text{-}\mathsf{Alg}(\mathsf{C}))^{op} \to \mathsf{Vect}_{\mathbf{F}}.$$

We refer the reader to [19], [1] for details on non-abelian derived functors and "non-abelian" Ext-groups. We also recommend the appendix in [8] for a short exposition of the topic.

Definition 6.2.1. Let $A_*, A'_* \in \mathcal{U}^h$ be unstable $\mathcal{A}^h_p$-algebras. We define the *s-th unstable* Ext-*group* $\mathrm{Ext}^s_{\mathcal{U}^h}(A_*, A'_*)$ be the next formula

$$\mathrm{Ext}^s_{\mathcal{U}^h}(A_*, A'_*) = \mathrm{Ext}^s_{\mathbb{T}_{\mathcal{U}^h}}(A_*, A'_*),$$

where $\mathbb{T}_{\mathcal{U}^h} = \mathsf{oblv}_{\mathcal{U}^h} \circ F_{\mathcal{U}^h}$ is the monad which defines unstable $\mathcal{A}^h_p$-algebras, see (6.1.17). Similarly,

$$\mathrm{Ext}^s_{\mathcal{M}^h}(M_*, M'_*) = \mathrm{Ext}^s_{\mathbb{T}_{\mathcal{M}^h}}(M_*, M'_*),\ M_*, M'_* \in \mathcal{M}^h,$$

and

$$\mathrm{Ext}^s_{\mathcal{M}^h_0}(M_*, M'_*) = \mathrm{Ext}^s_{\mathbb{T}_{\mathcal{M}^h_0}}(M_*, M'_*),\ M_*, M'_* \in \mathcal{M}^h_0.$$

Remark 6.2.2. The categories $\mathcal{M}^h$, $\mathcal{M}^h_0$ are abelian and they have enough projectives. Therefore unstable Ext-groups $\mathrm{Ext}^s_{\mathcal{M}^h}(M_*, M'_*)$ and $\mathrm{Ext}^s_{\mathcal{M}^h_0}(N_*, N'_*)$ can be computed by the following formulas

$$\mathrm{Ext}^s_{\mathcal{M}^h}(M_*, M'_*) \cong H^s\big(\operatorname{Hom}_{\mathcal{M}^h}(P_\bullet(M_*), M'_*)\big),\ M_*, M'_* \in \mathcal{M}^h$$

and

$$\mathrm{Ext}^s_{\mathcal{M}^h_0}(N_*, N'_*) \cong H^s\big(\operatorname{Hom}_{\mathcal{M}^h_0}(\widetilde{P}_\bullet(N_*), N'_*)\big),\ N_*, N'_* \in \mathcal{M}^h_0,$$

where $P_\bullet(M_*) \to M_*$ is a *projective* resolution of M_* in $\mathcal{M}^h$, and $\widetilde{P}_\bullet(N_*) \to N_*$ is a projective resolution of N_* in $\mathcal{M}^h_0$.

By Theorem 3.2.10, the category $\mathsf{s}_0\mathsf{CoAlg}^{tr}$ of reduced simplicial truncated coalgebras has a simplicial model structure. Therefore the *derived mapping space*

$$\operatorname{map}_{s\mathcal{CA}_0}(C_\bullet, D_\bullet) \in \mathsf{sSet}_*,\ C_\bullet, D_\bullet \in \mathsf{s}_0\mathsf{CoAlg}^{tr}$$

is defined, see [33, Section 17]. We recall that $\operatorname{map}_{s\mathcal{CA}_0}(C_\bullet, D_\bullet)$ is a pointed simplicial Kan complex which is defined by

$$\operatorname{map}_{s\mathcal{CA}_0}(C_\bullet, D_\bullet) = \operatorname{Map}_{\mathsf{s}_0\mathsf{CoAlg}^{tr}}(C_\bullet, RD_\bullet),$$

where $\operatorname{Map}_{\mathsf{s}_0\mathsf{CoAlg}^{tr}}(C_\bullet, RD_\bullet)$ is a simplicial mapping set in the simplicial category $\mathsf{s}_0\mathsf{CoAlg}^{tr}$ and $D_\bullet \to RD_\bullet$ is a fibrant replacement of $D_\bullet$. We point out that the derived mapping space is well-defined up to a weak equivalence and preserves weak equivalences in both variables.

Theorem 6.2.3 (Bousfield-Kan)**.** *Let $C_\bullet, D_\bullet \in \mathsf{s}_0\mathsf{CoAlg}^{tr}$ be reduced simplicial truncated coalgebras of finite type. Then there is a completely convergent spectral sequence*

$$E^2_{s,t} = \mathrm{Ext}^s_{\mathcal{U}^h}(\widetilde{\pi}^*(D_\bullet), \Sigma^t\widetilde{\pi}^*(C_\bullet)) \Rightarrow \pi_{t-s} \operatorname{map}_{s\mathcal{CA}_0}(C_\bullet, D_\bullet).$$

Here $d_r\colon E^r_{s,t} \to E^r_{s+r,t+r-1}$.

Proof. Recall from Proposition 2.2.15 that the category CoAlg^{tr} is comonadic over $\mathsf{Vect}_{\mathbf{F}}$; we denote by

$$\mathbb{R}\colon \mathsf{Vect}_{\mathbf{F}} \to \mathsf{Vect}_{\mathbf{F}}$$

the resulting comonad $\mathbb{R} = \mathsf{oblv} \circ \mathrm{Sym}^{tr}$. As $\mathbb{R}(0) = 0$, we extend $\mathbb{R}$ degreewise to the comonad

$$\mathbb{R}\colon \mathsf{s}_0\mathsf{Vect}_{\mathbf{F}} \to \mathsf{s}_0\mathsf{Vect}_{\mathbf{F}}$$

on the category of reduced simplicial vector spaces. Then we have

$$\mathsf{CoAlg}_{\mathbb{R}}(\mathsf{s}_0\mathsf{Vect}_{\mathbf{F}}) \cong \mathsf{s}_0\mathsf{CoAlg}^{tr};$$

and for $D_\bullet \in \mathsf{s}_0\mathsf{CoAlg}^{tr}$ we consider the cobar-construction

$$\mathbb{R}^\bullet(D_\bullet) = \mathrm{Sym}^{tr} \circ \mathbb{R}^{\circ\bullet} \circ \mathsf{oblv}(D_\bullet) \in \mathsf{cs}_0\mathsf{CoAlg}^{tr}.$$

There is a natural map $D_\bullet \to \mathbb{R}^\bullet(D_\bullet)$ which induces a weak equivalence:

$$D_\bullet \xrightarrow{\simeq} \mathrm{Tot}\,\mathbb{R}^\bullet(D_\bullet) \in \mathsf{s}_0\mathsf{CoAlg}^{tr},$$

where $\mathrm{Tot}\,\mathbb{R}^\bullet(D_\bullet) \in \mathsf{s}_0\mathsf{CoAlg}^{tr}$ is the totalization of the cosimplicial object $\mathbb{R}^\bullet(D_\bullet)$, see [33, Definition 18.6.3]. Thus we obtain weak equivalences of derived mapping spaces:

$$\mathrm{map}_{s\mathcal{C}\mathcal{A}_0}(C_\bullet, D_\bullet) \xrightarrow{\simeq} \mathrm{map}_{s\mathcal{C}\mathcal{A}_0}(C_\bullet, \mathrm{Tot}\,\mathbb{R}^\bullet(D_\bullet)) \xrightarrow{\simeq} \mathrm{Tot}\,\mathrm{map}_{s\mathcal{C}\mathcal{A}_0}(C_\bullet, \mathbb{R}^\bullet(D_\bullet)).$$

By [29, Proposition VIII.1.15] (see also [11, Chapter X]), there is a spectral sequence

$$\begin{aligned} E^2_{s,t} = \pi^s\pi_t\,\mathrm{map}_{s\mathcal{C}\mathcal{A}_0}(C_\bullet, \mathbb{R}^\bullet(D_\bullet)) &\Rightarrow \pi_{t-s}\,\mathrm{Tot}\,\mathrm{map}_{s\mathcal{C}\mathcal{A}_0}(C_\bullet, \mathbb{R}^\bullet(D_\bullet)) \\ &\cong \pi_{t-s}\,\mathrm{map}_{s\mathcal{C}\mathcal{A}_0}(C_\bullet, D_\bullet) \end{aligned} \tag{6.2.4}$$

associated with the cosimplicial simplicial set $\mathrm{map}_{s\mathcal{C}\mathcal{A}_0}(C_\bullet, \mathbb{R}^\bullet(D_\bullet)) \in \mathsf{csSet}_*$.

Next, we will compute the second page of the spectral sequence (6.2.4). Namely, we show that there is an isomorphism

$$\pi^s\pi_t\,\mathrm{map}_{s\mathcal{C}\mathcal{A}_0}(C_\bullet, \mathbb{R}^\bullet(D_\bullet)) \cong \mathrm{Ext}^s_{\mathcal{U}^h}(\widetilde{\pi}^*(D_\bullet), \Sigma^t\widetilde{\pi}^*(C_\bullet)). \tag{6.2.5}$$

Indeed, we have following isomorphisms of cosimplicial vector spaces

$$\begin{aligned} \pi_t\,\mathrm{map}_{s\mathcal{C}\mathcal{A}_0}(C_\bullet, \mathbb{R}^\bullet(D_\bullet)) &\cong \pi_0\,\mathrm{Map}_{\mathsf{s}_0\mathsf{CoAlg}^{tr}}(\Sigma^t C_\bullet, \mathrm{Sym}^{tr} \circ \mathbb{R}^{\circ\bullet} \circ \mathsf{oblv}_{\mathbb{R}}(D_\bullet)) \\ &\cong \pi_0\,\mathrm{Map}_{\mathsf{s}_0\mathsf{Vect}_{\mathbf{F}}}(\mathsf{oblv}_{\mathbb{R}}(\Sigma^t C_\bullet), \mathbb{R}^{\circ\bullet} \circ \mathsf{oblv}_{\mathbb{R}}(D_\bullet)) \\ &\cong \mathrm{Hom}_{\mathsf{Vect}^{gr}_{\mathbf{F}}}(\widetilde{\pi}_*(\Sigma^t C_\bullet), \widetilde{\pi}_*(\mathbb{R}^{\circ\bullet} \circ \mathsf{oblv}_{\mathbb{R}}(D_\bullet))) \\ &\cong \mathrm{Hom}_{\mathsf{Vect}^{gr}_{\mathbf{F}}}(\widetilde{\pi}^*(\mathbb{R}^{\circ\bullet} \circ \mathsf{oblv}_{\mathbb{R}}(D_\bullet)), \widetilde{\pi}^*(\Sigma^t C_\bullet)) \\ &\cong \mathrm{Hom}_{\mathsf{Vect}^{gr}_{\mathbf{F}}}(\mathbb{T}^{\circ\bullet}_{\mathcal{U}^h} \circ \mathsf{oblv}_{\mathcal{U}^h}(\widetilde{\pi}^*(D_\bullet)), \mathsf{oblv}_{\mathcal{U}^h}(\widetilde{\pi}^*(\Sigma^t C_\bullet))) \\ &\cong \mathrm{Hom}_{\mathbb{T}_{\mathcal{U}^h}}(F_{\mathcal{U}^h} \circ \mathbb{T}^{\circ\bullet}_{\mathcal{U}^h} \circ \mathsf{oblv}_{\mathcal{U}^h}(\widetilde{\pi}^*(D_\bullet)), \widetilde{\pi}^*(\Sigma^t C_\bullet)) \\ &= \mathrm{Hom}_{\mathbb{T}_{\mathcal{U}^h}}(\mathbb{T}_{\mathcal{U}^h,\bullet}(\widetilde{\pi}^*(D_\bullet)), \widetilde{\pi}^*(\Sigma^t C_\bullet)) \\ &\cong \mathrm{Hom}_{\mathbb{T}_{\mathcal{U}^h}}(\mathbb{T}_{\mathcal{U}^h,\bullet}(\widetilde{\pi}^*(D_\bullet)), \Sigma^t\widetilde{\pi}^*(C_\bullet)) \end{aligned}$$

Here the fourth isomorphism follows from the assumption that simplicial coalgebras $C_\bullet, D_\bullet \in \mathsf{s}_0\mathsf{CoAlg}^{tr}$ are of finite type, the fifth isomorphism follows from Theorem 6.1.21, and the last one follows from Example 6.1.12; all other isomorphisms are induced by various adjunctions. By Definition 6.2.1, this implies the isomorphism (6.2.5).

Finally, the spectral sequence (6.2.4) converges completely by applying the complete convergence lemma, see [29, Lemma VI.2.20] and [11, IX.5.4]. Indeed, by Remark 6.1.19, all entries $E^2_{s,t}$ on the second page are finite dimensional vector spaces, and so

$$\lim_r{}^1 E^r_{s,t} = 0,\ t-s\geq 1. \qquad \square$$

Definition 6.2.6. A simplicial restricted Lie algebra $L_\bullet$ is called *of finite type* if its homology groups $\widetilde{H}_*(L_\bullet;\mathbf{F})$ is a graded vector space of finite type.

Corollary 6.2.7. *Let $L_\bullet, L'_\bullet \in \mathsf{sLie}^r$ be $\mathbf{F}$-complete simplicial restricted Lie algebras of finite type. Then there is a completely convergent spectral sequence*

$$E^2_{s,t} = \mathrm{Ext}^s_{\mathcal{U}^h}(\widetilde{H}^*(L_\bullet;\mathbf{F}), \Sigma^t\widetilde{H}^*(L'_\bullet;\mathbf{F})) \Rightarrow \pi_{t-s}\,\mathrm{map}_{s\mathcal{L}_\xi}(L'_\bullet, L_\bullet).$$

Here $d_r\colon E^r_{s,t}\to E^r_{s+r,t+r-1}$.

In particular, there is a completely converging spectral sequence

$$E^2_{s,t} = \mathrm{Ext}^s_{\mathcal{U}^h}(\widetilde{H}^*(L_\bullet;\mathbf{F}), \Sigma^{t+1}\mathbf{F}) \Rightarrow \pi_{t-s}(L_\bullet) \tag{6.2.8}$$

by taking $L'_\bullet = L_\xi(L^r(\mathbf{F}))$ in the spectral sequence of Corollary 6.2.7.

Proof. By Theorem 3.2.26, there is a weak equivalence of derived mapping spaces

$$\mathrm{map}_{s\mathcal{L}_\xi}(L'_\bullet, L_\bullet) \simeq \mathrm{map}_{s\mathcal{CA}_0}(\overline{W}U^r(L'_\bullet), \overline{W}U^r(L_\bullet)).$$

By Example 6.1.10 and Theorem 6.2.3, we obtain the required spectral sequence. $\square$

6.3 Unstable Koszul resolutions. Recall that the *lambda algebra* Λ is the associative bigraded algebra over $\mathbf{F}_p$ generated by the elements $\lambda_a, a\geq 1$ of bidegree $|\lambda_a| = (2a(p-1)-1,1)$ and $\mu_a, a\geq 0$ of bidegree $|\mu_a| = (2a(p-1),1)$ (resp. $\lambda_a, a\geq 0$ of bidegree $|\lambda_a|=(a,1)$) subject to the following Adem-type relations:

1. If p is odd, $b\geq pa$, and $\varepsilon=0,1$, then

$$\lambda_a\nu^\varepsilon_b = \sum_{i=0}^{a+b}(-1)^{i+a+\varepsilon}\binom{(p-1)(b-i)-\varepsilon}{i-pa}\nu^\varepsilon_{a+b-i}\lambda_i \tag{6.3.1}$$
$$+ (1-\varepsilon)\sum_{l\geq 0}^{a+b}(-1)^{i+a+1}\binom{(p-1)(b-i)-1}{i-pa}\lambda_{a+b-i}\mu_i.$$

2. If p is odd, $b>pa$, and $\varepsilon=0,1$, then

$$\mu_a\nu^\varepsilon_b = \sum_{i=1}^{a+b}(-1)^{i+a}\binom{(p-1)(b-i)-1}{i-pa-1}\mu_{a+b-i}\nu^\varepsilon_i. \tag{6.3.2}$$

3. If $p=2$ and $b>2a$, then

$$\lambda_a\lambda_b = \sum_{i=1}^{a+b}\binom{b-i-1}{i-2a-1}\lambda_{a+b-i}\lambda_i. \tag{6.3.3}$$

Here we set $\nu_a^0 = \mu_a, a \geq 0$ and $\nu_a^1 = \lambda_a, a > 0$. Notice that we use the definition of the lambda algebra from [69, Definition 7.1], but not from the original paper [10].

In this section we compute *abelian* unstable Ext-groups

$$\mathrm{Ext}^s_{\mathcal{M}^h}(W, \Sigma^t \mathbf{F}) \;\; (\text{resp. } \mathrm{Ext}^s_{\mathcal{M}^h_0}(W, \Sigma^t \mathbf{F}))$$

in terms of the algebra Λ. Here $W \in \mathsf{Vect}^{gr}_{\mathbf{F}}$ is a graded vector space, which is considered as a left $\mathcal{A}^h_p$-module equipped with the trivial action. In order to calculate these Ext-groups, we construct a free resolution $K_\bullet(W) \in \mathcal{M}^h$ (resp. $K^0_\bullet(W) \in \mathcal{M}^h_0$) of the (resp. strongly) unstable $\mathcal{A}^h_p$-module W, see Remark 6.2.2.

The homogenized Steenrod algebra $\mathcal{A}^h_p$ has a Poincaré-Birkhoff-Witt (PBW) basis over $\mathbf{F}_p$ given by admissible monomials

$$B = \{St^{i_1} \cdot \ldots \cdot St^{i_k} \mid i_j \geq p i_{j+1}, k \geq 1\},$$

see Remark 6.1.19. We refer the reader to [56, Section 5] and [54, Chapter 4] for detailed accounts on algebras with a PBW basis. Therefore $\mathcal{A}^h_p$ is a Koszul algebra with respect to the weight grading, see [56, Theorem 5.3] and [54, Theorem 4.3.1].

We denote by $\mathcal{K}^*_p = \mathrm{Ext}^*_{\mathcal{A}^h_p}(\mathbf{F}_p, \mathbf{F}_p)$ the Koszul dual algebra for $\mathcal{A}^h_p$. Here the star in $\mathcal{K}^*_p$ stands for the weight grading. We recall from [56, Sections 7.1-7.2] that $\mathcal{K}^*_p$ is a bigraded algebra generated by the elements

$$Pr_i \in \mathcal{K}^1_p, \; |Pr_i| = (i, 1),$$

where Pr_i is dual to St^i, $i > 0$. Notice that S. Priddy used a different notation in [56]: if $p = 2$, σ_a is the dual to Sq^a; if p is odd, π_a is the dual to P^a and ρ_b is the dual to βP^b. In short,

$$Pr_i = \begin{cases} \sigma_a & \text{if } p = 2 \text{ and } i = a, a > 0 \\ \pi_a & \text{if } p > 2 \text{ and } i = 2a(p-1), a > 0 \\ \rho_a & \text{if } p > 2 \text{ and } i = 2a(p-1)+1, a \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore, there is an anti-isomorphism

$$\Phi_p \colon \mathcal{K}^*_p \to \Lambda \tag{6.3.4}$$

given by

$$\Phi_p(Pr_i) = \begin{cases} \lambda_{a-1} & \text{if } p = 2 \text{ and } i = a, a > 0 \\ \lambda_a & \text{if } p > 2 \text{ and } i = 2a(p-1), a > 0 \\ \mu_a & \text{if } p > 2 \text{ and } i = 2a(p-1)+1, a \geq 0. \end{cases}$$

The algebras $\mathcal{K}^*_p$ and Λ are bigraded, however, the map Φ_p does not preserve the bidegree: if $|x| = (m, n), x \in \mathcal{K}^*_p$, then $|\Phi_p(x)| = (m - n, n)$.

We say that a sequence $J = (a_1, \ldots, a_k)$ is *orthogonally admissible* if

$$a_j < p a_{j+1}, \; 1 \leq j \leq k-1.$$

The Koszul dual algebra $\mathcal{K}^*_p$ has a PBW basis given by orthogonally admissible monomials:

$$K(B) = \{Pr_J = Pr_{a_1} \cdot \ldots \cdot Pr_{a_k} \mid J = (a_1, \ldots, a_k) \text{ is orthogonally admissible}\}.$$

We denote by $\mathcal{K}_{p,*}$ the linear dual to $\mathcal{K}^*_p$. The graded vector space $\mathcal{K}_{p,*}$ is spanned by

$$K^\vee(B) = \{Pr^J \mid J = (a_1, \ldots, a_k) \text{ is orthogonally admissible}\},$$

where $Pr^J \in \mathcal{K}_{p,*}$ is dual to $Pr_J \in \mathcal{K}^*_p$.

Finally, we recall from [54, Section 2.3] that the trivial $\mathcal{A}^h_p$-module $\mathbf{F}_p$ has the *Koszul resolution* $K_\bullet$ by free $\mathcal{A}^h_p$-modules:

$$K_\bullet = (\ldots \to \mathcal{A}^h_p \otimes \mathcal{K}_{p,3} \xrightarrow{d_3} \mathcal{A}^h_p \otimes \mathcal{K}_{p,2} \xrightarrow{d_2} \mathcal{A}^h_p \otimes \mathcal{K}_{p,1} \xrightarrow{d_1} \mathcal{A}^h_p \to \mathbf{F}_p \to 0). \tag{6.3.5}$$

Here the differential d_s is given by:

$$d_s(Pr^J) = \sum_{i\geq 1} St^i \otimes (Pr^J \cdot Pr_i), \tag{6.3.6}$$

where $J = (a_1, \ldots, a_s)$ is an orthogonally admissible sequence and we consider $\mathcal{K}_{p,*}$ as a right $\mathcal{K}^*_p$-module.

We now construct an (resp. strongly) unstable analog of the Koszul resolution (6.3.5). Let $J = (i_1, \ldots, i_k)$ be an orthogonally admissible sequence. We define the *excess* $e(J)$ of J as follows:

$$e(J) = e(i_1, \ldots, i_k) = i_k \text{ if } k > 0 \text{ and } e(\emptyset) = 0.$$

Lemma 6.3.7. *Suppose that J is orthogonally admissible sequence and $i \geq 1$. Then*

$$Pr^J \cdot Pr_a = \sum_{J'} c_{J'} Pr^{J'},\ c_{J'} \neq 0 \in \mathbf{F}_p,$$

where all J' are orthogonally admissible and $e(J') \geq e(J)$. □

Proof. Recall that the Koszul dual algebra $\mathcal{K}^*_p$ is quadratic and all relations have a form

$$Pr_i Pr_j = \sum_{(i',j')} c_{i',j'} Pr_{i'} Pr_{j'},\ i < pj,\ c_{i',j'} \neq 0 \in \mathbf{F}_p,$$

where $j' > j$ and $i' \geq pj'$, see [56, Section 7]. In other words, each sequence (i', j') succeeds (i, j) in the reverse lexicographical order.

Now, let $I = (i_1, \ldots, i_k)$ be any sequence. By an inductive argument and the previous paragraph, we observe that

$$Pr_I = Pr_{i_1} \cdot \ldots \cdot Pr_{i_k} = \sum_{I'} c_{I'} Pr_{I'},\ c_{I'} \neq 0 \in \mathbf{F}_p,$$

where all sequences I' are orthogonally admissible and each I' either succeeds or equal I with respect to the reverse lexicographical order. This implies the lemma. □

Definition 6.3.8. Let $W \in \mathsf{Vect}^{gr}_{\mathbf{F}}$ be a graded vector space. We denote by $\mathcal{K}^*_p \widehat{\otimes} W$ (resp. $\mathcal{K}^*_p \widetilde{\otimes} W$) the vector subspace of $\mathcal{K}^*_p \otimes W$ spanned by elements

$$Pr_J \otimes w,\ \ Pr_J \in K(B),\ w \in W,$$

where $e(J) \leq (p-1)|w|$ (resp. $e(J) < (p-1)|w|$). Dually, we denote by $\mathcal{K}_{p,*} \widehat{\otimes} W$ (resp. $\mathcal{K}_{p,*} \widetilde{\otimes} W$) the subspace of $\mathcal{K}_{p,*} \otimes W$ spanned by elements

$$Pr^J \otimes w,\ \ Pr^J \in K^\vee(B),\ w \in W,$$

where $e(J) \leq (p-1)|w|$ (resp. $e(J) < (p-1)|w|$). We point out that $|Pr_J \otimes w| = |Pr^J \otimes w| = (|J| + |w|, l(J))$.

Remark 6.3.9. If $p = 2$, then we have

$$\mathcal{K}_p^*\widetilde{\otimes}\Sigma^l\mathbf{F} = \Sigma\mathcal{K}_p^*\widehat{\otimes}\Sigma^{l-1}\mathbf{F},\ l > 0.$$

Similarly, if p is odd and $l = 2n + 1$ is odd, then we have

$$\mathcal{K}_p^*\widetilde{\otimes}\Sigma^l\mathbf{F} = \mathcal{K}_p^*\widehat{\otimes}\Sigma^l\mathbf{F},$$

and if $l = 2n$ is even, then

$$\mathcal{K}_p^*\widetilde{\otimes}\Sigma^l\mathbf{F} = \Sigma\mathcal{K}_p^*\widehat{\otimes}\Sigma^{l-1}\mathbf{F}.$$

Remark 6.3.10. Recall that a monomial of the algebra Λ in the λ's and μ's (resp. the λ's) is called *admissible* if

- p is odd and whenever $\lambda_a\lambda_b$ or $\lambda_a\mu_b$ occurs, we have $b < pa$,
- p is odd and whenever $\mu_a\lambda_b$ or $\mu_a\mu_b$ occurs, we have $b \leq pa$, and
- $p = 2$ and whenever $\lambda_a\lambda_b$ occurs, we have $b \leq pa$.

In other words, a monomial $y \in \Lambda$ is admissible if and only if $y = \Phi_p(x)$, $x \in \mathcal{K}_p^*$ and x is orthogonally admissible. We also recall that the algebra Λ is filtered by vector subspaces $\Lambda(l), l \geq 0$, where $\Lambda(l)$ is spanned by the admissible monomials beginning with

- λ_a with $a \leq n$ or μ_a with $a < n$ if $l = 2n$ and p is odd ([31, (1.15)]),
- λ_a with $a \leq n$ or μ_a with $a \leq n$ if $l = 2n + 1$ and p is odd ([31, (1.16)]),
- λ_a with $a < l$ if $p = 2$ ([9, p. 459]).

Then we observe that the anti-isomorphism $\Phi_p\colon \mathcal{K}_p^* \to \Lambda$ maps the vector subspace

$$\Sigma^{-l}\mathcal{K}_p^*\widehat{\otimes}\Sigma^l\mathbf{F} \subset \mathcal{K}_p^*$$

onto $\Lambda(l) \subset \Lambda$.

Remark 6.3.11. Let $W \in \mathsf{Vect}_{\mathbf{F}}^{gr}$ be a graded vector space. Following [69, Definition 9.3], let us denote by $W\widehat{\otimes}\Lambda$ the subspace of $W \otimes \Lambda$ spanned by $w \otimes y \in W \otimes \Lambda$ such that $y \in \Lambda$ is an admissible monomial beginning with ν_a^ε (resp. λ_a) such that $2a - \varepsilon < |w|$ (resp. $a < |w|$). Similarly, $W\widetilde{\otimes}\Lambda$ is the subspace of $W\widehat{\otimes}\Lambda$ spanned by $w \otimes y$, $y \in \Lambda$ is an admissible monomials beginning with ν_a^ε (resp. λ_a) such that $2a < |w|$ (resp. $a + 1 < |w|$). Vector spaces $W\widehat{\otimes}\Lambda$ and $W\widetilde{\otimes}\Lambda$ are bigraded:

$$|w \otimes y| = (|w|, 0) + |y|,\ w \in W,\ y \in \Lambda.$$

Remarks 6.3.9 and 6.3.10 show that the isomorphism Φ_p defines (after suitable shifts) following isomorphisms

$$\hat{\Phi}_p\colon \mathcal{K}_p^*\widehat{\otimes}W \xrightarrow{\cong} W\widehat{\otimes}\Lambda,\quad \tilde{\Phi}_p\colon \mathcal{K}_p^*\widetilde{\otimes}W \xrightarrow{\cong} W\widetilde{\otimes}\Lambda,\quad W \in \mathsf{Vect}_{\mathbf{F}}^{gr}.$$

Furthermore, if $|x| = (m, n)$, $x \in \mathcal{K}_p^*\widehat{\otimes}W$ (resp. $\mathcal{K}_p^*\widetilde{\otimes}W$), then $|\hat{\Phi}_p(x)| = |\tilde{\Phi}_p(x)| = (m - n, n)$.

We introduce more notation. Let $y = Pr^J \in K^\vee(B)$ be an orthogonally admissible monomial. We write $y(w), w \in W, W \in \mathsf{Vect}_{\mathbf{F}}^{gr}$ for the following element of $\mathcal{K}_{p,*}\widehat{\otimes}W$ (resp. $\mathcal{K}_{p,*}\widetilde{\otimes}W$):

$$y(w) = \begin{cases} Pr^J \otimes w & \text{if } e(J) \leq (p-1)|w| \text{ (resp. } e(J) < (p-1)|w|\text{)}, \\ 0, & \text{otherwise.} \end{cases}$$

We extend this notation linearly to any element $y \in \mathcal{K}_{p,*}$. Given the differential (6.3.6), we produce a map of free unstable $\mathcal{A}_p^h$-modules

$$d_s^{un}\colon F_{\mathcal{M}^h}(\mathcal{K}_{p,s}\widehat{\otimes}W^{(s)}) \to F_{\mathcal{M}^h}(\mathcal{K}_{p,s-1}\widehat{\otimes}W^{(s-1)}) \tag{6.3.12}$$

given on the generators $Pr^J \otimes w$ by the following formula:

$$d_s^{un}(Pr^J \otimes w) = \sum_{i\geq 1} St^i((Pr^J \cdot Pr_i)(w)).$$

Here $St^i(x) = 0$ if $i > (p-1)|x|$ and $W^{(s)}$ is the Frobenius twist of W, see Definition 2.2.1. The Frobenius twist is necessary since the operations $St^i \in \mathcal{A}_p^h$ are only semi-linear (Definition 6.1.7). We also define by the same formula a map of free strongly unstable $\mathcal{A}_p^h$-modules:

$$d_s^{sun}\colon F_{\mathcal{M}_0^h}(\mathcal{K}_{p,s}\widetilde{\otimes} W^{(s)}) \to F_{\mathcal{M}_0^h}(\mathcal{K}_{p,s}\widetilde{\otimes} W^{(s-1)}), \tag{6.3.13}$$

where $St^i(x) = 0$ if $i \geq (p-1)|x|$. By Lemma 6.3.7 and $d_{s-1}\circ d_s = 0$, we have

$$d_{s-1}^{un}\circ d_s^{un} = 0 \ \ (\text{resp. } d_{s-1}^{sun}\circ d_s^{sun} = 0),\ s > 0,$$

and so we obtained the *unstable Koszul complex*

$$K_\bullet(W) = (F_{\mathcal{M}^h}(\mathcal{K}_{p,\bullet}\widehat{\otimes} W^{(\bullet)}), d^{un}) \tag{6.3.14}$$

for the unstable $\mathcal{A}_p^h$-module $W \in \mathsf{Vect}_{\mathbf{F}}^{gr} \subset \mathcal{M}^h$. Similarly, we have the *strongly unstable Koszul complex*

$$K_\bullet^0(W) = (F_{\mathcal{M}_0^h}(\mathcal{K}_{p,\bullet}\widetilde{\otimes} W^{(\bullet)}), d^{sun}) \tag{6.3.15}$$

for the strongly unstable $\mathcal{A}_p^h$-module $W \in \mathsf{Vect}_{\mathbf{F}}^{gr} \subset \mathcal{M}_0^h$.

Proposition 6.3.16. *Both complexes $K_\bullet(W)$ and $K_\bullet^0(W)$ are acyclic for all $W \in \mathsf{Vect}_{\mathbf{F}}^{gr}$.*

Proof. We prove the proposition for $K_\bullet(W)$ only; the argument for $K_\bullet^0(W)$ is almost identical, and we leave it to the reader to complete the details. By Remark 6.1.19, the complex $K_\bullet(W)$ has a basis $St^I(Pr^J \otimes w)$, where $I = (i_1,\ldots,i_k)$ is an admissible sequence, $J = (a_1,\ldots,a_s)$ is an orthogonally admissible sequence, $e(J) \leq (p-1)|w|$, and

$$e(I) \leq (p-1)(|J| + |w|) = (p-1)(a_1 + \ldots + a_s + |w|).$$

We denote by $K_\bullet(W)_n, n\geq 0$ the vector subspace of $K_\bullet(W)$ spanned by the elements of the form $St^I(Pr^J\otimes w)$ such that $l(I)+l(J) = n$. Note that the differential (6.3.12) preserves each $K_\bullet(W)_n, n\geq 0$.

Let us consider the lexicographical order on the set $\mathcal{I}_n$ of sequences of length n: $\alpha = (\alpha_1,\ldots,\alpha_n) \preceq \beta = (\beta_1,\ldots,\beta_n) \in \mathcal{I}_n$ if and only if there exists h such that

$$\alpha_1 = \beta_1,\ldots,\alpha_{h-1} = \beta_{h-1} \ \text{ and } \ \alpha_h < \beta_h.$$

Define an $\mathcal{I}_n$-valued decreasing filtration $F^\alpha K_\bullet(W)_n, \alpha\in\mathcal{I}_n$ on the vector space $K_\bullet(W)_n$ by the rule:

$$F^\alpha K_\bullet(W)_n = \operatorname{span}(St^I(Pr^J\otimes w) \mid (I,J)\succeq \alpha),$$

where (I,J) is the concatenation of the sequences I and J. By the Adem relations (6.1.2), (6.1.3), (6.1.4) in $\mathcal{A}_p^h$, and the relations [56, Section 7] in $\mathcal{K}_p^*$, the differential d^{un} preserves subspaces $F^\alpha K_\bullet(W)_n$, $\alpha\in\mathcal{I}_n$.

A straightforward computation also shows that the associated graded complex $gr^\alpha K_\bullet(W)_n = F^\alpha/F^{\alpha'}$ (where α' succeeds α in $\mathcal{I}_n$) has the induced differential given by the rule

$$d(St^I(Pr^J\otimes w)) = St^I St^{a_1}(Pr^{J_0}\otimes w)$$

provided $J = (a_1, J_0)$ and the sequence (I, a_1) is admissible; and $d(St^I(Pr^J\otimes w))$ is zero otherwise. Hence the complex $gr^\alpha K_\bullet(W)_n$ is acyclic for each $\alpha\in\mathcal{I}_n$ and each $n\geq 0$. In this way, the original complex $K_\bullet(W)$ is also acyclic. □

By combining Proposition 6.3.16 with Remarks 6.2.2 and 6.3.11, we obtain immediately the following statement.

Corollary 6.3.17. *Let $W \in \mathsf{Vect}^{gr}_{\mathbf{F}}$ be a graded vector space. Then there are natural isomorphisms:*

$$\operatorname{Ext}^s_{\mathcal{M}^h}(W, \Sigma^t \mathbf{F}) \cong (\mathcal{K}^s_p \widehat{\otimes} (W^*)^{(s)})_t \xrightarrow{\hat{\Phi}_p} ((W^*)^{(s)} \widehat{\otimes} \Lambda_s)_{t-s},\ t \geq s \geq 0,$$

$$\operatorname{Ext}^s_{\mathcal{M}^h_0}(W, \Sigma^t \mathbf{F}) \cong (\mathcal{K}^s_p \widetilde{\otimes} (W^*)^{(s)})_t \xrightarrow{\tilde{\Phi}_p} ((W^*)^{(s)} \widetilde{\otimes} \Lambda_s)_{t-s},\ t \geq s \geq 0. \qquad \square$$

6.4 Free restricted Lie algebra. In this section we sketch how to compute the homotopy groups $\pi_* L^r(V_\bullet)$ of the free simplicial restricted Lie algebra $L^r(V_\bullet), V_\bullet \in \mathsf{sVect}^{ft}_{\mathbf{F}}$ in terms of the algebra Λ by using the spectral sequence (6.2.8). Of course, the description of $\pi_* L^r(V_\bullet)$ is well-known since the seventies (see [9, Theorem 8.5] and [69, Proposition 13.2]), so our computation is only illustrative.

By Proposition 2.1.8, there is an isomorphism

$$\pi_*(L^r(V_\bullet)) \cong \pi_*\Big(\bigoplus_{n\geq 1} L^r_n(V_\bullet)\Big) \cong \bigoplus_{n\geq 1} \pi_*\left((\mathbf{Lie}_n \otimes V_\bullet^{\otimes n})^{\Sigma_n}\right). \tag{6.4.1}$$

We write $\pi_{*,n}(L^r(V_\bullet))$ for the direct summand $\pi_*((\mathbf{Lie}_n \otimes V_\bullet^{\otimes n})^{\Sigma_n})$ in $\pi_*(L^r(V_\bullet))$.

Recall that Lie is the category of Lie algebras over $\mathbf{F}$, see Section 1.4.

Lemma 6.4.2. *The forgetful functor $\mathsf{oblv}\colon \mathsf{Lie} \to \mathsf{Vect}_{\mathbf{F}}$ admits a left adjoint*

$$L\colon \mathsf{Vect}_{\mathbf{F}} \to \mathsf{Lie}$$

such that

$$\mathsf{oblv} \circ L(V) \cong \bigoplus_{n\geq 1} L_n(V),\quad V \in \mathsf{Vect}_{\mathbf{F}}$$

and $L_n(V)$ is the image of $(\mathbf{Lie}_n \otimes V^{\otimes n})_{\Sigma_n}$ under the norm map

$$(\mathbf{Lie}_n \otimes V^{\otimes n})_{\Sigma_n} \to (\mathbf{Lie}_n \otimes V^{\otimes n})^{\Sigma_n}.$$

Proof. [24, Proposition 1.2.16]. $\square$

Remark 6.4.3. Note that if $p \neq 2$, then $L_n(V) = (\mathbf{Lie}_n \otimes V^{\otimes n})_{\Sigma_n}$.

Proposition 6.4.4. *Let $V_\bullet \in \mathsf{sVect}_{\mathbf{F}}$ be a connected simplicial vector space, i.e. $\pi_0(V_\bullet) = 0$. Then*

1. *$\pi_i(L^r(V_\bullet))$ is a free $\mathbf{F}\{\xi\}$-module for each $i \geq 0$;*
2. *The ξ-adic completion $\widehat{\pi}_i(L^r(V_\bullet))$ is isomorphic as a vector space to the infinite direct product*

$$\widehat{\pi}_i(L^r(V_\bullet)) \cong \prod_{n\geq 1} \pi_{i,n}(L^r(V_\bullet)),\ i \geq 0.$$

Proof. We write

$$\iota_n\colon L_n(V_\bullet) \to (\mathbf{Lie}_n \otimes V_\bullet^{\otimes n})^{\Sigma_n} = L^r_n(V_\bullet)$$

for the map induced by the norm map. The p-operation $\xi\colon L^r(V_\bullet) \to L^r(V_\bullet)$ induces maps

$$\xi_n\colon L^r_n(V_\bullet) \to L^r_{pn}(V_\bullet),\ n \geq 1$$

Together, ι_{pn} and ξ_n give a splitting of simplicial sets

$$\xi_n + \iota_{pn} \colon L^r_n(V_\bullet) \times L_{pn}(V_\bullet) \xrightarrow{\cong} L^r_{pn}(V_\bullet). \tag{6.4.5}$$

Furthermore, the map $\iota_n \colon L_n(V_\bullet) \to L^r_n(V_\bullet)$ is an isomorphism if $p \nmid n$.

Let us denote by $c \in \mathbf{N}$ the *connectivity* of $V_\bullet$; i.e. c is the largest integer such that $\pi_i(V_\bullet) = 0$ for all $i < c$. (Note that this definition is consistent with the definition of connectivity which is used in the subsequent references, but it differs from Definition 4.1.1 used before.) By the assumption, $c \geq 1$. By the Curtis theorem, the simplicial vector space $L_n(V_\bullet)$ is $(c + \lceil \log_2 n \rceil)$-connected (see [15] for the original proof and [36] for a shorter one). We fix an integer N such that $i < c + \lceil \log_2 N \rceil$ and $N > p$. Then the splitting (6.4.5) implies that

$$\xi = \xi_{n*} \colon \pi_{i,n}(L^r(V_\bullet)) \to \pi_{i,pn}(L^r(V_\bullet))$$

is a monomorphism for all n and ξ_{n*} is an isomorphism as soon as $pn > N$. Furthermore, $\pi_{i,n}(L^r(V_\bullet)) = 0$ for $n > N$ and $p \nmid n$.

Let $\mathcal{I}$ be the set of integers m such that $N/p < m \leq N$. Given $n \in \mathbf{N}$, we denote by $\nu_p(n)$ its p-valuation and we set $k(n) = np^{-\nu_p(n)}$. We construct a monomorphism of left $\mathbf{F}\{\xi\}$-modules

$$i = \oplus i_n \colon \pi_i(L^r(V_\bullet)) = \bigoplus_{n \geq 1} \pi_{i,n}(L^r(V_\bullet)) \hookrightarrow \bigoplus_{m \in \mathcal{I}} M_{\nu_p(m)} \otimes_{\mathbf{F}} \pi_{i,m}(L^r(V_\bullet)), \tag{6.4.6}$$

where $M_k \in \mathsf{Mod}_{\mathbf{F}\{\xi\}}, k \geq 0$ is the left cyclic submodule of the twisted Laurent polynomials $\mathbf{F}\{\xi^{\pm}\}$ generated by ξ^{-k}, see Section 5.2. Since $M_k \cong \mathbf{F}\{\xi\}, k \geq 0$, the monomorphism (6.4.6) implies the first part of the proposition.

We define the components i_n of (6.4.6) as follows. If $k(n) > N$, then we have $\pi_{i,n}(L^r(V_\bullet)) = 0$; and so we set $i_n = 0$. If $k(n) \leq N$, then there exists a unique $d_n \in \mathbf{Z}$ such that $m(n) = np^{d_n} \in \mathcal{I}$; and then we set

$$i_n \colon \pi_{i,n}(L^r(V_\bullet)) \to M_{\nu_p(m(n))} \otimes_{\mathbf{F}} \pi_{i,m(n)}(L^r(V_\bullet)),$$
$$i_n(x) = \xi^{-d_n} \otimes \xi^{d_n}(x).$$

The map i_n is well-defined by the Curtis theorem, and this is straightforward to check that $i = \oplus i_n$ is a monomorphism of left $\mathbf{F}\{\xi\}$-modules.

We prove the second part. By the isomorphism (6.4.1) and the first part of the proposition, the ξ-adic completion $\widehat{\pi}_i(L^r(V_\bullet))$ is the subset of the product $\prod_{n\geq 1} \pi_{i,n}(L^r(V_\bullet))$ consisting of all sequences that ξ-adically converge to zero. By the monomorphism (6.4.6), the free $\mathbf{F}\{\xi\}$-module $\pi_i(L^r(V_\bullet))$ is generated by the subspace

$$\bigoplus_{n \leq N} \pi_{i,n}(L^r(V_\bullet)$$

such that all elements of higher weights are obtained by iterating the p-operation ξ, which multiply the weight by p. Therefore, any sequence in $\prod_{n\geq 1} \pi_{i,n}(L^r(V_\bullet))$ is ξ-adically convergent to zero. □

The last proposition together with Corollary 5.3.11 and Lemma 5.2.2 immediately imply the following statement.

Corollary 6.4.7. *Let $V_\bullet \in \mathsf{sVect}_{\mathbf{F}}$ be a connected simplicial vector space, i.e. $\pi_0(V_\bullet) = 0$. Then the homotopy groups $\pi_*(L_\xi L^r(V_\bullet))$ of the $\mathbf{F}$-completion $L_\xi L^r(V_\bullet)$ are isomorphic as vector spaces to*

$$\pi_*(L_\xi L^r(V_\bullet)) \cong \widehat{\pi}_*(L^r(V_\bullet)) \cong \prod_{n \geq 1} \pi_{*,n}(L^r(V_\bullet)).$$ □

In this way, the spectral sequence (6.2.8) looks as follows

$$E^2_{s,t} = \mathrm{Ext}^s_{\mathcal{U}^h}(\Sigma\pi^*(V_\bullet), \Sigma^{t+1}\mathbf{F}) \Rightarrow \prod_{n\geq 1} \pi_{t-s,n}(L^r(V_\bullet)). \tag{6.4.8}$$

We compute the second page $E^2_{s,t}$ provided the homotopy groups $\pi_*(V_\bullet)$ are a one-dimensional vector space and show that the spectral sequence (6.4.8) degenerates in this case. The case $\dim \pi_*(V_\bullet) > 1$ will be discussed in Remark 6.4.16.

Let $\pi_l(V_\bullet) \cong \mathbf{F}, l > 0$ and $\pi_*(V_\bullet) = 0, * \neq l$. Then there are three cases: $p = 2$ and l is any; p is odd, l is even; p is odd, l is odd. We consider them separately.

6.4.1 $p = 2$ or p is odd and l is even. In these cases, the unstable $\mathcal{A}^h_p$-algebra $\Sigma^{l+1}\mathbf{F}$ is equal to $\mathcal{F}(\Sigma^{l+1}\mathbf{F})$, where $\Sigma^{l+1}\mathbf{F}$ is considered as an unstable $\mathcal{A}^h_p$-module (Remark 6.1.18). Therefore we have an isomorphism

$$\mathrm{Ext}^s_{\mathcal{U}^h}(\Sigma^{l+1}\mathbf{F}, \Sigma^{t+1}\mathbf{F}) \cong \mathrm{Ext}^s_{\mathcal{M}^h}(\Sigma^{l+1}\mathbf{F}, \Sigma^{t+1}\mathbf{F}),$$

and so we can directly apply Corollary 6.3.17.

Corollary 6.4.9. *Let $V_\bullet \in \mathsf{sVect}_{\mathbf{F}}$ be a simplicial vector space of finite type such that $\pi_l(V_\bullet) \cong \mathbf{F}, l \geq 1$ and $\pi_*(V_\bullet) = 0, * \neq l$. Suppose that either $p = 2$ or p is odd and l is even. Then the spectral sequence (6.4.8):*

$$E^2_{s,t} = \mathrm{Ext}^s_{\mathcal{U}^h}(\Sigma\pi^*(V_\bullet), \Sigma^{t+1}\mathbf{F}) \Rightarrow \prod_{n\geq 1} \pi_{t-s,n}(L^r(V_\bullet)). \tag{6.4.10}$$

degenerates at the second page. Moreover, $\pi_{*,n}(L^r(V_\bullet)) = 0$ if $n \neq p^h, h \in \mathbf{N}$ and

$$\mathrm{Ext}^s_{\mathcal{U}^h}(\Sigma\pi^*(V_\bullet), \Sigma^{t+1}\mathbf{F}) \cong \pi_{t-s,p^s}(L^r(V_\bullet)),\ t \geq s \geq 0. \tag{6.4.11}$$

Proof. By Corollary 6.3.17, there are isomorphisms:

$$E^2_{s,t} \cong (\Sigma\pi_*(V_\bullet)^{(s)} \widehat{\otimes} \Lambda_s)_{t+1-s} \cong \pi_*(V_\bullet)^{(s)} \otimes \Lambda_{t-l-s,s}(l+1)$$

First, assume that the field $\mathbf{F}$ is algebraically closed, $\mathbf{F} = \overline{\mathbf{F}}$. Then there are no non-trivial natural transformations

$$(-)^{(s)} \to (-)^{(s')} \colon \mathsf{Vect}_{\mathbf{F}} \to \mathsf{Vect}_{\mathbf{F}},\ s \neq s'$$

between different Frobenius twists. Since the differentials $d_r, r \geq 2$ of (6.4.10) are natural in $V_\bullet$, they are all zeros.

Since the spectral sequence (6.4.10) is completely convergent, we obtain a natural isomorphism:

$$\prod_{s\geq 0} \mathrm{Ext}^s_{\mathcal{U}^h}(\Sigma\pi^*(V_\bullet), \Sigma^{t+1}\mathbf{F}) \cong \prod_{n\geq 1} \pi_{t-s,n}(L^r(V_\bullet)). \tag{6.4.12}$$

The multiplicative group $\mathbf{F}^\times$ acts on $V_\bullet$ by multiplication, so both sides are $\mathbf{F}^\times$-representations. We derive the isomorphism (6.4.11) by comparing isotypic components.

If the field $\mathbf{F}$ is not algebraically closed, then we have isomorphisms:

$$\mathrm{Ext}^s_{\mathcal{U}^h}(\Sigma\pi^*(V_\bullet), \Sigma^{t+1}\mathbf{F}) \otimes_{\mathbf{F}} \overline{\mathbf{F}} \cong \mathrm{Ext}^s_{\mathcal{U}^h}(\Sigma\pi^*(V_\bullet \otimes_{\mathbf{F}} \overline{\mathbf{F}}), \Sigma^{t+1}\overline{\mathbf{F}}),\ t \geq s \geq 0$$
$$\pi_{*,n}(L^r(V_\bullet)) \otimes_{\mathbf{F}} \overline{\mathbf{F}} \cong \pi_{*,n}(L^r(V_\bullet \otimes_{\mathbf{F}} \overline{\mathbf{F}})),\ n \geq 0$$
$$\mathrm{Ext}^s_{\mathcal{U}^h}(\Sigma\pi^*(V_\bullet \otimes_{\mathbf{F}} \overline{\mathbf{F}}), \Sigma^{t+1}\overline{\mathbf{F}}) \cong \pi_{t-s,p^s}(L^r(V_\bullet \otimes_{\mathbf{F}} \overline{\mathbf{F}})),\ t \geq s \geq 0.$$

These isomorphisms imply that the isomorphism (6.4.11) holds even for non-algebraically closed fields. Finally, the isomorphism (6.4.11) implies the isomorphism (6.4.12) of direct products, which gives the degeneration of the spectral sequence (6.4.10). ☐

6.4.2 p is odd, l is odd. In this case, we have

$$\widetilde{H}^*(L^r(V_\bullet);\mathbf{F}) \cong \mathbf{F}[x]/x^2,\ |x| = l+1.$$

This unstable $\mathcal{A}_p^h$-algebra is not $\mathcal{F}(M_*)$ for any $M_* \in \mathcal{M}^h$, which slightly complicates the problem.

Let A_* be a non-unital graded commutative $\mathbf{F}$-algebra. We denote by $AQ_q(A_*) \in \mathsf{Vect}_{\mathbf{F}}^{gr}$ the *q-th André-Quillen homology* of A_*; i.e. $AQ_*(A_*)$ is the left non-abelian derived functor of

$$A_* \mapsto AQ_0(A_*) = A_*/A_*^2.$$

We refer the reader to [1] and [61] for further details. If $A_* \in \mathcal{U}^h$ is an unstable $\mathcal{A}_p^h$-algebra, then $AQ_q(A_*) \in \mathcal{M}_0^h, q \geq 0$ are strongly unstable $\mathcal{A}_p^h$-modules (Definition 6.1.14). Similar to [47, Theorem 2.5] and [26, Theorem 4.3], we obtain the strongly convergent (Grothendieck) spectral sequence:

$$E_2^{r,q} = \mathrm{Ext}^r_{\mathcal{M}_0^h}(AQ_q(A_*),\Sigma^{t+1}\mathbf{F}) \Rightarrow \mathrm{Ext}^{r+q}_{\mathcal{U}^h}(A_*,\Sigma^{t+1}\mathbf{F}),\ A_* \in \mathcal{U}^h,\ t \geq 0. \tag{6.4.13}$$

Lemma 6.4.14. *Suppose that* $\mathrm{char}(\mathbf{F}) > 2$. *Let* $V_\bullet \in \mathsf{sVect}_{\mathbf{F}}$ *be a simplicial vector space such that* $\pi_l(V_\bullet) \cong \mathbf{F}$, *$l$ is odd,* $l \geq 1$, *and* $\pi_*(V_\bullet) = 0$ *if* $* \neq l$. *Then the spectral sequence* (6.4.13) *degenerates at the second page, there are isomorphisms:*

$$E_2^{r,0} \cong (\mathcal{K}_p^r\widetilde{\otimes}(\Sigma\pi_*(V_\bullet))^{(r)})_{t+1},\ \ E_2^{r,1} \cong (\mathcal{K}_p^r\widetilde{\otimes}((\Sigma\pi_*(V_\bullet))^{\otimes 2})^{(r)})_{t+1},\ r \geq t \geq 0,$$

and there is a natural (in $V_\bullet$) splitting

$$\mathrm{Ext}^s_{\mathcal{U}^h}(\Sigma\pi^*(V_\bullet),\Sigma^{t+1}\mathbf{F}) \cong \mathrm{Ext}^s_{\mathcal{M}_0^h}(\Sigma\pi^*(V_\bullet),\Sigma^{t+1}\mathbf{F}) \oplus \mathrm{Ext}^{s-1}_{\mathcal{M}_0^h}((\Sigma\pi^*(V_\bullet))^{\otimes 2},\Sigma^{t+1}\mathbf{F}).$$

Proof. There exists a pushout square

$$\begin{array}{ccc} \mathbf{F}[(\Sigma\pi^*(V_\bullet))^{\otimes 2}] & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ \mathbf{F}[\Sigma\pi^*(V_\bullet)] & \longrightarrow & \Sigma\pi^*(V_\bullet) \end{array}$$

of non-unital graded commutative $\mathbf{F}$-algebras. Here $\mathbf{F}[\Sigma\pi^*(V_\bullet)]$ and $\mathbf{F}[(\Sigma\pi^*(V_\bullet))^{\otimes 2}]$ are free non-unital algebras, and the left vertical arrow is induced by the inclusion $(\Sigma\pi^*(V_\bullet))^{\otimes 2} \subset \mathbf{F}[\Sigma\pi^*(V_\bullet)]$. Since the André-Quillen homology maps pushout squares to long exact sequences, we have

$$AQ_0(\Sigma\pi^*(V_\bullet)) \cong \Sigma\pi^*(V_\bullet),\ \ AQ_1(\Sigma\pi^*(V_\bullet)) \cong (\Sigma\pi^*(V_\bullet))^{\otimes 2},$$

and $AQ_q(\Sigma\pi^*(V_\bullet)) = 0$ if $q > 1$.

By Corollary 6.3.17, there are isomorphisms

$$E_2^{r,0} \cong (\mathcal{K}_p^r\widetilde{\otimes}(\Sigma\pi_*(V_\bullet))^{(r)})_{t+1},\ \ E_2^{r,1} \cong (\mathcal{K}_p^r\widetilde{\otimes}((\Sigma\pi_*(V_\bullet))^{\otimes 2})^{(r)})_{t+1},\ r \geq t \geq 0,$$

and $E_2^{r,q} = 0$ if $q > 1$. By the same argument with $\mathbf{F}^\times$-action as in Corollary 6.4.9, we get that the spectral sequence (6.4.13) degenerates at the second page and there is a natural splitting:

$$\mathrm{Ext}^s_{\mathcal{U}^h}(\Sigma\pi^*(V_\bullet),\Sigma^{t+1}\mathbf{F}) \cong E_2^{s,0} \oplus E_2^{s-1,1}.$$

□

Using the computation in Lemma 6.4.14 we obtain an analog of Corollary 6.4.9 for odd l as well. The proof of Corollary 6.4.15 is absolutely parallel to the proof of Corollary 6.4.9, so we leave it to the reader to complete the details.

Corollary 6.4.15. *Suppose that* $\mathrm{char}(\mathbf{F}) > 2$. *Let* $V_\bullet \in \mathsf{sVect}_\mathbf{F}$ *be a simplicial vector space such that* $\pi_l(V_\bullet) \cong \mathbf{F}$, l *is odd,* $l \geq 1$, *and* $\pi_*(V_\bullet) = 0$ *if* $* \neq l$. *Then the spectral sequence* (6.4.8):

$$E^2_{s,t} = \mathrm{Ext}^s_{\mathcal{U}^h}(\Sigma\pi^*(V_\bullet), \Sigma^{t+1}\mathbf{F}) \Rightarrow \prod_{n\geq 1} \pi_{t-s,n}(L^r(V_\bullet)).$$

degenerates at the second page. Moreover, $\pi_{*,n}(L^r(V_\bullet)) = 0$ *if* $n \neq p^h$ *or* $n \neq 2p^h$, $h \in \mathbf{N}$, *and there are isomorphisms*

$$\mathrm{Ext}^s_{\mathcal{U}^h}(\Sigma\pi^*(V_\bullet), \Sigma^{t+1}\mathbf{F}) \cong \pi_{t-s,p^s}(L^r(V_\bullet)) \oplus \pi_{t-s,2p^{s-1}}(L^r(V_\bullet)),$$

$$\pi_{t-s,p^s}(L^r(V_\bullet)) \cong \mathrm{Ext}^s_{\mathcal{M}^h_0}(\Sigma\pi^*(V_\bullet), \Sigma^{t+1}\mathbf{F}) \cong \pi_*(V_\bullet)^{(s)} \otimes \Lambda_{t-l-s,s}(l),$$

$$\begin{aligned}\pi_{t-s,2p^{s-1}}(L^r(V_\bullet)) &\cong \mathrm{Ext}^{s-1}_{\mathcal{M}^h_0}((\Sigma\pi^*(V_\bullet))^{\otimes 2}, \Sigma^{t+1}\mathbf{F}) \\ &\cong ((\pi_*(V_\bullet))^{\otimes 2})^{(s-1)} \otimes \Lambda_{t-2l-s,s-1}(2l+1)\end{aligned}$$

for $t \geq s \geq 0$. □

Remark 6.4.16. Let $V_\bullet \in \mathsf{sVect}^{ft}_\mathbf{F}$ be a simplicial vector space of finite type such that $\dim \pi_*(V_\bullet) > 1$. Then one still can apply spectral sequences (6.4.8) and (6.4.13) in order to compute the homotopy groups $\pi_*(L^r(V_\bullet))$ of the free simplicial restricted Lie algebra $L^r(V_\bullet) \in \mathsf{sLie}^r$. However, the André-Quillen homology groups of the trivial algebra $\Sigma\pi^*(V_\bullet)$ are quite large in this case, see [27] and [3, Theorem 8.18]. Therefore we can not expect that any of two spectral sequences (6.4.8) or (6.4.13) degenerate at the second page and an additional expertise is required.

Remark 6.4.17. Nevertheless, one can use the (algebraic) Hilton-Milnor theorem to compute $\pi_*(L^r(V_\bullet))$, $V_\bullet \in \mathsf{sVect}_\mathbf{F}$. Let $V_{1,\bullet}, V_{2,\bullet} \in \mathsf{sVect}_\mathbf{F}$ be simplicial vector spaces and let L be the free Lie algebra over the integers on two symbols i_1 and i_2 and let $B \subset L$ be the Hall basis for L, see [70, p. 512]. Then a word $w \in B$ is an iterated Lie bracket

$$w = [i_{j_1}[i_{j_2}, \ldots, i_{j_s}]],$$

where $j_t \in \{1, 2\}$, $1 \leq t \leq s$. We associate to w the iterated tensor product

$$w(V_{1,\bullet}, V_{2,\bullet}) = V_{j_1,\bullet} \otimes \ldots \otimes V_{j_s,\bullet}$$

and the canonical inclusion

$$l_w \colon w(V_{1,\bullet}, V_{2,\bullet}) \to \mathsf{oblv} \circ L^r(V_{1,\bullet} \oplus V_{2,\bullet})$$

such that $l_w(v_1^{j_1} \otimes \ldots \otimes v_s^{j_s}) = [v_1^{j_1}[v_2^{j_2}, \ldots, v_s^{j_s}]]$, where $v_t^{j_t} \in V_{j_t,\bullet}$. These maps determine the map

$$l \colon \bigoplus_{w\in B} \mathsf{oblv} \circ L^r(w(V_{1,\bullet}, V_{2,\bullet})) \to \mathsf{oblv} \circ L^r(V_{1,\bullet} \oplus V_{2,\bullet}),$$

which is an isomorphism, see e.g. [52, Example 8.7.4]. Therefore one can reduce the problem of calculating $\pi_*(L^r(V_\bullet))$, $\dim \pi_*(V_\bullet) > 1$ to the considered case $\dim \pi_*(V_\bullet) = 1$.

Remark 6.4.18. Let $V_* \in \mathsf{Vect}^{gr}_{\mathbf{F}}$ be a graded vector space. Let us denote by $L^r(V_*) \in \mathsf{sLie}^r$ the simplicial vector space generated by $\Gamma V_* \in \mathsf{sVect}_{\mathbf{F}}$, where Γ is the inverse of the normalized chain complex functor N, see Section 1.4. By Proposition 2.1.8, there is a canonical embedding $V_* \subset \pi_*(L^r(V_*))$. By Proposition 6.4.4, we observe that $\xi(v) \in \pi_*(L^r(V_*))$ is non-zero for any $v \in V_* \subset \pi_*(L^r(V_*))$. Finally, Corollaries 6.4.9 and 6.4.15 imply that $\xi(v)$ is a non-zero multiple of $v \otimes \mu_0 \in \pi_{*,p}(L^r(V_*))$ (resp. $v \otimes \lambda_0 \in \pi_{*,2}(L^r(V_*))$).

Remark 6.4.19. Let $x \in \pi_m L^r(\Sigma^n \mathbf{F})$ be a homotopy class

$$x \in \pi_m L^r(\Sigma^n \mathbf{F}) = [L^r(\Sigma^m \mathbf{F}), L^r(\Sigma^n \mathbf{F})].$$

Consider cofiber sequences

$$L^r(\Sigma^m \mathbf{F}) \xrightarrow{x} L^r(\Sigma^n \mathbf{F}) \to \mathrm{cofib}(x),$$

$$L^r(\Sigma^n \mathbf{F}) \to \mathrm{cofib}(x) \to L^r(\Sigma^{m+1} \mathbf{F}).$$

The second one implies that $\widetilde{H}^q(\mathrm{cofib}(x); \mathbf{F}) = 0$ if $q \neq n+1, m+2$, and there are canonical generators:

$$h_{n+1} \in \widetilde{H}^{n+1}(\mathrm{cofib}(x); \mathbf{F}) \cong \widetilde{H}^{n+1}(L^r(\Sigma^n \mathbf{F}); \mathbf{F}),$$
$$h_{m+2} \in \widetilde{H}^{m+2}(\mathrm{cofib}(x); \mathbf{F}) \cong \widetilde{H}^{m+2}(L^r(\Sigma^{m+1} \mathbf{F}); \mathbf{F}).$$

We say that a cohomology operation P *detects* x if $P(h_{n+1}) = h_{m+2}$. Let $\iota_n \in \pi_n L^r(\Sigma^n \mathbf{F})$ be the canonical generator. Then Corollary 6.4.9 implies

- if $p = 2$, then the element $\iota_n \otimes \lambda_i \in \pi_{n+i}(L^r(\Sigma^n \mathbf{F}))$, $0 \leq i \leq n$ is detected by Sq^{i+1}.
- if p is odd and $n = 2k$, then the element

  $$\iota_n \otimes \lambda_i \in \pi_{n+2i(p-1)-1}(L^r(\Sigma^n \mathbf{F})),\ 1 \leq i \leq k$$

  is detected by the Steenrod operation P^i.
- if p is odd and $n = 2k$, then the element

  $$\iota_n \otimes \mu_i \in \pi_{n+2i(p-1)}(L^r(\Sigma^n \mathbf{F})),\ 0 \leq i \leq k$$

  is detected by the Steenrod operation βP^i.

This characterization of the generators in the algebra Λ seems to be folklore. However, the only case covered in the literature is $p = 2$, see [57, p. 515]; the case of odd primes seems to be missing.

Acknowledgments

The author is grateful to Mark Behrens, Dmitry Kaledin, Kirill Magidson, and Artem Prikhodko for many helpful conversations. The author also thanks the Max Planck Institute for Mathematics in Bonn for its hospitality and financial support.

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