



∞ -type theories

Hoang Kim Nguyen^a and Taichi Uemura^{bc}

^aUniversity of Regensburg, Universitätsstraße 31, 93047 Regensburg, Germany

^bUniversity of Amsterdam, Science Park 107, 1098 XG Amsterdam, Netherlands

^cStockholm University, SE-106 91 Stockholm, Sweden

Abstract

We introduce ∞ -type theories as an ∞ -categorical generalization of the categorical definition of type theories introduced by the second named author. We establish analogous results to the previous work including the construction of initial models of ∞ -type theories, the construction of internal languages of models of ∞ -type theories, and the theory-model correspondence for ∞ -type theories. Some structured $(\infty, 1)$ -categories are naturally regarded as models of some ∞ -type theories. Thus, since every (1-categorical) type theory is in particular an ∞ -type theory, ∞ -type theories provide a unified framework for connections between type theories and $(\infty, 1)$ -categorical structures. As an application we prove Kapulkin and Lumsdaine’s conjecture that the dependent type theory with intensional identity types gives internal languages for $(\infty, 1)$ -categories with finite limits.

Communicated by: Carlos Tschudi Simpson.

Received: 2nd May, 2022. Accepted: 30th December, 2024.

MSC: 03B38; 18N60.

Keywords: ∞ -type theory, type theory, $(\infty, 1)$ -category, representable map, initial model, internal language, coherence problem.

1. Introduction

Type theory and *higher category theory* are closely related: dependent type theories with intensional identity types provide a syntactic way of reasoning about $(\infty, 1)$ -categories. This is known as the family of *internal language conjectures* and has led for example to syntactic developments of classical material in homotopy theory such as the homotopy groups of spheres [9, 10, 32] and the Blakers-Massey Theorem [21], just to name a few. These proofs often lead to new perspectives on classical material and their nature makes them applicable to a wider class of $(\infty, 1)$ -categories, importing ideas from the homotopy theory of spaces to other $(\infty, 1)$ -categories, see for example

Email addresses: hoang-kim.nguyen@klinik.uni-regensburg.de (Nguyen)

t.uemura00@gmail.com (Uemura)

© Nguyen and Uemura, 2025, under a Creative Commons Attribution 4.0 International License.

DOI: [10.21136/HS.2025.04](https://doi.org/10.21136/HS.2025.04)

[1] and [2]. One of the main appeals of type theory for higher category theory and homotopy theory is thus the usage of this type theoretic language to reason in a synthetic way. On the other hand, higher categories will be useful for the study of type theories. For example, one can expect a conceptual proof of Voevodsky’s homotopy canonicity conjecture that any closed term of the type of natural numbers is homotopic to a numeral using a higher dimensional analogue of the Freyd cover [30].

However, internal language conjectures are still open problems in *homotopy type theory* [45]. The advantage of type-theoretic languages, that a lot of equations *strictly* hold in type theories so that a lot of trivial homotopies in $(\infty, 1)$ -categories can be eliminated, is at the same time the main difficulty of internal language conjectures. One has to justify interpreting strict equality in type theories as homotopies in $(\infty, 1)$ -categories. This is an ∞ -dimensional version of the *coherence problem* in the categorical semantics of type theories.

An internal language conjecture should be formulated as an equivalence between an $(\infty, 1)$ -category of theories and an $(\infty, 1)$ -category of structured $(\infty, 1)$ -categories. Currently, only a few internal language conjectures have been made precise. Kapulkin and Lumsdaine [27] made precise formulations of the simplest cases and conjectured that the $(\infty, 1)$ -category of theories over Martin-Löf type theory with intensional identity types (and dependent function types with function extensionality) is equivalent to the $(\infty, 1)$ -category of small $(\infty, 1)$ -categories with finite limits (and pushforwards). In this paper, we prove Kapulkin and Lumsdaine’s conjecture by introducing a novel ∞ -dimensional generalization of type theories which we call *∞ -type theories*.

The basic strategy for proving Kapulkin and Lumsdaine’s conjecture is to decompose the equivalence to be proved into smaller pieces. An existing approach is to introduce 1-categorical presentations of $(\infty, 1)$ -categories with finite limits. It had already been shown by Szumiło [44] that $(\infty, 1)$ -categories with finite limits are equivalent to categories of fibrant objects in the sense of Brown [8]. Kapulkin and Szumiło [29] then proved that categories of fibrant objects are equivalent to Joyal’s tribes [24]. Tribes are considered as 1-categorical models of the type theory, but a full proof of the equivalence between tribes and theories has not yet been achieved.

Although this approach is natural for those who know the homotopical interpretation of intensional identity types [6, 3, 42], 1-categorical models of intensional identity types are not convenient to work with. A problem is that 1-categories of 1-categorical models of type theories need not be rich enough to calculate the $(\infty, 1)$ -categories they present. It is also unclear if this approach can be generalized to internal language conjectures for richer type theories.

In this paper we seek another path. The key idea is to introduce a notion of *∞ -type theories*, an ∞ -dimensional generalization of type theories. Intuitively, an ∞ -type theory is a kind of type theory but equality is like homotopies rather than strict one. Ordinary type theories are considered as truncated ∞ -type theories in the sense that all homotopies are trivial.

Our proof strategy is as follows. Let $\mathbb{I}$ denote the type theory with intensional identity types. We introduce an ∞ -type theory $\mathbb{I}_\infty$ which is analogous to $\mathbb{I}$ but without truncation. Because $\mathbb{I}_\infty$ is already a higher dimensional object, it is straightforward to interpret $\mathbb{I}_\infty$ in $(\infty, 1)$ -categories with finite limits. The internal language conjecture is then reduced to a coherence problem between $\mathbb{I}$ and $\mathbb{I}_\infty$: how to interpret $\mathbb{I}$ in models of $\mathbb{I}_\infty$. Although this coherence problem is as difficult as the original internal language conjecture, this reduction is an important step. Since the problem is now formulated in the language of ∞ -type theories and related concepts, our proof strategy is easily generalized to internal language conjectures for richer type theories. When we extend $\mathbb{I}$ by some type constructors, we just extend $\mathbb{I}_\infty$ in the same way.

A solution to coherence problems in the 1-categorical semantics of type theories given by

Hofmann [20] is to replace a “non-split” model, in which equality between types is up to isomorphism, by an equivalent “split” model, in which equality between types is strict. In our approach, models of $\mathbb{I}_\infty$ are like non-split models of $\mathbb{I}$, so we consider replacing a model of $\mathbb{I}_\infty$ by an equivalent model of $\mathbb{I}$. Splitting techniques for $(\infty, 1)$ -categorical structures have not yet been fully developed except for some presentable $(\infty, 1)$ -categories considered by Gepner [19] and Shulman [43]. Since we have to split small $(\infty, 1)$ -categories which are usually non-presentable, their results cannot directly apply. However, as he already mentioned in [43, Remark 1.4], Shulman’s result on splitting presentable $(\infty, 1)$ -toposes can be used for splitting small $(\infty, 1)$ -categories by embedding them into presheaf $(\infty, 1)$ -toposes.

Organization In Section 2 we fix notations and remember some concepts in $(\infty, 1)$ -category theory. Relevant concepts to this paper are ∞ -cosmoi [41], *compactly generated* $(\infty, 1)$ -categories, *exponentiable* arrows, and *representable maps* between right fibrations.

We introduce the notion of an ∞ -type theory in Section 3. It is defined as an $(\infty, 1)$ -category with a certain structure, generalizing the categorical definition of type theories introduced by the second named author [47]. There are two important notions around ∞ -type theories: *models* and *theories*. The notion of models we have in mind is a generalization of categories with families [15] and, equivalently, natural models [5, 16]. The notion of theories is close to the essentially algebraic definitions of theories given by Garner [18], Isaev [22], and Voevodsky [49].

In Section 4 we prove ∞ -analogue of the main results of the previous work [47]. Given an ∞ -type theory $\mathbb{T}$, we construct a functor that assigns to each model of $\mathbb{T}$ a $\mathbb{T}$ -theory called the *internal language* of the model. The internal language functor has a fully faithful left adjoint which constructs a *syntactic model* from a $\mathbb{T}$ -theory. We further characterize the image of the left adjoint.

We study some concrete ∞ -type theories in Section 5. The most basic example is $\mathbb{E}_\infty$, the ∞ -analogue of Martin-Löf type theory with *extensional* identity types. We show that $\mathbb{E}_\infty$ -theories are equivalent to $(\infty, 1)$ -categories with finite limits (Theorem 5.15), which is an ∞ -analogue of the result of Clairambault and Dybjer [13]. This is to be an intermediate step toward Kapulkin and Lumsdaine’s conjecture, but it also has an interesting corollary. One can derive a new universal property of the $(\infty, 1)$ -category of small $(\infty, 1)$ -categories with finite limits from a universal property of $\mathbb{E}_\infty$ (Corollary 5.21). We also study a couple of examples of ∞ -type theories with dependent function types. Finally in Section 6 we prove Kapulkin and Lumsdaine’s conjecture.

2. Preliminaries

2.1 ∞ -categories For concreteness, we will work with ∞ -categories, also called *quasicategories* in the literature, [23, 36, 12] as models for $(\infty, 1)$ -categories. An ∞ -category is a simplicial set satisfying certain horn filling conditions. We recollect some standard definitions and notations.

Definition 2.1. 1. Given an ∞ -category $\mathcal{C}$ and a simplicial set A , we denote $\mathbf{Fun}(A, \mathcal{C})$ the internal hom of simplicial sets, which is itself an ∞ -category and models the ∞ -category of functors and natural transformations.

2. For an ∞ -category $\mathcal{C}$, we denote by $\mathcal{C}^\simeq$ the space of objects of $\mathcal{C}$ modeled by the largest Kan complex contained in $\mathcal{C}$. Furthermore we write $k(\mathcal{C}, \mathcal{D}) := \mathbf{Fun}(\mathcal{C}, \mathcal{D})^\simeq$.

3. For an ∞ -category $\mathcal{C}$, we denote by $C^{\triangleright}$ the join $\mathcal{C} \star \Delta^0$.
4. We denote by $\mathbf{Cat}_{\infty}$ the ∞ -category of small ∞ -categories. This is obtained as the homotopy coherent nerve of the simplicial category with objects given by small ∞ -categories and hom simplicial sets given by $k(\mathcal{C}, \mathcal{D})$.
5. We denote by $\mathbf{CAT}_{\infty}$ the ∞ -category of (possibly large) ∞ -categories obtained in a similar way.
6. We denote by $\mathbf{S}$ the ∞ -category of small spaces obtained as the homotopy coherent nerve of the simplicial category with objects small Kan complexes and hom simplicial sets given by the internal hom of simplicial sets. By a subspace of a space A we mean a (-1) -truncated map $B \rightarrow A$ in $\mathbf{S}$.

Although we chose to work with ∞ -categories, we will primarily use the language of the formal category theory of ∞ -categories as expressed by ∞ -cosmoi. Therefore, most of our constructions, statements and proofs are independent of the model.

2.2 ∞ -cosmoi An ∞ -cosmos [41] is, roughly, a complete $(\infty, 2)$ -category with enough structure to do formal category theory. More concretely, an ∞ -cosmos $\mathcal{K}$ is a simplicially enriched category such that for any pair of objects $\mathcal{C}, \mathcal{D} \in \mathcal{K}$, the hom simplicial set $\mathcal{K}(\mathcal{C}, \mathcal{D})$ is an ∞ -category. $\mathcal{K}$ is also equipped a class of morphisms called isofibrations, and all small $(\infty, 1)$ -categorical limits are constructible from products and pullbacks of isofibrations. Moreover, $\mathcal{K}$ has cotensors with small simplicial sets $A \pitchfork \mathcal{C}$ characterized by the equivalence (isomorphism, in fact) of ∞ -categories

$$\mathcal{K}(\mathcal{D}, A \pitchfork \mathcal{C}) \simeq \mathbf{Fun}(A, \mathcal{K}(\mathcal{D}, \mathcal{C})).$$

Given an ∞ -category, we may take its *homotopy category*, which is just an ordinary category. Applying this to the hom spaces of an ∞ -cosmos gives rise to a 2-category. Adjunctions and equivalences in ∞ -cosmoi are then defined in the usual way using this 2-category.

Example 2.2. We denote by $\mathfrak{Cat}_{\infty}$ the ∞ -cosmos of (possibly large) ∞ -categories. That is, $\mathfrak{Cat}_{\infty}$ is the simplicial category with objects (possibly large) ∞ -categories and hom simplicial sets $\mathbf{Fun}(\mathcal{C}, \mathcal{D})$. The cotensor $A \pitchfork \mathcal{C}$ in $\mathfrak{Cat}_{\infty}$ is given by the functor ∞ -category $\mathbf{Fun}(A, \mathcal{C})$ and adjunctions and equivalences agree with the standard notions of ∞ -categories.

Cartesian fibrations and right fibrations in ∞ -cosmoi are characterized by analogy with those in complete 2-categories. Here we prefer to work with versions of these concepts that are invariant under equivalence. The following definition coincides with Riehl and Verity's when F is an isofibration.

Definition 2.3. A functor $F: \mathcal{C} \rightarrow \mathcal{D}$ in an ∞ -cosmos $\mathcal{K}$ is said to be a *cartesian fibration* if the functor

$$(\mathrm{ev}_1, F_*): \Delta^1 \pitchfork \mathcal{C} \rightarrow \mathcal{C} \times_{\mathcal{D}} (\Delta^1 \pitchfork \mathcal{D})$$

has a right adjoint with invertible counit. A *fibred functor* between cartesian fibrations is a morphism in $\mathcal{K}^{\rightarrow}$ that commutes with the right adjoint of (ev_1, F_*) . A cartesian fibration is a *right fibration* if (ev_1, F_*) is an equivalence.

For a small ∞ -category $\mathcal{C}$, we denote by $\mathbf{CartFib}_{\mathcal{C}} \subset \mathbf{Cat}_{\infty}/\mathcal{C}$ the ∞ -category of cartesian fibrations over $\mathcal{C}$ and fibred functors over $\mathcal{C}$. We denote by $\mathbf{RFib}_{\mathcal{C}} \subset \mathbf{CartFib}_{\mathcal{C}}$ the full subcategory spanned by the right fibrations over $\mathcal{C}$. Note that any functor between right fibrations over $\mathcal{C}$ is automatically a fibred functor, so $\mathbf{RFib}_{\mathcal{C}}$ is a full subcategory of $\mathbf{Cat}_{\infty}/\mathcal{C}$. We write $\mathbf{RFib} \subset \mathbf{Cat}_{\infty}^{\rightarrow}$ for the full subcategory spanned by the right fibrations.

2.3 Compactly generated ∞ -categories

Definition 2.4 (Lurie [36, Definition 5.5.7.1 and Theorem 5.5.1.1]). An ∞ -category $\mathcal{C}$ is said to be *compactly generated* if it is an ω -accessible localization of $\mathbf{Fun}(\mathcal{D}^{\mathrm{op}}, \mathbf{S})$, that is, a reflective full subcategory of $\mathbf{Fun}(\mathcal{D}^{\mathrm{op}}, \mathbf{S})$ closed under filtered colimits, for some small ∞ -category $\mathcal{D}$. The subcategory of $\mathbf{CAT}_{\infty}$ spanned by the compactly generated ∞ -categories and ω -accessible right adjoints is denoted by $\mathbf{Pr}_{\omega}^R$. We will moreover denote by $\mathfrak{Pr}_{\omega}^R \subset \mathfrak{CAT}_{\infty}$ the locally full subcategory spanned by the compactly generated ∞ -categories and ω -accessible right adjoints.

Recall [36, Proposition 5.5.7.6] that $\mathbf{Pr}_{\omega}^R \subset \mathbf{CAT}_{\infty}$ is closed under small limits. By definition, compactly generated ∞ -categories are closed in $\mathfrak{CAT}_{\infty}$ under cotensors with small simplicial sets. Hence, the subcategory $\mathfrak{Pr}_{\omega}^R \subset \mathfrak{CAT}_{\infty}$ is an ∞ -cosmos, and the inclusion $\mathfrak{Pr}_{\omega}^R \rightarrow \mathfrak{CAT}_{\infty}$ preserves the structures of ∞ -cosmoi and reflects equivalences.

Example 2.5. The ∞ -category $\mathbf{S}$ of small spaces is compactly generated. The ∞ -category $\mathbf{Cat}_{\infty}$ of small ∞ -categories is compactly generated, and the functor $k(\Delta^n, -) : \mathbf{Cat}_{\infty} \rightarrow \mathbf{S}$ sending an ∞ -category $\mathcal{C}$ to the space of n -cells of $\mathcal{C}$ is an ω -accessible right adjoint. This is because $\mathbf{Cat}_{\infty}$ is regarded as an ω -accessible localization of $\mathbf{Fun}(\Delta^{\mathrm{op}}, \mathbf{S})$ using the equivalence of quasicategories and complete Segal spaces [25].

Example 2.6. For a small ∞ -category $\mathcal{C}$, the ∞ -category $\mathbf{CartFib}_{\mathcal{C}}$ of cartesian fibrations over $\mathcal{C}$ and fibred functors over $\mathcal{C}$ is compactly generated as $\mathbf{CartFib}_{\mathcal{C}} \simeq \mathbf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Cat}_{\infty})$. The forgetful functor $\mathbf{CartFib}_{\mathcal{C}} \rightarrow \mathbf{Cat}_{\infty}/\mathcal{C}$ is an ω -accessible right adjoint. To see this, observe that this forgetful functor is the right derived functor of the forgetful functor

$$\mathbf{SSet}^+/\mathcal{C}^{\sharp} \rightarrow \mathbf{SSet}/\mathcal{C}$$

which is a right Quillen functor with respect to the cartesian model structure and the slice model structure of the Joyal model structure on $\mathbf{SSet}$ [36, Proposition 3.1.5.2] or [37, Proposition 3.1.18]. The functor $\mathbf{SSet}^+/\mathcal{C}^{\sharp} \rightarrow \mathbf{SSet}/\mathcal{C}$ preserves filtered colimits, and filtered colimits are homotopy colimits in both model structures, from which it follows that the right derived functor preserves filtered colimits.

Example 2.7. We define a subcategory $\mathbf{LAdj} \subset \mathbf{Cat}_{\infty}^{\rightarrow}$ to be the pullback

$$\begin{array}{ccc} \mathbf{LAdj} & \xrightarrow{\quad\quad\quad} & \mathbf{CartFib}_{\Delta^1} \\ \downarrow & & \downarrow \\ \mathbf{Cat}_{\infty}^{\rightarrow} & \xrightarrow[\simeq]{} \mathbf{coCartFib}_{\Delta^1} \longrightarrow & \mathbf{Cat}_{\infty}/\Delta^1. \end{array}$$

By construction, $\mathbf{LAdj}$ is compactly generated, and the forgetful functor $\mathbf{LAdj} \rightarrow \mathbf{Cat}_{\infty}^{\rightarrow}$ is a conservative, ω -accessible right adjoint. Since a functor $F : \mathcal{C} \rightarrow \Delta^1$ that is both a cocartesian fibration and a cartesian fibration can be identified with an adjunction between the fibers over 0 and 1, the ∞ -category $\mathbf{LAdj}$ can be described as follows:

- the objects are the functors $F : \mathcal{C} \rightarrow \mathcal{D}$ that have a right adjoint F^* ;
- the morphisms $(F_1 : \mathcal{C}_1 \rightarrow \mathcal{D}_1) \rightarrow (F_2 : \mathcal{C}_2 \rightarrow \mathcal{D}_2)$ are the squares

$$\begin{array}{ccc} \mathcal{C}_1 & \xrightarrow{G} & \mathcal{C}_2 \\ F_1 \downarrow & & \downarrow F_2 \\ \mathcal{D}_1 & \xrightarrow{H} & \mathcal{D}_2 \end{array}$$

satisfying the Beck-Chevalley condition: the canonical natural transformation

$$\begin{array}{ccc} \mathcal{C}_1 & \xrightarrow{G} & \mathcal{C}_2 \\ F_1^* \uparrow & \searrow & \uparrow F_2^* \\ \mathcal{D}_1 & \xrightarrow{H} & \mathcal{D}_2 \end{array}$$

is invertible.

We use Example 2.7 to verify that an ∞ -category whose objects are small ∞ -categories with a certain structure defined by adjunction is compactly generated.

Example 2.8. For a finite simplicial set A , we define $\mathbf{Lex}_\infty^{(A)}$ to be the pullback

$$\begin{array}{ccc} \mathbf{Lex}_\infty^{(A)} & \longrightarrow & \mathbf{LAdj} \\ \downarrow & & \downarrow \\ \mathbf{Cat}_\infty & \xrightarrow{\mathcal{C} \mapsto (\delta: \mathcal{C} \rightarrow A \pitchfork \mathcal{C})} & \mathbf{Cat}_\infty^{\rightarrow}. \end{array}$$

$\mathbf{Lex}_\infty^{(A)}$ is the ∞ -category of small ∞ -categories with limits of shape A . We define the ∞ -category $\mathbf{Lex}_\infty$ of small left exact ∞ -categories to be the wide pullback of $\mathbf{Lex}_\infty^{(A)}$ over $\mathbf{Cat}_\infty$ for all finite simplicial sets A . By construction, $\mathbf{Lex}_\infty^{(A)}$ and $\mathbf{Lex}_\infty$ are compactly generated, and the forgetful functors to $\mathbf{Cat}_\infty$ are conservative, ω -accessible right adjoints.

We remark that codomain functors are always cartesian fibrations in $\mathfrak{Pr}_\omega^R$.

Proposition 2.9. *For a compactly generated ∞ -category $\mathcal{C}$, the functor $\mathrm{cod} : \mathcal{C}^{\rightarrow} \rightarrow \mathcal{C}$ is a cartesian fibration in $\mathfrak{Pr}_\omega^R$.*

Proof. Recall that finite limits commute with filtered colimits in any compactly generated ∞ -category $\mathcal{C}$. This implies that $\mathcal{C}$ is finitely complete in the ∞ -cosmos $\mathfrak{Pr}_\omega^R$ (that is, the diagonal functor $\mathcal{C} \rightarrow A \pitchfork \mathcal{C}$ has a right adjoint for every finite simplicial set A). Hence, the codomain functor $\mathcal{C}^{\rightarrow} \rightarrow \mathcal{C}$ is a cartesian fibration. $\square$

2.4 Exponentiable arrows

Definition 2.10. An arrow $u : x \rightarrow y$ in a left exact ∞ -category $\mathcal{C}$ is said to be *exponentiable* if the pullback functor $u^* : \mathcal{C}/y \rightarrow \mathcal{C}/x$ has a right adjoint. If this is the case, we refer to the right adjoint of u^* as the *pushforward along u* and denote it by $u_* : \mathcal{C}/x \rightarrow \mathcal{C}/y$.

Definition 2.11. For an exponentiable arrow $u : x \rightarrow y$ in a left exact ∞ -category $\mathcal{C}$, the *associated polynomial functor* $\mathbf{P}_u : \mathcal{C} \rightarrow \mathcal{C}$ is the composite

$$\mathcal{C} \xrightarrow{x^*} \mathcal{C}/x \xrightarrow{u_*} \mathcal{C}/y \xrightarrow{y_!} \mathcal{C}$$

where x^* is the pullback along $x \rightarrow 1$ and $y_!$ is the forgetful functor.

Recall that polynomials can be *composed* [17, 50]: given two exponentiable arrows $u_1 : x_1 \rightarrow y_1$ and $u_2 : x_2 \rightarrow y_2$, we have an exponentiable arrow $u_1 \otimes u_2$ such that $\mathbf{P}_{u_1 \otimes u_2} \simeq \mathbf{P}_{u_1} \circ \mathbf{P}_{u_2}$. We may also concretely define $u_1 \otimes u_2$ as follows: $\mathrm{cod}(u_1 \otimes u_2) = \mathbf{P}_{u_1} y_2$; $\mathrm{dom}(u_1 \otimes u_2)$ is the pullback

$$\begin{array}{ccc} \mathrm{dom}(u_1 \otimes u_2) & \longrightarrow & x_2 \\ \downarrow & & \downarrow u_2 \\ \mathbf{P}_{u_1} y_2 \times_{y_1} x_1 & \xrightarrow{\mathrm{ev}} & y_2; \end{array}$$

$u_1 \otimes u_2$ is the composite $\mathrm{dom}(u_1 \otimes u_2) \rightarrow \mathbf{P}_{u_1} y_2 \times_{y_1} x_1 \rightarrow \mathbf{P}_{u_1} y_2 = \mathrm{cod}(u_1 \otimes u_2)$.

2.5 Representable maps of right fibrations We review the notion of a *representable map* of right fibrations, which is a generalization of a representable map of discrete fibrations over a 1-category. We think of a representable map of right fibrations as an ∞ -categorical analogue of a natural model of type theory [5] and a category with families [15].

Definition 2.12. We say a map $f : A \rightarrow B$ of right fibrations over an ∞ -category $\mathcal{C}$ is *representable* if it has a right adjoint.

Proposition 2.13. *Let $\pi : A \rightarrow \mathcal{C}$ be a right fibration between ∞ -categories. A functor $f : B \rightarrow A$ is a right fibration if and only if the composite $\pi f : B \rightarrow \mathcal{C}$ is. Consequently, we have a canonical equivalence of ∞ -categories*

$$\mathbf{RFib}_{\mathcal{C}}/A \simeq \mathbf{RFib}_A.$$

Remark 2.14. This proposition would be false if we chose Joyal’s definition of right fibration [23]. The right fibrations in our sense (Definition 2.3) are those functors that are equivalent to right fibrations in Joyal’s sense.

Proof. By definition, a functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is a right fibration if and only if the square

$$\begin{array}{ccc} \Delta^1 \pitchfork \mathcal{C} & \xrightarrow{\Delta^1 \pitchfork F} & \Delta^1 \pitchfork \mathcal{D} \\ \text{ev}_1 \downarrow & & \downarrow \text{ev}_1 \\ \mathcal{C} & \xrightarrow{F} & \mathcal{D} \end{array}$$

is a pullback. Then the claim follows from the cancellation property of pullback squares. $\square$

Corollary 2.15. *A representable map $f : A \rightarrow B$ of right fibrations over an ∞ -category $\mathcal{C}$ is exponentiable, and the pushforward along f is given by the pullback along the right adjoint $\delta : B \rightarrow A$ of f .*

$$\begin{array}{ccc} & f^* & \\ & \curvearrowright & \\ \mathbf{RFib}_{\mathcal{C}}/A \simeq \mathbf{RFib}_A & \perp & \mathbf{RFib}_B \simeq \mathbf{RFib}_{\mathcal{C}}/B \\ & \curvearrowleft & \\ & \delta^* & \end{array}$$

Corollary 2.16. *Representable maps of right fibrations over an ∞ -category $\mathcal{C}$ are stable under pullbacks: if*

$$\begin{array}{ccc} A_1 & \xrightarrow{g} & A_2 \\ f_1 \downarrow & & \downarrow f_2 \\ B_1 & \xrightarrow{h} & B_2 \end{array}$$

is a pullback in $\mathbf{RFib}_{\mathcal{C}}$ and f_2 is representable, then f_1 is representable. Moreover, if this is the case, the square satisfies the Beck-Chevalley condition.

Proof. By Proposition 2.13, the functor h is a right fibration. Thus, the right adjoint of f_2 lifts

to a fibred right adjoint of f_1 .

$$\begin{array}{ccc}
 A_1 & \xrightarrow{f_1} & B_1 \\
 \downarrow g & \lrcorner & \downarrow h \\
 A_2 & \xrightarrow{f_2} & B_2
 \end{array}$$

(The diagram shows two squares. The top square has vertices A_1 (top-left), B_1 (top-right), A_2 (bottom-left), and B_2 (bottom-right). Arrows are $f_1: A_1 \rightarrow B_1$ (top), $g: A_1 \rightarrow A_2$ (left), $h: B_1 \rightarrow B_2$ (right), and $f_2: A_2 \rightarrow B_2$ (bottom). A curved arrow labeled $\lrcorner$ goes from A_1 to B_1 . The bottom square has vertices A_2 (top-left), B_2 (top-right), A_1 (bottom-left), and B_1 (bottom-right). Arrows are $f_2: A_2 \rightarrow B_2$ (top), $g: A_2 \rightarrow A_1$ (left), $h: B_2 \rightarrow B_1$ (right), and $f_1: A_1 \rightarrow B_1$ (bottom). A curved arrow labeled $\lrcorner$ goes from A_2 to B_2 .)

□

Proposition 2.17. *A map $f : A \rightarrow B$ of right fibrations over an ∞ -category $\mathcal{C}$ is representable if and only if, for any object $y \in \mathcal{C}$ and any map $b : \mathcal{C}/y \rightarrow B$, the pullback b^*A is a representable right fibration over $\mathcal{C}$.*

Proof. For a map $b : \mathcal{C}/y \rightarrow B$, an arrow $u : fa \rightarrow b$ in B for some $a \in A$ corresponds to a square

$$\begin{array}{ccc}
 \mathcal{C}/x & \xrightarrow{a} & A \\
 u \downarrow & & \downarrow f \\
 \mathcal{C}/y & \xrightarrow{b} & B.
 \end{array} \tag{1}$$

(a, u) is a universal arrow from f to b if and only if Eq. (1) is a pullback. □

3. ∞ -type theories

We introduce notions of an ∞ -type theory, a theory over an ∞ -type theory and a model of an ∞ -type theory, translating the previous work of the second author [47] into the language of ∞ -categories. The idea is to extend the functorial semantics of algebraic theories [31]. Algebraic theories are identified with categories with finite products, and models of an algebraic theory are identified with functors into the category of sets preserving finite products. For type theories, it is natural to identify models of a type theory with functors into presheaf categories, because (extensions of) natural models [5] and categories with families [15] are diagrams in presheaf categories. Since representable maps of presheaves play a special role in the natural model semantics, some arrows in the source category should be specified to be sent to representable maps. This motivates the following definitions.

Definition 3.1. An ∞ -category with representable maps is a pair $(\mathcal{C}, R)$ where $\mathcal{C}$ is an ∞ -category and $R \subseteq k(\Delta^1, \mathcal{C})$ is a subspace of the space of arrows of $\mathcal{C}$ satisfying the conditions below. Arrows in R are called *representable arrows*.

1. $\mathcal{C}$ has finite limits.
2. All the identities are representable and representable arrows are closed under composition.
3. Representable arrows are stable under pullbacks.
4. Representable arrows are exponentiable.

A morphism of ∞ -categories with representable maps is a functor preserving representable arrows, finite limits and pushforwards along representable arrows. We denote by $\mathbf{REP}_\infty$ the ∞ -category of large ∞ -categories with representable maps and their morphisms.

Example 3.2. For a small ∞ -category $\mathcal{C}$, the ∞ -category $\mathbf{RFib}_{\mathcal{C}}$ of small right fibrations over $\mathcal{C}$ is an ∞ -category with representable maps in which a map is representable if it has a right adjoint.

Definition 3.3. An ∞ -type theory is an ∞ -category with representable maps whose underlying ∞ -category is small. A *morphism of ∞ -type theories* is a morphism of ∞ -categories with representable maps. By an n -type theory for $1 \leq n < \infty$, we mean an ∞ -type theory whose underlying ∞ -category is an n -category.

Example 3.4. The type theories in the sense of the previous work [47] are the 1-type theories.

Definition 3.5. Let $\mathbb{T}$ be an ∞ -type theory.

- A *model of $\mathbb{T}$* consists of an ∞ -category $\mathcal{M}(\star)$ with a terminal object and a morphism of ∞ -categories with representable maps $\mathcal{M} : \mathbb{T} \rightarrow \mathbf{RFib}_{\mathcal{M}(\star)}$.
- A *theory over $\mathbb{T}$* or a $\mathbb{T}$ -theory is a left exact functor $K : \mathbb{T} \rightarrow \mathbf{S}$.

Example 3.6. We will construct in Section 3.1 a presentable ∞ -category $\mathbf{TT}_{\infty}$ of ∞ -type theories and their morphisms, so we have various *free constructions* of ∞ -type theories. For example, there is an ∞ -type theory $\mathbb{G}_{\infty}$ freely generated by one representable arrow $\partial : E \rightarrow U$ justified in Example 3.11. The universal property of $\mathbb{G}_{\infty}$ asserts that a morphism $\mathbb{G}_{\infty} \rightarrow \mathcal{C}$ of ∞ -categories with representable maps is completely determined by the image of the representable arrow $\partial \in \mathbb{G}_{\infty}$. Thus, a model of $\mathbb{G}_{\infty}$ consists of the following data:

- an ∞ -category $\mathcal{M}(\star)$ with a terminal object;
- a representable map $\mathcal{M}(\partial) : \mathcal{M}(E) \rightarrow \mathcal{M}(U)$ of right fibrations over $\mathcal{M}(\star)$.

In other words, a model of $\mathbb{G}_{\infty}$ is an ∞ -categorical analogue of a natural model [5, 16]. One may think of an object $\Gamma \in \mathcal{M}(\star)$ as a context, a map $A : \mathcal{M}(\star)/\Gamma \rightarrow \mathcal{M}(U)$ as a type over Γ , and a map $a : \mathcal{M}(\star)/\Gamma \rightarrow \mathcal{M}(E)$ as a term over Γ . The representability of $\mathcal{M}(\partial)$ is used for modeling context comprehension: for a map $A : \mathcal{M}(\star)/\Gamma \rightarrow \mathcal{M}(U)$, the representing object for $A^*\mathcal{M}(E)$ is thought of as the context $(\Gamma, x : A)$ with x a fresh variable.

It is not simple to describe a $\mathbb{G}_{\infty}$ -theory, but we could say that the ∞ -category of $\mathbb{G}_{\infty}$ -theories is an ∞ -analogue of the category of generalized algebraic theories [11]. Indeed, the second named author showed in [46] that the category of generalized algebraic theories is equivalent to the category of left exact functors $\mathbb{G} \rightarrow \mathbf{Set}$ where $\mathbb{G}$ is the left exact category freely generated by an exponentiable arrow.

In Sections 3.1 to 3.3 below, we will construct an ∞ -category $\mathbf{TT}_{\infty}$ of ∞ -type theories, an ∞ -category $\mathbf{Th}(\mathbb{T})$ of $\mathbb{T}$ -theories and an ∞ -category $\mathbf{Mod}(\mathbb{T})$ of models of $\mathbb{T}$. These ∞ -categories are constructed inside the ∞ -cosmos $\mathfrak{Pr}_{\omega}^R$ of compactly generated ∞ -categories and ω -accessible right adjoints. In Section 3.4 we give a universal property of $\mathbf{Mod}(\mathbb{T})$ as an object of $\mathbf{CAT}_{\infty}/\mathbf{Lex}_{\infty}^{(\emptyset)}$ from which for example it follows that the assignment $\mathbb{T} \mapsto \mathbf{Mod}(\mathbb{T})$ takes colimits to limits. In Section 3.5 we see that a slice of the underlying ∞ -category of an ∞ -type theory is naturally equipped with a structure of ∞ -type theory and has a useful universal property.

3.1 The ∞ -category of ∞ -type theories We construct an ∞ -category $\mathbf{TT}_{\infty}$ of ∞ -type theories and their morphisms.

Definition 3.7. Let $\mathbf{Cat}_\infty^+$ be the ∞ -category of *marked ∞ -categories* defined by the pullback

$$\begin{array}{ccc} \mathbf{Cat}_\infty^+ & \longrightarrow & (\mathbf{S}^\rightarrow)_{\leq -1} \\ \downarrow & & \downarrow \text{cod} \\ \mathbf{Cat}_\infty & \xrightarrow{k(\Delta^1, -)} & \mathbf{S} \end{array}$$

where $(\mathbf{S}^\rightarrow)_{\leq -1}$ denotes the full subcategory of $\mathbf{S}^\rightarrow$ spanned by the (-1) -truncated maps of spaces which is an ω -accessible localization of $\mathbf{S}^\rightarrow$. $\mathbf{Cat}_\infty^+$ is the ∞ -category of small ∞ -categories equipped with a subspace of arrows called *marked arrows*. We define $\mathbf{Lex}_\infty^+$ to be the full subcategory of $\mathbf{Lex}_\infty \times_{\mathbf{Cat}_\infty} \mathbf{Cat}_\infty^+$ spanned by the left exact ∞ -categories with a class of arrows closed under composition and stable under pullbacks.

The inclusion $\mathbf{Lex}_\infty^+ \rightarrow \mathbf{Lex}_\infty \times_{\mathbf{Cat}_\infty} \mathbf{Cat}_\infty^+$ has a left adjoint by taking the closure of the marked arrows under composition and pullbacks, and $\mathbf{Lex}_\infty^+$ is closed in $\mathbf{Lex}_\infty \times_{\mathbf{Cat}_\infty} \mathbf{Cat}_\infty^+$ under filtered colimits. Hence, $\mathbf{Lex}_\infty^+$ is compactly generated, and the inclusion $\mathbf{Lex}_\infty^+ \rightarrow \mathbf{Lex}_\infty \times_{\mathbf{Cat}_\infty} \mathbf{Cat}_\infty^+$ is an ω -accessible right adjoint.

Let $(\mathcal{C}, R)$ be an object of $\mathbf{Lex}_\infty^+$. Since $\mathcal{C}$ has finite limits, we have a functor $\theta(\mathcal{C}, R)$ between isofibrations over R whose fiber over $(u : x \rightarrow y) \in R$ is the pullback functor $u^* : \mathcal{C}/y \rightarrow \mathcal{C}/x$. An ∞ -type theory is nothing but an object $(\mathcal{C}, R)$ of $\mathbf{Lex}_\infty^+$ such that $\theta(\mathcal{C}, R)$ has a fiberwise right adjoint. We show that this condition is equivalent to the condition that the functor has a right adjoint.

Proposition 3.8. *Let*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ & \searrow & \swarrow \\ & A & \end{array}$$

be a functor between isofibrations in $\mathfrak{CAT}_\infty$ such that A is an ∞ -groupoid. The following are equivalent:

1. *the functor $F : \mathcal{C} \rightarrow \mathcal{D}$ has a right adjoint;*
2. *for every point $a \in A$, the functor between fibers $F_a : \mathcal{C}_a \rightarrow \mathcal{D}_a$ has a right adjoint.*

Proof. Suppose that each $F_a : \mathcal{C}_a \rightarrow \mathcal{D}_a$ has a right adjoint G_a with counit $\varepsilon_{a,y} : F_a(G_a(y)) \rightarrow y$. It suffices to see that $\varepsilon_{a,y}$ is universal in $\mathcal{D}$. Let $x \in \mathcal{C}_{a'}$ be an object in another fiber and consider the induced map

$$\mathcal{C}(x, G_a(y)) \rightarrow \mathcal{D}(F_{a'}(x), y).$$

This is a map over $A(a', a)$, and thus it suffices to show that this is fiberwise an equivalence. Since A is an ∞ -groupoid and since $\mathcal{C} \rightarrow A$ and $\mathcal{D} \rightarrow A$ are isofibrations, the fibers over $p \in A(a', a)$ are equivalent to the fibers over $\text{id} \in A(a, a)$, but the map between the fibers over id is the equivalence $\mathcal{C}_a(x, G_a(y)) \simeq \mathcal{D}_a(G_a(x), y)$.

Suppose that F has a right adjoint $G : \mathcal{D} \rightarrow \mathcal{C}$ with counit $\varepsilon : FG \Rightarrow \text{id}$. Since A is an ∞ -groupoid, the natural transformation

$$\begin{array}{ccccc} \mathcal{D} & \xrightarrow{G} & \mathcal{C} & & \\ & \searrow \varepsilon & \downarrow F & \searrow & \\ & & \mathcal{D} & \xrightarrow{\quad} & A \end{array}$$

is invertible. Then, since $\mathcal{C} \rightarrow A$ and $\mathcal{D} \rightarrow A$ are isofibrations, one can replace G and ε by a functor $G' : \mathcal{D} \rightarrow \mathcal{C}$ and a natural transformation $\varepsilon' : FG' \Rightarrow \text{id}$, respectively, over A . Then G' and ε' give a fiberwise right adjoint of F . $\square$

Remark 3.9. The proposition also holds more generally when A is an ∞ -category. See [35, Proposition 7.3.2.1].

The functor $\theta(\mathcal{C}, R)$ is constructed as follows. Since $\mathcal{C}$ has finite limits, the functor $(\Delta^1 \times \Delta^1) \pitchfork \mathcal{C} \rightarrow \Lambda_2^2 \pitchfork \mathcal{C}$ sending a square to its bottom and right edges has a right adjoint. Composing the right adjoint and the functor $(\Delta^1 \times \Delta^1) \pitchfork \mathcal{C} \rightarrow \Lambda_1^2 \pitchfork \mathcal{C}$ sending a square to its bottom and left edges, we have a functor

$$\theta' : \Lambda_2^2 \pitchfork \mathcal{C} \rightarrow \Lambda_1^2 \pitchfork \mathcal{C}$$

over $\Delta^{\{1,2\}} \pitchfork \mathcal{C}$. The functor $\theta(\mathcal{C}, R)$ is then the pullback of θ' along the inclusion $R \rightarrow \Delta^{\{1,2\}} \pitchfork \mathcal{C}$. This construction is functorial and preserves limits and filtered colimits, yielding a functor $\theta : \mathbf{Lex}_\infty^+ \rightarrow \Delta^2 \pitchfork \mathbf{Cat}_\infty$ in $\mathfrak{Pr}_\omega^R$.

Definition 3.10. We define $\mathbf{TT}_\infty$ to be the pullback

$$\begin{array}{ccc} \mathbf{TT}_\infty & \xrightarrow{\quad} & \mathbf{LAdj} \\ \downarrow & & \downarrow \\ \mathbf{Lex}_\infty^+ & \xrightarrow[\theta]{} \Delta^2 \pitchfork \mathbf{Cat}_\infty & \longrightarrow \Delta^{\{0,1\}} \pitchfork \mathbf{Cat}_\infty. \end{array}$$

By Proposition 3.8, the objects of $\mathbf{TT}_\infty$ are precisely the ∞ -type theories. It is also straightforward to see that the morphisms of $\mathbf{TT}_\infty$ are precisely the morphisms of ∞ -type theories.

We can now justify free constructions of ∞ -type theories.

Example 3.11. Given a marked ∞ -category $\mathcal{D} \in \mathbf{Cat}_\infty^+$, let $\langle \mathcal{D} \rangle$ denote the image of $\mathcal{D}$ by the left adjoint of the forgetful functor $\mathbf{TT}_\infty \rightarrow \mathbf{Cat}_\infty^+$. For an ∞ -type theory $\mathbb{T}$, a morphism $\langle \mathcal{D} \rangle \rightarrow \mathbb{T}$ corresponds to a functor $\mathcal{D} \rightarrow \mathbb{T}$ sending marked arrows to representable arrows. Every ∞ -category $\mathcal{C}$ is regarded as a marked ∞ -category in two ways: $\mathcal{C}^\#$ where all of the arrows are marked; $\mathcal{C}^b$ where none of the arrows are marked. We write $\langle \alpha \vdash \beta \rangle$ for $\langle \{\alpha \leftarrow \beta\}^b \rangle$, $\langle \alpha \vdash^\# \beta \rangle$ for $\langle \{\alpha \leftarrow \beta\}^\# \rangle$, and $\langle \alpha \vdash \xi : \beta \rangle$ for $\langle S^b \rangle$ where S is the ∞ -category $\{ \begin{array}{c} \beta \\ \nearrow \xi \\ \alpha \end{array} \}$. The ∞ -type theory $\mathbb{G}_\infty$ in Example 3.6 is the same as $\langle \alpha \vdash^\# \beta \rangle$.

Example 3.12. Let $\mathbb{T}$ be an ∞ -type theory and $u : y \rightarrow x$ an arrow in $\mathbb{T}$ which we regard as a morphism $u : \langle \alpha \vdash \beta \rangle \rightarrow \mathbb{T}$ of ∞ -type theories. Consider the ∞ -type theory $\mathbb{T}'$ defined by the following pushout.

$$\begin{array}{ccc} \langle \alpha \vdash \beta \rangle & \xrightarrow{u} & \mathbb{T} \\ \downarrow & & \downarrow \\ \langle \alpha \vdash \xi : \beta \rangle & \longrightarrow & \mathbb{T}' \end{array}$$

The universal property of $\mathbb{T}'$ is that a morphism $\mathbb{T}' \rightarrow \mathbb{U}$ of ∞ -type theories corresponds to a pair (F, v) consisting of a morphism $F : \mathbb{T} \rightarrow \mathbb{U}$ and a section v of $F(u) : F(y) \rightarrow F(x)$. Thus, $\mathbb{T}'$ is the ∞ -type theory obtained from $\mathbb{T}$ by *freely adjoining a section of u* . This operation subsumes the following operations.

Adjoining an arrow Let $x, y \in \mathbb{T}$ be objects. Then freely adjoining a section of the projection $x \times y \rightarrow x$ is equivalent to freely adjoining an arrow $x \rightarrow y$.

Adjoining a homotopy Let $x \in \mathbb{T}$ and $y, y' \in \mathbb{T}/x$ be objects and let $u, v : y \rightarrow y'$ be morphisms in $\mathbb{T}/x$. Let $z \rightarrow y$ be the equalizer of u and v in $\mathbb{T}/x$. Then, freely adjoining a section of $z \rightarrow y$ is equivalent to freely adjoining a homotopy between u and v over x .

Inverting an arrow Let $u : x \rightarrow y$ be an arrow in $\mathbb{T}$. By adjoining arrows and homotopies, we can freely adjoin a left inverse and a right inverse of u . The resulting ∞ -type theory $\mathbb{T}'$ satisfies that a morphism $\mathbb{T}' \rightarrow \mathbb{U}$ of ∞ -type theories corresponds to a morphism $F : \mathbb{T} \rightarrow \mathbb{U}$ such that $F(u)$ is invertible.

3.2 The ∞ -category of theories over an ∞ -type theory

Definition 3.13. For an ∞ -type theory $\mathbb{T}$, we define $\mathbf{Th}(\mathbb{T})$ to be the full subcategory of $\mathbf{Fun}(\mathbb{T}, \mathbf{S})$ spanned by the functors preserving finite limits.

By definition, $\mathbf{Th}(\mathbb{T})$ is compactly generated, and the inclusion $\mathbf{Th}(\mathbb{T}) \rightarrow \mathbf{Fun}(\mathbb{T}, \mathbf{S})$ is an ω -accessible right adjoint. The ∞ -category $\mathbf{Th}(\mathbb{T})$ has the following alternative definitions:

- $\mathbf{Th}(\mathbb{T})$ is the cocompletion of $\mathbb{T}^{\text{op}}$ under filtered colimits;
- $\mathbf{Th}(\mathbb{T})$ is the ω -free cocompletion of $\mathbb{T}^{\text{op}}$, that is, the initial cocomplete ∞ -category equipped with a functor from $\mathbb{T}^{\text{op}}$ preserving finite colimits.

3.3 The ∞ -category of models of an ∞ -type theory We construct an ∞ -category $\mathbf{Mod}(\mathbb{T})$ of models of an ∞ -type theory $\mathbb{T}$. The following description of $\mathbf{Mod}(\mathbb{T})$ is based on unpublished work by John Bourke and the second named author on the 2-category of 1-models of a 1-type theory.

Let $\mathbb{T}$ be an ∞ -type theory. Recall that a functor to a slice ∞ -category $F' : \mathcal{C} \rightarrow \mathcal{D}/y$ corresponds to a functor $F : \mathcal{C}^{\triangleright} \rightarrow \mathcal{D}$ that sends $\star \in \mathcal{C}^{\triangleright}$ to y . Then a model $\mathcal{M}$ of $\mathbb{T}$ can be regarded as a functor $\mathcal{M} : \mathbb{T}^{\triangleright} \rightarrow \mathbf{Cat}_{\infty}$ satisfying the following conditions:

1. $\mathcal{M}(\star)$ has a terminal object;
2. for every object $x \in \mathbb{T}$, the functor $\mathcal{M}(x) \rightarrow \mathcal{M}(\star)$ is a right fibration;
3. for every finite diagram $x : A \rightarrow \mathbb{T}$, the canonical functor $\mathcal{M}(\lim_A x) \rightarrow \lim_{A^{\triangleright}} \mathcal{M}x^{\triangleright}$ is an equivalence;
4. for every representable arrow $u : x \rightarrow y$ in $\mathbb{T}$, the functor $\mathcal{M}(u) : \mathcal{M}(x) \rightarrow \mathcal{M}(y)$ has a right adjoint $\delta_u : \mathcal{M}(y) \rightarrow \mathcal{M}(x)$;
5. for every pair of arrows $u : x \rightarrow y$ and $v : y \rightarrow z$ with v representable, the canonical functor $\mathcal{M}(v_*x) \rightarrow \delta_v^* \mathcal{M}(x)$ is an equivalence (recall that the pushforward along $\mathcal{M}(v)$ in $\mathbf{RFib}_{\mathcal{M}(\star)}$ is given by the pullback along δ_v).

From this description, we will define $\mathbf{Mod}(\mathbb{T})$ as a subcategory of $\mathbf{Fun}(\mathbb{T}^{\triangleright}, \mathbf{Cat}_{\infty})$.

Definition 3.14. We define $\mathbf{Mod}_1(\mathbb{T})$ to be the pullback

$$\begin{array}{ccc} \mathbf{Mod}_1(\mathbb{T}) & \longrightarrow & \mathbf{Lex}_{\infty}^{(\emptyset)} \\ \downarrow & & \downarrow \\ \mathbf{Fun}(\mathbb{T}^{\triangleright}, \mathbf{Cat}_{\infty}) & \xrightarrow{\text{ev}_{\star}} & \mathbf{Cat}_{\infty}. \end{array}$$

Definition 3.15. For an object $x \in \mathbb{T}$, we define $\mathbf{Mod}_2^x(\mathbb{T})$ to be the pullback

$$\begin{array}{ccc} \mathbf{Mod}_2^x(\mathbb{T}) & \longrightarrow & \mathbf{RFib} \\ \downarrow & & \downarrow \\ \mathbf{Fun}(\mathbb{T}^\triangleright, \mathbf{Cat}_\infty) & \xrightarrow{\text{ev}_{(x \rightarrow *)}} & \mathbf{Cat}_\infty^\rightarrow \end{array}$$

and $\mathbf{Mod}_2(\mathbb{T})$ to be the wide pullback of $\mathbf{Mod}_2^x(\mathbb{T})$ over $\mathbf{Fun}(\mathbb{T}^\triangleright, \mathbf{Cat}_\infty)$ for all objects $x \in \mathbb{T}$.

Definition 3.16. For a finite diagram $x : A \rightarrow \mathbb{T}$, we define $\mathbf{Mod}_3^{(A,x)}(\mathbb{T})$ to be the pullback

$$\begin{array}{ccc} \mathbf{Mod}_3^{(A,x)}(\mathbb{T}) & \longrightarrow & \mathbf{Cat}_\infty^\simeq \\ \downarrow & & \downarrow \\ \mathbf{Fun}(\mathbb{T}^\triangleright, \mathbf{Cat}_\infty) & \xrightarrow{(\text{ev}_{\lim_A x} \Rightarrow \lim_{A^\triangleright} \text{ev}_{x^\triangleright})} & \mathbf{Cat}_\infty^\rightarrow \end{array}$$

and $\mathbf{Mod}_3(\mathbb{T})$ to be the wide pullback of $\mathbf{Mod}_3^{(A,x)}(\mathbb{T})$ over $\mathbf{Fun}(\mathbb{T}^\triangleright, \mathbf{Cat}_\infty)$ for all finite diagrams $(A, x : A \rightarrow \mathbb{T})$.

Definition 3.17. For a representable arrow $u : x \rightarrow y$ in $\mathbb{T}$, we define $\mathbf{Mod}_4^u(\mathbb{T})$ to be the pullback

$$\begin{array}{ccc} \mathbf{Mod}_4^u(\mathbb{T}) & \longrightarrow & \mathbf{LAdj} \\ \downarrow & & \downarrow \\ \mathbf{Fun}(\mathbb{T}^\triangleright, \mathbf{Cat}_\infty) & \xrightarrow{\text{ev}_u} & \mathbf{Cat}_\infty^\rightarrow \end{array}$$

and $\mathbf{Mod}_4(\mathbb{T})$ to be the wide pullback of $\mathbf{Mod}_4^u(\mathbb{T})$ over $\mathbf{Fun}(\mathbb{T}^\triangleright, \mathbf{Cat}_\infty)$ for all representable arrows u in $\mathbb{T}$.

Definition 3.18. We denote by $\mathbf{Mod}_{-5}(\mathbb{T})$ the wide pullback of $\mathbf{Mod}_1(\mathbb{T})$, $\mathbf{Mod}_2(\mathbb{T})$, $\mathbf{Mod}_3(\mathbb{T})$ and $\mathbf{Mod}_4(\mathbb{T})$ over $\mathbf{Fun}(\mathbb{T}^\triangleright, \mathbf{Cat}_\infty)$. By construction, $\mathbf{Mod}_{-5}(\mathbb{T})$ is the ∞ -category of functors $\mathcal{M} : \mathbb{T}^\triangleright \rightarrow \mathbf{Cat}_\infty$ satisfying Items 1 to 4.

Definition 3.19. For a pair of composable arrows $u : x \rightarrow y$ and $v : y \rightarrow z$ in $\mathbb{T}$ with v representable, we define $\mathbf{Mod}_5^{(u,v)}(\mathbb{T})$ to be the pullback

$$\begin{array}{ccc} \mathbf{Mod}_5^{(u,v)}(\mathbb{T}) & \longrightarrow & \mathbf{Cat}_\infty^\simeq \\ \downarrow & & \downarrow \\ \mathbf{Mod}_{-5}(\mathbb{T}) & \xrightarrow{(\text{ev}_{v*x} \Rightarrow \delta_v^* \text{ev}_x)} & \mathbf{Cat}_\infty^\rightarrow \end{array}$$

and $\mathbf{Mod}(\mathbb{T})$ to be the wide pullback of $\mathbf{Mod}_5^{(u,v)}(\mathbb{T})$ over $\mathbf{Mod}_{-5}(\mathbb{T})$ for all pairs (u, v) of composable arrows in $\mathbb{T}$ with v representable.

By construction, the ∞ -category $\mathbf{Mod}(\mathbb{T})$ is compactly generated, and the forgetful functor $\mathbf{Mod}(\mathbb{T}) \rightarrow \mathbf{Fun}(\mathbb{T}^\triangleright, \mathbf{Cat}_\infty)$ is a conservative, ω -accessible right adjoint. Moreover, the objects of $\mathbf{Mod}(\mathbb{T})$ are the models of $\mathbb{T}$ and the morphisms in $\mathbf{Mod}(\mathbb{T})$ are described as follows. Let $\mathcal{M}$ and $\mathcal{N}$ be models of $\mathbb{T}$ and $F : \mathcal{M} \Rightarrow \mathcal{N} : \mathbb{T}^\triangleright \rightarrow \mathbf{Cat}_\infty$ be a natural transformation. Then F is in $\mathbf{Mod}(\mathbb{T})$ if and only if the following conditions hold:

- the component $F(\star) : \mathcal{M}(\star) \rightarrow \mathcal{N}(\star)$ preserves terminal objects;

- for any representable arrow $u : x \rightarrow y$ in $\mathbb{T}$, the square

$$\begin{array}{ccc} \mathcal{M}(x) & \xrightarrow{F(x)} & \mathcal{N}(x) \\ \mathcal{M}(u) \downarrow & & \downarrow \mathcal{N}(u) \\ \mathcal{M}(y) & \xrightarrow{F(y)} & \mathcal{N}(y) \end{array}$$

satisfies the Beck-Chevalley condition.

3.4 Universal property of $\mathbf{Mod}(\mathbb{T})$ We give a universal property of $\mathbf{Mod}(\mathbb{T})$ seen as an object of $\mathbf{CAT}_\infty / \mathbf{Lex}_\infty^{(\emptyset)}$. A consequence is that the assignment $\mathbb{T} \mapsto \mathbf{Mod}(\mathbb{T})$ takes colimits of ∞ -type theories to limits of ∞ -categories over $\mathbf{Lex}_\infty^{(\emptyset)}$ (Corollary 3.25).

Definition 3.20. For a functor $\mathcal{C} : \mathcal{I} \rightarrow \mathbf{Cat}_\infty$, we denote by $\mathbf{Fun}(\mathcal{I}, \mathbf{RFib})_{\mathcal{C}}$ the full subcategory of $\mathbf{Fun}(\mathcal{I}, \mathbf{Cat}_\infty) / \mathcal{C}$ spanned by the natural transformations $\pi : A \Rightarrow \mathcal{C} : \mathcal{I} \rightarrow \mathbf{Cat}_\infty$ whose components are right fibrations. In other words, $\mathbf{Fun}(\mathcal{I}, \mathbf{RFib})_{\mathcal{C}}$ is the fiber of the functor $\mathbf{Fun}(\mathcal{I}, \mathrm{cod}) : \mathbf{Fun}(\mathcal{I}, \mathbf{RFib}) \rightarrow \mathbf{Fun}(\mathcal{I}, \mathbf{Cat}_\infty)$ over the object $\mathcal{C} \in \mathbf{Fun}(\mathcal{I}, \mathbf{Cat}_\infty)$. We say a map $f : A \rightarrow B$ in $\mathbf{Fun}(\mathcal{I}, \mathbf{RFib})_{\mathcal{C}}$ is *representable* if every component $f(i) : A(i) \rightarrow B(i)$ is a representable map of right fibrations over $\mathcal{C}(i)$ and if every naturality square

$$\begin{array}{ccc} A(i) & \xrightarrow{A(\alpha)} & A(j) \\ f(i) \downarrow & & \downarrow f(j) \\ B(i) & \xrightarrow{B(\alpha)} & B(j) \end{array}$$

satisfies the Beck-Chevalley condition.

Proposition 3.21. *Representable maps in $\mathbf{Fun}(\mathcal{I}, \mathbf{RFib})_{\mathcal{C}}$ are closed under composition and stable under pullbacks.*

Proof. Let

$$\begin{array}{ccc} A_1 & \xrightarrow{g} & A_2 \\ f_1 \downarrow & & \downarrow f_2 \\ B_1 & \xrightarrow{h} & B_2 \end{array}$$

be a pullback in $\mathbf{Fun}(\mathcal{I}, \mathbf{RFib})_{\mathcal{C}}$ and suppose that f_2 is representable. By Corollary 2.16, every $f_1(i) : A_1(i) \rightarrow B_1(i)$ is a representable map of right fibrations over $\mathcal{C}(i)$, and the square

$$\begin{array}{ccc} A_1(i) & \xrightarrow{g(i)} & A_2(i) \\ f_1(i) \downarrow & & \downarrow f_2(i) \\ B_1(i) & \xrightarrow{h(i)} & B_2(i) \end{array}$$

satisfies the Beck-Chevalley condition. It remains to show that, for any arrow $\alpha : i \rightarrow j$ in $\mathcal{I}$, the square

$$\begin{array}{ccc} A_1(i) & \xrightarrow{A_1(\alpha)} & A_1(j) \\ f_1(i) \downarrow & & \downarrow f_1(j) \\ B_1(i) & \xrightarrow{B_1(\alpha)} & B_1(j) \end{array}$$

satisfies the Beck-Chevalley condition. Since a map of right fibrations over a fixed base is conservative, it suffices to show that the composite of squares

$$\begin{array}{ccccc} A_1(i) & \xrightarrow{A_1(\alpha)} & A_1(j) & \xrightarrow{g(j)} & A_2(j) \\ f_1(i) \downarrow & & \downarrow f_1(j) & & \downarrow f_2(j) \\ B_1(i) & \xrightarrow{B_1(\alpha)} & B_1(j) & \xrightarrow{h(j)} & B_2(j) \end{array}$$

satisfies the Beck-Chevalley condition, but this is true by the Beck-Chevalley condition for f_2 . $\square$

Proposition 3.22. *A representable map in $\mathbf{Fun}(\mathcal{I}, \mathbf{RFib})_{\mathcal{C}}$ is exponentiable, and the pushforward is given by the pullback along the right adjoint.*

Proof. The same as Corollary 2.15. $\square$

Proposition 3.23. *For a functor $\mathcal{C} : \mathcal{I} \rightarrow \mathbf{Cat}_{\infty}$, the ∞ -category $\mathbf{Fun}(\mathcal{I}, \mathbf{RFib})_{\mathcal{C}}$ together with the class of representable maps is an ∞ -category with representable maps. For a functor $F : \mathcal{I}' \rightarrow \mathcal{I}$, the functor $F^* : \mathbf{Fun}(\mathcal{I}, \mathbf{RFib})_{\mathcal{C}} \rightarrow \mathbf{Fun}(\mathcal{I}', \mathbf{RFib})_{\mathcal{C}F}$ defined by the precomposition of F is a morphism of ∞ -categories with representable maps.*

Proof. By definition. $\square$

Let $\mathbb{T}$ be an ∞ -type theory and $\mathcal{C} : \mathcal{I} \rightarrow \mathbf{Lex}_{\infty}^{(\emptyset)}$ a functor. We have equivalences

$$\begin{aligned} & \mathbf{CAT}_{\infty} / \mathbf{Lex}_{\infty}^{(\emptyset)}((\mathcal{I}, \mathcal{C}), (\mathbf{Fun}(\mathbb{T}^{\flat}, \mathbf{Cat}_{\infty}), \mathrm{ev}_{\star})) \\ \simeq & \quad \{\text{transposition}\} \\ & \{\star\} / \mathbf{CAT}_{\infty}((\mathbb{T}^{\flat}, \{\star\}), (\mathbf{Fun}(\mathcal{I}, \mathbf{Cat}_{\infty}), \mathcal{C})) \\ \simeq & \quad \{\text{adjunction of join and slice}\} \\ & \mathbf{CAT}_{\infty}(\mathbb{T}, \mathbf{Fun}(\mathcal{I}, \mathbf{Cat}_{\infty}) / \mathcal{C}). \end{aligned}$$

Proposition 3.24. *Let $\mathbb{T}$ be an ∞ -type theory and $\mathcal{C} : \mathcal{I} \rightarrow \mathbf{Lex}_{\infty}^{(\emptyset)}$ a functor. A functor $F : \mathcal{I} \rightarrow \mathbf{Fun}(\mathbb{T}^{\flat}, \mathbf{Cat}_{\infty})$ over $\mathbf{Lex}_{\infty}^{(\emptyset)}$ factors through $\mathbf{Mod}(\mathbb{T})$ if and only if its transpose $\tilde{F} : \mathbb{T} \rightarrow \mathbf{Fun}(\mathcal{I}, \mathbf{Cat}_{\infty}) / \mathcal{C}$ factors through $\mathbf{Fun}(\mathcal{I}, \mathbf{RFib})_{\mathcal{C}}$ and is a morphism of ∞ -categories with representable maps. Consequently, we have an equivalence*

$$\mathbf{CAT}_{\infty} / \mathbf{Lex}_{\infty}^{(\emptyset)}((\mathcal{I}, \mathcal{C}), (\mathbf{Mod}(\mathbb{T}), \mathrm{ev}_{\star})) \simeq \mathbf{REP}_{\infty}(\mathbb{T}, \mathbf{Fun}(\mathcal{I}, \mathbf{RFib})_{\mathcal{C}}).$$

Proof. Immediate from the definition of models of $\mathbb{T}$. $\square$

Corollary 3.25. $\mathbf{Mod} : \mathbf{TT}_{\infty}^{\mathrm{op}} \rightarrow \mathbf{CAT}_{\infty} / \mathbf{Lex}_{\infty}^{(\emptyset)}$ preserves limits.

3.5 Slice ∞ -type theories

Definition 3.26. For an ∞ -category with representable maps $\mathcal{C}$ and an object $x \in \mathcal{C}$, we regard the slice $\mathcal{C}/x$ as an ∞ -category with representable maps in which an arrow is representable if it is representable in $\mathcal{C}$.

The goal of this subsection is to show that $\mathcal{C}/x$ is the ∞ -category with representable maps obtained from $\mathcal{C}$ by freely adjoining a global section of x (Proposition 3.28). We first recall a universal property of a slice of a left exact ∞ -category.

Proposition 3.27. *Let $\mathcal{C}$ be a left exact ∞ -category and $x \in \mathcal{C}$ an object. We denote by $x^* : \mathcal{C} \rightarrow \mathcal{C}/x$ the pullback functor along $x \rightarrow 1$ and $\delta_x : 1 \rightarrow x^*x$ the arrow in $\mathcal{C}/x$ which is the diagonal arrow $x \rightarrow x \times x$ in $\mathcal{C}$. For a left exact ∞ -category $\mathcal{D}$ and a left exact functor $F : \mathcal{C} \rightarrow \mathcal{D}$, the map*

$$\mathcal{C}/\mathbf{LEX}_\infty(\mathcal{C}/x, \mathcal{D}) \ni G \mapsto G(\delta_x) \in \mathcal{D}(1, Fx) \quad (2)$$

is an equivalence of spaces.

Proof. An object of $\mathcal{C}/\mathbf{LEX}_\infty(\mathcal{C}/x, \mathcal{D})$ is a left exact functor $G : \mathcal{C}/x \rightarrow \mathcal{D}$ equipped with an invertible natural transformation $\sigma : G \circ x^* \Rightarrow F$. By the adjunction $x_! \dashv x^*$, such a natural transformation σ corresponds to a natural transformation $\tilde{\sigma} : G \Rightarrow F \circ x_!$. One can check that $\sigma : G \circ x^* \Rightarrow F$ is invertible if and only if $\tilde{\sigma} : G \Rightarrow F \circ x_!$ is a cartesian natural transformation, that is, any naturality square is a pullback. Therefore, the statement is equivalent to that, given a global section $u : 1 \rightarrow Fx$, the space of pairs (G, σ) consisting of a left exact functor $G : \mathcal{C}/x \rightarrow \mathcal{D}$ and a cartesian natural transformation $\sigma : G \Rightarrow F \circ x_!$ extending u is contractible.

Since $\mathcal{D}$ has finite limits, the evaluation at the terminal object of $\mathcal{C}/x$ defines a cartesian fibration $\mathbf{Fun}(\mathcal{C}/x, \mathcal{D}) \rightarrow \mathcal{D}$ in which a natural transformation $\sigma : G_1 \Rightarrow G_2$ is cartesian if and only if the naturality square

$$\begin{array}{ccc} G_1 y & \xrightarrow{\sigma} & G_2 y \\ \downarrow & & \downarrow \\ G_1 1 & \xrightarrow{\sigma} & G_2 1 \end{array}$$

is a pullback for any object $y \in \mathcal{C}/x$, which is equivalent to that σ is a cartesian natural transformation. Therefore, given a functor $G_2 : \mathcal{C}/x \rightarrow \mathcal{D}$ and an arrow $u : x' \rightarrow G_2 1$, the space of pairs (G_1, σ) consisting of a functor $G_1 : \mathcal{C}/x \rightarrow \mathcal{D}$ and a cartesian natural transformation σ extending u is contractible. When $G_2 = F \circ x_!$, the functor G_1 must preserve pullbacks since G_2 does. If, in addition, $x' \simeq 1$, then G_1 preserves terminal objects, and thus it is left exact. We conclude that, given an arrow $u : 1 \rightarrow Fx \simeq F(x_! 1)$, the space of pairs (G, σ) consisting of a left exact functor $G : \mathcal{C}/x \rightarrow \mathcal{D}$ and a cartesian natural transformation $\sigma : G \Rightarrow F \circ x_!$ extending u is contractible, as we have a unique cartesian lift $G \Rightarrow F \circ x_!$ and G must be left exact. $\square$

From this proof, we can extract the inverse of the map (2): it is given by

$$\mathcal{D}(1, Fx) \ni u \mapsto u^* \circ F/x \in \mathcal{C}/\mathbf{LEX}_\infty(\mathcal{C}/x, \mathcal{D})$$

where $F/x : \mathcal{C}/x \rightarrow \mathcal{D}/Fx$ is the functor induced by F .

Proposition 3.28. *Let $\mathcal{C}$ be an ∞ -category with representable maps and $x \in \mathcal{C}$ an object.*

1. *The functor $x^* : \mathcal{C} \rightarrow \mathcal{C}/x$ is a morphism of ∞ -categories with representable maps.*
2. *For an ∞ -category with representable maps $\mathcal{D}$ and a morphism $F : \mathcal{C} \rightarrow \mathcal{D}$, the map*

$$\mathcal{C}/\mathbf{REP}_\infty(\mathcal{C}/x, \mathcal{D}) \ni G \mapsto G(\delta_x) \in \mathcal{D}(1, Fx)$$

is an equivalence of spaces.

Proof. Item 1 is because pullbacks preserve all limits and all pushforwards. Item 2 follows from Proposition 3.27, because $u^* \circ F/x : \mathcal{C}/x \rightarrow \mathcal{D}$ is a morphism of ∞ -categories with representable maps for every global section $u : 1 \rightarrow Fx$. $\square$

Proposition 3.28 can be reformulated as follows. Let $\langle \square \rangle$ be the free ∞ -type theory generated by one object $\square$ and $\langle \tilde{\square} : 1 \rightarrow \square \rangle$ the free ∞ -type theory generated by one object $\square$ and one global section $\tilde{\square}$ of $\square$. By definition, a morphism $\langle \square \rangle \rightarrow \mathcal{C}$ corresponds to an object of $\mathcal{C}$, and a morphism $\langle \tilde{\square} : 1 \rightarrow \square \rangle \rightarrow \mathcal{C}$ corresponds to a pair (x, u) consisting of an object x of $\mathcal{C}$ and a global section $u : 1 \rightarrow x$. Then, for an ∞ -category with representable maps $\mathcal{C}$ and an object $x \in \mathcal{C}$, we can form a square

$$\begin{array}{ccc} \langle \square \rangle & \xrightarrow{x} & \mathcal{C} \\ \downarrow & & \downarrow x^* \\ \langle \tilde{\square} : 1 \rightarrow \square \rangle & \xrightarrow{\delta_x} & \mathcal{C}/x. \end{array} \quad (3)$$

Proposition 3.28 is equivalent to that, for any ∞ -category with representable maps, the diagram

$$\begin{array}{ccc} \mathbf{REP}_{\infty}(\mathcal{C}/x, \mathcal{D}) & \longrightarrow & \mathbf{REP}_{\infty}(\langle \tilde{\square} : 1 \rightarrow \square \rangle, \mathcal{D}) \\ \downarrow & & \downarrow \\ \mathbf{REP}_{\infty}(\mathcal{C}, \mathcal{D}) & \longrightarrow & \mathbf{REP}_{\infty}(\langle \square \rangle, \mathcal{D}) \end{array}$$

induced by Eq. (3) is a pullback of spaces. In other words:

Proposition 3.29. *For an ∞ -category with representable maps $\mathcal{C}$ and an object $x \in \mathcal{C}$, Eq. (3) is a pushout in $\mathbf{REP}_{\infty}$.*

Using Proposition 3.24 and its corollary, we have the following description of $\mathbf{Mod}(\mathbb{T}/x)$ for an ∞ -type theory $\mathbb{T}$ and an object $x \in \mathbb{T}$. We first observe:

1. $\mathbf{Mod}(\langle \square \rangle) \simeq \mathbf{RFib}'$ where $\mathbf{RFib}'$ is the base change of $\mathbf{RFib}$ along the forgetful functor $\mathbf{Lex}_{\infty}^{(\emptyset)} \rightarrow \mathbf{Cat}_{\infty}$;
2. $\mathbf{Mod}(\langle \tilde{\square} : 1 \rightarrow \square \rangle) \simeq \mathbf{RFib}'_{\bullet}$ where $\mathbf{RFib}'_{\bullet}$ is the ∞ -category of right fibrations $A \rightarrow \mathcal{C}$ with a terminal object in $\mathcal{C}$ and a global section $a : \mathcal{C} \rightarrow A$.

These follow from Proposition 3.24, for example

$$\begin{aligned} & \mathbf{CAT}_{\infty}/\mathbf{Lex}_{\infty}^{(\emptyset)}((\mathcal{I}, \mathcal{C}), (\mathbf{Mod}(\langle \square \rangle), \mathrm{ev}_{\star})) \\ & \simeq \mathbf{REP}_{\infty}(\langle \square \rangle, \mathbf{Fun}(\mathcal{I}, \mathbf{RFib})_{\mathcal{C}}) \\ & \simeq (\mathbf{Fun}(\mathcal{I}, \mathbf{RFib})_{\mathcal{C}})^{\simeq} \\ & \simeq \mathbf{CAT}_{\infty}/\mathbf{Lex}_{\infty}^{(\emptyset)}((\mathcal{I}, \mathcal{C}), (\mathbf{RFib}', \mathrm{cod})) \end{aligned}$$

for any $\mathcal{C} : \mathcal{I} \rightarrow \mathbf{Lex}_{\infty}^{(\emptyset)}$. Then, by Corollary 3.25 and Proposition 3.29, we have:

Proposition 3.30. *For any ∞ -type theory $\mathbb{T}$ and any object $x \in \mathbb{T}$, we have a pullback*

$$\begin{array}{ccc} \mathbf{Mod}(\mathbb{T}/x) & \longrightarrow & \mathbf{RFib}'_{\bullet} \\ \downarrow & & \downarrow \\ \mathbf{Mod}(\mathbb{T}) & \xrightarrow{x^*} \mathbf{Mod}(\langle \square \rangle) \xrightarrow{\simeq} & \mathbf{RFib}'. \end{array}$$

4. The theory-model correspondence

Given an ∞ -type theory $\mathbb{T}$, we establish an adjunction between the ∞ -category of $\mathbb{T}$ -theories and the ∞ -category of models of $\mathbb{T}$. The right adjoint assigns an *internal language* to each model

of $\mathbb{T}$, and the left adjoint assigns a *syntactic model* to each $\mathbb{T}$ -theory. Not all models of $\mathbb{T}$ are syntactic ones. We give a characterization of syntactic models.

All the results in this section are ∞ -categorical analogues of results from the previous work of the second author [47], but proofs are simplified and improved.

- In the previous work $(2, 1)$ -categorical (co)limits are distinguished from 1-categorical (co)limits, but there is no such difference in the ∞ -categorical setting.
- In the previous work the left adjoint of the internal language functor is made by hand, but in this work we construct the internal language functor inside the ∞ -cosmos $\mathfrak{Pr}_\omega^R$, so it has a left adjoint by definition. Therefore, all we have to do is to analyze the unit and counit of the adjunction.

Let $\mathbb{T}$ be an ∞ -type theory. Since the base ∞ -category $\mathcal{M}(\star)$ of a model $\mathcal{M}$ of $\mathbb{T}$ has a terminal object $1 : \Delta^0 \rightarrow \mathcal{M}(\star)$, we have a natural transformation

$$\begin{array}{ccc} \mathbf{Mod}(\mathbb{T}) & \longrightarrow & \mathbf{Fun}(\mathbb{T}^\triangleright, \mathbf{Cat}_\infty) \\ \downarrow & \nearrow 1 & \downarrow \text{ev}_\star \\ \Delta^0 & \xrightarrow{\Delta^0} & \mathbf{Cat}_\infty. \end{array}$$

Since $\mathbf{Fun}(\mathbb{T}^\triangleright, \mathbf{Cat}_\infty)$ is the pullback

$$\begin{array}{ccc} \mathbf{Fun}(\mathbb{T}^\triangleright, \mathbf{Cat}_\infty) & \longrightarrow & \mathbf{Fun}(\mathbb{T}, \mathbf{Cat}_\infty) \rightarrow \\ \text{ev}_\star \downarrow & & \downarrow \text{cod} \\ \mathbf{Cat}_\infty & \xrightarrow{\delta} & \mathbf{Fun}(\mathbb{T}, \mathbf{Cat}_\infty), \end{array}$$

the functor $\text{ev}_\star : \mathbf{Fun}(\mathbb{T}^\triangleright, \mathbf{Cat}_\infty) \rightarrow \mathbf{Cat}_\infty$ is a cartesian fibration in $\mathfrak{Pr}_\omega^R$. Thus, the natural transformation 1 induces an ω -accessible right adjoint $1^* : \mathbf{Mod}(\mathbb{T}) \rightarrow (\Delta^0)^* \mathbf{Fun}(\mathbb{T}^\triangleright, \mathbf{Cat}_\infty) \simeq \mathbf{Fun}(\mathbb{T}, \mathbf{Cat}_\infty)$. By the definition of a model of $\mathbb{T}$, the functor $1^* : \mathbf{Mod}(\mathbb{T}) \rightarrow \mathbf{Fun}(\mathbb{T}, \mathbf{Cat}_\infty)$ factors through $\mathbf{Th}(\mathbb{T}) = \mathbf{Lex}(\mathbb{T}, \mathbf{S}) \subset \mathbf{Fun}(\mathbb{T}, \mathbf{Cat}_\infty)$. We denote this functor $\mathbf{Mod}(\mathbb{T}) \rightarrow \mathbf{Th}(\mathbb{T})$ by L . By definition, $L(\mathcal{M})$ is the composite

$$\mathbb{T} \xrightarrow{\mathcal{M}} \mathbf{RFib}_{\mathcal{M}(\star)} \xrightarrow{1^*} \mathbf{S}$$

where 1^*A is the fiber over $1 \in \mathcal{M}(\star)$ for a right fibration A over $\mathcal{M}(\star)$. As the functor $L : \mathbf{Mod}(\mathbb{T}) \rightarrow \mathbf{Th}(\mathbb{T})$ lies in $\mathfrak{Pr}_\omega^R$, it has a left adjoint $\mathcal{F} : \mathbf{Th}(\mathbb{T}) \rightarrow \mathbf{Mod}(\mathbb{T})$.

Definition 4.1. For a model $\mathcal{M}$ of $\mathbb{T}$, the $\mathbb{T}$ -theory $L(\mathcal{M})$ is called the *internal language* of $\mathcal{M}$. For a theory K over $\mathbb{T}$, we call $\mathcal{F}(K)$ the *syntactic model* generated by a $\mathbb{T}$ -theory K .

In this section, we prove the following:

1. the unit of the adjunction $\mathcal{F} \dashv L$ is invertible, so the functor $\mathcal{F} : \mathbf{Th}(\mathbb{T}) \rightarrow \mathbf{Mod}(\mathbb{T})$ is fully faithful;
2. the essential image of $\mathcal{F} : \mathbf{Th}(\mathbb{T}) \rightarrow \mathbf{Mod}(\mathbb{T})$ is the class of democratic models of $\mathbb{T}$ defined below.

Consequently, the adjunction $\mathcal{F} \dashv L$ induces an equivalence between $\mathbf{Th}(\mathbb{T})$ and the full subcategory of $\mathbf{Mod}(\mathbb{T})$ spanned by the democratic models of $\mathbb{T}$. We define the notion of a democratic model in Section 4.1. The components of the unit $\eta : \text{id} \Rightarrow L\mathcal{F}$ are completely determined by the components at the representable $\mathbb{T}$ -theories $\mathbb{T}(x, -)$, because $\mathbf{Th}(\mathbb{T})$ is the cocompletion of $\mathbb{T}^{\text{op}}$

under filtered colimits and the right adjoint $L : \mathbf{Mod}(\mathbb{T}) \rightarrow \mathbf{Th}(\mathbb{T})$ preserves filtered colimits. We thus study in details the syntactic model generated by a representable $\mathbb{T}$ -theory. In Section 4.2 we concretely describe the initial model of $\mathbb{T}$ which is the syntactic model generated by the initial $\mathbb{T}$ -theory $\mathbb{T}(1, -)$. We then generalize it in Section 4.3 to a description of the syntactic model generated by an arbitrary representable $\mathbb{T}$ -theory. Finally we prove the main results in Section 4.4.

4.1 Democratic models For a model $\mathcal{M}$ of an ∞ -type theory, we think of an object $\Gamma \in \mathcal{M}(\star)$ as a context (see Example 3.6), but contexts from the syntax of type theory satisfy an additional property: every context is obtained from the empty context by context comprehension. A model of an ∞ -type theory satisfying this property is said to be *democratic*, generalizing the notion of a democratic category with families [13].

Definition 4.2. Let $\mathcal{M}$ be a model of $\mathbb{T}$, $u : x \rightarrow y$ a representable arrow in $\mathbb{T}$, $\Gamma \in \mathcal{M}(\star)$ an object and $b : \mathcal{M}(\star)/\Gamma \rightarrow \mathcal{M}(y)$ a map. Let $\delta_u : \mathcal{M}(y) \rightarrow \mathcal{M}(x)$ be the right adjoint of $\mathcal{M}(u)$. Then the counit $p_u(b) : \mathcal{M}(u)(\delta_u(b)) \rightarrow b$ is a pullback square

$$\begin{array}{ccc} \mathcal{M}(\star)/\{b\}_u & \xrightarrow{\delta_u(b)} & \mathcal{M}(x) \\ p_u(b) \downarrow & & \downarrow \mathcal{M}(u) \\ \mathcal{M}(\star)/\Gamma & \xrightarrow{b} & \mathcal{M}(y). \end{array}$$

We refer to the object $\{b\}_u$ the *context comprehension of b with respect to u* .

Definition 4.3. Let $\mathcal{M}$ be a model of $\mathbb{T}$. The class of *contextual objects* of $\mathcal{M}$ is the smallest replete class of objects of $\mathcal{M}(\star)$ containing the terminal object and closed under context comprehension.

In other words, the contextual objects of $\mathcal{M}$ are inductively defined as follows:

- the terminal object $1 \in \mathcal{M}(\star)$ is contextual;
- if $\Gamma \in \mathcal{M}(\star)$ is a contextual object, $u : x \rightarrow y$ is a representable arrow in $\mathbb{T}$ and $b : \mathcal{M}(\star)/\Gamma \rightarrow \mathcal{M}(y)$ is a map, then the context comprehension $\{b\}_u$ is contextual;
- if $\Gamma \in \mathcal{M}(\star)$ is a contextual object and $\Gamma \simeq \Delta$, then Δ is contextual.

Definition 4.4. We call a model $\mathcal{M}$ *democratic* if all the objects of $\mathcal{M}(\star)$ are contextual. We denote by $\mathbf{Mod}^{\text{dem}}(\mathbb{T})$ the full subcategory of $\mathbf{Mod}(\mathbb{T})$ spanned by the democratic models.

One can always find a largest democratic model contained in an arbitrary model of $\mathbb{T}$.

Definition 4.5. For a model $\mathcal{M}$ of $\mathbb{T}$, we define a model $\mathcal{M}^\heartsuit$ of $\mathbb{T}$ called the *heart* of $\mathcal{M}$ as follows:

- the base ∞ -category $\mathcal{M}^\heartsuit(\star)$ is the full subcategory of $\mathcal{M}(\star)$ spanned by the contextual objects;
- the functor $\mathcal{M}^\heartsuit : \mathbb{T} \rightarrow \mathbf{RFib}_{\mathcal{M}^\heartsuit(\star)}$ is the composite with the pullback along the inclusion $\mathcal{M}^\heartsuit(\star) \rightarrow \mathcal{M}(\star)$

$$\mathbb{T} \xrightarrow{\mathcal{M}} \mathbf{RFib}_{\mathcal{M}(\star)} \longrightarrow \mathbf{RFib}_{\mathcal{M}^\heartsuit(\star)}.$$

$\mathcal{M}^\heartsuit$ is indeed a model of $\mathbb{T}$, and the inclusion $\mathcal{M}^\heartsuit \hookrightarrow \mathcal{M}$ is a morphism of models of $\mathbb{T}$. By definition, the functor $\mathcal{M}^\heartsuit : \mathbb{T} \rightarrow \mathbf{RFib}_{\mathcal{M}^\heartsuit(\star)}$ preserves finite limits. Since $\mathcal{M}^\heartsuit(\star)$ is

closed under context comprehension, for a representable arrow $u : x \rightarrow y$ in $\mathbb{T}$, the composite $\mathcal{M}^\heartsuit(y) \hookrightarrow \mathcal{M}(y) \xrightarrow{\delta_u} \mathcal{M}(x)$ factors through $\mathcal{M}^\heartsuit(x) \hookrightarrow \mathcal{M}(x)$

$$\begin{array}{ccc} \mathcal{M}^\heartsuit(y) & \cdots \cdots \rightarrow & \mathcal{M}^\heartsuit(x) \\ \downarrow & & \downarrow \\ \mathcal{M}(y) & \xrightarrow{\delta_u} & \mathcal{M}(x). \end{array}$$

This means that $\mathcal{M}^\heartsuit(u) : \mathcal{M}^\heartsuit(x) \rightarrow \mathcal{M}^\heartsuit(y)$ has a right adjoint, and the square

$$\begin{array}{ccc} \mathcal{M}^\heartsuit(x) & \hookrightarrow & \mathcal{M}(x) \\ \mathcal{M}^\heartsuit(u) \downarrow & & \downarrow \mathcal{M}(u) \\ \mathcal{M}^\heartsuit(y) & \hookrightarrow & \mathcal{M}(y) \end{array}$$

satisfies the Beck-Chevalley condition. Since the pushforward along $\mathcal{M}^\heartsuit(u)$ is given by the pullback along its right adjoint δ_u , we see that $\mathcal{M}^\heartsuit : \mathbb{T} \rightarrow \mathbf{RFib}_{\mathcal{M}^\heartsuit(\star)}$ preserves pushforwards along representable maps.

Proposition 4.6. *For a democratic model $\mathcal{M}$ of $\mathbb{T}$ and an arbitrary model $\mathcal{N}$ of $\mathbb{T}$, the inclusion $\mathcal{N}^\heartsuit \hookrightarrow \mathcal{N}$ induces an equivalence of spaces*

$$\mathbf{Mod}^{\text{dem}}(\mathbb{T})(\mathcal{M}, \mathcal{N}^\heartsuit) \simeq \mathbf{Mod}(\mathbb{T})(\mathcal{M}, \mathcal{N}).$$

In other words, $(-)^{\heartsuit}$ is a right adjoint of the inclusion $\mathbf{Mod}^{\text{dem}}(\mathbb{T}) \hookrightarrow \mathbf{Mod}(\mathbb{T})$.

Proof. Because any morphism of models of $\mathbb{T}$ preserves contextual objects, any morphism $\mathcal{M} \rightarrow \mathcal{N}$ from a democratic model $\mathcal{M}$ factors through $\mathcal{N}^\heartsuit$. $\square$

4.2 The initial model

Definition 4.7. Recall that the Yoneda embedding $\mathbf{y}_{\mathbb{T}} : \mathbb{T} \rightarrow \mathbf{RFib}_{\mathbb{T}}$ preserves all existing limits and pushforwards. Therefore, the pair $(\mathbb{T}, \mathbf{y}_{\mathbb{T}})$ is regarded as a model of $\mathbb{T}$. We define the *initial model* $\mathcal{I}(\mathbb{T})$ to be the heart of the model $(\mathbb{T}, \mathbf{y}_{\mathbb{T}})$.

The goal of this subsection is to show that $\mathcal{I}(\mathbb{T})$ is indeed an initial object of $\mathbf{Mod}(\mathbb{T})$.

By definition, the model $\mathcal{I}(\mathbb{T})$ is described as follows:

- the base ∞ -category is $\mathbb{T}_r$, the full subcategory of $\mathbb{T}$ spanned by the objects x such that the arrow $x \rightarrow 1$ is representable;
- $\mathcal{I}(\mathbb{T})(y) = \mathbb{T}_r/y$ defined by the pullback

$$\begin{array}{ccc} \mathbb{T}_r/y & \hookrightarrow & \mathbb{T}/y \\ \downarrow & & \downarrow \\ \mathbb{T}_r & \hookrightarrow & \mathbb{T} \end{array}$$

for $y \in \mathbb{T}$.

Alternatively, the functor $\mathcal{I}(\mathbb{T}) : \mathbb{T} \rightarrow \mathbf{RFib}_{\mathbb{T}_r}$ is defined as the left Kan extension of the Yoneda embedding $\mathbf{y}_{\mathbb{T}_r} : \mathbb{T}_r \rightarrow \mathbf{RFib}_{\mathbb{T}_r}$ along the inclusion $\mathbb{T}_r \hookrightarrow \mathbb{T}$.

$$\begin{array}{ccc} \mathbb{T}_r & \xrightarrow{\mathbf{y}_{\mathbb{T}_r}} & \mathbf{RFib}_{\mathbb{T}_r} \\ \downarrow & \searrow \simeq & \uparrow \\ \mathbb{T} & \xrightarrow{\quad} & \mathcal{I}(\mathbb{T}) \end{array}$$

Theorem 4.8. *For an ∞ -type theory $\mathbb{T}$, the model $\mathcal{I}(\mathbb{T})$ is an initial object in $\mathbf{Mod}(\mathbb{T})$.*

Proof. We first note that, since $\mathbf{Mod}(\mathbb{T})$ has finite limits, it suffices to show that $\mathcal{I}(\mathbb{T})$ is an initial object in the homotopy category of $\mathbf{Mod}(\mathbb{T})$ [38, Proposition 2.2.2], that is, for any model $\mathcal{M}$ of $\mathbb{T}$, there exists a morphism $\mathcal{I}(\mathbb{T}) \rightarrow \mathcal{M}$ and any two morphisms $\mathcal{I}(\mathbb{T}) \rightarrow \mathcal{M}$ are equivalent.

Let $\mathcal{M}$ be a model of $\mathbb{T}$. Suppose that we have a morphism $G : \mathcal{I}(\mathbb{T}) \rightarrow \mathcal{M}$. It is regarded as a pair $(G(\star), G)$ consisting of a functor $G(\star) : \mathbb{T}_r \rightarrow \mathcal{M}(\star)$ and a natural transformation $G : \mathcal{I}(\mathbb{T}) \Rightarrow G(\star)^* \mathcal{M} : \mathbb{T} \rightarrow \mathbf{RFib}_{\mathbb{T}_r}$.

The Beck-Chevalley condition for a representable arrow $v : y \rightarrow z$ in $\mathbb{T}$ means that, for any object $(u : x \rightarrow z) \in \mathbb{T}_r/z$, the square

$$\begin{array}{ccc} \mathcal{M}(\star)/G(\star)(u^*y) & \xrightarrow{G(u^*y)(v^*u)} & \mathcal{M}(y) \\ u^*v \downarrow & & \downarrow \mathcal{M}(v) \\ \mathcal{M}(\star)/G(\star)(x) & \xrightarrow{G(\star)(u)} & \mathcal{M}(z) \end{array}$$

is a pullback. From the special case when z is the terminal object, we see that the canonical map $G(y)(\mathrm{id}_y) : \mathcal{M}(\star)/G(\star)(y) \rightarrow \mathcal{M}(y)$ is an equivalence for every object $y \in \mathbb{T}_r$. In other words, the diagram

$$\begin{array}{ccc} \mathbb{T}_r & \xrightarrow{G(\star)} & \mathcal{M}(\star) \\ \downarrow & & \downarrow \mathbf{y}_{\mathcal{M}(\star)} \\ \mathbb{T} & \xrightarrow{\mathcal{M}} & \mathbf{RFib}_{\mathcal{M}(\star)} \end{array}$$

commutes up to equivalence, and we have an equivalence

$$\begin{array}{ccc} \mathbb{T}_r \xrightarrow{G(\star)} \mathcal{M}(\star) & & \mathbb{T}_r \xrightarrow{G(\star)} \mathcal{M}(\star) \\ \downarrow & & \downarrow \mathbf{y}_{\mathcal{M}(\star)} \\ \mathbb{T} \xrightarrow{\mathcal{M}} \mathbf{RFib}_{\mathcal{M}(\star)} & \simeq & \mathbf{RFib}_{\mathcal{M}(\star)} \\ \downarrow G & & \downarrow G(\star)^* \\ \mathcal{I}(\mathbb{T}) \rightarrow \mathbf{RFib}_{\mathbb{T}_r} & & \mathbf{RFib}_{\mathbb{T}_r} \end{array}$$

where $G(\star)_1$ is the natural transformation $G(\star)_{x,y} : \mathbb{T}_r(x, y) \rightarrow \mathcal{M}(\star)(G(\star)(x), G(\star)(y))$. Hence, $G(\star) : \mathbb{T}_r \rightarrow \mathcal{M}(\star)$ is uniquely determined, and then $G : \mathcal{I}(\mathbb{T}) \Rightarrow G(\star)^* \mathcal{M} : \mathbb{T} \rightarrow \mathbf{RFib}_{\mathbb{T}_r}$ is uniquely determined because $\mathcal{I}(\mathbb{T}) : \mathbb{T} \rightarrow \mathbf{RFib}_{\mathbb{T}_r}$ is the left Kan extension of the Yoneda embedding $\mathbb{T}_r : \mathbb{T}_r \rightarrow \mathbf{RFib}_{\mathbb{T}_r}$ along the inclusion $\mathbb{T}_r \hookrightarrow \mathbb{T}$. This shows that morphisms $\mathcal{I}(\mathbb{T}) \rightarrow \mathcal{M}$ are unique up to equivalence.

It suffices now to construct a morphism $F : \mathcal{I}(\mathbb{T}) \rightarrow \mathcal{M}$ of models of $\mathbb{T}$. We first construct a functor $F(\star) : \mathbb{T}_r \rightarrow \mathcal{M}(\star)$. For an object $x \in \mathbb{T}_r$, the map $\mathcal{M}(x) \rightarrow \mathcal{M}(1) \simeq \mathcal{M}(\star)$ of right fibrations over $\mathcal{M}(\star)$ is representable. Thus, since $\mathcal{M}(\star)$ has a terminal object, the right fibration $\mathcal{M}(x)$ is representable. Hence, the restriction of $\mathcal{M} : \mathbb{T} \rightarrow \mathbf{RFib}_{\mathcal{M}(\star)}$ along the inclusion $\mathbb{T}_r \rightarrow \mathbb{T}$ factors as a functor $F(\star) : \mathbb{T}_r \rightarrow \mathcal{M}(\star)$ followed by the Yoneda embedding.

$$\begin{array}{ccc} \mathbb{T}_r & \xrightarrow{F(\star)} & \mathcal{M}(\star) \\ \downarrow & & \downarrow \mathbf{y}_{\mathcal{M}(\star)} \\ \mathbb{T} & \xrightarrow{\mathcal{M}} & \mathbf{RFib}_{\mathcal{M}(\star)} \end{array}$$

We then define a natural transformation $F : \mathcal{I}(\mathbb{T}) \Rightarrow F(\star)^* \mathcal{M} : \mathbb{T} \rightarrow \mathbf{RFib}_{\mathbb{T}_r}$ to be the one whose restriction to $\mathbb{T}_r$ is the natural transformation $F(\star) : \mathbf{y}_{\mathbb{T}_r} \Rightarrow F(\star)^* \mathcal{M} F(\star) : \mathbb{T}_r \rightarrow \mathbf{RFib}_{\mathbb{T}_r}$.

$$\begin{array}{ccc}
 \mathbb{T}_r & \xrightarrow{F(\star)} & \mathcal{M}(\star) \\
 \downarrow & & \downarrow \mathbf{y}_{\mathcal{M}(\star)} \\
 \mathbb{T} & \xrightarrow{\mathcal{M}} & \mathbf{RFib}_{\mathcal{M}(\star)} \simeq \\
 \searrow \mathcal{I}(\mathbb{T}) & \nearrow F & \downarrow F(\star)^* \\
 & & \mathbf{RFib}_{\mathbb{T}_r}
 \end{array}
 \quad
 \begin{array}{ccc}
 \mathbb{T}_r & \xrightarrow{F(\star)} & \mathcal{M}(\star) \\
 \searrow \mathbf{y}_{\mathbb{T}_r} & \nearrow F(\star)_1 & \downarrow \mathbf{y}_{\mathcal{M}(\star)} \\
 & & \mathbf{RFib}_{\mathcal{M}(\star)} \\
 & & \downarrow F(\star)^* \\
 & & \mathbf{RFib}_{\mathbb{T}_r}
 \end{array}$$

In order to show that F is a morphism of models of $\mathbb{T}$, it remains to prove that $F(\star) : \mathbb{T}_r \rightarrow \mathcal{M}(\star)$ preserves terminal objects and that F satisfies the Beck-Chevalley condition for representable arrows. The first claim is clear by definition. For the second, we have to show that, for any representable arrow $v : y \rightarrow z$ in $\mathbb{T}$, the square

$$\begin{array}{ccc}
 \mathbb{T}_r/y & \xrightarrow{F(y)} & \mathcal{M}(y) \\
 v \downarrow & & \downarrow \mathcal{M}(v) \\
 \mathbb{T}_r/z & \xrightarrow{F(z)} & \mathcal{M}(z)
 \end{array}$$

satisfies the Beck-Chevalley condition. It suffices to show that, for any arrow $u : x \rightarrow z$ with $x \in \mathbb{T}_r$, the composite of squares

$$\begin{array}{ccccc}
 \mathbb{T}_r/u^*y & \xrightarrow{v^*u} & \mathbb{T}_r/y & \xrightarrow{F(y)} & \mathcal{M}(y) \\
 u^*v \downarrow & & \downarrow v & & \downarrow \mathcal{M}(v) \\
 \mathbb{T}_r/x & \xrightarrow{u} & \mathbb{T}_r/z & \xrightarrow{F(z)} & \mathcal{M}(z)
 \end{array} \tag{4}$$

satisfies the Beck-Chevalley condition. By the definition of F , Eq. (4) is equivalent to

$$\begin{array}{ccccccc}
 \mathbb{T}_r/u^*y & \xrightarrow{F(\star)} & \mathcal{M}(\star)/F(\star)(u^*y) & \xrightarrow{\simeq} & \mathcal{M}(u^*y) & \xrightarrow{\mathcal{M}(v^*u)} & \mathcal{M}(y) \\
 u^*v \downarrow & & F(\star)(u^*v) \downarrow & & \downarrow \mathcal{M}(u^*v) & & \downarrow \mathcal{M}(v) \\
 \mathbb{T}_r/x & \xrightarrow{F(\star)} & \mathcal{M}(\star)/F(\star)(x) & \xrightarrow{\simeq} & \mathcal{M}(x) & \xrightarrow{\mathcal{M}(u)} & \mathcal{M}(z).
 \end{array} \tag{5}$$

The right square of Eq. (5) satisfies the Beck-Chevalley condition by Corollary 2.16. The middle square satisfies the Beck-Chevalley condition as the horizontal maps are equivalences. The Beck-Chevalley condition for the left square asserts that $F(\star)$ preserves pullbacks of representable arrows in $\mathbb{T}_r$, which is true by the definition of $F(\star)$. $\square$

4.3 Syntactic models generated by representable theories We describe the model $\mathcal{F}(\mathbf{y}(x))$ for $x \in \mathbb{T}$, where $\mathbf{y} : \mathbb{T}^{\text{op}} \rightarrow \mathbf{Th}(\mathbb{T}) \subset \mathbf{Fun}(\mathbb{T}, \mathbf{S})$ is the Yoneda embedding.

Proposition 4.9. *For an object x of $\mathbb{T}$, we have a pullback*

$$\begin{array}{ccc}
 \mathbf{Mod}(\mathbb{T}/x) & \longrightarrow & \mathbf{y}(x)/\mathbf{Th}(\mathbb{T}) \\
 \downarrow & & \downarrow \\
 \mathbf{Mod}(\mathbb{T}) & \xrightarrow{\quad \text{L} \quad} & \mathbf{Th}(\mathbb{T}).
 \end{array}$$

Proof. Recall (Proposition 3.30) that we have a pullback

$$\begin{array}{ccc} \mathbf{Mod}(\mathbb{T}/x) & \longrightarrow & \mathbf{RFib}'_{\bullet} \\ \downarrow & & \downarrow \\ \mathbf{Mod}(\mathbb{T}) & \xrightarrow{x^*} & \mathbf{RFib}' \end{array}$$

For an object $(A \rightarrow \mathcal{C}) \in \mathbf{RFib}'_{\bullet}$, the fiber of $\mathbf{RFib}'_{\bullet}$ over $(A \rightarrow \mathcal{C})$ is the space of global sections of A . Since the base ∞ -category $\mathcal{C}$ has a terminal object 1 , that space is equivalent to the fiber of A over 1 . In other words, we have a pullback

$$\begin{array}{ccc} \mathbf{RFib}'_{\bullet} & \longrightarrow & 1/\mathbf{S} \\ \downarrow & & \downarrow \\ \mathbf{RFib}' & \xrightarrow{1^*} & \mathbf{S} \end{array}$$

By the definition of L , the composite $1^* \circ x^*$ is equivalent to the composite

$$\mathbf{Mod}(\mathbb{T}) \xrightarrow{L} \mathbf{Th}(\mathbb{T}) \xrightarrow{\text{ev}_x} \mathbf{S}.$$

By Yoneda, we have a pullback

$$\begin{array}{ccc} \mathbf{y}(x)/\mathbf{Th}(\mathbb{T}) & \longrightarrow & 1/\mathbf{S} \\ \downarrow & & \downarrow \\ \mathbf{Th}(\mathbb{T}) & \xrightarrow{\text{ev}_x} & \mathbf{S}, \end{array}$$

and then we get a pullback as in the statement. $\square$

By Proposition 4.9, $\mathbf{Mod}(\mathbb{T}/x)$ is equivalent to the comma ∞ -category $(\mathbf{y}(x) \downarrow L)$. Since $\mathcal{F}(\mathbf{y}(x))$ is the image of the initial object of $(\mathbf{y}(x) \downarrow L)$ by the forgetful functor $(\mathbf{y}(x) \downarrow L) \rightarrow \mathbf{Mod}(\mathbb{T})$, it is obtained from the initial model $\mathcal{I}(\mathbb{T}/x)$ of $\mathbb{T}/x$ by restricting the morphism of ∞ -categories with representable maps $\mathcal{I}(\mathbb{T}/x) : \mathbb{T}/x \rightarrow \mathbf{RFib}_{\mathcal{I}(\mathbb{T}/x)(*)}$ along $x^* : \mathbb{T} \rightarrow \mathbb{T}/x$. We thus have a concrete description of $\mathcal{F}(\mathbf{y}(x))$ as follows:

- the base ∞ -category $\mathcal{F}(\mathbf{y}(x))(*)$ is the full subcategory of $\mathbb{T}/x$ spanned by the representable arrows over x ;
- for objects $y \in \mathbb{T}$ and $(u : x' \rightarrow x) \in \mathcal{F}(\mathbf{y}(x))(*)$, the fiber of $\mathcal{F}(\mathbf{y}(x))(y)$ over u is $\mathbb{T}/x(u, x^*y) \simeq \mathbb{T}(x', y)$.

4.4 The equivalence of theories and democratic models

Proposition 4.10. *The unit of the adjunction $\mathcal{F} \dashv L : \mathbf{Th}(\mathbb{T}) \rightarrow \mathbf{Mod}(\mathbb{T})$ is invertible. Consequently, the left adjoint $\mathcal{F} : \mathbf{Th}(\mathbb{T}) \rightarrow \mathbf{Mod}(\mathbb{T})$ is fully faithful.*

Proof. Since both functors $\mathcal{F}$ and L preserve filtered colimits, it suffices to show that the unit $\eta_K : K \rightarrow L(\mathcal{F}(K))$ is invertible for every representable functor $K : \mathbb{T} \rightarrow \mathbf{S}$. From the description of $\mathcal{F}(\mathbf{y}(x))$ in Section 4.3, we have that $L(\mathcal{F}(\mathbf{y}(x)))(y) \simeq \mathbb{T}(x, y) = \mathbf{y}(x)(y)$ and $\eta_{\mathbf{y}(x)}$ is just the identity. $\square$

Proposition 4.11. *The functor $\mathcal{F} : \mathbf{Th}(\mathbb{T}) \rightarrow \mathbf{Mod}(\mathbb{T})$ factors through $\mathbf{Mod}^{\text{dem}}(\mathbb{T}) \subset \mathbf{Mod}(\mathbb{T})$.*

Proof. Since $\mathbf{Mod}^{\text{dem}}(\mathbb{T}) \subset \mathbf{Mod}(\mathbb{T})$ is a coreflective subcategory by Proposition 4.6, it is closed under colimits. Thus, it suffices to show that $\mathcal{F}(\mathbf{y}(x))$ is democratic for every $x \in \mathbb{T}$, but this follows from the description of $\mathcal{F}(\mathbf{y}(x))$ in Section 4.3. $\square$

Proposition 4.12. *The restriction of $L : \mathbf{Mod}(\mathbb{T}) \rightarrow \mathbf{Th}(\mathbb{T})$ to $\mathbf{Mod}^{\text{dem}}(\mathbb{T}) \subset \mathbf{Mod}(\mathbb{T})$ is conservative.*

We first show the following lemma.

Lemma 4.13. *Let $F : \mathcal{M} \rightarrow \mathcal{N}$ be a morphism of models of $\mathbb{T}$ such that $L(F) : L(\mathcal{M}) \rightarrow L(\mathcal{N})$ is an equivalence of $\mathbb{T}$ -theories. Then, the map*

$$F(x)_{\Gamma} : \mathcal{M}(x)_{\Gamma} \rightarrow \mathcal{N}(x)_{F(\star)(\Gamma)}$$

is an equivalence of spaces for any contextual object $\Gamma \in \mathcal{M}(\star)$ and any object $x \in \mathbb{T}$.

Proof. By induction on the contextual object $\Gamma \in \mathcal{M}(\star)$. When $\Gamma = 1$, the map $F(x)_1$ is an equivalence by assumption. Suppose that $\Gamma = \{a\}_u$ for some contextual object $\Gamma' \in \mathcal{M}(\star)$, representable arrow $u : y \rightarrow z$ in $\mathbb{T}$ and map $a : \mathcal{M}(\star)/\Gamma' \rightarrow \mathcal{M}(z)$. Since $\mathcal{M} : \mathbb{T} \rightarrow \mathbf{RFib}_{\mathcal{M}(\star)}$ commutes with the polynomial functor $\mathbf{P}_u$, the maps $\mathcal{M}(\star)/\{a\}_u \rightarrow \mathcal{M}(x)$ correspond to the sections of $\mathcal{M}(\mathbf{P}_u x) \rightarrow \mathcal{M}(z)$ over $a : \mathcal{M}(\star)/\Gamma' \rightarrow \mathcal{M}(z)$. Thus, $\mathcal{M}(x)_{\{a\}_u}$ is the pullback

$$\begin{array}{ccc} \mathcal{M}(x)_{\{a\}_u} & \longrightarrow & \mathcal{M}(\mathbf{P}_u x)_{\Gamma'} \\ \downarrow & & \downarrow \\ \Delta^0 & \xrightarrow{a} & \mathcal{M}(z)_{\Gamma'}. \end{array}$$

By the induction hypothesis, $F(\mathbf{P}_u x)_{\Gamma'}$ and $F(z)_{\Gamma'}$ are equivalences, and thus $F(x)_{\{a\}_u}$ is an equivalence. $\square$

Proof of Proposition 4.12. Let $F : \mathcal{M} \rightarrow \mathcal{N}$ be a morphism between democratic models of $\mathbb{T}$ and suppose that $L(F) : L(\mathcal{M}) \rightarrow L(\mathcal{N})$ is an equivalence of $\mathbb{T}$ -theories. We show that F is an equivalence of models of $\mathbb{T}$. Since the forgetful functor $\mathbf{Mod}(\mathbb{T}) \rightarrow \mathbf{Fun}(\mathbb{T}^{\text{p}}, \mathbf{Cat}_{\infty})$ is conservative, it suffices to show that $F(x) : \mathcal{M}(x) \rightarrow \mathcal{N}(x)$ is an equivalence of ∞ -categories for every object $x \in \mathbb{T}^{\text{p}}$. Lemma 4.13 implies that the square

$$\begin{array}{ccc} \mathcal{M}(x) & \xrightarrow{F(x)} & \mathcal{N}(x) \\ \downarrow & & \downarrow \\ \mathcal{M}(\star) & \xrightarrow{F(\star)} & \mathcal{N}(\star) \end{array}$$

is a pullback for every $x \in \mathbb{T}$. It remains to show that the functor $F(\star) : \mathcal{M}(\star) \rightarrow \mathcal{N}(\star)$ is fully faithful and essentially surjective.

We show by induction on Δ that $F(\star) : \mathcal{M}(\star)(\Gamma, \Delta) \rightarrow \mathcal{N}(\star)(F(\star)(\Gamma), F(\star)(\Delta))$ is an equivalence of spaces for any objects $\Gamma, \Delta \in \mathcal{M}(\star)$. The case when $\Delta = 1$ is trivial. Suppose that $\Delta = \{a\}_u$ for some object $\Delta' \in \mathcal{M}(\star)$, representable arrow $u : x \rightarrow y$ in $\mathbb{T}$ and map $a : \mathcal{M}(\star)/\Delta' \rightarrow \mathcal{M}(y)$. By definition, we have a pullback

$$\begin{array}{ccc} \mathcal{M}(\star)(\Gamma, \{a\}_u) & \longrightarrow & \mathcal{M}(x)_{\Gamma} \\ \downarrow & & \downarrow \mathcal{M}(u)_{\Gamma} \\ \mathcal{M}(\star)(\Gamma, \Delta') & \xrightarrow{f \mapsto f^* a} & \mathcal{M}(y)_{\Gamma}. \end{array}$$

Then, by the induction hypothesis and Lemma 4.13, the map

$$F(\star) : \mathcal{M}(\star)(\Gamma, \{a\}_u) \rightarrow \mathcal{N}(\star)(F(\star)(\Gamma), F(\star)(\{a\}_u))$$

is an equivalence.

Finally, we show by induction on Δ that, for any object $\Delta \in \mathcal{N}(\star)$, there exists an object $\Gamma \in \mathcal{M}(\star)$ such that $F(\star)(\Gamma) \simeq \Delta$. The case when $\Delta = 1$ is trivial. Suppose that $\Delta = \{b\}_u$ for some object $\Delta' \in \mathcal{N}(\star)$, representable arrow $u : x \rightarrow y$ in $\mathbb{T}$ and map $b : \mathcal{N}(\star)/\Delta' \rightarrow \mathcal{N}(y)$. By the induction hypothesis, we have an object $\Gamma' \in \mathcal{M}(\star)$ such that $F(\star)(\Gamma') \simeq \Delta'$. By Lemma 4.13, we have a map $a : \mathcal{M}(\star)/\Gamma' \rightarrow \mathcal{M}(y)$ such that $F(y)_{\Gamma'}(a) \simeq b$. Then $F(\star)(\{a\}_u) \simeq \{b\}_u$. $\square$

Theorem 4.14. *For an ∞ -type theory, the restriction of $L : \mathbf{Mod}(\mathbb{T}) \rightarrow \mathbf{Th}(\mathbb{T})$ to $\mathbf{Mod}^{\text{dem}}(\mathbb{T}) \subset \mathbf{Mod}(\mathbb{T})$ is an equivalence*

$$\mathbf{Mod}^{\text{dem}}(\mathbb{T}) \simeq \mathbf{Th}(\mathbb{T}).$$

Proof. By Proposition 4.11, the functor $L : \mathbf{Mod}^{\text{dem}}(\mathbb{T}) \rightarrow \mathbf{Th}(\mathbb{T})$ has the left adjoint $\mathcal{F}$. By Proposition 4.10, the unit of this adjunction is invertible. By Proposition 4.12 and the triangle identities, the counit is also invertible. $\square$

5. Correspondence between type-theoretic structures and categorical structures

We discuss a correspondence between type-theoretic structures and categorical structures. Given an ∞ -category $\mathcal{C}$ whose objects are small ∞ -categories equipped with a certain structure and morphisms are structure-preserving functors, we try to find an ∞ -type theory $\mathbb{T}$ such that $\mathbf{Th}(\mathbb{T}) \simeq \mathcal{C}$. Such an ∞ -type theory $\mathbb{T}$ can be understood in a couple of ways. Type-theoretically, $\mathbb{T}$ provides *internal languages* for ∞ -categories in $\mathcal{C}$. We will find type-theoretic structures corresponding to categorical structures like finite limits and pushforwards. Categorically, $\mathbb{T}$ gives a *presentation* of the ∞ -category $\mathcal{C}$ as a localization of a presheaf ∞ -category. Such a presentation has the advantage that the ∞ -type theory $\mathbb{T}$ often has a simple universal property from which one can derive a universal property of $\mathcal{C}$ (see Corollary 5.21 for example).

The fundamental example of such an ∞ -category $\mathcal{C}$ is $\mathcal{C} = \mathbf{Lex}_{\infty}$, the ∞ -category of small left exact ∞ -categories. In Section 5.3, we introduce an ∞ -type theory $\mathbb{E}_{\infty}$ which is an ∞ -analogue of Martin-Löf type theory with extensional identity types. The main result of this section is to establish an equivalence $\mathbf{Th}(\mathbb{E}_{\infty}) \simeq \mathbf{Lex}_{\infty}$, and this is a higher analogue of the result of Clairambault and Dybjer [13]. To do this, we need two preliminaries: one is the *representable map classifier* of right fibrations over a left exact ∞ -category (Section 5.1) which is used for constructing a democratic model of $\mathbb{E}_{\infty}$ out of a left exact ∞ -category; the other is the notion of *univalence* in ∞ -categories with representable maps (Section 5.2) which for example makes a type constructor unique up to contractible choice. We also give two other examples $\mathcal{C} = \mathbf{LCCC}_{\infty}$, the ∞ -category of small locally cartesian closed ∞ -categories (Section 5.4), and $\mathcal{C} = \mathbf{TT}_{\infty}$, the ∞ -category of ∞ -type theories (Section 5.5). The latter example shows that the notion of ∞ -type theories itself can be written in the ∞ -type-theoretic language.

5.1 The representable map classifier In this preliminary subsection, we review a *representable map classifier* over a left exact ∞ -category $\mathcal{C}$, that is, a classifying object for the class of representable maps of right fibrations over $\mathcal{C}$.

Definition 5.1. Let S denote the category

$$\begin{array}{ccc} & & 0 \\ & \nearrow & \downarrow \\ 1 & \xlongequal{\quad} & 1. \end{array}$$

The inclusion $\Delta^1 = \{0 \rightarrow 1\} \rightarrow S$ induces a functor $S \pitchfork \mathcal{C} \rightarrow \Delta^1 \pitchfork \mathcal{C} = \mathcal{C}^\rightarrow$ for an ∞ -category $\mathcal{C}$. Note that $S \pitchfork \mathcal{C}$ is the ∞ -category of sections in $\mathcal{C}$.

Definition 5.2. Let $\mathcal{C}$ be a left exact ∞ -category. We define $\mathcal{R}_{\mathcal{C}}$ to be the largest right fibration over $\mathcal{C}$ contained in the cartesian fibration $\mathrm{cod} : \mathcal{C}^\rightarrow \rightarrow \mathcal{C}$ and $\rho_{\mathcal{C}} : \tilde{\mathcal{R}}_{\mathcal{C}} \rightarrow \mathcal{R}_{\mathcal{C}}$ to be the pullback

$$\begin{array}{ccc} \tilde{\mathcal{R}}_{\mathcal{C}} & \hookrightarrow & S \pitchfork \mathcal{C} \\ \rho_{\mathcal{C}} \downarrow & & \downarrow \\ \mathcal{R}_{\mathcal{C}} & \hookrightarrow & \mathcal{C}^\rightarrow. \end{array}$$

That is, $\mathcal{R}_{\mathcal{C}}$ is the wide subcategory of $\mathcal{C}^\rightarrow$ whose morphisms are the pullback squares. We refer to $\mathcal{R}_{\mathcal{C}}$ as the *representable map classifier* over $\mathcal{C}$ and $\rho_{\mathcal{C}}$ as the *generic representable map* of right fibrations over $\mathcal{C}$ because of the following proposition.

Proposition 5.3. *Let $\mathcal{C}$ be a left exact ∞ -category.*

1. $\rho_{\mathcal{C}} : \tilde{\mathcal{R}}_{\mathcal{C}} \rightarrow \mathcal{R}_{\mathcal{C}}$ is a representable map of right fibrations over $\mathcal{C}$.
2. For any right fibration A over $\mathcal{C}$, the map

$$\mathbf{RFib}_{\mathcal{C}}(A, \mathcal{R}_{\mathcal{C}}) \rightarrow (\mathbf{RFib}_{\mathcal{C}}/A)_r$$

defined by the pullback of $\rho_{\mathcal{C}}$ is an equivalence, where $(\mathbf{RFib}_{\mathcal{C}}/A)_r$ denotes the subspace of $(\mathbf{RFib}_{\mathcal{C}}/A)^{\simeq}$ spanned by the representable maps over A .

Proof. We first observe that $\rho_{\mathcal{C}} : \tilde{\mathcal{R}}_{\mathcal{C}} \rightarrow \mathcal{R}_{\mathcal{C}}$ is a right fibration, that is, the functor

$$(\mathrm{ev}_1, (\rho_{\mathcal{C}})_*) : \Delta^1 \pitchfork \tilde{\mathcal{R}}_{\mathcal{C}} \rightarrow \tilde{\mathcal{R}}_{\mathcal{C}} \times_{\mathcal{R}_{\mathcal{C}}} (\Delta^1 \pitchfork \mathcal{R}_{\mathcal{C}})$$

is an equivalence. Since $\Delta^1 \pitchfork \mathcal{R}_{\mathcal{C}}$ is a subcategory of $(\Delta^1 \times \Delta^1) \pitchfork \mathcal{C}$ whose objects are the pullback squares, this follows from the universal property of pullbacks.

For the representability of $\rho_{\mathcal{C}}$, we use Proposition 2.17. Let $x \in \mathcal{C}$ be an object and $\kappa_y : \mathcal{C}/x \rightarrow \mathcal{R}_{\mathcal{C}}$ a map which corresponds to an arrow $y \rightarrow x$ in $\mathcal{C}$. We show that $\kappa_y^* \tilde{\mathcal{R}}_{\mathcal{C}}$ is representable by y . Since the diagonal map $y \rightarrow y \times_x y$ is a section of the first projection, it determines a map $\delta : \mathcal{C}/y \rightarrow \tilde{\mathcal{R}}_{\mathcal{C}}$ such that the diagram

$$\begin{array}{ccc} \mathcal{C}/y & \xrightarrow{\delta} & \tilde{\mathcal{R}}_{\mathcal{C}} \\ \downarrow & & \downarrow \rho_{\mathcal{C}} \\ \mathcal{C}/x & \xrightarrow{\kappa_y} & \mathcal{R}_{\mathcal{C}} \end{array}$$

commutes. This square is a pullback. Indeed, for an object $(u : z \rightarrow x) \in \mathcal{C}/x$, the fiber of $\rho_{\mathcal{C}}$ over $\kappa_y(u)$ is the space of sections of $z \times_x y \rightarrow z$ which is equivalent to the space of sections of $y \rightarrow x$ over u .

For the second claim, observe that $(\mathbf{RFib}_{\mathcal{C}}/\mathrm{colim}_{i \in \mathcal{I}} A_i)_r \simeq \lim_{i \in \mathcal{I}} (\mathbf{RFib}_{\mathcal{C}}/A_i)_r$ for any diagram $(A_i)_{i \in \mathcal{I}}$ in $\mathbf{RFib}_{\mathcal{C}}$. Indeed, since $\mathbf{RFib}_{\mathcal{C}}$ is an ∞ -topos, we have $(\mathbf{RFib}_{\mathcal{C}}/\mathrm{colim}_{i \in \mathcal{I}} A_i)^{\simeq} \simeq$

$\lim_{i \in \mathcal{I}} (\mathbf{RFib}_{\mathcal{C}}/A_i)^\simeq$, and this equivalence is restricted to representable maps by Proposition 2.17. Then it is enough to show that the map in the statement is an equivalence in the case when A is representable by some $x \in \mathcal{C}$. By definition $\mathbf{RFib}_{\mathcal{C}}(\mathcal{C}/x, \mathcal{R}_{\mathcal{C}}) \simeq (\mathcal{C}/x)^\simeq$. By Proposition 2.17, $(\mathbf{RFib}_{\mathcal{C}}/(\mathcal{C}/x))_r$ is the space of arrows $y \rightarrow x$ of which the pullback along an arbitrary arrow $z \rightarrow x$ exists, but since $\mathcal{C}$ has pullbacks this is $(\mathcal{C}/x)^\simeq$. $\square$

Remark 5.4. The representable map classifier in $\mathbf{RFib}_{\mathcal{C}}$ exists even when $\mathcal{C}$ is not left exact. In the above proof, we have seen that $(\mathbf{RFib}_{\mathcal{C}}/\mathrm{colim}_{i \in \mathcal{I}} A_i)_r \simeq \lim_{i \in \mathcal{I}} (\mathbf{RFib}_{\mathcal{C}}/A_i)_r$ and that $(\mathbf{RFib}_{\mathcal{C}}/A)_r$ is essentially small. Then, by [36, Proposition 5.5.2.2], the functor $\mathbf{RFib}_{\mathcal{C}}^{\mathrm{op}} \ni A \mapsto (\mathbf{RFib}_{\mathcal{C}}/A)_r \in \mathbf{S}$ is representable, and the representing object is the representable map classifier. From the concrete construction given in Definition 5.2, the construction of the representable map classifier in the case when $\mathcal{C}$ is left exact is moreover functorial: any left exact functor $F : \mathcal{C} \rightarrow \mathcal{D}$ induces a map of right fibrations $\mathcal{R}_{\mathcal{C}} \rightarrow \mathcal{R}_{\mathcal{D}}$ over F .

5.2 Univalent representable arrows In this preliminary subsection, we extend the notion of a univalent map in a (presentable) locally cartesian closed ∞ -category [19, 39, 40] to a notion of a univalent representable arrow in an ∞ -categories with representable maps.

Definition 5.5. For objects x and y of an ∞ -category $\mathcal{C}$ with finite products, let $\mathrm{Map}(x, y) \rightarrow \mathcal{C}$ denote the right fibration whose fiber over z is $\mathcal{C}/z(x \times z, y \times z) \simeq \mathcal{C}(x \times z, y)$. It is defined by the pullback

$$\begin{array}{ccc} \mathrm{Map}(x, y) & \longrightarrow & \mathcal{C}/y \\ \downarrow & & \downarrow \\ \mathcal{C} & \xrightarrow{(- \times x)} & \mathcal{C}. \end{array}$$

If $\mathrm{Map}(x, y)$ is representable, we denote by $\underline{\mathrm{Map}}(x, y)$ the representing object. We define $\mathrm{Eq}(x, y)$ to be the subfibration of $\mathrm{Map}(x, y)$ spanned by the equivalences $x \times z \simeq y \times z$. If $\mathrm{Eq}(x, y)$ is representable, we denote by $\underline{\mathrm{Eq}}(x, y)$ the representing object.

Definition 5.6. Let $u : x \rightarrow y$ be an arrow in a left exact ∞ -category $\mathcal{C}$. We regard $u \times y : x \times y \rightarrow y \times y$ and $y \times u : y \times x \rightarrow y \times y$ as objects of $\mathcal{C}/y \times y$ and denote by $\mathrm{Eq}(u)$ the right fibration $\mathrm{Eq}(u \times y, y \times u) \rightarrow \mathcal{C}/y \times y$. If $\mathrm{Eq}(u)$ representable, we denote by $\underline{\mathrm{Eq}}(u)$ the representing object.

By definition, an arrow $z \rightarrow \underline{\mathrm{Eq}}(u)$ corresponds to a triple (v_1, v_2, w) consisting of arrows $v_1, v_2 : z \rightarrow y$ and an equivalence $w : v_1^* x \simeq v_2^* x$ over z .

Definition 5.7. Let $u : x \rightarrow y$ be an arrow in a left exact ∞ -category $\mathcal{C}$ such that $\mathrm{Eq}(u)$ is representable. We have a section $|\mathrm{id}| : y \rightarrow \underline{\mathrm{Eq}}(u)$ over the diagonal $\delta : y \rightarrow y \times y$ corresponding to the identity $\mathrm{id} : x \rightarrow x$. We say u is *univalent* if the arrow $|\mathrm{id}| : y \rightarrow \underline{\mathrm{Eq}}(u)$ is an equivalence.

Proposition 5.8. Let $u : x \rightarrow y$ be an arrow in a left exact ∞ -category $\mathcal{C}$ such that $\mathrm{Eq}(u)$ is representable. Let $\kappa_u : \mathcal{C}/y \rightarrow \mathcal{R}_{\mathcal{C}}$ be the map corresponding to u by Yoneda. The following are equivalent:

1. u is univalent:
2. the square

$$\begin{array}{ccc} \mathcal{C}/y & \xrightarrow{\kappa_u} & \mathcal{R}_{\mathcal{C}} \\ \delta \downarrow & & \downarrow \delta \\ \mathcal{C}/y \times y & \xrightarrow{\kappa_u \times \kappa_u} & \mathcal{R}_{\mathcal{C}} \times_{\mathcal{C}} \mathcal{R}_{\mathcal{C}} \end{array}$$

is a pullback;

3. $\kappa_u : \mathcal{C}/y \rightarrow \mathcal{R}_{\mathcal{C}}$ is a (-1) -truncated map of right fibrations over $\mathcal{C}$. Equivalently, for any object $z \in \mathcal{C}$, the map $\mathcal{C}(z, y) \rightarrow (\mathcal{C}/z)^{\simeq}$ defined by the pullback of u is (-1) -truncated.

Proof. The same proof as [19, Proposition 3.8 (1)–(3)] works only assuming the representability of $\text{Eq}(u)$. $\square$

Example 5.9. For any left exact ∞ -category $\mathcal{C}$, the generic representable map $\rho_{\mathcal{C}} : \widetilde{\mathcal{R}}_{\mathcal{C}} \rightarrow \mathcal{R}_{\mathcal{C}}$ is a univalent representable map in $\mathbf{RFib}_{\mathcal{C}}$ by Proposition 5.3.

Proposition 5.10. *Let x and y be objects in a left exact ∞ -category $\mathcal{C}$ and suppose that $x \times z$ and $y \times z$ are exponentiable in $\mathcal{C}/z$ for any object $z \in \mathcal{C}$.*

1. *The right fibration $\text{Eq}(x, y) \rightarrow \mathcal{C}$ is representable.*
2. *Let $\mathcal{D}$ be a left exact ∞ -category and $F : \mathcal{C} \rightarrow \mathcal{D}$ a left exact functor. If F sends $x \times z$ and $y \times z$ to exponentiable objects over Fz and commutes with exponentiation by $x \times z$ and $y \times z$ for any $z \in \mathcal{C}$, then the canonical arrow $F(\text{Eq}(x, y)) \rightarrow \text{Eq}(F(x), F(y))$ is an equivalence.*

Proof. The right fibration $\text{Eq}(x, y)$ is equivalent to the right fibration $\text{BiInv}(x, y)$ of bi-invertible arrows whose fiber over $z \in \mathcal{C}$ is the space of tuples $(u, v, \eta, w, \varepsilon)$ consisting of arrows $u : x \times z \rightarrow y \times z$ and $v, w : y \times z \rightarrow x \times z$ over z and homotopies $\eta : vu \simeq \text{id}$ and $\varepsilon : uw \simeq \text{id}$ over z . The right fibration $\text{BiInv}(x, y)$ is representable by the exponentiability of $x \times z$ and $y \times z$. The second assertion is clear from the construction of the representing object for $\text{BiInv}(x, y)$. $\square$

Corollary 5.11. *Let $u : x \rightarrow y$ be a representable arrow in an ∞ -category with representable maps $\mathcal{C}$.*

1. *The right fibration $\text{Eq}(u) \rightarrow \mathcal{C}/y \times y$ is representable.*
2. *If u is univalent, so is Fu for any morphism of ∞ -categories with representable maps $F : \mathcal{C} \rightarrow \mathcal{D}$.*

5.3 Left exact ∞ -categories We define an ∞ -type theory $\mathbb{E}_{\infty}$ whose theories are equivalent to small left exact ∞ -categories.

Definition 5.12. Let $\mathcal{C}$ be an ∞ -category with representable maps and $\partial : E \rightarrow U$ a representable arrow in $\mathcal{C}$.

- A *1-type structure* on ∂ is a pullback square of the form

$$\begin{array}{ccc} 1 & \xrightarrow{*} & E \\ \parallel & & \downarrow \partial \\ 1 & \xrightarrow{1} & U. \end{array} \quad (6)$$

- A *Σ -type structure* on ∂ is a pullback square of the form

$$\begin{array}{ccc} \text{dom}(\partial \otimes \partial) & \xrightarrow{\text{p}} & E \\ \partial \otimes \partial \downarrow & & \downarrow \partial \\ \text{cod}(\partial \otimes \partial) & \xrightarrow{\Sigma} & U \end{array} \quad (7)$$

where $\otimes$ is the composition of polynomials (Section 2.4).

- An *ld-type structure* on ∂ is a pullback square of the form

$$\begin{array}{ccc} E & \xrightarrow{\text{refl}} & E \\ \delta \downarrow & & \downarrow \partial \\ E \times_U E & \xrightarrow{\text{Id}} & U. \end{array} \quad (8)$$

Proposition 5.13. *Let $\partial : E \rightarrow U$ be a univalent representable arrow in an ∞ -category with representable maps $\mathcal{C}$. Then 1-type structures, Σ -type structures and ld-type structures are unique up to contractible choice. Moreover, we have the following:*

1. ∂ has a 1-type structure if and only if all the identity arrows are pullbacks of ∂ ;
2. ∂ has a Σ -type structure if and only if pullbacks of ∂ are closed under composition;
3. ∂ has an ld-type structure if and only if pullbacks of ∂ are closed under equalizers: if $u : x \rightarrow y$ is a pullback of ∂ and $v_1, v_2 : x' \rightarrow x$ are arrows such that $uv_1 \simeq uv_2$, then the equalizer $x'' \rightarrow x'$ of v_1 and v_2 in $\mathcal{C}/y$ is a pullback of ∂ .

Proof. The uniqueness follows from Item 3 of Proposition 5.8. The rests are straightforward. $\square$

Definition 5.14. By a *left exact universe* in an ∞ -category with representable maps $\mathcal{C}$ we mean a univalent representable arrow $\partial : E \rightarrow U$ equipped with a 1-type structure, a Σ -type structure and an ld-type structure. We denote by $\mathbb{E}_\infty$ the initial ∞ -type theory equipped with a left exact universe $\partial : E \rightarrow U$ constructed from $\mathbb{G}_\infty$ by freely adjoining suitable arrows and homotopies (Example 3.12).

Theorem 5.15. *The functor $\text{ev}_\star : \mathbf{Mod}^{\text{dem}}(\mathbb{E}_\infty) \rightarrow \mathbf{Cat}_\infty$ factors through $\mathbf{Lex}_\infty$ and induces an equivalence*

$$\mathbf{Mod}^{\text{dem}}(\mathbb{E}_\infty) \simeq \mathbf{Lex}_\infty.$$

Lemma 5.16. *An arrow in $\mathbb{E}_\infty$ is representable if and only if it is a pullback of $\partial : E \rightarrow U$.*

Proof. Let $\mathbb{E}'_\infty$ denote the ∞ -category with representable maps whose underlying ∞ -category is the same as $\mathbb{E}_\infty$ and representable arrows are the pullbacks of ∂ . As ∂ is equipped with a 1-type structure and a Σ -type structure, the representable arrows in $\mathbb{E}'_\infty$ include all the identities and are closed under composition by Proposition 5.13, so $\mathbb{E}'_\infty$ is indeed an ∞ -category with representable maps. By the initiality of $\mathbb{E}_\infty$, the inclusion $\mathbb{E}'_\infty \rightarrow \mathbb{E}_\infty$ has a section, and thus $\mathbb{E}'_\infty \simeq \mathbb{E}_\infty$. $\square$

Definition 5.17. Let $\mathcal{M}$ be a model of an ∞ -type theory $\mathbb{T}$. By a *display map* we mean an arrow $f : \Delta \rightarrow \Gamma$ in $\mathcal{M}(\star)$ that is equivalent over Γ to $p_u : \{b\}_u \rightarrow \Gamma$ for some representable arrow $u : x \rightarrow y$ in $\mathbb{T}$ and map $b : \mathcal{M}(\star)/\Gamma \rightarrow \mathcal{M}(y)$. By definition, display maps are stable under pullbacks.

Lemma 5.18. *Let $\mathcal{M}$ be a model of $\mathbb{E}_\infty$. An arrow $f : \Gamma_1 \rightarrow \Gamma_2$ in $\mathcal{M}(\star)$ is a display map if and only if there exists a pullback of the form*

$$\begin{array}{ccc} \mathcal{M}(\star)/\Gamma_1 & \xrightarrow{\quad} & \mathcal{M}(E) \\ f \downarrow & & \downarrow \mathcal{M}(\partial) \\ \mathcal{M}(\star)/\Gamma_2 & \xrightarrow{\quad} & \mathcal{M}(U). \end{array}$$

Proof. By Lemma 5.16. $\square$

Lemma 5.19. *Let $\mathcal{D}$ be a pullback-stable class of arrows in an ∞ -category $\mathcal{C}$. Suppose that $\mathcal{D}$ contains all the identity and is closed under composition and equalizers. Then, for arrows $u : x \rightarrow y$ and $v : y \rightarrow z$, if v and vu are in $\mathcal{D}$, so is u .*

Proof. Replacing $\mathcal{C}$ by $\mathcal{C}/z$, we may assume without loss of generality that z is a terminal object. Let $p : \alpha \rightarrow x \times y$ be the equalizer of $u\pi_1, \pi_2 : x \times y \rightarrow y$ which is in $\mathcal{D}$ by assumption. It fits into the pullback

$$\begin{array}{ccc} \alpha & \xrightarrow{\quad} & y \\ p \downarrow & & \downarrow \delta \\ x \times y & \xrightarrow{u \times \text{id}} & y \times y. \end{array}$$

By pasting this on the pullback square

$$\begin{array}{ccc} x \times y & \xrightarrow{u \times \text{id}} & y \times y \\ \pi_1 \downarrow & & \downarrow \pi_1 \\ x & \xrightarrow{u} & y, \end{array}$$

we see that the composite $\pi_1 p$ is invertible. Thus, u is equivalent (in $\mathcal{C}^{\rightarrow}$) to $u\pi_1 p$ which is homotopic to $\pi_2 p$ by construction. Since π_2 and p are in $\mathcal{D}$, we conclude that u is in $\mathcal{D}$. $\square$

Lemma 5.20. *Let $\mathcal{M}$ be a democratic model of $\mathbb{E}_\infty$. Then every arrow in $\mathcal{M}(\star)$ is a display map.*

Proof. By Proposition 5.13 and Lemma 5.18, the class of display maps in $\mathcal{M}(\star)$ contains all the identity arrows and is closed under composition and equalizers. Since $\mathcal{M}$ is democratic, the unique arrow $x \rightarrow 1$ is a composite of display maps and thus a display map for any object $x \in \mathcal{M}(\star)$. Then apply Lemma 5.19 with $z = 1$. $\square$

Proof of Theorem 5.15. Since display maps are stable under pullbacks and morphisms of models commute with pullbacks of display maps, Lemma 5.20 implies that the base ∞ -category of a democratic model of $\mathbb{E}_\infty$ has all finite limits and that any morphism between democratic models of $\mathbb{E}_\infty$ commutes with finite limits in the base ∞ -categories. In other words, the functor $\text{ev}_\star : \mathbf{Mod}^{\text{dem}}(\mathbb{E}_\infty) \rightarrow \mathbf{Cat}_\infty$ factors through $\mathbf{Lex}_\infty$.

Let $\mathcal{C}$ be a left exact ∞ -category. We define a model $\mathcal{R}_\mathcal{C}$ of $\mathbb{E}_\infty$ by setting $\mathcal{R}_\mathcal{C}(\star)$ to be $\mathcal{C}$ and $\mathcal{R}_\mathcal{C}(\partial) : \mathcal{R}_\mathcal{C}(E) \rightarrow \mathcal{R}_\mathcal{C}(U)$ to be the generic representable map of right fibrations over $\mathcal{C}$. We have seen in Example 5.9 that the generic representable map is univalent. Since $\mathcal{C}$ has finite limits, representable maps of right fibrations over $\mathcal{C}$ are closed under equalizers. Thus, by Proposition 5.13, $\mathcal{R}_\mathcal{C}$ is indeed a model of $\mathbb{E}_\infty$. Since the construction of the generic representable map for a left exact ∞ -category is functorial, the assignment $\mathcal{C} \mapsto \mathcal{R}_\mathcal{C}$ is part of a functor

$$\mathcal{R} : \mathbf{Lex}_\infty \rightarrow \mathbf{Mod}(\mathbb{E}_\infty).$$

The model $\mathcal{R}_\mathcal{C}$ is democratic as the map $\mathcal{C}/x \rightarrow \mathcal{C}/1$ is representable for every object $x \in \mathcal{C}$.

We show that the functor $\mathcal{R} : \mathbf{Lex}_\infty \rightarrow \mathbf{Mod}^{\text{dem}}(\mathbb{E}_\infty)$ is an inverse of $\text{ev}_\star : \mathbf{Mod}^{\text{dem}}(\mathbb{E}_\infty) \rightarrow \mathbf{Lex}_\infty$. By definition, $\text{ev}_\star \circ \mathcal{R} \simeq \text{id}$. To show the other equivalence $\mathcal{R} \circ \text{ev}_\star \simeq \text{id}$, let $\mathcal{M}$ be a democratic model of $\mathbb{E}_\infty$. Since $\mathcal{R}_{\mathcal{M}(\star)}(\partial)$ is the generic representable map, we have a unique pullback

$$\begin{array}{ccc} \mathcal{M}(E) & \xrightarrow{\quad} & \mathcal{R}_{\mathcal{M}(\star)}(E) \\ \mathcal{M}(\partial) \downarrow & & \downarrow \mathcal{R}_{\mathcal{M}(\star)}(\partial) \\ \mathcal{M}(U) & \xrightarrow{f} & \mathcal{R}_{\mathcal{M}(\star)}(U). \end{array}$$

It suffices to show that f is an equivalence of right fibrations over $\mathcal{M}(\star)$. Since $\mathcal{M}(\partial)$ is univalent, the map f is (-1) -truncated [19, Corollary 3.10]. Recall that the objects of $\mathcal{R}_{\mathcal{M}(\star)}(U)$ are the arrows of $\mathcal{M}(\star)$. Lemma 5.18 implies that the essential image of f is the class of display maps in $\mathcal{M}(\star)$. Then, by Lemma 5.20, the map f is essentially surjective and thus an equivalence. $\square$

Consider the image of the arrow $\partial : E \rightarrow U$ by the inclusion

$$\mathbb{E}_\infty \rightarrow \mathbf{Th}(\mathbb{E}_\infty)^{\text{op}} \simeq \mathbf{Mod}^{\text{dem}}(\mathbb{E}_\infty)^{\text{op}} \simeq \mathbf{Lex}_\infty^{\text{op}}.$$

For a left exact ∞ -category $\mathcal{C}$, we have

$$\begin{aligned} \mathbf{Th}(\mathbb{E}_\infty)(\mathbf{y}(U), \mathbf{L}(\mathcal{R}_\mathcal{C})) &\simeq \mathcal{R}_\mathcal{C}(U)_1 \simeq \mathcal{C}^\simeq \\ \mathbf{Th}(\mathbb{E}_\infty)(\mathbf{y}(E), \mathbf{L}(\mathcal{R}_\mathcal{C})) &\simeq \mathcal{R}_\mathcal{C}(E)_1 \simeq (1/\mathcal{C})^\simeq. \end{aligned}$$

Hence, the object U corresponds to the free left exact ∞ -category $\langle \square \rangle$ generated by an object $\square$, the object E corresponds to the free left exact ∞ -category $\langle \tilde{\square} : 1 \rightarrow \square \rangle$ generated by an object $\square$ and a global section $\tilde{\square} : 1 \rightarrow \square$, and the arrow $\partial : E \rightarrow U$ corresponds to the inclusion $\iota : \langle \square \rangle \rightarrow \langle \tilde{\square} : 1 \rightarrow \square \rangle$. Since $\mathbf{y}(x)/\mathbf{Th}(\mathbb{E}_\infty) \simeq \mathbf{Th}(\mathbb{E}_\infty/x)$, we see that the inclusion ι becomes an exponentiable arrow in $\mathbf{Lex}_\infty^{\text{op}}$. This makes $\mathbf{Lex}_\infty^{\text{op}}$ an ∞ -category with representable maps in which the representable arrows are the pullbacks of ι , and ι is a left exact universe in $\mathbf{Lex}_\infty^{\text{op}}$. Since $\mathbf{Th}(\mathbb{E}_\infty)$ is the ω -free cocompletion of $\mathbb{E}_\infty^{\text{op}}$, the universal property of $\mathbb{E}_\infty$ gives the following universal property of $\mathbf{Lex}_\infty$.

Corollary 5.21. *Let $\mathcal{C}$ be an ∞ -category with representable maps that has all small limits and $u : x \rightarrow y$ a left exact universe in $\mathcal{C}$. Then there exists a unique morphism of ∞ -categories with representable maps $F : \mathbf{Lex}_\infty^{\text{op}} \rightarrow \mathcal{C}$ that sends ι to u and preserves small limits.*

Proof. By the definition of $\mathbb{E}_\infty$, we have a unique morphism of ∞ -categories with representable maps $\bar{F} : \mathbb{E}_\infty \rightarrow \mathcal{C}$ that sends ∂ to u , which uniquely extends to a limit-preserving functor $F : \mathbf{Lex}_\infty^{\text{op}} \simeq \mathbf{Th}(\mathbb{E}_\infty)^{\text{op}} \rightarrow \mathcal{C}$. The functor F sends pushforwards along ι to pushforwards along u because the pushforward functors preserve limits and every object of $\mathbf{Th}(\mathbb{E}_\infty)^{\text{op}}$ is a limit of objects from $\mathbb{E}_\infty$. $\square$

5.4 Locally cartesian closed ∞ -categories We define an ∞ -type theory $\mathbb{E}_\infty^\Pi$ whose theories are equivalent to small locally cartesian closed ∞ -categories.

Definition 5.22. Let $\mathcal{C}$ be an ∞ -category with representable maps and $\partial : E \rightarrow U$ a representable arrow in $\mathcal{C}$. A Π -type structure on ∂ is a pullback square of the form

$$\begin{array}{ccc} \mathbf{P}_\partial E & \xrightarrow{\lambda} & E \\ \mathbf{P}_\partial \partial \downarrow & & \downarrow \partial \\ \mathbf{P}_\partial U & \xrightarrow[\Pi]{} & U. \end{array} \tag{9}$$

The following is straightforward.

Proposition 5.23. *Let $\partial : E \rightarrow U$ be a univalent representable arrow in an ∞ -category with representable maps. Then Π -type structures on ∂ are unique up to contractible choice. Moreover, there exists a Π -type structure on ∂ if and only if pullbacks of ∂ are closed under pushforwards along pullbacks of ∂ .*

Definition 5.24. Let $\mathbb{E}_\infty^\Pi$ denote the initial ∞ -type theory equipped with a left exact universe $\partial : E \rightarrow U$ with a Π -type structure constructed from $\mathbb{E}_\infty$ by freely adjoining suitable arrows and homotopies (Example 3.12).

Theorem 5.25. The functor $\text{ev}_\star : \mathbf{Mod}^{\text{dem}}(\mathbb{E}_\infty^\Pi) \rightarrow \mathbf{Cat}_\infty$ factors through the ∞ -category $\mathbf{LCCC}_\infty$ of small locally cartesian closed ∞ -categories and induces an equivalence

$$\mathbf{Mod}^{\text{dem}}(\mathbb{E}_\infty^\Pi) \simeq \mathbf{LCCC}_\infty.$$

Proof. Lemma 5.16 holds for $\mathbb{E}_\infty^\Pi$: an arrow in $\mathbb{E}_\infty^\Pi$ is representable if and only if it is a pullback of ∂ . It follows from this that the restriction of a democratic model of $\mathbb{E}_\infty^\Pi$ along the inclusion $\mathbb{E}_\infty \rightarrow \mathbb{E}_\infty^\Pi$ is a democratic model of $\mathbb{E}_\infty$. Thus, by Theorem 5.15, it suffices to show that the composite $\mathbf{Mod}^{\text{dem}}(\mathbb{E}_\infty^\Pi) \rightarrow \mathbf{Mod}^{\text{dem}}(\mathbb{E}_\infty) \xrightarrow{\text{ev}_\star} \mathbf{Lex}_\infty$ factors through $\mathbf{LCCC}_\infty$ and gives rise a pullback square

$$\begin{array}{ccc} \mathbf{Mod}^{\text{dem}}(\mathbb{E}_\infty^\Pi) & \xrightarrow{\quad \quad \quad} & \mathbf{LCCC}_\infty \\ \downarrow & & \downarrow \\ \mathbf{Mod}^{\text{dem}}(\mathbb{E}_\infty) & \xrightarrow[\text{ev}_\star]{\simeq} & \mathbf{Lex}_\infty. \end{array}$$

It suffices to show the following:

1. an object $\mathcal{M}$ in $\mathbf{Mod}^{\text{dem}}(\mathbb{E}_\infty)$ is in $\mathbf{Mod}^{\text{dem}}(\mathbb{E}_\infty^\Pi)$ if and only if $\mathcal{M}(\star)$ is in $\mathbf{LCCC}_\infty$;
2. for objects $\mathcal{M}, \mathcal{N} \in \mathbf{Mod}^{\text{dem}}(\mathbb{E}_\infty^\Pi)$, a morphism $F : \mathcal{M} \rightarrow \mathcal{N}$ in $\mathbf{Mod}^{\text{dem}}(\mathbb{E}_\infty)$ is in $\mathbf{Mod}^{\text{dem}}(\mathbb{E}_\infty^\Pi)$ if and only if $F(\star) : \mathcal{M}(\star) \rightarrow \mathcal{N}(\star)$ is in $\mathbf{LCCC}_\infty$.

Item 1. By Proposition 5.23, an object $\mathcal{M} \in \mathbf{Mod}^{\text{dem}}(\mathbb{E}_\infty)$ is in $\mathbf{Mod}^{\text{dem}}(\mathbb{E}_\infty^\Pi)$ if and only if representable maps of right fibrations over $\mathcal{M}(\star)$ are closed under pushforwards along representable maps. This is equivalent to that the ∞ -category $\mathcal{M}(\star)$ is locally cartesian closed.

Item 2. A morphism $F : \mathcal{M} \rightarrow \mathcal{N}$ in $\mathbf{Mod}^{\text{dem}}(\mathbb{E}_\infty)$ between objects from $\mathbf{Mod}^{\text{dem}}(\mathbb{E}_\infty^\Pi)$ is in $\mathbf{Mod}^{\text{dem}}(\mathbb{E}_\infty^\Pi)$ if and only if it commutes with Π -type structures. Observe that $\mathcal{M}(\Pi) : \mathcal{M}(\mathbf{P}_\partial U) \rightarrow \mathcal{M}(U)$ sends a pair of composable arrows $u : x \rightarrow y$ and $v : y \rightarrow z$ in $\mathcal{M}(\star)$ to the pushforward $v_* u : v_* x \rightarrow z$. Thus, F commutes with Π -type structures if and only if $F(\star) : \mathcal{M}(\star) \rightarrow \mathcal{N}(\star)$ commutes with pushforwards. $\square$

5.5 ∞ -type theories We define an ∞ -type theory $\mathbb{R}_\infty$ whose theories are equivalent to ∞ -type theories.

Definition 5.26. Let $\partial_1 : E_1 \rightarrow U_1$, $\partial_2 : E_2 \rightarrow U_2$ and $\partial_3 : E_3 \rightarrow U_3$ be representable arrows in an ∞ -category with representable maps. A $(\partial_1, \partial_2, \partial_3)$ - Π -type structure is a pullback of the form

$$\begin{array}{ccc} \mathbf{P}_{\partial_1} E_2 & \xrightarrow{\lambda} & E_3 \\ \mathbf{P}_{\partial_1} \partial_2 \downarrow & & \downarrow \partial_3 \\ \mathbf{P}_{\partial_1} U_2 & \xrightarrow[\Pi]{} & U_3. \end{array} \tag{10}$$

Note that if ∂_3 is univalent, then $(\partial_1, \partial_2, \partial_3)$ - Π -type structure are unique up to contractible choice, and there exists a $(\partial_1, \partial_2, \partial_3)$ - Π -type structure if and only if the pushforward of a pullback of ∂_2 along a pullback of ∂_1 is a pullback of ∂_3 .

Definition 5.27. We denote by $\mathbb{R}_\infty$ the initial ∞ -type theory equipped with the following data:

- a left exact universe $\partial : E \rightarrow U$;

- a (-1) -truncated arrow $R \hookrightarrow U$. We denote by ∂_R the pullback of ∂ along the inclusion $R \hookrightarrow U$;
- a 1-type structure and a Σ -type structure on ∂_R ;
- a $(\partial_R, \partial, \partial)$ - Π -type structure.

Note that the inclusion $R \hookrightarrow U$ automatically commutes with 1-type structures and Σ -type structures because of univalence. Such a $\mathbb{R}_\infty$ is constructed as follows. We begin with the pushout

$$\begin{array}{ccc} \langle \{\alpha\} \rangle & \xrightarrow{U} & \mathbb{E}_\infty \\ \downarrow & & \downarrow \\ \langle \alpha \vdash \beta \rangle & \longrightarrow & \mathbb{T}. \end{array}$$

Let $R \in \mathbb{T}$ be the image of β by the morphism $\langle \alpha \vdash \beta \rangle \rightarrow \mathbb{T}$. We can force $R \rightarrow U$ to be (-1) -truncated by freely inverting the diagonal arrow $R \rightarrow R \times_U R$. We then freely adjoin the rest of arrows and homotopies.

Definition 5.28. Let $\mathcal{M}$ be a model of $\mathbb{R}_\infty$. We say an arrow in $\mathcal{M}(\star)$ is *representable* if it is a context comprehension with respect to ∂_R . Using the $(\partial_R, \partial, \partial)$ - Π -type structure, we see that the pushforward of a display map along a representable map exists and is a display map. In particular, if $\mathcal{M}$ is democratic, then $\mathcal{M}(\star)$ is an ∞ -type theory and, for any morphism $F : \mathcal{M} \rightarrow \mathcal{N}$ between democratic models, $F(\star) : \mathcal{M}(\star) \rightarrow \mathcal{N}(\star)$ is a morphism of ∞ -type theories. Hence, we have a functor

$$\text{ev}_\star : \mathbf{Mod}^{\text{dem}}(\mathbb{R}_\infty) \rightarrow \mathbf{TT}_\infty$$

Theorem 5.29. *The functor $\text{ev}_\star : \mathbf{Mod}^{\text{dem}}(\mathbb{R}_\infty) \rightarrow \mathbf{TT}_\infty$ is an equivalence.*

Proof. Similar to Theorem 5.15. For an ∞ -type theory $\mathcal{C}$, the representable map classifier $\mathcal{R}_\mathcal{C}(U)$ has the full subfibration $\mathcal{R}_\mathcal{C}(R) \subset \mathcal{R}_\mathcal{C}(U)$ spanned by the representable arrows in $\mathcal{C}$, which defines a democratic model of $\mathbb{R}_\infty$. $\square$

6. Internal languages for left exact ∞ -categories

In this section, we show Kapulkin and Lumsdaine's conjecture that the ∞ -category of small left exact ∞ -categories is a localization of the category of theories over Martin-Löf type theory with intensional identity types [27].

We first introduce a structure of *intensional identity types* in the context of ∞ -type theory.

Definition 6.1. Let $\mathcal{C}$ be an ∞ -category with representable maps and $\partial : E \rightarrow U$ a representable arrow in $\mathcal{C}$. An *ld⁺-type structure* on ∂ is a commutative square of the form

$$\begin{array}{ccc} E & \xrightarrow{\text{refl}} & E \\ \delta \downarrow & & \downarrow \partial \\ E \times_U E & \xrightarrow{\text{Id}} & U \end{array} \quad (11)$$

equipped with a section $\text{elim}_{\text{Id}^+}$ of the induced arrow

$$(\text{refl}^*, \partial_*) : (\text{Id}^* E \Rightarrow_U U^* E) \rightarrow (E \Rightarrow_U U^* E) \times_{(E \Rightarrow_U U^* U)} (\text{Id}^* E \Rightarrow_U U^* U),$$

where $\Rightarrow_U$ is the exponential in the slice $\mathcal{C}/U$.

The codomain of the arrow $(\text{refl}^*, \partial_*)$ classifies *lifting problems* for refl against ∂ , and the section $\text{elim}_{\text{Id}^+}$ is considered as a *uniform solution* to the lifting problems. See [6, 5, 28] for how this definition is related to syntactically presented intensional identity types. We note that for an Id -type structure (Id, refl) , the arrow $(\text{refl}^*, \partial_*)$ is invertible, and thus any Id -type structure is uniquely extended to an Id^+ -type structure.

Let $\mathbb{I}$ denote the 1-type theory freely generated by a representable arrow $\partial : E \rightarrow U$ equipped with a 1-type structure, a Σ -type structure, and an Id^+ -type structure. Kapulkin and Lumsdaine [27] conjectured that the ∞ -category $\mathbf{Lex}_\infty$ is a localization of the 1-category $\mathbf{Th}(\mathbb{I})$. Strictly, they work with contextual categories with a unit type, Σ -types, and intensional identity types instead of theories over $\mathbb{I}$ in our sense, but it is straightforward to see that those contextual categories are equivalent to democratic models of $\mathbb{I}$. They also gave a specific functor $\mathbf{Th}(\mathbb{I}) \rightarrow \mathbf{Lex}_\infty$ and conjectured that it is a localization functor. We prove their conjecture using the theory of ∞ -type theories and the equivalence $\mathbf{Th}(\mathbb{E}_\infty) \simeq \mathbf{Lex}_\infty$.

We construct the functor $\mathbf{Th}(\mathbb{I}) \rightarrow \mathbf{Lex}_\infty \simeq \mathbf{Th}(\mathbb{E}_\infty)$ differently from Kapulkin and Lumsdaine. A first attempt is to construct a morphism between $\mathbb{I}$ and $\mathbb{E}_\infty$, but this fails: since the generating representable arrow ∂ is not univalent in $\mathbb{I}$, we do not have a morphism $\mathbb{E}_\infty \rightarrow \mathbb{I}$; since ∂ is not 0-truncated in $\mathbb{E}_\infty$, we do not have a morphism $\mathbb{I} \rightarrow \mathbb{E}_\infty$. We thus introduce an intermediate ∞ -type theory $\mathbb{I}_\infty$ defined as the free ∞ -type theory generated by the same data as $\mathbb{I}$ but without truncatedness. Then $\mathbb{I}$ is the universal 1-type theory under $\mathbb{I}_\infty$, and $\mathbb{E}_\infty$ is the universal ∞ -type theory under $\mathbb{I}_\infty$ inverting the morphisms $\text{refl} : E \rightarrow \text{Id}^* E$ and $|\text{id}| : U \rightarrow \underline{\text{Eq}}(\partial)$. We thus have a span of ∞ -type theories

$$\mathbb{I} \xleftarrow{\tau} \mathbb{I}_\infty \xrightarrow{\gamma} \mathbb{E}_\infty. \quad (12)$$

Since any morphism $F : \mathbb{T} \rightarrow \mathbb{T}'$ between ∞ -type theories induces an adjunction $F_! \dashv F^* : \mathbf{Th}(\mathbb{T}) \rightarrow \mathbf{Th}(\mathbb{T}')$ as $\mathbf{Th}(\mathbb{T}) = \mathbf{Lex}(\mathbb{T}, \mathbf{S})$, we have a functor

$$\mathbf{Th}(\mathbb{I}) \xrightarrow{\tau^*} \mathbf{Th}(\mathbb{I}_\infty) \xrightarrow{\gamma_!} \mathbf{Th}(\mathbb{E}_\infty). \quad (13)$$

We define the *weak equivalences* in $\mathbf{Th}(\mathbb{I})$ to be the morphisms inverted by the functor $\gamma_! \tau^*$ and write $L(\mathbf{Th}(\mathbb{I}))$ for the localization by the weak equivalences.

Theorem 6.2. *The functor $\gamma_! \tau^* : \mathbf{Th}(\mathbb{I}) \rightarrow \mathbf{Th}(\mathbb{E}_\infty)$ induces an equivalence of ∞ -categories*

$$L(\mathbf{Th}(\mathbb{I})) \simeq \mathbf{Th}(\mathbb{E}_\infty).$$

Moreover, the composite $\mathbf{Th}(\mathbb{I}) \xrightarrow{\gamma_! \tau^*} \mathbf{Th}(\mathbb{E}_\infty) \xrightarrow{\simeq} \mathbf{Lex}_\infty$ coincides with the functor considered by Kapulkin and Lumsdaine [27, Conjecture 3.7].

Remark 6.3. The construction of the functor $\gamma_! \tau^* : \mathbf{Th}(\mathbb{I}) \rightarrow \mathbf{Th}(\mathbb{E}_\infty)$ is easily generalized to extensions with type-theoretic structures such as Π -types, (higher) inductive types, and universes. For example, if we extend $\mathbb{I}$ with Π -types, then we have a span

$$\mathbb{I}^\Pi \xleftarrow{\tau} \mathbb{I}_\infty^\Pi \xrightarrow{\gamma} \mathbb{E}_\infty^\Pi$$

by extending $\mathbb{I}_\infty$ with Π -types. We expect that similar results to Theorem 6.2 can be proved for a wide range of extensions of $\mathbb{I}$, which is left as future work. See Section 6.2 for discussion.

6.1 Proof of the theorem This subsection is devoted to the proof of Theorem 6.2. We use Cisinski's results on localizations of ∞ -categories [12, Chapter 7]. We first give the category $\mathbf{Th}(\mathbb{I})$ the structure of a *category with weak equivalences and cofibrations* (we recall the definition below) and show that the functor $\gamma_! \tau^*$ is *right exact*. We then show that the functor $\gamma_! \tau^*$ satisfies the *left approximation property* (also recalled below), which implies that the induced functor on localization is an equivalence.

Definition 6.4. A *category with weak equivalences and cofibrations* is a category $\mathcal{C}$ equipped with two classes of arrows called *weak equivalences* and *cofibrations* satisfying the conditions below. An object $x \in \mathcal{C}$ is *cofibrant* if the arrow $0 \rightarrow x$ is a cofibration. An arrow is a *trivial cofibration* if it is both a weak equivalence and a cofibration.

1. $\mathcal{C}$ has an initial object.
2. All the identities are trivial cofibrations, and weak equivalences and cofibrations are closed under composition.
3. The weak equivalences satisfy the *2-out-of-3* property: if u and v are a composable pair of arrows and if two of u , v , and vu are weak equivalences, then so is the rest.
4. (Trivial) cofibrations are stable under pushouts along arbitrary arrows between cofibrant objects: if $x, x' \in \mathcal{C}$ are cofibrant objects, $i : x \rightarrow y$ is a (trivial) cofibration, and $u : x \rightarrow x'$ is an arbitrary arrow, then the pushout $u_! y$ exists and the arrow $x' \rightarrow u_! y$ is a (trivial) cofibration.
5. Any arrow $u : x \rightarrow y$ with cofibrant domain factors into a cofibration $x \rightarrow y'$ followed by a weak equivalence $y' \rightarrow y$.

Definition 6.5. Let $\mathcal{C}$ be a category with weak equivalences and cofibrations and $\mathcal{D}$ an ∞ -category with finite colimits. A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is *right exact* if it sends trivial cofibrations between cofibrant objects to invertible arrows and preserves initial objects and pushouts of cofibrations along arrows between cofibrant objects. A right exact functor $F : \mathcal{C} \rightarrow \mathcal{D}$ has the *left approximation property* if the following conditions hold:

1. an arrow in $\mathcal{C}$ is a weak equivalence if and only if it becomes invertible in $\mathcal{D}$;
2. for any cofibrant object $x \in \mathcal{C}$ and any arrow $u : F(x) \rightarrow y$ in $\mathcal{D}$, there exists an arrow $u' : x \rightarrow y'$ in $\mathcal{C}$ such that $F(y') \simeq y$ under $F(x)$.

Proposition 6.6. Any right exact functor $F : \mathcal{C} \rightarrow \mathcal{D}$ with the left approximation property induces an equivalence $L(\mathcal{C}) \simeq \mathcal{D}$.

Proof. By [12, Proposition 7.6.15]. □

Our first task will be to show that $\mathbf{Th}(\mathbb{I})$ admits the structure of a category with weak equivalences and cofibrations. We have already defined the weak equivalences in $\mathbf{Th}(\mathbb{I})$ as those morphisms inverted by $\gamma_! \tau^*$. We define the cofibrations in $\mathbf{Th}(\mathbb{I})$ as follows. Recall that $\mathbf{y} : \mathbb{I}^{\text{op}} \rightarrow \mathbf{Th}(\mathbb{I})$ is the Yoneda embedding and $\mathbf{P}_\partial : \mathbb{I} \rightarrow \mathbb{I}$ is the polynomial functor associated with $\partial : E \rightarrow U$.

Definition 6.7. The *generating cofibrations* in $\mathbf{Th}(\mathbb{I})$ are the following morphisms:

- $\mathbf{y}(\mathbf{P}_\partial^n(1)) \rightarrow \mathbf{y}(\mathbf{P}_\partial^n(U))$ for $n \geq 0$;
- $\mathbf{y}(\mathbf{P}_\partial^n(\partial)) : \mathbf{y}(\mathbf{P}_\partial^n(U)) \rightarrow \mathbf{y}(\mathbf{P}_\partial^n(E))$ for $n \geq 0$.

The class of *cofibrations* in $\mathbf{Th}(\mathbb{I})$ is the closure of the generating cofibrations under retracts, pushouts along arbitrary morphisms, and transfinite composition. Cofibrations in $\mathbf{Th}(\mathbb{I}_\infty)$ and

$\mathbf{Th}(\mathbb{E}_\infty)$ are defined in the same way. Note that the functors $\tau_!$ and $\gamma_!$ preserve generating cofibrations.

Remark 6.8. Our choice of generating cofibrations in $\mathbf{Th}(\mathbb{I})$ coincides with the choice by Kapulkin and Lumsdaine [27]. That is, $\mathbf{y}(\mathbf{P}_\partial^n(1))$ is the theory freely generated by a context of length n , $\mathbf{y}(\mathbf{P}_\partial^n(U))$ is the theory freely generated by a type over a context of length n , and $\mathbf{y}(\mathbf{P}_\partial^n(E))$ is the theory freely generated by a term over a context of length n . This is verified as follows. Let K be an $\mathbb{I}$ -theory and let $\mathcal{M}$ be the democratic model of $\mathbb{I}$ corresponding to K via the equivalence $\mathbf{Th}(\mathbb{I}) \simeq \mathbf{Mod}^{\text{dem}}(\mathbb{I})$. By construction, a morphism $\mathbf{y}x \rightarrow K$ correspond to a global section $\mathcal{M}(\star) \rightarrow \mathcal{M}(x)$ for any object $x \in \mathbb{I}$. Then, by the universal property of $\mathbf{P}_\partial$, a morphism $\Gamma : \mathbf{y}(\mathbf{P}_\partial^n(1)) \rightarrow K$ corresponds to a list of maps

$$\begin{aligned} \Gamma_1 &: \mathcal{M}(\star)/\{\Gamma_0\} \rightarrow \mathcal{M}(U) \\ \Gamma_2 &: \mathcal{M}(\star)/\{\Gamma_1\} \rightarrow \mathcal{M}(U) \\ &\vdots \\ \Gamma_n &: \mathcal{M}(\star)/\{\Gamma_{n-1}\} \rightarrow \mathcal{M}(U) \end{aligned} \tag{14}$$

where $\{\Gamma_0\} = 1$ and $\mathcal{M}(\star)/\{\Gamma_{i+1}\} \simeq \Gamma_i^* \mathcal{M}(E)$. Since we think of maps of $\mathcal{M}(U)$ as types, such a list of maps can be regarded as a context of length n . Under this identification, an extension $\mathbf{y}(\mathbf{P}_\partial^n(U)) \rightarrow K$ of Γ corresponds to a map $\mathcal{M}(\star)/\{\Gamma_n\} \rightarrow \mathcal{M}(U)$, that is, a type over Γ . Similarly, an extension $\mathbf{y}(\mathbf{P}_\partial^n(E)) \rightarrow K$ of Γ corresponds to a map $\mathcal{M}(\star)/\{\Gamma_n\} \rightarrow \mathcal{M}(E)$, that is, a term over Γ . Hence, morphisms from $\mathbf{y}(\mathbf{P}_\partial^n(1))$, $\mathbf{y}(\mathbf{P}_\partial^n(U))$, and $\mathbf{y}(\mathbf{P}_\partial^n(E))$ correspond to contexts, types, and terms, respectively. In this view, a cofibration in $\mathbf{Th}(\mathbb{I})$ is an extension by types and terms, but without any equation. In particular, cofibrant $\mathbb{I}$ -theories are those freely generated by types and terms.

Theorem 6.9. *The classes of cofibrations and weak equivalences endow $\mathbf{Th}(\mathbb{I})$ with the structure of a category with weak equivalences and cofibrations.*

By definition, $\mathbf{Th}(\mathbb{I})$ satisfies Items 1 to 3 of Definition 6.4, and cofibrations are stable under arbitrary pushouts. To make $\mathbf{Th}(\mathbb{I})$ a category with weak equivalences and cofibrations, it remains to verify the stability of trivial cofibrations under pushouts and the factorization axiom. The former is true if the functor $\gamma_! \tau^* : \mathbf{Th}(\mathbb{I}) \rightarrow \mathbf{Th}(\mathbb{E}_\infty)$ preserves initial objects and pushouts of cofibrations along morphisms between cofibrant objects. Note that this also implies that the functor $\gamma_! \tau^*$ must be right exact. Since $\gamma_!$ preserves all colimits, it suffices to show that τ^* has this property. For the latter, we introduce the notion of *trivial fibration*.

Definition 6.10. A morphism $f : K \rightarrow L$ in $\mathbf{Th}(\mathbb{I})$ is a *trivial fibration* if it has the right lifting property against cofibrations: for any commutative square

$$\begin{array}{ccc} A & \xrightarrow{g} & K \\ i \downarrow & & \downarrow f \\ B & \xrightarrow{h} & L \end{array}$$

in which i is a cofibration, there exists a morphism $k : B \rightarrow K$ such that $fk = h$ and $ki = g$. Trivial fibrations in $\mathbf{Th}(\mathbb{I}_\infty)$ and $\mathbf{Th}(\mathbb{E}_\infty)$ are defined in the same way. By a standard argument in model category theory, f is a trivial fibration if and only if it has the right lifting property against generating cofibrations.

By the *small object argument*, we know that any morphism in $\mathbf{Th}(\mathbb{I})$ factors into a cofibration followed by a trivial fibration. Thus, to show that $\mathbf{Th}(\mathbb{I})$ satisfies the factorization axiom it is enough to show that trivial fibrations are inverted by $\gamma_! \tau^*$. In conclusion, theorem 6.9 will follow from the following two propositions.

Proposition 6.11. *The functor $\tau^* : \mathbf{Th}(\mathbb{I}) \rightarrow \mathbf{Th}(\mathbb{I}_\infty)$ preserves initial objects and pushouts of cofibrations along morphisms between cofibrant objects.*

Proposition 6.12. *Trivial fibrations in $\mathbf{Th}(\mathbb{I})$ are inverted by the functor $\gamma_! \tau^* : \mathbf{Th}(\mathbb{I}) \rightarrow \mathbf{Th}(\mathbb{I}_\infty)$.*

We begin by proving Proposition 6.12. It can be broken into the following two lemmas.

Lemma 6.13. *In $\mathbf{Th}(\mathbb{I}_\infty)$, the trivial fibrations are precisely the invertible morphisms. Equivalently, all the morphisms are cofibrations.*

Lemma 6.14. *For any $\mathbb{I}$ -theory K , the unit $\tau^* K \rightarrow \gamma^* \gamma_! \tau^* K$ is a trivial fibration.*

Proof of Proposition 6.12. Let $f : K \rightarrow L$ be a trivial fibration in $\mathbf{Th}(\mathbb{I})$. Consider the naturality square

$$\begin{array}{ccc} \tau^* K & \xrightarrow{\eta_{\tau^* K}} & \gamma^* \gamma_! \tau^* K \\ \tau^* f \downarrow & & \downarrow \gamma^* \gamma_! \tau^* f \\ \tau^* L & \xrightarrow{\eta_{\tau^* L}} & \gamma^* \gamma_! \tau^* L \end{array}$$

where η is the unit of the adjunction $\gamma_! \dashv \gamma^*$. By an adjoint argument, $\tau^* f$ is a trivial fibration. By Lemma 6.14, $\eta_{\tau^* K}$ and $\eta_{\tau^* L}$ are trivial fibrations. Since the domains of the generating cofibrations are cofibrant, it follows that $\gamma^* \gamma_! \tau^* f$ is a trivial fibration. Then, again by an adjoint argument, $\gamma_! \tau^* f$ is a trivial fibration and thus invertible by Lemma 6.13. $\square$

Lemma 6.13 is straightforward.

Proof of Lemma 6.13. Since the representable arrow ∂ in $\mathbb{I}_\infty$ has an **Id**-type structure, the diagonal $E \rightarrow E \times_U E$ is a pullback of ∂ . This implies that the codiagonal $\mathbf{y}(\mathbf{P}_\partial^n(E \times_U E)) \rightarrow \mathbf{y}(\mathbf{P}_\partial^n(E))$ in $\mathbf{Th}(\mathbb{I}_\infty)$ is a cofibration for $n \geq 0$. Similarly, the univalence of ∂ implies that the codiagonal $\mathbf{y}(\mathbf{P}_\partial^n(U \times U)) \rightarrow \mathbf{y}(\mathbf{P}_\partial^n(U))$ in $\mathbf{Th}(\mathbb{I}_\infty)$ is a cofibration for $n \geq 0$. Hence, for any generating cofibration $i : A \rightarrow B$ in $\mathbf{Th}(\mathbb{I}_\infty)$, the codiagonal $B +_A B \rightarrow B$ is a cofibration, and thus cofibrations in $\mathbf{Th}(\mathbb{I}_\infty)$ are closed under codiagonal. It then follows from the dual of Lemma 5.19 that cofibrations in $\mathbf{Th}(\mathbb{I}_\infty)$ have the right cancellation property: for a composable pair of morphisms f and g , if f and gf are cofibrations, then so is g . Therefore, it suffices to show that all the objects of $\mathbf{Th}(\mathbb{I}_\infty)$ are cofibrant. One can show that $\mathbf{y}(\mathbf{P}_\partial^n(U))$'s and $\mathbf{y}(\mathbf{P}_\partial^n(E))$'s generate $\mathbf{Th}(\mathbb{I}_\infty)$ under colimits. Since they are cofibrant, it then suffices to show that cofibrant objects are closed under colimits. Since colimits are constructible from coproducts and pushouts [36, Proposition 4.4.2.6], it is enough to show that cofibrant objects are closed under coproducts and pushouts. Cofibrant objects are closed under coproducts since cofibrations are closed under pushouts and transfinite composition. Let $f : X \rightarrow A$ and $g : X \rightarrow B$ be morphisms between cofibrant objects. By the right cancellation property of cofibrations, f is a cofibration. Then the morphism $B \rightarrow A +_X B$ is a cofibration, and thus $A +_X B$ is cofibrant. $\square$

For Proposition 6.11 and Lemma 6.14, we need analysis of the functors τ^* and $\gamma_!$. We work with democratic models instead of theories via the equivalence $\mathbf{Mod}^{\text{dem}}(\mathbb{T}) \simeq \mathbf{Th}(\mathbb{T})$ (Theorem 4.14). We first note that the functors $\tau^* : \mathbf{Mod}(\mathbb{I}) \rightarrow \mathbf{Mod}(\mathbb{I}_\infty)$ and $\gamma^* : \mathbf{Mod}(\mathbb{E}_\infty) \rightarrow \mathbf{Mod}(\mathbb{I}_\infty)$ are fully faithful. More precisely, the models of $\mathbb{I}$ are the models $\mathcal{M}$ of $\mathbb{I}_\infty$ such that $\mathcal{M}(U)$ and $\mathcal{M}(E)$ are 0-truncated objects in $\mathbf{RFib}_{\mathcal{M}(\star)}$, and the models of $\mathbb{E}_\infty$ are the models $\mathcal{M}$ of $\mathbb{I}_\infty$ such that the map $\mathcal{M}(\text{refl}) : \mathcal{M}(E) \rightarrow \mathcal{M}(\text{Id}^*E)$ is invertible and the representable map $\mathcal{M}(\partial)$ is a univalent. It is also clear from this description that the functors τ^* and γ^* preserve democratic models. Hence, we may identify the functors $\tau^* : \mathbf{Th}(\mathbb{I}) \rightarrow \mathbf{Th}(\mathbb{I}_\infty)$ and $\gamma^* : \mathbf{Th}(\mathbb{E}_\infty) \rightarrow \mathbf{Th}(\mathbb{I}_\infty)$ with the inclusions $\mathbf{Mod}^{\text{dem}}(\mathbb{I}) \subset \mathbf{Mod}^{\text{dem}}(\mathbb{I}_\infty)$ and $\mathbf{Mod}^{\text{dem}}(\mathbb{E}_\infty) \subset \mathbf{Mod}^{\text{dem}}(\mathbb{I}_\infty)$, respectively.

To prove Lemma 6.14, we concretely describe $\gamma_! \mathcal{M} \in \mathbf{Mod}^{\text{dem}}(\mathbb{E}_\infty)$ for a democratic model $\mathcal{M}$ of $\mathbb{I}$. Recall that identity types induce notions of *homotopy* and *homotopy equivalence*. Let $\mathcal{M}$ be a democratic model of $\mathbb{I}$ and let $\Gamma \in \mathcal{M}(\star)$ be an object and $A : \mathcal{M}(\star)/\Gamma \rightarrow \mathcal{M}(U)$ a map. The *path object* $P(A) \in \mathcal{M}(\star)$ is defined by the pullback

$$\begin{array}{ccc} \mathcal{M}(\star)/P(A) & \xrightarrow{\quad\quad\quad} & \mathcal{M}(E) \\ \downarrow & & \downarrow \\ \mathcal{M}(\star)/(\{A\} \times_\Gamma \{A\}) & \longrightarrow \mathcal{M}(E \times_U E) \xrightarrow{\mathcal{M}(\text{Id})} & \mathcal{M}(U). \end{array}$$

Since the proof of Lemma 5.18 uses only 1-type and Σ -type structures, the same is true for this $\mathcal{M}$ and in particular every object $\Gamma \in \mathcal{M}(\star)$ is written as $\{B\}$ for some map $B : \mathcal{M}(\star) \rightarrow \mathcal{M}(U)$. Thus, a path object $P(\Gamma)$ can be chosen for any object $\Gamma \in \mathcal{M}(\star)$. For morphisms $f_1, f_2 : \Delta \rightarrow \Gamma$ in $\mathcal{M}(\star)$, a *homotopy* from f_1 to f_2 is a lift of $(f_1, f_2) : \Delta \rightarrow \Gamma \times \Gamma$ along $P(\Gamma) \rightarrow \Gamma \times \Gamma$. A morphism $f : \Gamma \rightarrow \Delta$ is a *homotopy equivalence* if there exists an inverse of f up to homotopy. One can check that the notion of homotopy equivalence is independent of the choice of path objects and that the first (or second) projection $P(A) \rightarrow \{A\}$ is a homotopy equivalence for any $A : \mathcal{M}(\star)/\Gamma \rightarrow \mathcal{M}(U)$.

Proposition 6.15 (Avigad, Kapulkin, and Lumsdaine [4, Theorem 3.2.5]). *Let $\mathcal{M}$ be a democratic model of $\mathbb{I}$. Then $\mathcal{M}(\star)$ is equipped with a structure of category with weak equivalences and fibrations (the dual notion of Definition 6.4) where the weak equivalences are the homotopy equivalences and the fibrations are the display maps. Moreover, all the objects of $\mathcal{M}(\star)$ are fibrant.*

Corollary 6.16. *For any democratic model $\mathcal{M}$ of $\mathbb{I}$, the localization $L(\mathcal{M}(\star))$ has finite limits and satisfies the universal property that a functor $F : \mathcal{M}(\star) \rightarrow \mathcal{C}$ to a left exact ∞ -category induces a left exact functor $L(\mathcal{M}(\star)) \rightarrow \mathcal{C}$ if and only if F sends homotopy equivalences to invertible arrows and preserves terminal objects and pullbacks of display maps.*

Proof. By [12, Proposition 7.5.6]. □

The construction $\mathcal{M} \mapsto L(\mathcal{M}(\star))$ is the one considered by Kapulkin and Lumsdaine [27, Conjecture 3.7], and thus the following lemma implies the second assertion of Theorem 6.2.

Lemma 6.17. *The functor*

$$\gamma_! : \mathbf{Mod}^{\text{dem}}(\mathbb{I}) \subset \mathbf{Mod}^{\text{dem}}(\mathbb{I}_\infty) \rightarrow \mathbf{Mod}^{\text{dem}}(\mathbb{E}_\infty) \simeq \mathbf{Lex}_\infty$$

is naturally equivalent to the functor $\mathcal{M} \mapsto L(\mathcal{M}(\star))$.

Proof. For a left exact ∞ -category $\mathcal{C}$, let $\mathcal{R}_{\mathcal{C}}$ denote the corresponding democratic model of $\mathbb{E}_{\infty}$. Recall that $\mathcal{R}_{\mathcal{C}}(\star) = \mathcal{C}$, that $\mathcal{R}_{\mathcal{C}}(U) \subset \mathcal{C}^{\rightarrow}$ is the wide subcategory whose arrows are the pullback squares, and that $\mathcal{R}_{\mathcal{C}}(E) = \mathcal{R}_{\mathcal{C}}(U) \times_{\mathcal{C}^{\rightarrow}} (S \rhd \mathcal{C})$ is the ∞ -category of sections in $\mathcal{C}$. Let $\mathcal{M}$ be a democratic model of $\mathbb{I}$. We construct a natural equivalence

$$\mathbf{Mod}^{\text{dem}}(\mathbb{I}_{\infty})(\mathcal{M}, \mathcal{R}_{\mathcal{C}}) \simeq \mathbf{Lex}_{\infty}(L(\mathcal{M}(\star)), \mathcal{C}) \quad (15)$$

so that $L(\mathcal{M}(\star))$ satisfies the same universal property as $\gamma_! \mathcal{M}$.

We first observe that the forgetful map

$$\mathbf{Mod}^{\text{dem}}(\mathbb{I}_{\infty})(\mathcal{M}, \mathcal{R}_{\mathcal{C}}) \rightarrow \mathbf{Cat}_{\infty}(\mathcal{M}(\star), \mathcal{C}) \quad (16)$$

is (-1) -truncated. That is, the base functor $F(\star) : \mathcal{M}(\star) \rightarrow \mathcal{C}$ uniquely determines the rest of the components of a morphism $F : \mathcal{M} \rightarrow \mathcal{R}_{\mathcal{C}}$ of models of $\mathbb{I}_{\infty}$. To see this, observe that context comprehension defines functors $\mathcal{M}(U) \rightarrow \mathcal{M}(\star)^{\rightarrow}$ and $\mathcal{M}(E) \rightarrow S \rhd \mathcal{M}(\star)$ over $\mathcal{M}(\star)$ and that the components $F(U)$ and $F(E)$ of a morphism $F : \mathcal{M} \rightarrow \mathcal{C}$ are uniquely determined by the commutativity of the following diagram.

$$\begin{array}{ccccc} \mathcal{M}(E) & \xrightarrow{\quad F(E) \quad} & \mathcal{R}_{\mathcal{C}}(E) & & \\ \downarrow & \searrow & \downarrow & \swarrow & \\ & S \rhd \mathcal{M}(\star) & \xrightarrow{\quad S \rhd F(\star) \quad} & S \rhd \mathcal{C} & \\ & \downarrow & \downarrow & \downarrow & \\ \mathcal{M}(U) & \xrightarrow{\quad F(U) \quad} & \mathcal{R}_{\mathcal{C}}(U) & & \\ & \searrow & \downarrow & \swarrow & \\ & \mathcal{M}(\star)^{\rightarrow} & \xrightarrow{\quad F(\star)^{\rightarrow} \quad} & \mathcal{C}^{\rightarrow} & \end{array} \quad (17)$$

Homotopies expressing commutation of F with type-theoretic structures are also unique because those type-theoretic structures on $\mathcal{R}_{\mathcal{C}}$ are defined by universal properties in $\mathcal{C}$.

We then show that the image of the map (16) is precisely those functors sending homotopy equivalences to invertible arrows and preserving terminal objects and pullbacks of display maps. Then we get the equivalence (15) by Corollary 6.16. Let $F(\star) : \mathcal{M}(\star) \rightarrow \mathcal{C}$ be a functor. To extend $F(\star)$ to a morphism $\mathcal{M} \rightarrow \mathcal{R}_{\mathcal{C}}$, the functor $F(\star)$ must preserve terminal objects. There exist maps $F(U)$ and $F(E)$ as in the diagram (17) if and only if $F(\star)$ preserves pullbacks of display maps. Thus, $F(\star)$ extends to a morphism $\mathcal{M} \rightarrow \mathcal{R}_{\mathcal{C}}$ of models of $\mathbb{G}_{\infty}$ if and only if it preserves terminal objects and pullbacks of display maps. Suppose that $F(\star)$ extends to a morphism of models of $\mathbb{G}_{\infty}$. It automatically commutes with 1-type structures and Σ -type structures because they are defined by the identity arrows and composition, respectively, on $\mathcal{C}$. If F commutes with Id^+ -type structures, then $F(\star)$ sends a path object $P(\Gamma) \rightarrow \Gamma \times \Gamma$ of an object $\Gamma \in \mathcal{M}(\star)$ to the diagonal arrow $F(\star)(\Gamma) \rightarrow F(\star)(\Gamma) \times F(\star)(\Gamma)$, from which it follows that $F(\star)$ sends homotopy equivalences to invertible arrows. Conversely, suppose that $F(\star)$ sends homotopy equivalences to invertible arrows. Then it sends the first projection $P(A) \rightarrow \{A\}$ to an invertible arrow for any $A : \mathcal{M}(\star)/\Gamma \rightarrow \mathcal{M}(U)$. This implies that $F(\star)$ sends the path object $P(A) \rightarrow \{A\} \times_{\Gamma} \{A\}$ to the diagonal $F(\star)(\{A\}) \rightarrow F(\star)(\{A\}) \times_{F(\star)(\Gamma)} F(\star)(\{A\})$, which means that F commutes with Id^+ -type structures. $\square$

We characterize trivial fibrations of democratic models of $\mathbb{I}_{\infty}$ in the same way as Kapulkin and Lumsdaine [27].

Lemma 6.18. *A morphism $F : \mathcal{M} \rightarrow \mathcal{N}$ in $\mathbf{Mod}^{\text{dem}}(\mathbb{I}_\infty)$ is a trivial fibration if and only if the following conditions are satisfied:*

Type lifting *for any object $\Gamma \in \mathcal{M}(\star)$ and any map $A : \mathcal{N}(\star)/F(\Gamma) \rightarrow \mathcal{N}(U)$, there exists a map $A' : \mathcal{M}(\star)/\Gamma \rightarrow \mathcal{M}(U)$ such that $F(A') \simeq A$;*

Term lifting *for any object $\Gamma \in \mathcal{M}(\star)$, any map $A : \mathcal{M}(\star)/\Gamma \rightarrow \mathcal{M}(U)$, and any map $a : \mathcal{N}(\star)/F(\Gamma) \rightarrow \mathcal{N}(E)$ over $F(A)$, there exists a map $a' : \mathcal{M}(\star)/\Gamma \rightarrow \mathcal{M}(E)$ over A such that $F(a') \simeq a$ over $F(A)$.*

Proof. Let K be the $\mathbb{I}_\infty$ -theory corresponding to $\mathcal{M}$, that is, $K(x)$ is the space of global sections of $\mathcal{M}(x)$ for $x \in \mathbb{I}_\infty$. As we saw in Remark 6.8, a morphism $\Gamma : \mathbf{y}(\mathbf{P}_\partial^n(1)) \rightarrow K$ corresponds to a list of maps (14), and extensions $\mathbf{y}(\mathbf{P}_\partial^n(U)) \rightarrow K$ and $\mathbf{y}(\mathbf{P}_\partial^n(E)) \rightarrow K$ of Γ correspond to maps $\mathcal{M}(\star)/\{\Gamma_n\} \rightarrow \mathcal{M}(U)$ and $\mathcal{M}(\star)/\{\Gamma_n\} \rightarrow \mathcal{M}(E)$, respectively. Then, type lifting and term lifting implies the right lifting property against the generating cofibrations. The converse is also true because, since $\mathcal{M}$ is democratic, any object of $\mathcal{M}(\star)$ is of the form $\{\Gamma_n\}$ for some list of maps (14). $\square$

Proof of Lemma 6.14. We check type lifting and term lifting along the unit $\eta : \mathcal{M} \rightarrow \gamma_! \mathcal{M} \simeq L(\mathcal{M}(\star))$ for a democratic model $\mathcal{M}$ of $\mathbb{I}$. Type lifting is immediate because any object in $L(\mathcal{M}(\star))/\eta(\Gamma)$ is represented by a fibration $A \rightarrow \Gamma$ in $\mathcal{M}(\star)$. For term lifting, we also need the fact that $\mathcal{M}(\star)$ is not only a category of fibrant objects but also a *tribe* [24] and in particular a *path category* [48]. In this special case, a section of $\eta(A) \rightarrow \eta(\Gamma)$ in $L(\mathcal{M}(\star))$ for a fibration $A \rightarrow \Gamma$ in $\mathcal{M}(\star)$ is represented by a section in $\mathcal{M}(\star)$ by [48, Corollary 2.19]. $\square$

This concludes the proof that trivial fibrations are weak equivalences in $\mathbf{Th}(\mathbb{I})$. It remains to show Proposition 6.11, which follows from the following theorem.

Theorem 6.19. *Any cofibrant object of $\mathbf{Mod}^{\text{dem}}(\mathbb{I}_\infty)$ belongs to $\mathbf{Mod}^{\text{dem}}(\mathbb{I})$.*

Proof of Proposition 6.11. Initial objects are cofibrant, and the pushout of a cofibration along a morphism between cofibrant objects is cofibrant. Thus, by Theorem 6.19, $\mathbf{Mod}^{\text{dem}}(\mathbb{I}) \subset \mathbf{Mod}^{\text{dem}}(\mathbb{I}_\infty)$ is closed under these colimits. $\square$

Theorem 6.19 is the hardest part. We may think of this theorem as a form of *coherence problem*. A general democratic model of $\mathbb{I}_\infty$ may contain a lot of non-trivial homotopies, but Theorem 6.19 says that all the homotopies in a cofibrant democratic model of $\mathbb{I}_\infty$ are trivial.

A successful approach to coherence problems in the categorical semantics of type theory is to replace a non-split model by a split model [20, 34]. Following them, we construct, given a democratic model $\mathcal{M}$ of $\mathbb{I}_\infty$, a democratic model $\text{Sp}\mathcal{M}$ of $\mathbb{I}$ equipped with a trivial fibration $\varepsilon : \text{Sp}\mathcal{M} \rightarrow \mathcal{M}$. Then Theorem 6.19 follows from a retract argument.

The construction of $\text{Sp}\mathcal{M}$ crucially relies on Shulman’s result of replacing any (Grothendieck) ∞ -topos by a well-behaved model category called a *type-theoretic model topos* [43]. Let $\mathcal{M}$ be a democratic model of $\mathbb{I}_\infty$. Recall that it consists of a base ∞ -category $\mathcal{M}(\star)$, a representable map $\mathcal{M}(\partial) : \mathcal{M}(E) \rightarrow \mathcal{M}(U)$ of right fibrations over $\mathcal{M}(\star)$, and some other structures. Since the ∞ -category $\mathbf{RFib}_{\mathcal{M}(\star)}$ is an ∞ -topos, it is a localization $\gamma_{\mathcal{X}} : \mathcal{X} \rightarrow \mathbf{RFib}_{\mathcal{M}(\star)}$ of some type-theoretic model topos $\mathcal{X}$ [43, Theorem 11.1]. Then there exists a fibration $\partial_{\mathcal{X}} : E_{\mathcal{X}} \rightarrow U_{\mathcal{X}}$ between fibrant objects in $\mathcal{X}$ such that $\gamma_{\mathcal{X}}(\partial_{\mathcal{X}}) \simeq \mathcal{M}(\partial)$. We will choose $\partial_{\mathcal{X}}$ that has a 1-type structure, a Σ -type structure, and an Id^+ -type structure so that it induces a model of $\mathbb{I}$.

We remind the reader that the type-theoretic model topos $\mathcal{X}$ has nice properties by definition [43, Definition 6.1]: the underlying 1-category is a Grothendieck 1-topos, the cofibrations are the

monomorphisms, and the model structure is right proper. The right properness in particular implies that the localization functor $\gamma_{\mathcal{X}} : \mathcal{X} \rightarrow \mathbf{RFib}_{\mathcal{M}(\star)}$ preserves pullbacks of fibrations and pushforwards of fibrations along fibrations used in the definitions of 1-type, Σ -type, and Id^+ -type structures.

Lemma 6.20. *For any cofibration $i : A \rightarrow B$ between fibrant objects in $\mathcal{X}$ and for any fibration $p : Y \rightarrow X$ in $\mathcal{X}$, the induced map*

$$(i^*, p_*) : Y^B \rightarrow Y^A \times_{X^A} X^B$$

is a fibration.

Proof. By an adjoint argument, it suffices to show that for any trivial cofibration $i' : A' \rightarrow B'$, the induced map

$$(i', i) : (A' \times B) \amalg_{(A' \times A)} (B' \times A) \rightarrow B' \times B$$

is a trivial cofibration. Since the class of cofibrations are the class of monomorphisms in the Grothendieck 1-topos $\mathcal{X}$, the map (i', i) is a cofibration. Since A and B are fibrant and since the model structure is right proper, the maps $i' \times A : A' \times A \rightarrow B' \times A$ and $i' \times B : A' \times B \rightarrow B' \times B$ are weak equivalences. By 2-out-of-3, the map (i', i) is a weak equivalence. $\square$

Lemma 6.21. *For any choice of $\partial_{\mathcal{X}}$, there exists an Id^+ -type structure on $\partial_{\mathcal{X}}$ sent by $\gamma_{\mathcal{X}} : \mathcal{X} \rightarrow \mathbf{RFib}_{\mathcal{M}(\star)}$ to the Id^+ -type structure on $\mathcal{M}(\partial)$.*

Proof. Since all the objects in $\mathcal{X}$ are cofibrant and $\partial_{\mathcal{X}}$ is a fibration between fibrant objects, the commutative square (11) for $\mathcal{M}(\partial)$ can be lifted to one for $\partial_{\mathcal{X}}$. The map $\text{refl} : E \rightarrow \text{Id}^* E$ is a monomorphism in $\mathcal{X}$ and thus a cofibration. Applying Lemma 6.20 for the slice $\mathcal{X}/U$ instead of $\mathcal{X}$, we see that the induced map $(\text{refl}^*, \partial_*)$ is a fibration. The codomain of the map $(\text{refl}^*, \partial_*)$ is fibrant by the right properness. Hence, the section of $(\text{refl}^*, \partial_*)$ for $\mathcal{M}(\partial)$ can be lifted to one for $\partial_{\mathcal{X}}$. $\square$

Lemma 6.22. *One can choose $\partial_{\mathcal{X}}$ that has a 1-type structure and a Σ -type structure sent by $\gamma_{\mathcal{X}} : \mathcal{X} \rightarrow \mathbf{RFib}_{\mathcal{M}(\star)}$ to those structures on $\mathcal{M}(\partial)$.*

Proof. A 1-type structure and a Σ -type structure on ∂ are a pullback of the form

$$\begin{array}{ccc} \text{dom}(\partial^{\otimes n}) & \xrightarrow{\text{p}_n} & E \\ \partial^{\otimes n} \downarrow & & \downarrow \partial \\ \text{cod}(\partial^{\otimes n}) & \xrightarrow{\Sigma_n} & U \end{array}$$

for $n = 0$ and $n = 2$, respectively, where $\partial^{\otimes n}$ is the n -fold composition of the polynomial ∂ . Since $\mathcal{M}$ is a model of $\mathbb{I}_{\infty}$, the map $\mathcal{M}(\partial)$ is equipped with such pullbacks in $\mathbf{RFib}_{\mathcal{M}(\star)}$. However, they give rise to only *homotopy* pullbacks

$$\begin{array}{ccc} \text{dom}(\partial_{\mathcal{X}}^{\otimes n}) & \xrightarrow{\text{p}_n} & E_{\mathcal{X}} \\ \partial_{\mathcal{X}}^{\otimes n} \downarrow & & \downarrow \partial_{\mathcal{X}} \\ \text{cod}(\partial_{\mathcal{X}}^{\otimes n}) & \xrightarrow{\Sigma_n} & U_{\mathcal{X}} \end{array} \tag{18}$$

in $\mathcal{X}$, and thus $\partial_{\mathcal{X}}$ need not have 1-type and Σ -type structures.

The idea of fixing this issue is to replace $\partial_{\mathcal{X}}$ by another fibration $\partial'_{\mathcal{X}} : E'_{\mathcal{X}} \rightarrow U'_{\mathcal{X}}$ between fibrant objects such that the pullbacks of $\partial'_{\mathcal{X}}$ are the homotopy pullbacks of $\partial_{\mathcal{X}}$. Let κ be a regular cardinal such that $\partial_{\mathcal{X}}$ is κ -small. By [43, Theorem 5.22], there exists a fibration $\partial_{\mathcal{X}}^{\kappa} : E_{\mathcal{X}}^{\kappa} \rightarrow U_{\mathcal{X}}^{\kappa}$ between fibrant objects that classifies κ -small fibrations. Moreover, $\partial_{\mathcal{X}}^{\kappa}$ satisfies the univalence axiom with respect to the model structure. Since $\partial_{\mathcal{X}}$ is κ -small, we have a pullback

$$\begin{array}{ccc} E_{\mathcal{X}} & \xrightarrow{\quad \quad} & E_{\mathcal{X}}^{\kappa} \\ \partial_{\mathcal{X}} \downarrow & & \downarrow \partial_{\mathcal{X}}^{\kappa} \\ U_{\mathcal{X}} & \xrightarrow{\quad \quad \iota \quad} & U_{\mathcal{X}}^{\kappa}. \end{array}$$

Factor ι into a weak equivalence $\iota' : U_{\mathcal{X}} \rightarrow U'_{\mathcal{X}}$ followed by a fibration $\pi : U'_{\mathcal{X}} \rightarrow U_{\mathcal{X}}^{\kappa}$, and define $\partial'_{\mathcal{X}} : E'_{\mathcal{X}} \rightarrow U'_{\mathcal{X}}$ to be the pullback of $\partial_{\mathcal{X}}^{\kappa}$ along π . Since $\partial_{\mathcal{X}}^{\kappa}$ satisfies the univalence axiom, we can choose $U'_{\mathcal{X}}$ such that the maps $A \rightarrow U'_{\mathcal{X}}$ correspond to the triples (B_1, B_2, f) consisting of maps $B_1 : A \rightarrow U_{\mathcal{X}}$ and $B_2 : A \rightarrow U_{\mathcal{X}}^{\kappa}$ and a weak equivalence $f : B_1^* E_{\mathcal{X}} \rightarrow B_2^* E_{\mathcal{X}}^{\kappa}$ over A . In particular, we have a *generic* homotopy pullback from a κ -small fibration

$$\begin{array}{ccc} E'_{\mathcal{X}} & \longrightarrow & E_{\mathcal{X}} \\ \partial'_{\mathcal{X}} \downarrow & & \downarrow \partial_{\mathcal{X}} \\ U'_{\mathcal{X}} & \xrightarrow{\quad p \quad} & U_{\mathcal{X}} \end{array} \tag{19}$$

in the sense that any homotopy pullback from a κ -small fibration to $\partial_{\mathcal{X}}$ factors into a strict pullback followed by the homotopy pullback (19).

We now construct 1-type and Σ -type structures on $\partial'_{\mathcal{X}}$. There are homotopy pullbacks as in Eq. (18) for $n = 0$ and $n = 2$ sent by $\gamma_{\mathcal{X}} : \mathcal{X} \rightarrow \mathbf{RFib}_{\mathcal{M}(\star)}$ to the 1-type and Σ -type structures, respectively, on $\mathcal{M}(\partial)$. Since $\partial_{\mathcal{X}}$ is the pullback of $\partial'_{\mathcal{X}}$ along the weak equivalence $\iota' : U_{\mathcal{X}} \rightarrow U'_{\mathcal{X}}$, one can construct a commutative square

$$\begin{array}{ccc} \mathrm{dom}(\partial_{\mathcal{X}}^{\otimes n}) & \xrightarrow{\quad \quad} & \mathrm{dom}((\partial'_{\mathcal{X}})^{\otimes n}) \\ \partial_{\mathcal{X}}^{\otimes n} \downarrow & & \downarrow (\partial'_{\mathcal{X}})^{\otimes n} \\ \mathrm{cod}(\partial_{\mathcal{X}}^{\otimes n}) & \xrightarrow{\quad \quad} & \mathrm{cod}((\partial'_{\mathcal{X}})^{\otimes n}) \end{array}$$

in which the horizontal maps are weak equivalences. Then we have a homotopy pullback from $(\partial'_{\mathcal{X}})^{\otimes n}$ to $\partial_{\mathcal{X}}$, which factors into a strict pullback followed by the homotopy pullback (19) because the composition of polynomials preserves κ -smallness. By construction, this strict pullback is sent by $\gamma_{\mathcal{X}} : \mathcal{X} \rightarrow \mathbf{RFib}_{\mathcal{M}(\star)}$ to the 1-type structure on $\mathcal{M}(\partial)$ when $n = 0$ and to the Σ -type structure on $\mathcal{M}(\partial)$ when $n = 2$. $\square$

By the preceding lemmas, we can choose $\partial_{\mathcal{X}}$ that has a 1-type structure, a Σ -type structure, and an Id^+ -type structure. Then we have a morphism of 1-categories with representable maps $\mathbb{I} \rightarrow \mathcal{X}$, and we define $\mathrm{Sp}\mathcal{M}$ to be the heart of the model of $\mathbb{I}$ defined by the composite $\mathbb{I} \rightarrow \mathcal{X} \rightarrow \mathbf{RFib}_{\mathcal{X}}$ with the Yoneda embedding. Concretely, the base category $(\mathrm{Sp}\mathcal{M})(\star)$ is the full subcategory of $\mathcal{X}$ spanned by the objects Γ such that the map $\Gamma \rightarrow 1$ is a composite of pullbacks of $\partial_{\mathcal{X}}$, and the maps $(\mathrm{Sp}\mathcal{M})(\star)/\Gamma \rightarrow (\mathrm{Sp}\mathcal{M})(U)$ and $(\mathrm{Sp}\mathcal{M})(\star)/\Gamma \rightarrow (\mathrm{Sp}\mathcal{M})(E)$ are the maps $\Gamma \rightarrow U_{\mathcal{X}}$ and $\Gamma \rightarrow E_{\mathcal{X}}$, respectively, in $\mathcal{X}$.

Since the localization functor $\gamma_{\mathcal{X}} : \mathcal{X} \rightarrow \mathbf{RFib}_{\mathcal{M}(\star)}$ sends $\partial_{\mathcal{X}}$ to the representable map $\mathcal{M}(\partial)$ and preserves pullbacks of $\partial_{\mathcal{X}}$ along maps between fibrant objects, the restriction of $\gamma_{\mathcal{X}}$

to $(\mathrm{Sp} \mathcal{M})(\star)$ factors through the Yoneda embedding $\mathcal{M}(\star) \rightarrow \mathbf{RFib}_{\mathcal{M}(\star)}$. Let $\varepsilon_\star : (\mathrm{Sp} \mathcal{M})(\star) \rightarrow \mathcal{M}(\star)$ be the induced functor.

$$\begin{array}{ccc} (\mathrm{Sp} \mathcal{M})(\star) & \xrightarrow{\varepsilon_\star} & \mathcal{M}(\star) \\ \downarrow & & \downarrow \gamma \\ \mathcal{X} & \xrightarrow{\gamma_\mathcal{X}} & \mathbf{RFib}_{\mathcal{M}(\star)} \end{array}$$

The functor $\gamma_\mathcal{X}$ also induces maps $\varepsilon_U : (\mathrm{Sp} \mathcal{M})(U) \rightarrow \mathcal{M}(U)$ and $\varepsilon_E : (\mathrm{Sp} \mathcal{M})(E) \rightarrow \mathcal{M}(E)$ of right fibrations over $\varepsilon_\star$, and these define a morphism $\varepsilon : \mathrm{Sp} \mathcal{M} \rightarrow \mathcal{M}$ of models of $\mathbb{I}_\infty$.

Lemma 6.23. *The morphism $\varepsilon : \mathrm{Sp} \mathcal{M} \rightarrow \mathcal{M}$ is a trivial fibration.*

Proof. We verify type lifting and term lifting. To give type lifting, let $\Gamma \in (\mathrm{Sp} \mathcal{M})(\star)$ be an object and $A : \mathcal{M}(\star)/\varepsilon_\star(\Gamma) \rightarrow \mathcal{M}(U)$ a map. Since $\mathcal{M}(\star)/\varepsilon_\star(\Gamma) \simeq \gamma_\mathcal{X}(\Gamma)$ and $\mathcal{M}(U) \simeq \gamma_\mathcal{X}(U_\mathcal{X})$, the map A is represented by some map $\Gamma \rightarrow U_\mathcal{X}$ in $\mathcal{X}$, that is, a map $(\mathrm{Sp} \mathcal{M})(\star)/\Gamma \rightarrow (\mathrm{Sp} \mathcal{M})(U)$. Term lifting can be checked in the same way. $\square$

Proof of Theorem 6.19. Let $\mathcal{M}$ be a cofibrant democratic model of $\mathbb{I}_\infty$. Then we have a section of the trivial fibration $\varepsilon : \mathrm{Sp} \mathcal{M} \rightarrow \mathcal{M}$. Since $\mathbf{Mod}^{\mathrm{dem}}(\mathbb{I}) \subset \mathbf{Mod}^{\mathrm{dem}}(\mathbb{I}_\infty)$ is closed under retracts, $\mathcal{M}$ belongs to $\mathbf{Mod}^{\mathrm{dem}}(\mathbb{I})$. $\square$

In conclusion we have shown that $\mathbf{Th}(\mathbb{I})$ is a category with weak equivalences and cofibrations. Moreover, Proposition 6.11 also implies that $\gamma_! \tau^*$ is left exact. Thus, to show that this map induces an equivalence after localization, it is enough to show the left approximation property. Since the first axiom is satisfied by definition, we only have to show the second. But this is now an easy task using Lemmas 6.13 and 6.14 and Theorem 6.19.

Lemma 6.24. *For any cofibrant $\mathbb{I}$ -theory K and any morphism $f : \gamma_! \tau^* K \rightarrow L$ in $\mathbf{Th}(\mathbb{E}_\infty)$, there exists a morphism $f' : K \rightarrow L'$ in $\mathbf{Th}(\mathbb{I})$ such that $\gamma_! \tau^* L' \simeq L$ under $\gamma_! \tau^* K$.*

Proof. Let $f : \gamma_! \tau^* K \rightarrow L$ be a morphism in $\mathbf{Th}(\mathbb{E}_\infty)$ where K is a cofibrant $\mathbb{I}$ -theory. By Lemma 6.13 and the small object argument, f is written as a transfinite composite of pushouts of generating cofibrations. Thus, it suffices to prove the case when f is a pushout of a generating cofibration. Let us assume that f is a pushout of the form

$$\begin{array}{ccc} \gamma_! A & \xrightarrow{g} & \gamma_! \tau^* K \\ \gamma_! i \downarrow & & \downarrow f \\ \gamma_! B & \xrightarrow{h} & L \end{array} \quad (20)$$

where $i : A \rightarrow B$ is one of the generating cofibrations in $\mathbf{Th}(\mathbb{I}_\infty)$. Since A is cofibrant, the transpose $A \rightarrow \gamma^* \gamma_! \tau^* K$ of g factors through the unit $\tau^* K \rightarrow \gamma^* \gamma_! \tau^* K$ by Lemma 6.14. Let $g' : \tau_! A \rightarrow K$ be the transpose of the induced morphism $A \rightarrow \tau^* K$ and take the pushout

$$\begin{array}{ccc} \tau_! A & \xrightarrow{g'} & K \\ \tau_! i \downarrow & & \downarrow \\ \tau_! B & \longrightarrow & L'. \end{array} \quad (21)$$

By Theorem 6.19, the units $A \rightarrow \tau^* \tau_! A$ and $B \rightarrow \tau^* \tau_! B$ are invertible, and $\gamma_! \tau^*$ sends the pushout (21) to the pushout (20) since K is cofibrant. Hence, $\gamma_! \tau^* L'$ is equivalent to L under $\gamma_! \tau^* K$. $\square$

Proof of Theorem 6.2. By Theorem 6.9, the category $\mathbf{Th}(\mathbb{I})$ is a category with weak equivalences and cofibrations, and Proposition 6.11 implies that the functor $\gamma_! \tau^* : \mathbf{Th}(\mathbb{I}) \rightarrow \mathbf{Th}(\mathbb{E}_\infty)$ is right exact. We checked the left approximation property in Lemma 6.24. Thus, by Proposition 6.6, $\gamma_! \tau^*$ induces an equivalence $L(\mathbf{Th}(\mathbb{I})) \simeq \mathbf{Th}(\mathbb{E}_\infty)$. $\square$

6.2 Generalizations We end this section with discussion about generalizations of Theorem 6.2. Let $\tilde{\mathbb{I}}$ be an extension of $\mathbb{I}$ with some type-theoretic structures such as Π -types and (higher) inductive types, and we similarly define extensions $\tilde{\mathbb{I}}_\infty$ and $\tilde{\mathbb{E}}_\infty$ of $\mathbb{I}_\infty$ and $\mathbb{E}_\infty$, respectively. We have a span

$$\tilde{\mathbb{I}} \xleftarrow{\tau} \tilde{\mathbb{I}}_\infty \xrightarrow{\gamma} \tilde{\mathbb{E}}_\infty$$

and ask if the functor $\gamma_! \tau^* : \mathbf{Th}(\tilde{\mathbb{I}}) \rightarrow \mathbf{Th}(\tilde{\mathbb{E}}_\infty)$ induces an equivalence

$$L(\mathbf{Th}(\tilde{\mathbb{I}})) \simeq \mathbf{Th}(\tilde{\mathbb{E}}_\infty).$$

Most part of the proof of Theorem 6.2 works also for this case, but we have to modify Lemmas 6.17 and 6.22. For Lemma 6.17, we need to find a ∞ -categorical structure corresponding to $\tilde{\mathbb{E}}_\infty$ -theories and show that the localization $L(\mathcal{M}(\star))$ for a democratic model $\mathcal{M}$ of $\tilde{\mathbb{I}}$ has that structure. For example, in the case when $\tilde{\mathbb{I}}$ is the extension $\mathbb{I}^\Pi$ of $\mathbb{I}$ with Π -types satisfying function extensionality in the sense of [45, Section 2.9], we have $\mathbf{Th}(\mathbb{E}_\infty^\Pi) \simeq \mathbf{LCCC}_\infty$ (Theorem 5.25), and by the results of Kapulkin [26], $L(\mathcal{M}(\star))$ is indeed locally cartesian closed. When we extend $\mathbb{I}$ with (higher) inductive types, the corresponding ∞ -categorical structure will be some form of pullback-stable initial algebras. For Lemma 6.22, we have to choose the fibration $\partial_{\mathcal{X}}$ such that it also has the type-theoretic structures that $\tilde{\mathbb{I}}$ has. In the case of $\tilde{\mathbb{I}} = \mathbb{I}^\Pi$, one might want to choose the regular cardinal κ in the proof of Lemma 6.22 such that κ -small fibrations are closed under pushforwards. However, there is no guarantee of the existence of such a regular cardinal within the same Grothendieck universe, unless the Grothendieck universe is 1-accessible, that is, there are unboundedly many inaccessible cardinals [33]. Nevertheless, the existence of $\mathrm{Sp} \mathcal{M}$ in a larger universe is enough to prove Theorem 6.19, and thus we have the second part of Conjecture 3.7 of Kapulkin and Lumsdaine [27] under an extra assumption on universes.

Theorem 6.25. *Suppose that our ambient Grothendieck universe is 1-accessible or contained in a larger universe. Then the functor $\gamma_! \tau^* : \mathbf{Th}(\mathbb{I}^\Pi) \rightarrow \mathbf{Th}(\mathbb{E}_\infty^\Pi)$ induces an equivalence of ∞ -categories*

$$L(\mathbf{Th}(\mathbb{I}^\Pi)) \simeq \mathbf{Th}(\mathbb{E}_\infty^\Pi) \simeq \mathbf{LCCC}_\infty.$$

The current proof of Lemma 6.22 has some issues when generalizing it. As we have seen, it could cause a rise in universe levels. Furthermore, the same proof does not work when we extend $\mathbb{I}$ with (higher) inductive types, because having (higher) inductive types is not a closure property. One possible approach to these issues is to refine the construction of $\mathrm{Sp} \mathcal{M}$. The current construction does not depend on the choice of a type-theoretic model topos $\mathcal{X}$ that presents $\mathbf{RFib}_{\mathcal{M}(\star)}$, but there should be a convenient choice to work with. Another approach is to give a totally different proof of Theorem 6.19 without the use of $\mathrm{Sp} \mathcal{M}$. There has been a syntactic approach to coherence problems initiated by Curien [14]. In this approach, coherence problems are solved by rewriting techniques, and we expect that it works for a wide range of type-theoretic structures without a rise of universe levels. Of course, we first have to develop nice syntax for ∞ -type theories, and this is not obvious.

7. Conclusion and future work

We introduced ∞ -type theories as a higher dimensional generalization of type theories and as an application proved Kapulkin and Lumsdaine’s conjecture that the ∞ -category of small left exact ∞ -categories is a localization of the category of theories over Martin-Löf type theory with intensional identity types [27]. The technique developed in this paper also works for the internal language conjecture for locally cartesian closed ∞ -categories, but further generalization including (higher) inductive types is left as future work.

7.1 Syntax for ∞ -type theories Coherence problems are often solved by syntactic arguments [14]. Therefore, syntactic presentations of ∞ -type theories will be helpful for solving internal language conjectures for structured ∞ -categories. We have not figured out syntax for ∞ -type theories. Here we consider one possibility based on *logical frameworks*.

In the previous work [47], the author introduced a logical framework to define type theories. For every signature Σ in that logical framework, the syntactic category $\mathbb{R}(\Sigma)$ is naturally equipped with a structure of a category with representable maps and satisfies a certain universal property. To define ∞ -type theories syntactically, we modify the logical framework as follows:

- the new logical framework has intensional identity types instead of extensional identity types;
- dependent product types indexed over representable types satisfy the function extensionality axiom.

Remark 7.1. A similar kind of framework is used by Bocquet [7, Section 7] to represent space-valued models of a type theory.

Proposition 7.2. *Let Σ be a signature in this new logical framework.*

1. *The syntactic category $\mathbb{R}(\Sigma)$ is equipped with a structure of a fibration category.*
2. *The localization $L(\mathbb{R}(\Sigma))$ is equipped with a structure of an ∞ -category with representable maps.*

Proof. It is known [4] that the syntactic category of a type theory with intensional identity types is a fibration category. The second claim is proved in the same way as the fact that the localization of a locally cartesian closed fibration category is a locally cartesian closed ∞ -category [26, 12]. $\square$

We expect that the syntactic ∞ -category with representable maps $L(\mathbb{R}(\Sigma))$ satisfies a universal property analogous to [47, Theorem 5.17] so that the logical framework with intensional identity types provides syntactic presentations of ∞ -type theories.

Acknowledgements

Uemura was supported by the research programme “The Computational Content of Homotopy Type Theory” with project number 613.001.602, which is financed by the Netherlands Organisation for Scientific Research (NWO), and KAW Grant “Type Theory for Mathematics and Computer Science” investigated by Thierry Coquand and Peter LeFanu Lumsdaine.

References

- [1] M. Anel, G. Biedermann, E. Finster, and A. Joyal. Goodwillie’s calculus of functors and higher topos theory. *J. Topol.*, 11(4):1100–1132, 2018.
- [2] Mathieu Anel, Georg Biedermann, Eric Finster, and André Joyal. A generalized Blakers-Massey theorem. *J. Topol.*, 13(4):1521–1553, 2020.
- [3] Peter Arndt and Krzysztof Kapulkin. Homotopy-Theoretic Models of Type Theory. In Luke Ong, editor, *Typed Lambda Calculi and Applications: 10th International Conference, TLCA 2011, Novi Sad, Serbia, June 1-3, 2011. Proceedings*, pages 45–60, Berlin, Heidelberg, 2011. Springer Berlin Heidelberg.
- [4] Jeremy Avigad, Krzysztof Kapulkin, and Peter LeFanu Lumsdaine. Homotopy limits in type theory. *Mathematical Structures in Computer Science*, 25(5):1040–1070, 2015.
- [5] Steve Awodey. Natural models of homotopy type theory. *Mathematical Structures in Computer Science*, 28(2):241–286, 2018.
- [6] Steve Awodey and Michael A. Warren. Homotopy theoretic models of identity types. *Mathematical Proceedings of the Cambridge Philosophical Society*, 146(1):45–55, 2009.
- [7] Rafaël Bocquet. Coherence of strict equalities in dependent type theories, 2020. [arXiv:2010.14166v1](https://arxiv.org/abs/2010.14166v1).
- [8] Kenneth S. Brown. Abstract homotopy theory and generalized sheaf cohomology. *Transactions of the American Mathematical Society*, 186:419–458, 1973.
- [9] Guillaume Brunerie. *On the homotopy groups of spheres in homotopy type theory*. PhD thesis, Université de Nice, 2016.
- [10] Guillaume Brunerie. The James construction and $\pi_4(\mathbb{S}^3)$ in homotopy type theory. *J. Automat. Reason.*, 63(2):255–284, 2019.
- [11] J. W. Cartmell. *Generalised algebraic theories and contextual categories*. PhD thesis, Oxford University, 1978.
- [12] Denis-Charles Cisinski. *Higher Categories and Homotopical Algebra*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2019.
- [13] Pierre Clairambault and Peter Dybjer. The biequivalence of locally cartesian closed categories and Martin-Löf type theories. *Mathematical Structures in Computer Science*, 24(6):e240606, 2014.
- [14] Pierre-Louis Curien. Substitution up to Isomorphism. *Fundam. Inform.*, 19(1/2):51–85, 1993.
- [15] Peter Dybjer. Internal Type Theory. In Stefano Berardi and Mario Coppo, editors, *Types for Proofs and Programs: International Workshop, TYPES ’95 Torino, Italy, June 5–8, 1995 Selected Papers*, pages 120–134. Springer Berlin Heidelberg, Berlin, Heidelberg, 1996.
- [16] Marcelo Fiore. Discrete Generalised Polynomial Functors, 2012. Talk at ICALP 2012, <http://www.cl.cam.ac.uk/~mpf23/talks/ICALP2012.pdf>.

- [17] Nicola Gambino and Joachim Kock. Polynomial functors and polynomial monads. *Mathematical Proceedings of the Cambridge Philosophical Society*, 154(1):153–192, 2013.
- [18] Richard Garner. Combinatorial structure of type dependency. *Journal of Pure and Applied Algebra*, 219(6):1885 – 1914, 2015.
- [19] David Gepner and Joachim Kock. Univalence in locally cartesian closed ∞ -categories. *Forum Mathematicum*, 29(3), jan 2017.
- [20] Martin Hofmann. On the interpretation of type theory in locally cartesian closed categories. In Leszek Pacholski and Jerzy Tiuryn, editors, *Computer Science Logic*, pages 427–441, Berlin, Heidelberg, 1995. Springer Berlin Heidelberg.
- [21] Kuen-Bang Hou (Favonia), Eric Finster, Daniel R. Licata, and Peter LeFanu Lumsdaine. A mechanization of the Blakers-Massey connectivity theorem in homotopy type theory. In *Proceedings of the 31st Annual ACM-IEEE Symposium on Logic in Computer Science (LICS 2016)*, page 10. ACM, New York, 2016.
- [22] Valery Isaev. Algebraic Presentations of Dependent Type Theories, 2017. [arXiv:1602.08504v3](https://arxiv.org/abs/1602.08504v3).
- [23] André Joyal. Notes on quasi-categories, 2008. <http://www.math.uchicago.edu/~may/IMA/Joyal.pdf>.
- [24] André Joyal. Notes on clans and tribes, 2017. [arXiv:1710.10238v1](https://arxiv.org/abs/1710.10238v1).
- [25] André Joyal and Myles Tierney. Quasi-categories vs Segal spaces. In *Categories in algebra, geometry and mathematical physics*, volume 431 of *Contemp. Math.*, pages 277–326. Amer. Math. Soc., Providence, RI, 2007.
- [26] Krzysztof Kapulkin. Locally cartesian closed quasi-categories from type theory. *Journal of Topology*, 10(4):1029–1049, 2017.
- [27] Krzysztof Kapulkin and Peter LeFanu Lumsdaine. The homotopy theory of type theories. *Adv. Math.*, 337:1–38, 2018.
- [28] Krzysztof Kapulkin and Peter LeFanu Lumsdaine. The simplicial model of Univalent Foundations (after Voevodsky). *Journal of the European Mathematical Society*, 2021.
- [29] Krzysztof Kapulkin and Karol Szumiło. Internal languages of finitely complete $(\infty, 1)$ -categories. *Selecta Math. (N.S.)*, 25(2):Art. 33, 46, 2019.
- [30] J. Lambek and P. J. Scott. *Introduction to Higher Order Categorical Logic*. Cambridge University Press, 1986.
- [31] F. William Lawvere. *Functorial Semantics of Algebraic Theories*. PhD thesis, Columbia University, 1963.
- [32] Daniel R. Licata and Michael Shulman. Calculating the fundamental group of the circle in homotopy type theory. In *2013 28th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS 2013)*, page 223–232. IEEE Computer Soc., Los Alamitos, CA, 2013.

- [33] Giulio Lo Monaco. Dependent products and 1-inaccessible universes. *Theory and Applications of Categories*, 37(5):107–143, 2021.
- [34] Peter LeFanu Lumsdaine and Michael A. Warren. The Local Universes Model: An Overlooked Coherence Construction for Dependent Type Theories. *ACM Trans. Comput. Logic*, 16(3):23:1–23:31, jul 2015.
- [35] Jacob Lurie. *Higher Algebra*. 2009. <http://www.math.harvard.edu/~lurie/papers/HA.pdf>.
- [36] Jacob Lurie. *Higher Topos Theory*. Princeton University Press, 2009.
- [37] Hoang Kim Nguyen. *Theorems in Higher Category Theory and Applications*. PhD thesis, Universität Regensburg, 2019.
- [38] Hoang Kim Nguyen, George Raptis, and Christoph Schrade. Adjoint functor theorems for ∞ -categories. *Journal of the London Mathematical Society*, 2019.
- [39] Nima Rasekh. Complete Segal Objects, 2018. [arXiv:1805.03561v1](https://arxiv.org/abs/1805.03561).
- [40] Nima Rasekh. Univalence in Higher Category Theory, 2021. [arXiv:2103.12762v2](https://arxiv.org/abs/2103.12762).
- [41] Emily Riehl and Dominic Verity. *Elements of ∞ -Category Theory*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2022.
- [42] Michael Shulman. Univalence for inverse diagrams and homotopy canonicity. *Mathematical Structures in Computer Science*, 25(05):1203–1277, 2015.
- [43] Michael Shulman. All $(\infty, 1)$ -toposes have strict univalent universes, 2019. [arXiv:1904.07004v2](https://arxiv.org/abs/1904.07004).
- [44] Karol Szumilo. *Two Models for the Homotopy Theory of Cocomplete Homotopy Theories*. PhD thesis, University of Bonn, nov 2014.
- [45] The Univalent Foundations Program. *Homotopy Type Theory: Univalent Foundations of Mathematics*. Institute for Advanced Study, 2013.
- [46] Taichi Uemura. The universal exponentiable arrow. *Journal of Pure and Applied Algebra*, 226(7):106991, 2022.
- [47] Taichi Uemura. A general framework for the semantics of type theory. *Mathematical Structures in Computer Science*, pages 1–46, 2023.
- [48] Benno van den Berg and Ieke Moerdijk. Exact completion of path categories and algebraic set theory: Part I: Exact completion of path categories. *Journal of Pure and Applied Algebra*, 222(10):3137 – 3181, 2018.
- [49] Vladimir Voevodsky. B-systems, 2014. [arXiv:1410.5389v1](https://arxiv.org/abs/1410.5389).
- [50] Mark Weber. Polynomials in categories with pullbacks. *Theory and Applications of Categories*, 30(16):533–598, 2015.