

Unitary pseudonatural transformations

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Abstract

We suggest two approaches to a definition of unitarity for pseudonatural transformations between unitary pseudofunctors on pivotal dagger 2-categories. The first is to require that the 2-morphism components of the transformation be unitary. The second is to require that the dagger of the transformation be equal to its inverse. We show that the ‘inverse’ making these definitions equivalent is the right dual of the transformation in the 2-category $\text{Fun}(\mathcal{C}, \mathcal{D})$ of pseudofunctors $\mathcal{C} \rightarrow \mathcal{D}$, pseudonatural transformations, and modifications. We show that the subcategory $\text{Fun}_u(\mathcal{C}, \mathcal{D}) \subset \text{Fun}(\mathcal{C}, \mathcal{D})$ whose objects are unitary pseudofunctors and whose 1-morphisms are unitary pseudonatural transformations is a pivotal dagger 2-category. We apply these results to obtain a Morita-theoretical classification of unitary pseudonatural transformations between fibre functors on the category of representations of a compact quantum group.

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1. Introduction

1.1 Overview Natural transformations between functors are a crucial element of category theory. Let \mathcal{C}, \mathcal{D} be categories and $F, F' : \mathcal{C} \rightarrow \mathcal{D}$ be functors. We say that a natural transformation $\alpha : F \rightarrow F'$ is *invertible* if its components $\{\alpha_X\}_{X \in \text{Obj}(\mathcal{C})}$ are invertible in \mathcal{D} . If \mathcal{D} is a dagger category, then we say that an invertible natural transformation is *unitary* if its components are additionally unitary in \mathcal{D} .

Perhaps more naturally, these notions of invertibility may be defined with respect to the category $\text{Fun}(\mathcal{C}, \mathcal{D})$ of functors and natural transformations. An invertible natural transformation is just an invertible morphism in this category. If \mathcal{C}, \mathcal{D} are dagger categories, the subcategory of $\text{Fun}(\mathcal{C}, \mathcal{D})$ whose objects are unitary functors inherits a dagger structure; a unitary natural transformation is a unitary morphism in this dagger category.

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Just as natural transformations between functors are an important part of category theory, pseudonatural transformations between pseudofunctors are an important part of 2-category theory, which includes monoidal category theory. In this work we consider the generalisation of the aforementioned notions of invertibility to pseudonatural transformations.¹

Let \mathcal{C}, \mathcal{D} be 2-categories, and let $\text{Fun}(\mathcal{C}, \mathcal{D})$ be the 2-category of pseudofunctors $\mathcal{C} \rightarrow \mathcal{D}$, pseudonatural transformations and modifications. We consider invertibility of a pseudonatural transformation as a 1-morphism in $\text{Fun}(\mathcal{C}, \mathcal{D})$.

We could consider equivalences in $\text{Fun}(\mathcal{C}, \mathcal{D})$. However, we find that this notion of invertibility is too strong for our purposes. A weaker notion of invertibility of a 1-morphism in a 2-category is duality, a.k.a. adjunction. A 2-category is said to ‘have right (resp. left) duals’ when every 1-morphism has a chosen right (resp. left) dual. A coherent choice of left and right duals for every object is called a pivotal structure; a 2-category with a pivotal structure is called *pivotal*. Here we unpack the notion of duality for pseudonatural transformations (Definition 3.1) and show the following facts.

- If \mathcal{C} has left (resp. right) duals and \mathcal{D} has right (resp. left) duals, then $\text{Fun}(\mathcal{C}, \mathcal{D})$ has right (resp. left) duals (Corollary 3.4).
- If \mathcal{C}, \mathcal{D} are pivotal, then $\text{Fun}_p(\mathcal{C}, \mathcal{D})$ is also pivotal, where the subscript p represents restriction to pivotal functors. (Proposition 3.6).

If the 2-categories \mathcal{C}, \mathcal{D} additionally have a dagger structure, we restrict $\text{Fun}(\mathcal{C}, \mathcal{D})$ to unitary pseudofunctors. We now consider the notion of *unitarity* of a pseudonatural transformation. This consideration is motivated either physically, by the desire that the components of the transformation should be unitary in \mathcal{D} ; or categorically, by the desire that the 2-category $\text{Fun}(\mathcal{C}, \mathcal{D})$ should itself inherit a dagger structure (for general pseudonatural transformations, there is no obvious dagger structure on $\text{Fun}(\mathcal{C}, \mathcal{D})$).

We could say that a pseudonatural transformation is unitary when all its 2-morphism components are unitary in \mathcal{D} . This is our first definition of a *unitary pseudonatural transformation*. However, the more categorically natural way of specifying unitarity of a pseudonatural transformation is to say that its *dagger* is equal to its inverse (i.e. its right dual). When \mathcal{C}, \mathcal{D} are pivotal dagger (i.e. possessing compatible pivotal and dagger structures), we observe that there is a notion of the dagger of a pseudonatural transformation such that the following definitions of a unitary pseudonatural transformation are equivalent (Lemma 4.2):

- A pseudonatural transformation all of whose 2-morphism components are unitary.
- A pseudonatural transformation whose right dual is equal to its dagger.

Let $\text{Fun}_u(\mathcal{C}, \mathcal{D}) \subset \text{Fun}(\mathcal{C}, \mathcal{D})$ be the subcategory whose objects are unitary pseudofunctors and whose 1-morphisms are unitary pseudonatural transformations. The category $\text{Fun}_u(\mathcal{C}, \mathcal{D})$ inherits a dagger structure from \mathcal{D} . Moreover, pivotality comes ‘for free’, with no need to restrict to pivotal functors.

- Let \mathcal{C}, \mathcal{D} be pivotal dagger categories. Then the category $\text{Fun}_u(\mathcal{C}, \mathcal{D})$ is a pivotal dagger category. (Theorem 4.5.)

Our main motivation for this work is the study of unitary pseudonatural transformations between fibre functors on representation categories of compact quantum groups, which are the subject of the paper [11]. In particular, the results in this paper allow us to classify fibre functors and unitary pseudonatural transformations between them in terms of Morita theory in the 2-category

¹We remark that our results about duality generalise straightforwardly to oplax natural transformations, although for applications we did not require this level of generality.

$\text{Fun}_u(\text{Rep}(G), \text{Hilb})$. To show this we prove a more general result (Theorem 5.7) which relates equivalence classes of 1-morphisms out of an object in a pivotal dagger 2-category to unitary $*$ -isomorphism classes of special Frobenius algebras in its endomorphism category.

1.2 Structure In Section 2 we introduce necessary background material for the rest of this paper. In Section 2.3 we recall the basic theory of pseudonatural transformations. In Section 3 we discuss dualisability of pseudonatural transformations. In Section 4 we consider unitary pseudonatural transformations. In Section 5 we consider an application of our results to the study of fibre functors on representation categories of compact quantum groups.

2. Background: Pivotal dagger 2-categories

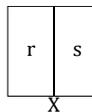
2.1 Diagrams for 2-categories Recall that every monoidal category is equivalent to a strict monoidal category [5]. This allows us to assume our monoidal categories are strict, allowing the use of a convenient and well-known diagrammatic calculus [9]. In 2-category theory, a similar strictification result holds — every weak 2-category is equivalent to a strict 2-category [4]. We can therefore also use a diagrammatic calculus in this case.

A monoidal category is precisely a 2-category with a single object, where 1-morphisms are the ‘objects’ of the monoidal category, 2-morphisms are the ‘morphisms’, and composition of 1-morphisms is the ‘monoidal product’. The 2-categorical diagrammatic calculus is nothing more than the diagrammatic calculus for monoidal categories enhanced with region labels. We briefly summarise this calculus now, closely following the exposition in [6]. More information can be found in e.g. [2].

Objects r, s, \dots of a 2-category are represented by labelled two-dimensional regions of a planar diagram:

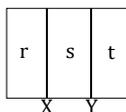


The 1-morphisms $X : r \rightarrow s$ are represented by edges, separating the region r on the left from the region s on the right:



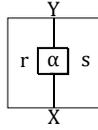
Edges corresponding to identity 1-morphisms $\text{id}_r : r \rightarrow r$ are invisible in the diagrammatic calculus.

The 1-morphisms compose from left to right. That is, for 1-morphisms $X : r \rightarrow s, Y : s \rightarrow t$, the composite $X \otimes Y : r \rightarrow t$ is represented as:

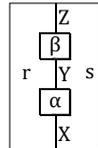


For two parallel 1-morphisms $X, Y : r \rightarrow s$, a 2-morphism $\alpha : X \rightarrow Y$ is represented by a vertex

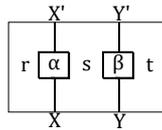
in the diagram, drawn as a box:



The 2-morphisms can compose in two ways, depending on their type. For parallel 1-morphisms $X, Y, Z : r \rightarrow s$, 2-morphisms $\alpha : X \rightarrow Y, \beta : Y \rightarrow Z$ can be composed ‘vertically’ to obtain a 2-morphism $\beta \circ \alpha : X \rightarrow Z$. This is represented by vertical juxtaposition in the diagram:



For 1-morphisms $X, X' : r \rightarrow S$ and $Y, Y' : s \rightarrow t$, 2-morphisms $\alpha : X \rightarrow X'$ and $\beta : Y \rightarrow Y'$ can be composed ‘horizontally’ to obtain a 2-morphism $\alpha \otimes \beta : X \otimes Y \rightarrow X' \otimes Y'$. This is represented by horizontal juxtaposition in the diagram:



As with 1-morphisms, the identity 2-morphisms $\text{id}_X : X \rightarrow X$ are invisible in the diagrammatic calculus.

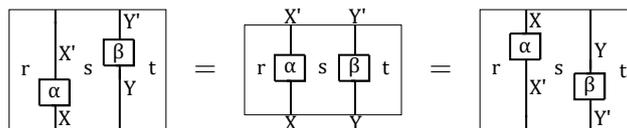
All 2-categories satisfy the *interchange law*. For any 1-morphisms $X, X', X'' : r \rightarrow s$ and $Y, Y', Y'' : s \rightarrow t$, and 2-morphisms $\alpha : X \rightarrow X', \alpha' : X' \rightarrow X'', \beta : Y \rightarrow Y', \beta' : Y' \rightarrow Y''$:

$$(\alpha' \circ \alpha) \otimes (\beta' \circ \beta) = (\alpha' \otimes \beta') \circ (\alpha \otimes \beta)$$

This corresponds to the following diagram having an unambiguous interpretation as a 2-morphism:



We also have the following *sliding equalities*, which may be obtained by taking some morphisms to be the identity in (1):



These equalities allow us to move 2-morphism boxes past each other provided there are no obstructions.

Before moving onto pseudofunctors, we give a first definition from 2-category theory. Equivalence is a strong notion of invertibility of a 1-morphism in a 2-category. From now on we will not draw an enclosing box around diagrams.

Definition 2.1. Let \mathcal{C} be a 2-category and let $X : r \rightarrow s$ be a 1-morphism in \mathcal{C} . We say that X is an *equivalence* if there exists a 1-morphism $X^{-1} : s \rightarrow r$, and invertible² 2-morphisms $\alpha : \text{id}_r \rightarrow X \otimes X^{-1}$ and $\beta : \text{id}_s \rightarrow X^{-1} \otimes X$. In diagrams, the equations for invertibility of α, β are as follows, where α^{-1}, β^{-1} are the inverse 2-morphisms:

$$\begin{array}{c} \boxed{\alpha^{-1}} \\ | \\ r \quad s \quad r \\ | \\ \boxed{\alpha} \end{array} = r, \quad \begin{array}{c} X \quad s \quad X^{-1} \\ | \\ \boxed{\alpha} \\ | \\ r \\ | \\ \boxed{\alpha^{-1}} \\ | \\ X \quad s \quad X^{-1} \end{array} = r \quad \begin{array}{c} X^{-1} \\ | \\ r \\ | \\ X^{-1} \end{array}, \quad \begin{array}{c} \boxed{\beta^{-1}} \\ | \\ s \quad r \quad s \\ | \\ \boxed{\beta} \end{array} = s, \quad \begin{array}{c} X^{-1} \quad r \quad X \\ | \\ \boxed{\beta} \\ | \\ s \\ | \\ \boxed{\beta^{-1}} \\ | \\ X^{-1} \quad r \quad X \end{array} = s \quad \begin{array}{c} X^{-1} \quad X \\ | \\ r \quad s \\ | \\ X^{-1} \quad X \end{array} \quad (2)$$

If there exists an equivalence $X : r \rightarrow s$ we say that the objects r and s are *equivalent* in \mathcal{C} .

2.2 Diagrams for pseudofunctors While our 2-categories are strictified, allowing us to use the diagrammatic calculus, we will consider functors between them which are not strict. For this, we use a graphical calculus of *functorial boxes* previously applied in the special case of monoidal functors [7].

Definition 2.2. Let \mathcal{C}, \mathcal{D} , be 2-categories. A *pseudofunctor* $F : \mathcal{C} \rightarrow \mathcal{D}$ consists of the following data.

- For each object r of \mathcal{C} , an object $F(r)$ of \mathcal{D} .
- For each hom-category $\mathcal{C}(r, s)$ of \mathcal{C} , a functor $F_{r,s} : \mathcal{C}(r, s) \rightarrow \mathcal{D}(F(r), F(s))$.

In the graphical calculus, we represent the effect of the functor $F_{r,s}$ by drawing a shaded box around 1- and 2-morphisms in $\mathcal{C}(r, s)$. For example, $X, Y : r \rightarrow s$ be 1-morphisms and $f : X \rightarrow Y$ a 2-morphism in \mathcal{C} . Then the 2-morphism $F(f) : F(X) \rightarrow F(Y)$ in $\mathcal{D}(F(r), F(s))$ is represented as:

$$\begin{array}{c} F(Y) \\ | \\ \boxed{F(f)} \\ | \\ F(X) \end{array} = \begin{array}{c} Y \\ | \\ \boxed{f} \\ | \\ X \end{array}$$

- For every pair of composable 1-morphisms $X : r \rightarrow s, Y : s \rightarrow t$ of \mathcal{C} , an invertible *multiplicator* 2-morphism $m_{X,Y} : F(X) \otimes_{\mathcal{D}} F(Y) \rightarrow F(X \otimes_{\mathcal{C}} Y)$. In the graphical calculus, these 2-morphisms and their inverses are represented as follows:

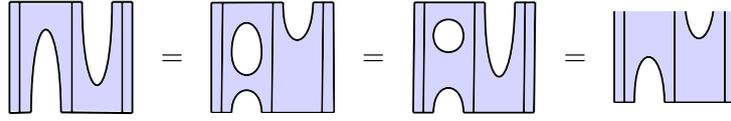
$$\begin{array}{c} X \otimes Y \\ | \\ \boxed{m_{X,Y}} \\ | \\ F(r) \quad F(r) \\ | \\ X \quad Y \\ | \\ F(s) \end{array} \quad \begin{array}{c} X \quad Y \\ | \\ \boxed{m_{X,Y}^{-1}} \\ | \\ F(r) \quad F(r) \\ | \\ X \otimes Y \end{array} \quad (3)$$

$$m_{X,Y} : F(X) \otimes_{\mathcal{D}} F(Y) \rightarrow F(X \otimes_{\mathcal{C}} Y) \quad m_{X,Y}^{-1} : F(X \otimes_{\mathcal{C}} Y) \rightarrow F(X) \otimes_{\mathcal{D}} F(Y)$$

- For every object r of \mathcal{C} , an invertible ‘unit’ 2-morphism $u_r : \text{id}_{F(r)} \rightarrow F(\text{id}_r)$. In the diagrammatic calculus, these 2-morphism and their inverses are represented as follows

²I.e. invertible in the Hom-categories $\mathcal{C}(r, r)$ and $\mathcal{C}(s, s)$. We sometimes call an invertible 2-morphism a *2-isomorphism*.

Proof. We prove the left equation; the right equation is proved similarly.



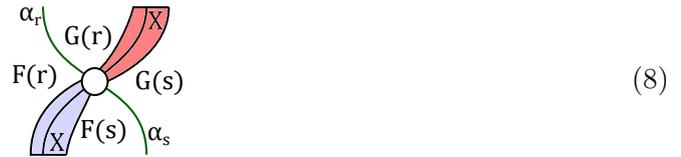
Here the first and third equalities are by invertibility of $m_{X,Y}$, and the second is by coassociativity. \square

With Lemma 2.3, the equations (5-7) are sufficient to deform functorial boxes topologically as required. From now on we will do this mostly without comment.

2.3 Pseudonatural transformations We now recall the definition of a pseudonatural transformation between pseudofunctors [4].

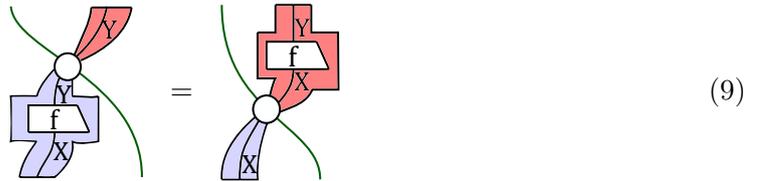
Definition 2.4. Let \mathcal{C}, \mathcal{D} be 2-categories, and let $F, G : \mathcal{C} \rightarrow \mathcal{D}$ be pseudofunctors (depicted by blue and red boxes respectively). A *pseudonatural transformation* $\alpha : F \rightarrow G$ is defined by the following data:

- For every object r of \mathcal{C} , a 1-morphism $\alpha_r : F(r) \rightarrow G(r)$ of \mathcal{D} (drawn as a green wire).
- For every 1-morphism $X : r \rightarrow s$ of \mathcal{C} , an invertible 2-morphism $\alpha_X : F(X) \otimes \alpha_s \rightarrow \alpha_r \otimes G(X)$ (drawn as a white vertex):



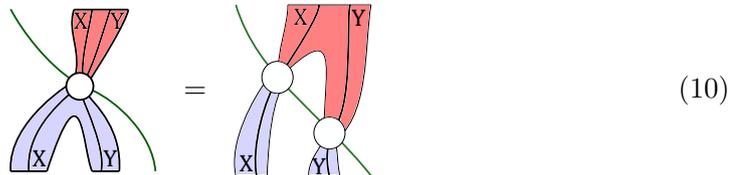
The 1-morphisms α_X must satisfy the following conditions:

- *Naturality.* For every 2-morphism $f : X \rightarrow Y$ in \mathcal{C} :

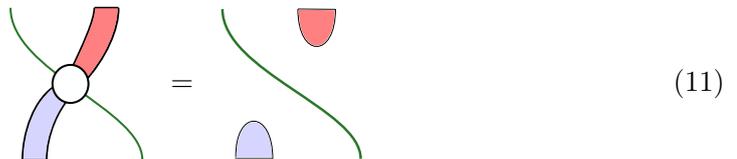


- *Monoidality.*

- For every pair of 1-morphisms $X : r \rightarrow s, Y : s \rightarrow t$ in \mathcal{C} :



- For every object r of \mathcal{C} :



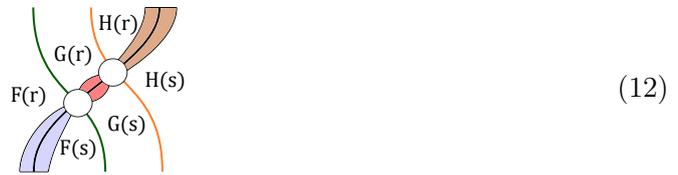
(Equation (10) already implies the analogous pullthroughs for the comultiplicators $\{m_{X,Y}^{-1}\}.$)

If $\alpha_r = \text{id}_{F(r)}$ for every object r of \mathcal{C} , we say that α is an *invertible icon* [3]. In particular, if \mathcal{C}, \mathcal{D} are one-object 2-categories and α is an invertible icon, we recover the standard notion of monoidal natural isomorphism.

Remark 2.5. The results we will prove in Section 3 extend to oplax natural transformations (i.e. where the 2-morphism components are not invertible). However, we did not need this level of generality for applications.

Pseudonatural transformations $\alpha : F \rightarrow G$ and $\beta : G \rightarrow H$ can be composed associatively. We define $\alpha \otimes \beta : F \rightarrow H$ as follows.

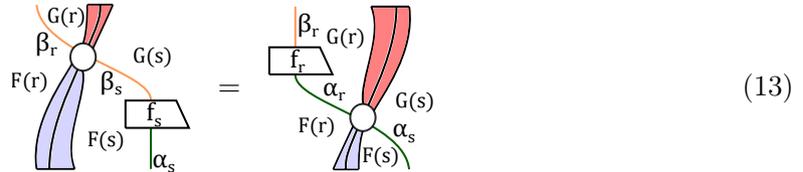
- For every object r of \mathcal{C} , $(\alpha \otimes \beta)_r := \alpha_r \otimes \beta_r$.
- For any 1-morphism $X : r \rightarrow s$ of \mathcal{C} , $(\alpha \otimes \beta)_X$ is defined as the following composite (we colour the β -wire orange, and the H -box brown):



There are also morphisms between pseudonatural transformations, known as *modifications* [4].

Definition 2.6. Let $\alpha, \beta : F \Rightarrow G$ be pseudonatural transformations between pseudofunctors $F, G : \mathcal{C} \rightarrow \mathcal{D}$. (We colour the α -wire green and the β -wire orange.) A *modification* $f : \alpha \rightarrow \beta$ is defined by the following data:

- For every object r of \mathcal{C} , a 2-morphism $f_r : \alpha_r \rightarrow \beta_r$ in \mathcal{D} , such that the 2-morphisms $\{f_r\}$ satisfy the following equation for all 1-morphisms $X : r \rightarrow s$ in \mathcal{C} :



Modifications can themselves be composed horizontally and vertically in an obvious way. Altogether, this compositional structure is again a 2-category.

Definition 2.7. Let \mathcal{C}, \mathcal{D} be 2-categories. The 2-category $\text{Fun}(\mathcal{C}, \mathcal{D})$ is defined as follows:

- Objects: pseudofunctors $F, G, \dots, \cdot : \mathcal{C} \rightarrow \mathcal{D}$.
- 1-morphisms: pseudonatural transformations $\alpha, \beta, \dots : F \rightarrow G$.
- 2-morphisms: modifications $f, g, \dots : \alpha \rightarrow \beta$.

As we are assuming that \mathcal{C} and \mathcal{D} are strict, strictness of $\text{Fun}(\mathcal{C}, \mathcal{D})$ follows.

2.4 Pivotal 2-categories In a 2-category the most general notion of invertibility of a 1-morphism is *duality*, also known as *adjunction*.

Definition 2.8. Let $X : r \rightarrow s$ be a 1-morphism in a 2-category. A *right dual* $[X^*, \eta, \epsilon]$ for X is:

- A 1-morphism $X^* : s \rightarrow r$.

In this work we are mostly interested in categories with compatible left and right duals. Such categories are called *pivotal*. Let \mathcal{C} be a 2-category with right duals. It is straightforward to check that the following defines an identity-on-objects pseudofunctor $\mathcal{C} \rightarrow \mathcal{C}$, which we call the *double duals* pseudofunctor:

- 1-morphisms $X : r \rightarrow s$ are taken to the double dual $X^{**} := (X^*)^*$.
- 2-morphisms $f : X \rightarrow Y$ are taken to the double transpose $f^{**} := (f^*)^*$.
- The multipliers $m_{X,Y}$ and unitors u_r are defined using the isomorphisms of Lemma 2.10.

Definition 2.12. We say that a 2-category \mathcal{C} with right duals is *pivotal* if there is an invertible icon (Definition 2.4) from the double duals pseudofunctor to the identity pseudofunctor.

Roughly, the existence of an invertible icon in Definition 2.12 comes down to the following statement:

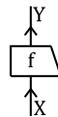
- For every 1-morphism $X : r \rightarrow s$, there is a 2-isomorphism $\iota_X : X^{**} \rightarrow X$.
- These $\{\iota_X\}$ are compatible with composition in \mathcal{C} .

In a pivotal 2-category, for any $X : r \rightarrow s$ the right dual X^* is also a left dual for X by the following cup and cap. Here and throughout we will indicate the double dual X^{**} in the diagrammatic calculus by an X -labelled wire with a double upwards-pointing arrow:

$$\begin{array}{c} \cup \\ \downarrow \end{array} := \begin{array}{c} \uparrow \\ \boxed{\iota_X} \\ \uparrow \\ \downarrow \end{array} \qquad \begin{array}{c} \downarrow \\ \cup \end{array} := \begin{array}{c} \downarrow \\ \downarrow \\ \boxed{\iota_X^{-1}} \\ \uparrow \\ \downarrow \end{array} \tag{18}$$

With these left duals, the left transpose of a 2-morphism is equal to the right transpose. Whenever we refer to a pivotal 2-category from now on, we suppose that the left duals are chosen in this way.

There is a very useful graphical calculus for these compatible dualities in a pivotal 2-category. To represent the transpose, we will modify our notation slightly. We now represent a morphism $f : X \rightarrow Y$ not by a rectangular box, but by a box where the right vertical edge is tilted:



We now represent the transpose by rotating the boxes, as though we had ‘yanked’ both ends of the wire in the RHS of (17):

$$\begin{array}{c} \downarrow X \\ \boxed{f} \\ \downarrow Y \end{array} := \begin{array}{c} \downarrow X \\ \boxed{f^*} \\ \downarrow Y \end{array}$$

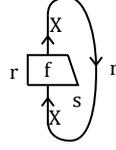
Using this notation, 2-morphisms now freely slide around cups and caps.

Lemma 2.13 ([1, Lemma 3.12, Lemma 3.26]). *Let \mathcal{C} be a pivotal 2-category and $f : X \rightarrow Y$ a 2-morphism. Then:*

$$\begin{array}{c} \boxed{f} \\ \downarrow \end{array} = \begin{array}{c} \downarrow \\ \boxed{f} \end{array} \qquad \begin{array}{c} \downarrow \\ \boxed{f} \\ \downarrow \end{array} = \begin{array}{c} \downarrow \\ \boxed{f} \\ \downarrow \end{array} \qquad \begin{array}{c} \downarrow \\ \downarrow \\ \boxed{f} \\ \downarrow \end{array} = \begin{array}{c} \downarrow \\ \downarrow \\ \boxed{f} \\ \downarrow \end{array} \qquad \begin{array}{c} \downarrow \\ \downarrow \\ \boxed{f} \\ \downarrow \end{array} = \begin{array}{c} \downarrow \\ \downarrow \\ \boxed{f} \\ \downarrow \end{array} \tag{19}$$

In a pivotal 2-category, we can define notions of *trace* and *dimension* for 1-morphisms.

Definition 2.14. Let $X : r \rightarrow s$ be an 1-morphism and let $f : X \rightarrow X$ be a 2-morphism in a pivotal 2-category \mathcal{C} . We define the *right trace* of f to be the following 2-morphism $\text{Tr}_R(f) : \text{id}_r \rightarrow \text{id}_r$:



We define the *right dimension* $\text{dim}_R(X)$ of the 1-morphism X to be $\text{Tr}_R(\text{id}_X)$. The *left trace* $\text{Tr}_L(f) : \text{id}_s \rightarrow \text{id}_s$ and *left dimension* $\text{dim}_L(X)$ are defined analogously using the right cup and left cap.

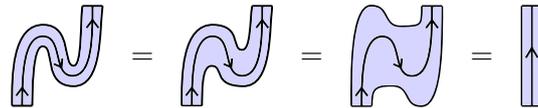
Pivotal functors. We now consider pseudofunctors between pivotal 2-categories. We first observe that the duals in \mathcal{C} induce duals in \mathcal{D} under a pseudofunctor $F : \mathcal{C} \rightarrow \mathcal{D}$.

Lemma 2.15 (Induced duals). *Let $X : r \rightarrow s$ be a 1-morphism in \mathcal{C} and $[X^*, \eta, \epsilon]$ a right dual. Then $F(X^*)$ is a right dual of $F(X)$ in \mathcal{D} with the following cup and cap:*



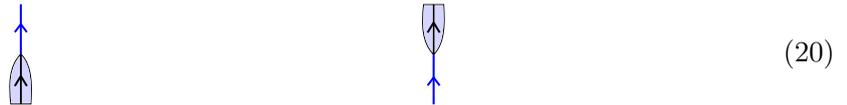
The analogous statement holds for left duals.

Proof. We show one of the snake equations (14) in the case of right duals; the others are all proved similarly.



Here the first equality is by Lemma 2.3, the second by (5) and the third by (7). \square

For any 1-morphism X of \mathcal{C} , then, we have two sets of left and right duals on $F(X)$; the first from the pivotal structure in \mathcal{C} by Lemma 2.15, and the second from the pivotal structure in \mathcal{D} . To depict both dualities in the graphical calculus, we here introduce elements of the graphical syntax which allow us to ‘zoom in’ and ‘zoom out’, representing $F(X)$ as a directed coloured wire rather than as a boxed wire:



We emphasise that these elements of the graphical calculus are semantically empty, simply switching between two ways of representing $F(X)$. We can now represent the duality corresponding to the pivotal structure in \mathcal{D} in the usual way on the directed coloured wire, writing $F(X)^*$ and $F(X)^{**}$ with a downwards and a double upwards arrow respectively, as usual.

We now define a pivotal pseudofunctor. Let \mathcal{C}, \mathcal{D} be pivotal 2-categories, and let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a pseudofunctor. By Lemma 2.10, for every 1-morphism $X : r \rightarrow s$ in \mathcal{C} we obtain two

2-isomorphisms $F_l, F_r : F(X^*) \rightarrow F(X)^*$, the first from the left duality and the second from the right duality:

$$\begin{array}{c}
 \text{Diagram of } F_l \text{ (box in U-shape)} \\
 = \\
 \text{Diagram of } F_l \text{ (cap and cup)}
 \end{array}
 \qquad
 \begin{array}{c}
 \text{Diagram of } F_r \text{ (box in U-shape)} \\
 = \\
 \text{Diagram of } F_r \text{ (cup and cap)}
 \end{array}
 \tag{21}$$

The following definition is inspired by the corresponding definition for monoidal functors [10, §1.7.5].

Definition 2.16. Let \mathcal{C}, \mathcal{D} be pivotal 2-categories, let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a pseudofunctor, and let $F_l, F_r : F(X^*) \rightarrow F(X)^*$ be the isomorphisms (21). We say that F is *pivotal* if $F_l = F_r =: P$.

In the graphical calculus we again here write these isomorphisms P and their inverses as ‘zoom ins’ and ‘zoom outs’, which this time are not semantically empty:

$$\begin{array}{c}
 \text{Zoom In Diagram} \\
 := \\
 \text{Zoom In Diagram}
 \end{array}
 \qquad
 \begin{array}{c}
 \text{Zoom Out Diagram} \\
 := \\
 \text{Zoom Out Diagram}
 \end{array}$$

2.5 Pivotal dagger 2-categories The final structure we will consider on a 2-category is a *dagger*. In this section we define a dagger 2-category and discuss compatibility with the various notions already introduced.

Definition 2.17. A dagger 2-category is a 2-category equipped with contravariant identity-on-objects functors $\dagger_{r,s} : \mathcal{C}(r, s) \rightarrow \mathcal{C}(r, s)$ for each pair of objects r, s , which are:

- *Involutive*: for any morphism $f : X \rightarrow Y$ in $\mathcal{C}(r, s)$, $\dagger_{r,s}(\dagger_{r,s}(f)) = f$. (This is to say that $\mathcal{C}(r, s)$ is a *dagger category*.)
- *Compatible with 1-morphism composition*: for any 1-morphisms $X, X' : r \rightarrow s$ and $Y, Y' : s \rightarrow t$, and 2-morphisms $\alpha : X \rightarrow X'$ and $\beta : Y \rightarrow Y'$, we have $(\alpha \otimes \beta)^{\dagger_{r,t}} = \alpha^{\dagger_{r,s}} \otimes \beta^{\dagger_{s,t}}$.

We call the image of a 2-morphism $f : X \rightarrow Y$ under $\dagger_{r,s}$ its *dagger*, and write it as f^\dagger .

In the graphical calculus, we represent the dagger of a 2-morphism by reflection in a horizontal axis, preserving the direction of any arrows:

$$\begin{array}{c}
 \begin{array}{c} \uparrow X \\ \boxed{f} \\ \uparrow Y \end{array} \\
 := \\
 \begin{array}{c} \uparrow X \\ \boxed{f^\dagger} \\ \uparrow Y \end{array}
 \end{array}
 \tag{22}$$

Definition 2.18. Let \mathcal{C} be a dagger 2-category. We say that a 2-morphism $\alpha : X \rightarrow Y$ in $\mathcal{C}(r, s)$ is an *isometry* if $\alpha^\dagger \circ \alpha = \text{id}_X$. We say that it is *unitary* if it is an isometry and additionally a *coisometry*, i.e. $\alpha \circ \alpha^\dagger = \text{id}_Y$.

Definition 2.19. Let \mathcal{C} be a dagger 2-category and let r, s be objects. We say that a 1-morphism $X : r \rightarrow s$ is a *dagger equivalence* if there exists some 1-morphism $X^{-1} : s \rightarrow r$ (called the weak inverse) and unitary 2-morphisms $\eta : \text{id}_s \rightarrow X^{-1} \otimes X$ and $\epsilon : X \otimes X^{-1} \rightarrow \text{id}_r$ witnessing an equivalence (Definition 2.1). It is a standard result that η, ϵ may be chosen such that $[X^{-1}, \eta, \epsilon]$ is a right dual for X (this is to say that any dagger equivalence can be promoted to an *adjoint dagger equivalence*).

We now give the condition for compatibility of dagger and pivotal structure.

Definition 2.20. Let \mathcal{C} be a pivotal 2-category which is also a dagger 2-category. We say that \mathcal{C} is a *pivotal dagger 2-category* when, for all 1-morphisms $X : r \rightarrow s$:

$$\downarrow \cup \downarrow = \left(\uparrow \cap \uparrow \right)^\dagger \qquad \uparrow \cap \uparrow = \left(\downarrow \cup \downarrow \right)^\dagger \quad (23)$$

Remark 2.21. For any object X in a dagger 2-category, a right dual $[X^*, \eta_X, \epsilon_X]$ is also a left dual $[X^*, \epsilon_X^\dagger, \eta_X^\dagger]$. This means that a dagger 2-category with right duals also has left duals. The pivotal structure gives another way to obtain left duals from right duals (18). The equation (23) implies that the left duals obtained from the dagger structure are the same as those obtained from the pivotal structure.

Practically, when taking the dagger of a cup or a cap in a pivotal dagger category, the equation (23) implies we should reflect the cup or cap in a horizontal axis, preserving the direction of the arrows.

The following result from the theory of pivotal dagger categories generalises immediately to pivotal dagger 2-categories, since the proof is entirely diagrammatic.

Lemma 2.22 ([1, Prop. 3.5.2, Prop. 3.5.3]). *Let \mathcal{C} be a pivotal dagger 2-category. Then the 2-isomorphism components ι_X of the invertible icon $\iota : **\mathcal{C} \rightarrow \text{id}_{\mathcal{C}}$ are unitary, and the following equality holds:*

$$\downarrow \cup \downarrow = \uparrow \cap \uparrow \quad \uparrow \cap \uparrow = \downarrow \cup \downarrow \quad (24)$$

For any morphism $f : X \rightarrow Y$, a pivotal dagger structure implies the following *conjugate 2-morphism* f_* is graphically well-defined:

$$\begin{array}{c} \downarrow Y \\ \boxed{f} \\ \downarrow X \end{array} := \begin{array}{c} \downarrow X \\ \boxed{f} \\ \downarrow Y \end{array} = \left[\begin{array}{c} \downarrow X \\ \boxed{f} \\ \downarrow Y \end{array} \right]^\dagger \quad (25)$$

Remark 2.23. In view of (23), in a pivotal dagger 2-category we also have sliding equations for the dagger and conjugate 2-morphisms obtained by taking the reflections of (19) in a horizontal axis.

Finally, we consider the notion of a unitary pseudofunctor between dagger 2-categories.

Definition 2.24. Let \mathcal{C}, \mathcal{D} be dagger 2-categories and let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a pseudofunctor. We say that F is *unitary* if the following hold:

- For any 2-morphism f , $F(f^\dagger) = F(f)^\dagger$:

$$\begin{array}{c} \boxed{Ff} \\ \downarrow FY \end{array} = \begin{array}{c} \boxed{f} \\ \downarrow Y \end{array}$$

- The multipliers $\{m_{X,Y}\}$ and unitors $\{u_r\}$ are all unitary 2-morphisms in \mathcal{D} .

Remark 2.25. The latter condition implies that our depiction of the inverses $\{m_{X,Y}^{-1}\}$ and $\{u_r^{-1}\}$ by reflection in a horizontal axis (3, 4) is consistent with the ‘horizontal flip’ calculus (22) of the dagger in \mathcal{D} .

3. Dualisable pseudonatural transformations

3.1 Duals We now consider invertibility of pseudonatural transformations. As we saw in Definition 2.8, the most general notion of invertibility of a 1-morphism in a 2-category is dualisability. The following definition is nothing more than an explicit statement of what it means for a 1-morphism in $\text{Fun}(\mathcal{C}, \mathcal{D})$ to have a dual (Definition 2.8).

Definition 3.1. Let $F, G : \mathcal{C} \rightarrow \mathcal{D}$ be pseudofunctors and $\alpha : F \rightarrow G$ a pseudonatural transformation. A *right dual* for α is a triple $[\alpha^*, \eta, \epsilon]$, where $\alpha^* : G \rightarrow F$ is a pseudonatural transformation and $\epsilon : \alpha \otimes \alpha^* \rightarrow \text{id}_F$ and $\eta : \text{id}_G \rightarrow \alpha^* \otimes \alpha$ are modifications, such that the following equations hold for any 1-morphism $X : r \rightarrow s$ in \mathcal{C} :

(26)

In the above equations we have drawn the α -wire in green with an upwards-facing arrow and the α^* -wire in blue with a downwards-facing arrow, as though α_r and α_r^* were dual 1-morphisms. This will be justified by Lemma 3.2. A *left dual* is defined analogously.

Lemma 3.2. Let $F, G : \mathcal{C} \rightarrow \mathcal{D}$ be pseudofunctors and $\alpha : F \rightarrow G$ a pseudonatural transformation with right dual $[\alpha^*, \eta, \epsilon]$. Then for each object r of \mathcal{C} , $[\alpha_r^*, \eta_r, \epsilon_r]$ is a right dual for α_r in \mathcal{D} . The analogous statement holds for left duals.

Proof. We prove the right snake equation for right duals; everything else may be proved similarly. For any object r of \mathcal{C} :

(27)

Here the first equation is by invertibility of the unitor u_r (4) for F ; the second by monoidality (11) of the pseudonatural transformation α on the 1-morphism $\text{id}_r : r \rightarrow r$ and invertibility of the unitor for G ; the third by (26); the fourth by monoidality of α and α^* on id_r ; and the last by invertibility of the unitors. \square

From this point forward, therefore, we will draw η_r and ϵ_r as a cup and cap. If \mathcal{C} has duals, we obtain explicit expressions for the left and right duals in $\text{Fun}(\mathcal{C}, \mathcal{D})$ whenever they exist.

Lemma 3.3. Let $F, G : \mathcal{C} \rightarrow \mathcal{D}$ be pseudofunctors, and suppose that \mathcal{C} has left duals. A pseudonatural transformation $\alpha : F \rightarrow G$ has a right dual in $\text{Fun}(\mathcal{C}, \mathcal{D})$ precisely when α_r has a right dual $[\alpha_r^*, \eta_r, \epsilon_r]$ in \mathcal{D} for each object r of \mathcal{C} . A right dual α^* is defined as follows:

- For each object r of \mathcal{C} , $(\alpha^*)_r = (\alpha_r)^*$ and the components of the modifications η, ϵ are $[\eta_r, \epsilon_r]$.
- For each 1-morphism $X : r \rightarrow s$ of \mathcal{C} , $(\alpha^*)_X$ is:

(28)

This statement also holds with ‘left’ and ‘right’ swapped, in which case a left dual ${}^*\alpha$ is defined as follows:

- For each object r of \mathcal{C} , $({}^*\alpha)_r = {}^*(\alpha_r)$ and the components of the modifications η, ϵ are $[\eta_r, \epsilon_r]$.
- For each 1-morphism $X : r \rightarrow s$ of \mathcal{C} , $({}^*\alpha)_X$ is defined as in (28), but with the opposite transposition.

Proof. We consider the case of the right dual α^* ; the argument for the left dual is similar.

If some α_r has no right dual, then nor can α by Lemma 3.2.

If every α_r has some right dual, then we must show firstly that α^* as defined is a pseudonatural transformation, and secondly that η, ϵ as defined are modifications satisfying the snake equations (14).

1. *Naturality of α^* .* (9) For all 2-morphisms $f : X \rightarrow Y$ in \mathcal{C} :

Here the first and third equalities use Lemma 2.13; the second equality is by naturality of α .

2. *Monoidality of α^* .* (10-11)

- For every pair of 1-morphisms $X : r \rightarrow s, Y : s \rightarrow t$ in \mathcal{C} , let $f : {}^*(X \otimes Y) \rightarrow {}^*Y \otimes {}^*X$ be the isomorphism of Lemmas 2.9 and 2.10. Then:

Here the first equality is by definition; the second by a snake equation for α_s ; the third by monoidality of α and some manipulation of functorial boxes; the fourth by Lemmas 2.9 and 2.10; the fifth by naturality of α ; and the sixth by definition.

- For every object r of \mathcal{C} :

Here the first equality is by definition, the second by monoidality of α and manipulation of functorial boxes, and the third by a snake equation for α_r . We have assumed for that the chosen left dual of id_r is $[\text{id}_r, \text{id}_{\text{id}_r}, \text{id}_{\text{id}_r}]$; in general one can use Lemma 2.10 and naturality of α as in (30).

3. Since η_r, ϵ_r already satisfy the snake equations for every r by assumption, we need only show that η, ϵ are modifications. For all $X : r \rightarrow s$ in \mathcal{C} :

$$(32)$$

$$(33)$$

Here, the first equalities are by definition, the second are by a snake equation for α_r^* or α_s^* , and the third are by naturality and monoidality of α .

□

Corollary 3.4. *If \mathcal{C} has left duals, and \mathcal{D} has right duals, then $\text{Fun}(\mathcal{C}, \mathcal{D})$ has right duals. This statement also holds with ‘left’ and ‘right’ swapped.*

3.2 Pivotality We now consider pivotality of $\text{Fun}(\mathcal{C}, \mathcal{D})$. Recall that a 2-category with right duals is *pivotal* (Definition 2.12) if there is an invertible icon (Definition 2.4) from the double duals pseudofunctor to the identity pseudofunctor.

We now show that $\text{Fun}(\mathcal{C}, \mathcal{D})$ inherits pivotality from \mathcal{C} and \mathcal{D} upon restriction to pivotal pseudofunctors.

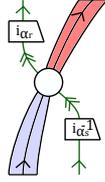
Definition 3.5. When \mathcal{C}, \mathcal{D} are pivotal we define $\text{Fun}_p(\mathcal{C}, \mathcal{D}) \subset \text{Fun}(\mathcal{C}, \mathcal{D})$ to be the subcategory whose objects are pivotal pseudofunctors.

Proposition 3.6. *Let \mathcal{C}, \mathcal{D} be pivotal 2-categories, and let $\iota : **_{\mathcal{D}} \rightarrow \text{id}_{\mathcal{D}}$ be the pivotal structure on \mathcal{D} .*

*The 2-category $\text{Fun}_p(\mathcal{C}, \mathcal{D})$ has a canonical structure of a pivotal 2-category. The pivotal structure $\hat{\iota} : **_{\text{Fun}(\mathcal{C}, \mathcal{D})} \rightarrow \text{id}_{\text{Fun}(\mathcal{C}, \mathcal{D})}$ assigns to every pseudonatural transformation $\alpha^{**} : F \rightarrow G$ the invertible modification $\hat{\iota}_\alpha : \alpha^{**} \rightarrow \alpha$ whose components are the 2-isomorphisms $\iota_{\alpha_r} : \alpha_r^{**} \rightarrow \alpha_r$ from the pivotal structure on \mathcal{D} .*

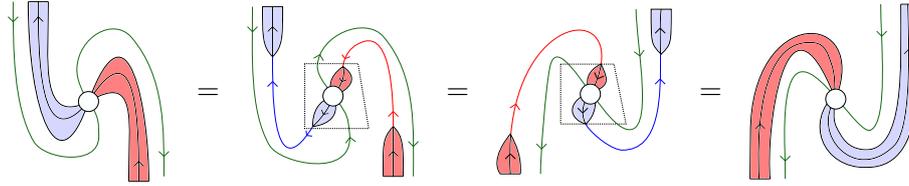
Proof. First we show that the $\hat{\iota}_\alpha$ are really modifications. Since $\{\iota_{\alpha_r}\}$ are 2-isomorphisms it is immediate that the $\hat{\iota}_\alpha$ -conjugate $(\alpha^{**})^{\hat{\iota}_\alpha}$ of α^{**} is a pseudonatural transformation $F \rightarrow G$, where

$(\alpha^{**})_r^{\hat{\iota}_\alpha} = \alpha_r$ for all objects r of \mathcal{C} , and $(\alpha^{**})_X^{\hat{\iota}_\alpha}$ is defined as follows for all $X : r \rightarrow s$:


(34)

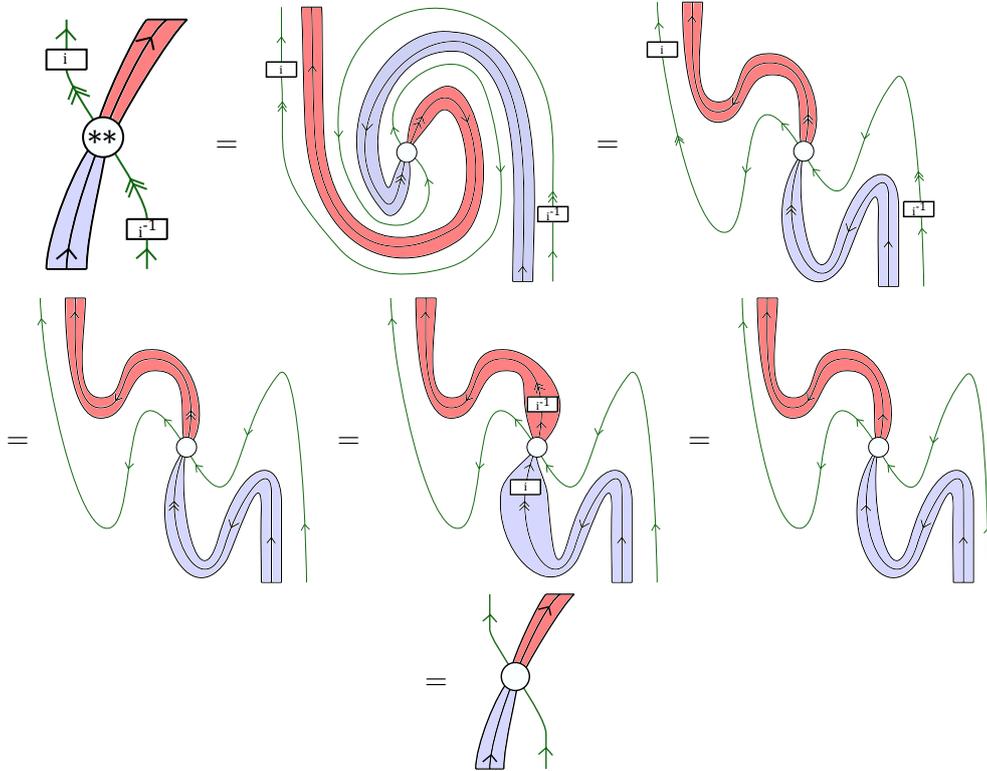
It is also clear that $\hat{\iota}_\alpha$ is a modification $\alpha^{**} \rightarrow (\alpha^{**})^{\hat{\iota}_\alpha}$.

We now show that $\hat{\iota}_\alpha$ has the right target, i.e. $(\alpha^{**})^{\hat{\iota}_\alpha} = \alpha$. We first observe that the chosen left dual of a pseudonatural transformation between pivotal functors is identical to its chosen right dual:


(35)

Here for the first and third equalities we used Lemma 2.10 and the ‘zoom out’ notation (20) to relate the duals in \mathcal{C} and \mathcal{D} . For the second equality we follow the custom in the setting of pivotal categories of appealing to an unproven but very plausible coherence theorem [9, Theorem 4.14]; it is not hard to prove the equality directly from the axioms, but we leave this to the reader. For the third equality we require that the pseudofunctors are pivotal.

Now for any $\alpha : F \rightarrow G$ and $X : r \rightarrow s$ in \mathcal{C} we have:



Here the first equality is by definition; the second uses (35); the third uses the definition (18) of the left duality in the pivotal 2-category \mathcal{D} ; the fourth uses naturality of α to insert ι^{-1} , where

$\iota : X^{**} \rightarrow X$ is the isomorphism from the pivotal structure in \mathcal{C} ; the fifth uses the definition (18) of the left duality in \mathcal{C} ; and the last uses the snake equations in \mathcal{C} and \mathcal{D} .

Finally, we need to show that $\hat{\iota}$ is an invertible icon $**_{\text{Fun}(\mathcal{C}, \mathcal{D})} \rightarrow \text{id}_{\text{Fun}(\mathcal{C}, \mathcal{D})}$.

- *Monoidality*: For every pair of pseudonatural transformations $\alpha : F \rightarrow G$, $\beta : G \rightarrow H$, we need $\hat{\iota}_{\alpha \otimes \beta} = \hat{\iota}_\alpha \otimes \hat{\iota}_\beta$. For each $X : r \rightarrow s$ this is implied by monoidality of $\iota : **_{\mathcal{D}} \rightarrow \text{id}_{\mathcal{D}}$. Indeed, we have:

$$(\hat{\iota}_{\alpha \otimes \beta})_r = \iota_{\alpha_r \otimes \beta_r} = \iota_{\alpha_r} \otimes \iota_{\beta_r} = (\hat{\iota}_\alpha)_r \otimes (\hat{\iota}_\beta)_r$$

- *Naturality*: We need that, for every modification $f : \alpha \rightarrow \beta$, $\hat{\iota}_\beta \circ f^{**} = f \circ \hat{\iota}_\alpha$. For each $X : r \rightarrow s$ this is implied by naturality of $\iota : **_{\mathcal{D}} \rightarrow \text{id}_{\mathcal{D}}$. Indeed, we have:

$$(f \circ \hat{\iota}_\alpha)_r = (f)_r \circ (\hat{\iota}_\alpha)_r = f_r \circ \iota_{\alpha_r} = \iota_{\beta_r} \circ f_r = (\hat{\iota}_\beta)_r \circ f_r = (\hat{\iota}_\beta \circ f)_r$$

□

4. Unitary pseudonatural transformations

We have considered the case where \mathcal{C}, \mathcal{D} are pivotal. We now consider the case where \mathcal{C}, \mathcal{D} are pivotal dagger and the pseudofunctors are unitary.

In this case, we get a new contravariant operation on pseudonatural transformations.

Lemma 4.1. *Let $F, G : \mathcal{C} \rightarrow \mathcal{D}$ be unitary pseudofunctors between pivotal dagger 2-categories. For any pseudonatural transformation $\alpha : F \rightarrow G$, there is a pseudonatural transformation $\alpha^\dagger : G \rightarrow F$ (its dagger), defined as follows:*

- For each object r of \mathcal{C} , $(\alpha^\dagger)_r = (\alpha_r)^*$.
- For each $X : r \rightarrow s$ in \mathcal{C} , $(\alpha^\dagger)_X$ is defined as follows:

(36)

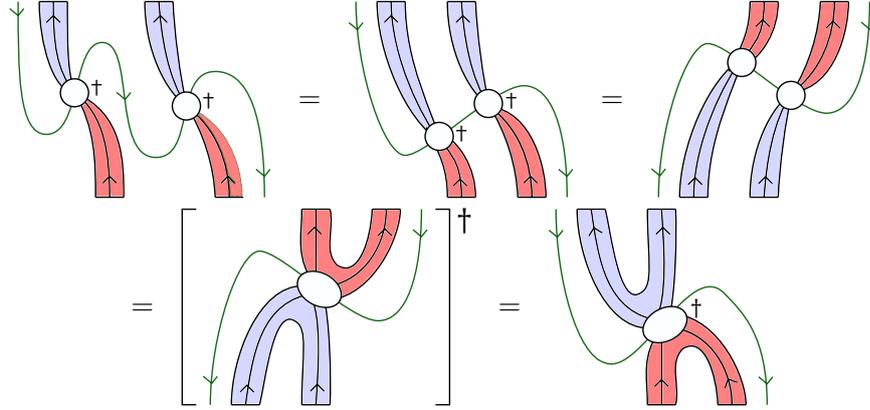
is also a pseudonatural transformation.

Proof. We must show naturality and monoidality.

- *Naturality.* For any $f : X \rightarrow Y$ in \mathcal{C} :

Here the first equality is by unitarity of G , the second equality is by naturality of α , and the third equality is by unitarity of F .

- *Monoidality.* For any $X : r \rightarrow s, Y : s \rightarrow t$ in \mathcal{C} :



Here the first and second equalities are by dagger pivotality of \mathcal{D} , the third equality is by monoidality of α , and the fourth equality is by unitarity of F, G and dagger pivotality of \mathcal{D} .

We leave the other monoidality condition (11) to the reader. □

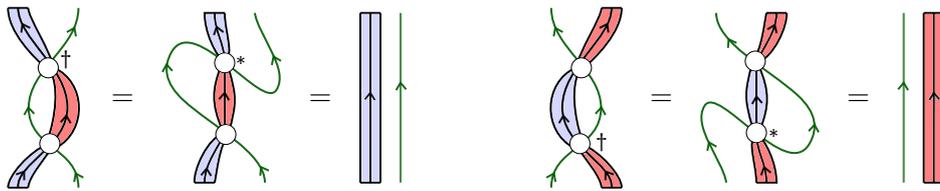
We would like $\text{Fun}(\mathcal{C}, \mathcal{D})$ to inherit the structure of a dagger 2-category. In general, however, there is no reason why the componentwise dagger of a modification $f : \alpha \rightarrow \beta$ — the only reasonable candidate for a dagger on $\text{Fun}(\mathcal{C}, \mathcal{D})$ — should yield a modification $f^\dagger : \beta \rightarrow \alpha$.

This problem is resolved by restriction to unitary pseudonatural transformations. There are two obvious ways to define unitarity. First, given that the dual is the ‘inverse’ of a pseudonatural transformation, we could ask that the dagger (36) of the transformation should be equal to the right dual (28). Alternatively, by analogy with the definition of unitary monoidal natural transformations, and motivated by physicality in quantum mechanics [12], we might demand that the components of the transformation be individually unitary in \mathcal{D} . In fact, these definitions are equivalent.

Lemma 4.2. *Let \mathcal{C}, \mathcal{D} be pivotal dagger 2-categories and let $\alpha : F \rightarrow G$ be a pseudonatural transformation between unitary pseudofunctors $F, G : \mathcal{C} \rightarrow \mathcal{D}$. The following are equivalent:*

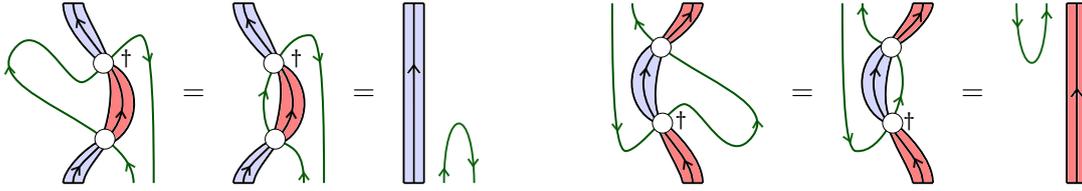
1. *There is an equality of pseudonatural transformations $\alpha^* = \alpha^\dagger$.*
2. *For all 1-morphisms $X : r \rightarrow s$ in \mathcal{C} , the component $\alpha_X : F(X) \otimes \alpha_s \rightarrow \alpha_r \otimes G(X)$ is unitary.*

Proof. (i) \Rightarrow (ii): For all $X : r \rightarrow s$ in \mathcal{C} , unitarity of α_X follows from right duality:



(ii) \Rightarrow (i): Unitarity of the components implies that $[\alpha^\dagger, \eta, \epsilon]$ is a right dual, where η, ϵ are the

cup and cap of the right dual $[\alpha^*, \eta, \epsilon]$, since for each component:



But this implies equality $\alpha^\dagger = \alpha^*$; indeed, since the cup and cap modifications are identical, the unique 2-isomorphism of Lemma 2.10 relating the two right duals in $\text{Fun}(\mathcal{C}, \mathcal{D})$ must be the identity. \square

We therefore make the following definition.

Definition 4.3. Let \mathcal{C}, \mathcal{D} be pivotal dagger 2-categories and let $F, G : \mathcal{C} \rightarrow \mathcal{D}$ be unitary pseudofunctors. Then a *unitary pseudonatural transformation (UPT)* $\alpha : F \rightarrow G$ is a pseudonatural transformation such that either of the following equivalent conditions are satisfied:

- There is an equality of pseudonatural transformations $\alpha^* = \alpha^\dagger$.
- For all 1-morphisms $X : r \rightarrow s$ in \mathcal{C} , the component $\alpha_X : F(X) \otimes \alpha_s \rightarrow \alpha_r \otimes G(X)$ is unitary.

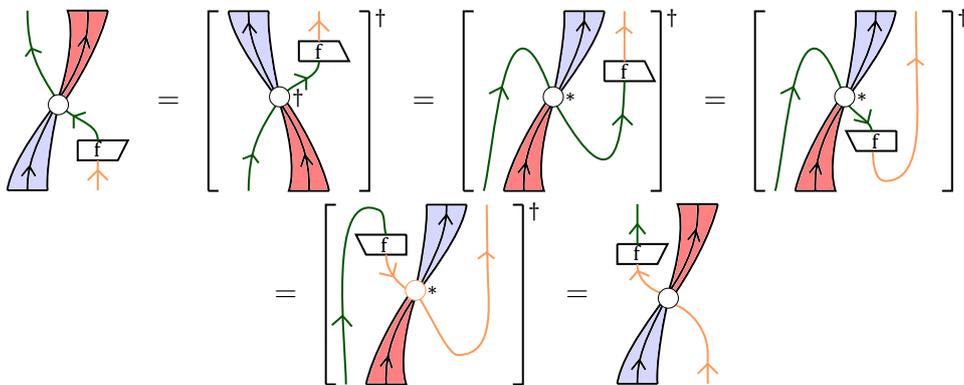
Definition 4.4. When \mathcal{C}, \mathcal{D} are pivotal dagger, let $\text{Fun}_u(\mathcal{C}, \mathcal{D}) \subset \text{Fun}(\mathcal{C}, \mathcal{D})$ be the subcategory whose objects are unitary pseudofunctors and whose 1-morphisms are UPTs.

We now show that $\text{Fun}_u(\mathcal{C}, \mathcal{D})$ is a dagger 2-category. Moreover, it is pivotal dagger, with no need to restrict to pivotal functors.

Theorem 4.5. Let \mathcal{C}, \mathcal{D} be pivotal dagger 2-categories. Then the 2-category $\text{Fun}_u(\mathcal{C}, \mathcal{D})$ is pivotal dagger, where:

- The dagger of a modification $f : \alpha \rightarrow \beta$ is defined on components as $(f^\dagger)_r = (f_r)^\dagger$.
- The pivotal structure $\hat{\iota} : **_{\text{Fun}_u(\mathcal{C}, \mathcal{D})} \rightarrow \text{id}_{\text{Fun}_u(\mathcal{C}, \mathcal{D})}$ assigns to every pseudonatural transformation $\alpha^{**} : F \rightarrow G$ the invertible modification $\hat{\iota}_\alpha : \alpha^{**} \rightarrow \alpha$ whose components are the 2-isomorphisms $\iota_{\alpha_r} : \alpha_r^{**} \rightarrow \alpha_r$ from the pivotal structure on \mathcal{D} .

Proof. We first show that f^\dagger is a modification $\beta \rightarrow \alpha$:



Here the second equality is by unitarity of α , and the fourth equality is by transposition in $\text{Fun}_u(\mathcal{C}, \mathcal{D})$.

$\text{Fun}_u(\mathcal{C}, \mathcal{D})$ is therefore a dagger 2-category. We now show that it is pivotal dagger. First we demonstrate that $\hat{\iota}$ is indeed a pivotal structure. Since by Lemma 4.2 we have $\alpha^* = \alpha^\dagger$, we have the following expression for the components of $\alpha^{**} = \alpha^{\dagger\dagger}$:

(37)

Here the last equality is by Lemma 2.22. We claim that $\hat{\iota}_\alpha : \alpha^{**} \rightarrow \alpha$ is a modification. Indeed, by (37) and unitarity of $\{\iota_X\}$ we clearly have:

The proof that $\hat{\iota}$ is an invertible icon $**_{\text{Fun}(\mathcal{C}, \mathcal{D})} \rightarrow \text{id}_{\text{Fun}(\mathcal{C}, \mathcal{D})}$, i.e. that the transformation is monoidal and natural, is given at the end of the proof of Proposition 3.6.

Finally, we must show that the duals of $\text{Fun}_u(\mathcal{C}, \mathcal{D})$ are dagger duals (23). This follows from the fact that the dagger of a modification is taken componentwise, and the cup and cap for each component come from the pivotal dagger structure in \mathcal{D} . \square

Corollary 4.6. *Let \mathcal{C}, \mathcal{D} be pivotal dagger 2-categories and let $\alpha : F_1 \rightarrow F_2$ be a UPT between pseudofunctors $\mathcal{C} \rightarrow \mathcal{D}$. Then the right dual α^* defined in Lemma 3.3 is equal to the left dual ${}^*\alpha$ defined in Lemma 3.3.*

Proof. For every 1-morphism $X : r \rightarrow s$ of \mathcal{C} the right dual UPT satisfies the following equation with respect to the double right dual UPT:

(38)

Postcomposing the leftmost and rightmost expressions by $\iota_{\alpha_r} \otimes \text{id}_{\alpha^*} \otimes \text{id}_{F_1(X)}$, we obtain the following pullthrough equation for the cup of the left duality:

(39)

A similar pullthrough equation can be obtained for the cap of the left duality. It follows that α^* is a left dual of α with the same cup and cap as the chosen left dual of α . We must therefore have $\alpha^* = {}^*\alpha$ by Lemma 2.10 in $\text{Fun}_u(\mathcal{C}, \mathcal{D})$. \square

5. Morita theory for fibre functors on $\text{Rep}(G)$

We finish by discussing an application of the results in this work. One reason for proving that $\text{Fun}_u(\mathcal{C}, \mathcal{D})$ is a pivotal dagger 2-category is that this provides an appropriate setting for Morita theory, which relates 1-morphisms out of an object r to Frobenius monoids in its pivotal dagger category of endomorphisms $\text{End}(r) := \text{Hom}(r, r)$.

Definition 5.1. A *monoid* in a monoidal dagger category is an object A with multiplication and unit morphisms, depicted as follows:

$$\begin{array}{ccc}
 \begin{array}{c} | \\ \circ \\ \swarrow \searrow \\ | \end{array} & & \begin{array}{c} | \\ \circ \end{array} \\
 m : A \otimes A \rightarrow A & & u : \mathbb{1} \rightarrow A
 \end{array} \tag{40}$$

These morphisms satisfy the following associativity and unitality equations:

$$\begin{array}{ccc}
 \begin{array}{c} | \\ \circ \\ \swarrow \searrow \\ \circ \swarrow \searrow \\ | \end{array} = \begin{array}{c} | \\ \circ \\ \swarrow \searrow \\ \circ \swarrow \searrow \\ | \end{array} & & \begin{array}{c} | \\ \circ \\ \swarrow \searrow \\ \circ \end{array} = | = \begin{array}{c} | \\ \circ \\ \swarrow \searrow \\ \circ \end{array}
 \end{array} \tag{41}$$

Analogously, a *comonoid* is an object A with a coassociative comultiplication $\delta : A \rightarrow A \otimes A$ and a counit $\epsilon : A \rightarrow \mathbb{1}$. The dagger of a monoid (A, m, u) is a comonoid $(A, m^\dagger, u^\dagger)$. A monoid (A, m, u) is called *Frobenius* if the monoid and adjoint comonoid structures are related by the following *Frobenius equation*:

$$\begin{array}{ccc}
 \begin{array}{c} | \\ \circ \\ \swarrow \searrow \\ \circ \swarrow \searrow \\ | \end{array} = \begin{array}{c} | \\ \circ \\ \swarrow \searrow \\ \circ \swarrow \searrow \\ | \end{array} = \begin{array}{c} | \\ \circ \\ \swarrow \searrow \\ \circ \swarrow \searrow \\ | \end{array}
 \end{array} \tag{42}$$

A Frobenius monoid is *special* if the following equation is satisfied:

$$\begin{array}{ccc}
 \begin{array}{c} | \\ \circ \\ \swarrow \searrow \\ \circ \swarrow \searrow \\ | \end{array} = |
 \end{array} \tag{43}$$

Let \mathcal{C} be a pivotal dagger 2-category. We say that \mathcal{C} is \mathbb{C} -linear if the 2-morphism sets are complex vector spaces such that horizontal and vertical composition of 2-morphisms are bilinear maps and the dagger is an antilinear map. We assume additionally that, for any 2-morphism $f : X \rightarrow Y$, $f^\dagger \circ f = 0$ implies $f = 0$. We say that an object r of \mathcal{C} is *simple* if $\text{Hom}(\text{id}_r, \text{id}_r) \cong \mathbb{C}$.

The variant of Morita theory consider is essentially as follows. Let \mathcal{C} be a \mathbb{C} -linear pivotal dagger 2-category, let s be a simple object, and let $X : r \rightarrow s$ be a 1-morphism. Let d_X be the nonzero scalar such that $\dim_L(X) = d_X \text{id}_s$. Observe that $\text{End}(r)$ is a monoidal dagger category. Then, making use of the left duality in the pivotal dagger 2-category, the object $X \otimes X^*$ in

$\text{End}(r)$ acquires the structure of a special Frobenius monoid with the following multiplication and unit morphisms:

$$\begin{array}{c} \frac{1}{\sqrt{d_X}} \end{array} \begin{array}{c} \begin{array}{c} \text{r} \quad \text{s} \quad \text{r} \\ \text{---} \end{array} \end{array} \quad \begin{array}{c} \sqrt{d_X} \end{array} \begin{array}{c} \begin{array}{c} \text{r} \quad \text{s} \quad \text{r} \\ \text{---} \end{array} \end{array} \quad (44)$$

We will see that 1-morphisms from r to simple objects can in fact be classified in terms of relations between their corresponding special Frobenius monoids.

We are here particularly interested in the pivotal dagger 2-category $\text{Fun}_u(\text{Rep}(G), \text{Hilb})$, where $\text{Rep}(G)$ is the pivotal dagger category of continuous finite-dimensional unitary representations of a compact quantum group G and Hilb is the category of Hilbert spaces and linear maps. We restrict to \mathbb{C} -linear unitary monoidal functors, which we call *fibre functors*. It is not important for our purposes here to discuss the definition of the category $\text{Rep}(G)$ (see e.g. [11, §2.3.2] for this). All that matters here is that $\text{Rep}(G)$ is a pivotal dagger category with a privileged *canonical fibre functor* $F : \text{Rep}(G) \rightarrow \text{Hilb}$.

In this case, since $\text{Rep}(G)$ and Hilb are one-object 2-categories, we obtain a simpler description of UPTs and modifications. In particular:

- Let F_1, F_2 be fibre functors on $\text{Rep}(G)$. A unitary pseudonatural transformation $(\alpha, H) : F_1 \rightarrow F_2$ is defined by the following data:
 - An Hilbert space H (drawn as a green wire).
 - For every object X of $\text{Rep}(G)$, a unitary $\alpha_X : F_1(X) \otimes H \rightarrow H \otimes F_2(X)$ (drawn as a white vertex):

$$\begin{array}{c} \text{X} \\ \text{---} \\ \text{X} \end{array} \quad (45)$$

These unitaries must obey the naturality and monoidality conditions (9-11). We call $\dim(H)$ the *dimension* of the UPT.

- Let $(\alpha, H), (\beta, H') : F_1 \rightarrow F_2$ be UPTs. (We colour the H -wire green and the H' -wire orange.) A modification $f : \alpha \rightarrow \beta$ is a linear map $f : H \rightarrow H'$ satisfying the following equation for all unitaries $\{\alpha_X, \beta_X\}$:

$$\begin{array}{c} \text{X} \\ \text{---} \\ \text{X} \end{array} \quad = \quad \begin{array}{c} \text{X} \\ \text{---} \\ \text{X} \end{array} \quad (46)$$

It is clear that $\text{Fun}_u(\text{Rep}(G), \text{Hilb})$ is \mathbb{C} -linear. Moreover, every object of $\text{Fun}(\text{Rep}(G), \text{Hilb})$ is simple.

In [11, §3] we characterised the category $\text{End}(F)$ of unitary pseudonatural transformations and modifications from the canonical fibre functor to itself, showing that it is isomorphic to the category $\text{Rep}(A_G)$ of finite-dimensional $*$ -representations of the *compact quantum group algebra* A_G associated to the compact quantum group G . Morita theory will therefore allow us to classify fibre functors accessible by a UPT from the canonical fibre functor, and UPTs from the canonical fibre functor, in terms of special Frobenius monoids in the category $\text{Rep}(A_G)$.

5.1 Classification of UPTs from the canonical fibre functor We begin with a technical definition.

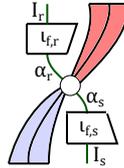
Definition 5.2. We say that a dagger 2-category has *split dagger idempotents* if, for any 1-morphism $X : r \rightarrow s$ and any 2-morphism $\alpha : X \rightarrow X$ such that $\alpha = \alpha^\dagger = \alpha^2$ (we call such 2-morphisms *dagger idempotent*), there exists a 1-morphism $V : r \rightarrow s$ and an isometry $\iota : V \rightarrow X$ such that $\iota \circ \iota^\dagger = \alpha$.

Lemma 5.3. *The category $\text{Fun}_u(\mathcal{C}, \mathcal{D})$ has split dagger idempotents if \mathcal{D} has split dagger idempotents.*

Proof. Let $\alpha : F_1 \rightarrow F_2$ be a UPT and let $f : \alpha \rightarrow \alpha$ be a dagger idempotent modification. Since for each object r of \mathcal{C} the component $f_r : \alpha_r \rightarrow \alpha_r$ is itself a dagger idempotent in \mathcal{D} , there exist objects I_r of \mathcal{D} and isometries $\iota_{f,r} : I_r \rightarrow \alpha_r$ such that:

$$\iota_{f,r}^\dagger \circ \iota_{f,r} = \text{id}_{I_r} \qquad \iota_{f,r} \circ \iota_{f,r}^\dagger = f_r \qquad (47)$$

Now we define a new UPT α^{l_f} whose components $\{\alpha_X^{l_f}\}$ are given as follows:



$$(48)$$

It is clear that this is a UPT and that ι_f , with components defined as $(\iota_f)_r = \iota_{f,r}$, is a modification $\alpha^{l_f} \rightarrow \alpha$ satisfying $\iota_f^\dagger \circ \iota_f = \text{id}_\alpha$ and $\iota_f \circ \iota_f^\dagger = f$. \square

It immediately follows that $\text{Fun}_u(\text{Rep}(G), \text{Hilb})$ has split dagger idempotents, since Hilb does. In order to classify UPTs from the canonical fibre functor we will need a notion of equivalence of 1-morphisms.

Definition 5.4. Let r, s, t be objects in a dagger 2-category \mathcal{C} . We say that two 1-morphisms $X : r \rightarrow s$ and $Y : r \rightarrow t$ are *equivalent* if there exists a dagger equivalence $E : t \rightarrow s$ and a unitary 2-morphism $\tau : X \rightarrow Y \otimes E$.

In $\text{Fun}_u(\text{Rep}(G), \text{Hilb})$ equivalence of UPTs can be put in more familiar terms.

Lemma 5.5. *Two UPTs $\alpha_1 : F \rightarrow F_1$ and $\alpha_2 : F \rightarrow F_2$ are equivalent in $\text{Fun}_u(\text{Rep}(G), \text{Hilb})$ if and only if there exists a unitary monoidal natural isomorphism $E : F_2 \rightarrow F_1$ and a unitary modification $\tau : \alpha_1 \rightarrow \alpha_2 \otimes E$.*

Proof. Suppose that there is an equivalence $\alpha_1 \cong \alpha_2$. Let $[\tilde{E} : t \rightarrow s, \tilde{E}^{-1}, \eta, \epsilon]$ be the data of the dagger equivalence $F_2 \rightarrow F_1$, and let $\tilde{\tau} : \alpha_1 \rightarrow \alpha_2 \otimes \tilde{E}$ be the unitary modification.

We first observe that η is a unitary modification $\text{id}_{F_1} \rightarrow \tilde{E}^{-1} \otimes \tilde{E}$. Considering underlying Hilbert spaces this yields a unitary map $\mathbb{C} \rightarrow H_{\tilde{E}^{-1}} \otimes H_{\tilde{E}}$, which implies that both these Hilbert spaces are one-dimensional. Therefore there is a unitary isomorphism $\omega : H_{\tilde{E}} \rightarrow \mathbb{C}$. Conjugating \tilde{E} by this unitary isomorphism we obtain a unitary monoidal natural isomorphism $E : F_2 \rightarrow F_1$ (i.e. a UPT whose underlying Hilbert space is \mathbb{C}). Then $\tau := (\text{id}_{\alpha_2} \otimes \omega) \circ \tilde{\tau}$ is a unitary modification $\alpha_1 \rightarrow \alpha_2 \otimes E$.

In the other direction, a unitary monoidal natural isomorphism is a dagger equivalence: the weak inverse is the actual inverse, and the unitary modifications witnessing the equivalence are trivial. \square

We now define a corresponding equivalence relation for Frobenius monoids.

Definition 5.6. Let A, B be Frobenius monoids in a monoidal dagger category. We say that a morphism $f : A \rightarrow B$ is a **-homomorphism* precisely when it satisfies the following equations:

$$\begin{array}{c} \circlearrowleft \\ | \\ f \\ | \\ \circlearrowright \end{array} = \begin{array}{c} \circlearrowleft \\ | \\ \circlearrowleft \\ | \\ f \\ | \\ \circlearrowright \\ | \\ f \\ | \\ \circlearrowright \end{array} \quad \begin{array}{c} \circlearrowleft \\ | \\ f \\ | \\ \circlearrowright \end{array} = \begin{array}{c} \circlearrowleft \\ | \\ \circlearrowright \end{array} \quad \begin{array}{c} \circlearrowleft \\ | \\ f^\dagger \\ | \\ \circlearrowright \end{array} = \begin{array}{c} \circlearrowleft \\ | \\ \circlearrowleft \\ | \\ \circlearrowright \\ | \\ \circlearrowleft \\ | \\ f \\ | \\ \circlearrowright \end{array} \quad (49)$$

We call a unitary *-homomorphism a *unitary *-isomorphism*. It is easy to check that a unitary *-isomorphism also obeys the following **-cohomomorphism* equations:

$$\begin{array}{c} \circlearrowright \\ | \\ \circlearrowleft \\ | \\ f \\ | \\ \circlearrowright \end{array} = \begin{array}{c} \circlearrowright \\ | \\ \circlearrowleft \\ | \\ f \\ | \\ \circlearrowright \\ | \\ \circlearrowleft \\ | \\ f \\ | \\ \circlearrowright \end{array} \quad \begin{array}{c} \circlearrowright \\ | \\ f \\ | \\ \circlearrowleft \end{array} = \begin{array}{c} \circlearrowright \\ | \\ \circlearrowleft \end{array} \quad \begin{array}{c} \circlearrowright \\ | \\ f^\dagger \\ | \\ \circlearrowleft \end{array} = \begin{array}{c} \circlearrowright \\ | \\ \circlearrowright \\ | \\ \circlearrowleft \\ | \\ \circlearrowright \\ | \\ \circlearrowleft \\ | \\ f \\ | \\ \circlearrowright \end{array} \quad (50)$$

Theorem 5.7. Let \mathcal{C} be a \mathbb{C} -linear pivotal dagger 2-category with split dagger idempotents. Let s, t be simple objects, and let $X : r \rightarrow s$ and $Y : r \rightarrow t$ be 1-morphisms. Then X and Y are equivalent in \mathcal{C} if and only if the special Frobenius monoids $X \otimes X^*$ and $Y \otimes Y^*$ in $\text{End}(r)$ are unitarily *-isomorphic.

Proof. Suppose that X and Y are equivalent by some dagger equivalence $[E, E^{-1}, \eta, \epsilon]$ and unitary 2-morphism $\tau : X \rightarrow Y \otimes E$. WLOG we may take $[E^{-1}, \eta, \epsilon]$ to be a right dual for E . We will show that $X \otimes X^*$ and $Y \otimes Y^*$ are unitarily *-isomorphic.

We first consider the relationship between the right dual $[E^{-1}, \eta, \epsilon]$ and the chosen right dual for E in the pivotal dagger 2-category \mathcal{C} . Let $u : E^* \rightarrow E^{-1}$ be the isomorphism relating the right duals E^* and E^{-1} by Lemma 2.10:

$$\begin{array}{c} \downarrow \downarrow \\ \eta \\ \uparrow \uparrow \end{array} = \begin{array}{c} \downarrow \downarrow \\ u \\ \uparrow \uparrow \end{array} \quad \begin{array}{c} \epsilon \\ \uparrow \uparrow \\ \downarrow \downarrow \end{array} = \begin{array}{c} \uparrow \uparrow \\ u^{-1} \\ \downarrow \downarrow \end{array} \quad (51)$$

(Here and throughout we draw the equivalence E and its duals with a blue wire, and the E^{-1} wire with a triangular arrow.) Let d_E be the scalar such that $\dim_R(E) = \frac{1}{d_E} \text{id}_t$. We first observe that

$$u^\dagger = \frac{1}{d_E} u^{-1}, \quad (52)$$

which can be seen by the following equalities:

$$\begin{array}{c} \downarrow \\ u \\ \downarrow \\ u \\ \downarrow \end{array} = \begin{array}{c} \downarrow \\ u \\ \downarrow \uparrow \uparrow \\ u \\ \downarrow \end{array} = \begin{array}{c} \downarrow \\ \eta \\ \downarrow \uparrow \uparrow \\ \eta \\ \downarrow \end{array} = \begin{array}{c} \downarrow \\ \uparrow \uparrow \\ \downarrow \end{array}$$

We can therefore make the following further observation:

$$\text{cup} = \text{diag}(\text{u}, \text{ε}) = \frac{1}{d_E} \text{lines} \quad (53)$$

We also note that

$$\dim_L(E) = d_E \text{id}_s, \quad (54)$$

which is seen as follows:

$$\text{id}_s = \text{diag}(\eta) = \text{diag}(\text{u}, \text{loop}) = \frac{1}{d_E} \text{loop} = \frac{1}{d_E} \dim_L(E)$$

Finally we consider the relationship between d_E and d_X, d_Y :

$$d_X \text{id}_s = \text{loop}_X = \text{diag}(\tau, Y) = \text{diag}(\tau, E) = d_Y d_E \text{id}_s$$

Here the second and fourth equalities are by unitarity of τ , and the third equality is by pulling τ around the cup and cap of the duality. It follows that:

$$d_E = \frac{d_X}{d_Y} \quad (55)$$

Now we can define our unitary $*$ -isomorphism $X \otimes X^* \rightarrow Y \otimes Y^*$. Consider the following 2-morphism:

$$\sqrt{\frac{d_X}{d_Y}} \text{diag}(\tau) = \text{diag}(\tau) \quad (56)$$

We show that this 2-morphism is a unitary $*$ -isomorphism. For unitarity:

$$\frac{d_X}{d_Y} \text{diag}(\tau) = \text{diag}(\tau) = \text{lines} \quad (57)$$

Here the first equality is by (53) and the second is by unitarity of τ .

$$\begin{array}{c} \frac{d_X}{d_Y} \\ \downarrow \\ \begin{array}{c} \uparrow \tau \\ \downarrow \tau \\ \uparrow \tau \\ \downarrow \tau \end{array} \end{array} = \begin{array}{c} \frac{d_X}{d_Y} \\ \downarrow \\ \begin{array}{c} \uparrow \tau \\ \downarrow \tau \end{array} \end{array} = \begin{array}{c} \uparrow \\ \downarrow \end{array} \quad (58)$$

Here the first equality is by unitarity of τ and the second is by (55).

For the first *-homomorphism condition:

$$\begin{array}{c} \frac{d_X}{d_Y} \cdot \frac{1}{\sqrt{d_Y}} \\ \downarrow \\ \begin{array}{c} \uparrow \tau \\ \downarrow \tau \\ \uparrow \tau \\ \downarrow \tau \end{array} \end{array} = \begin{array}{c} \frac{d_X}{d_Y} \cdot \frac{1}{\sqrt{d_Y}} \\ \downarrow \\ \begin{array}{c} \uparrow \tau \\ \downarrow \tau \\ \uparrow \tau \\ \downarrow \tau \end{array} \end{array} \\ \\ = \begin{array}{c} \frac{1}{\sqrt{d_Y}} \\ \downarrow \\ \begin{array}{c} \uparrow \tau \\ \downarrow \tau \\ \uparrow \tau \\ \downarrow \tau \end{array} \end{array} = \begin{array}{c} \frac{1}{\sqrt{d_Y}} \\ \downarrow \\ \begin{array}{c} \uparrow \tau \\ \downarrow \tau \\ \uparrow \tau \\ \downarrow \tau \end{array} \end{array} \\ \\ = \sqrt{\frac{d_X}{d_Y}} \cdot \frac{1}{\sqrt{d_X}} \quad (59)$$

Here the second equality is by (53) and the fourth equality is by unitarity of τ .

For the second *-homomorphism condition:

$$\begin{array}{c} \sqrt{\frac{d_X}{d_Y}} \cdot \sqrt{d_X} \\ \downarrow \\ \begin{array}{c} \uparrow \tau \\ \downarrow \tau \end{array} \end{array} = \begin{array}{c} \frac{d_X}{\sqrt{d_Y}} \\ \downarrow \\ \begin{array}{c} \uparrow \tau \\ \downarrow \tau \end{array} \end{array} = \begin{array}{c} \frac{d_X}{\sqrt{d_Y}} \\ \downarrow \\ \begin{array}{c} \uparrow \tau \\ \downarrow \tau \end{array} \end{array} = \sqrt{d_Y} \quad (60)$$

Here the second equality is by unitarity of τ and the third equality is by definition of d_E .

The third *-homomorphism condition is implied by unitarity and the first two *-homomorphism conditions.

One direction is therefore proved. For the other direction, let $f : X \otimes X^* \rightarrow Y \otimes Y^*$ be a unitary $*$ -isomorphism. We will now construct a dagger equivalence $E : t \rightarrow s$ and a unitary 2-morphism $\tau : X \rightarrow Y \otimes E$.

We first observe that the following modification $\tilde{f} : Y^* \otimes X \rightarrow Y^* \otimes X$ is a dagger idempotent:

$$\frac{1}{\sqrt{d_X d_Y}} \text{ (diagram of } \tilde{f} \text{)} \tag{61}$$

Indeed, we have the following equations for dagger idempotency. For idempotency:

$$\frac{1}{d_X d_Y} \text{ (diagram)} = \frac{1}{\sqrt{d_Y} (d_X)^{3/2}} \text{ (diagram)} = \frac{1}{\sqrt{d_X d_Y}} \text{ (diagram)} \tag{62}$$

Here the first equality is by the first $*$ -homomorphism condition (49). To see that the idempotent is dagger:

$$\frac{1}{\sqrt{d_X d_Y}} \text{ (diagram)} = \frac{1}{\sqrt{d_X d_Y}} \text{ (diagram)} = \frac{1}{\sqrt{d_X d_Y}} \text{ (diagram)} \tag{63}$$

Here the first equality is by the third $*$ -cohomomorphism condition (50).

Since dagger idempotents split, we obtain a new 1-morphism $E : t \rightarrow s$ and an isometry $\tilde{\tau} : E \rightarrow Y^* \otimes X$ satisfying $\tilde{\tau} \circ \tilde{\tau}^\dagger = \tilde{f}$, i.e.:

$$\begin{matrix} \uparrow \uparrow \\ \tilde{\tau} \\ \downarrow \downarrow \\ \uparrow \uparrow \\ \tilde{\tau} \\ \downarrow \downarrow \end{matrix} = \frac{1}{\sqrt{d_X d_Y}} \text{ (diagram of } \tilde{f} \text{)} \tag{64}$$

We will first show that E is a dagger equivalence. Indeed, we observe that

$$\dim_R(E) = \frac{d_Y}{d_X} \text{id}_t \tag{65}$$

by the following equalities:

$$\text{ (diagram)} = \text{ (diagram)} = \text{ (diagram)} = \frac{1}{\sqrt{d_X d_Y}} \text{ (diagram)} = \frac{1}{d_X} \text{ (diagram)} \tag{66}$$

Here the first equality is by the fact that $\tilde{\tau}$ is an isometry, the second equality is by sliding $\tilde{\tau}$ around the cup and cap, the third equality is by $\tilde{\tau} \circ \tilde{\tau}^\dagger = \tilde{f}$, and the fourth equality is by the second *-homomorphism condition (49). Likewise, we can show

$$\dim_L(E) = \frac{d_X}{d_Y} \text{id}_s; \quad (67)$$

for this we use the same technique with the second *-cohomomorphism condition (50).

We therefore propose that E^* is a weak inverse for E , with the following 2-morphisms witnessing the equivalence:

$$\sqrt{\frac{d_Y}{d_X}} \text{ (cup) } \quad \sqrt{\frac{d_X}{d_Y}} \text{ (cap) } \quad (68)$$

The equations (67) and (65) show that the 2-morphisms (68) are an isometry and a coisometry respectively. For unitarity we must show that they are also a coisometry and an isometry respectively.

For this we first observe the following decomposition of the unitary *-isomorphism f in terms of the isometry $\tilde{\tau}$, which follows straightforwardly from the definition of \tilde{f} and $\tilde{\tau}$:

$$\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \\ \text{Diagram 3} \\ \text{Diagram 4} \end{array} = \sqrt{d_X d_Y} \begin{array}{c} \text{Diagram 5} \\ \text{Diagram 6} \end{array} = \sqrt{d_X d_Y} \begin{array}{c} \text{Diagram 7} \\ \text{Diagram 8} \end{array} \quad (69)$$

It will also be useful to note the following expression of f^\dagger in terms of $\tilde{\tau}$ for later:

$$\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \\ \text{Diagram 3} \\ \text{Diagram 4} \end{array} = \sqrt{d_X d_Y} \begin{array}{c} \text{Diagram 5} \\ \text{Diagram 6} \end{array} = \sqrt{d_X d_Y} \begin{array}{c} \text{Diagram 7} \\ \text{Diagram 8} \end{array} \quad (70)$$

Here the second equality was by the third *-cohomomorphism equation (50).

Using (69), we now consider what the first *-homomorphism (49) and *-cohomomorphism (50) equations tell us about $\tilde{\tau}$. We begin with the first *-homomorphism equation:

$$\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \\ \text{Diagram 3} \\ \text{Diagram 4} \\ \text{Diagram 5} \\ \text{Diagram 6} \\ \text{Diagram 7} \\ \text{Diagram 8} \end{array} = \sqrt{d_Y} \begin{array}{c} \text{Diagram 9} \\ \text{Diagram 10} \end{array} = d_X \begin{array}{c} \text{Diagram 11} \\ \text{Diagram 12} \end{array} = \begin{array}{c} \text{Diagram 13} \\ \text{Diagram 14} \end{array} = \begin{array}{c} \text{Diagram 15} \\ \text{Diagram 16} \end{array} = \begin{array}{c} \text{Diagram 17} \\ \text{Diagram 18} \end{array} \quad (71)$$

Here for the first implication we bent the top left and top right legs down and precomposed with $\tilde{\tau}$ on the left and $\tilde{\tau}^*$ on the right, using the fact that $\tilde{\tau}$ is an isometry. For the second implication we bent the two rightmost legs upwards. For the third implication we took the transpose.

We now consider the first $*$ -cohomomorphism equation (the derivation of these implications is precisely as before):

(72)

These equations are all we need to show that the 2-morphisms (68) are unitary. Indeed, we show that the first is a coisometry:

(73)

Here the second equality is by (71); the third equality is by (70); the fourth equality is by unitarity of f ; and the last equality follows since $\tilde{\tau}$ is an isometry.

We similarly show that the second 2-morphism of (68) is an isometry:

$$\begin{aligned}
 & \frac{d_X}{d_Y} \begin{array}{c} \text{U-turn} \\ \text{U-turn} \end{array} = \frac{d_X}{(d_Y)^3} \begin{array}{c} \text{Loop} \text{ U-turn} \text{ Loop} \\ \text{U-turn} \end{array} \\
 & = d_X d_Y \begin{array}{c} \text{Diagram with } \tilde{\tau} \text{ boxes and arrows} \\ \text{Diagram with } f \text{ boxes and arrows} \end{array} \\
 & = \begin{array}{c} \text{Diagram with } \tilde{\tau} \text{ boxes and arrows} \\ \text{Diagram with } \tilde{\tau} \text{ boxes and arrows} \end{array} = \begin{array}{c} \text{Vertical arrow} \\ \text{Vertical arrow} \end{array} \quad (74)
 \end{aligned}$$

Here the second equality is by (72); the third equality is by (69); the fourth equality is by unitarity of f ; and the last equality follows since $\tilde{\tau}$ is an isometry.

We have therefore shown that E is a dagger equivalence. Lastly, we need to define a unitary 2-morphism $\tau : X \rightarrow Y \otimes E$. We define τ to be the following 2-morphism:

$$\sqrt{d_Y} \begin{array}{c} \text{U-turn} \\ \text{Box } \tilde{\tau} \end{array} \quad (75)$$

We need to show that τ is unitary. We already saw that it is a coisometry (72). To show that it is an isometry we consider the second *-cohomomorphism equation (50):

$$\begin{aligned}
 & \sqrt{d_X d_Y} \begin{array}{c} \text{U-turn} \\ \text{Box } \tilde{\tau} \end{array} = \sqrt{d_X} \begin{array}{c} \text{U-turn} \end{array} \\
 & \Leftrightarrow d_Y \begin{array}{c} \text{Box } \tilde{\tau} \\ \text{Box } \tilde{\tau} \end{array} = \text{Vertical arrow} \quad (76)
 \end{aligned}$$

Here the implication is by bending the bottom right leg upwards. The 2-morphism τ is therefore unitary and the result follows. \square

We are almost ready to classify UPTs from the canonical fibre functor F . To classify UPTs in terms of special Frobenius monoids, we need some intrinsic characterisation of those special Frobenius monoids in $\text{End}(F)$ which are *split*: that is, which arise as $\alpha \otimes \alpha^*$ for some UPT α whose source is F . In [11, Def. 4.10] (c.f. [8, Def. 3.2]) we define the notion of a *simple Frobenius monoid* in $\text{End}(F)$. For any simple Frobenius monoid A , we construct a fibre functor F' and a UPT $\alpha : F \rightarrow F'$ such that $A \cong \alpha \otimes \alpha^*$. In the other direction, every special Frobenius algebra $\alpha \otimes \alpha^*$ is a simple Frobenius monoid.

By Theorem 5.7 we therefore obtain the following classification.

Corollary 5.8. *Let G be a compact quantum group and let $F : \text{Rep}(G) \rightarrow \text{Hilb}$ be the canonical fibre functor. There is a bijective correspondence between:*

- Unitary $*$ -isomorphism classes of simple Frobenius monoids in $\text{End}(F) \cong \text{Rep}(A_G)$.
- Equivalence classes of UPTs whose source is the canonical fibre functor F .

5.2 Classification of fibre functors We have classified equivalence classes of UPTs from the canonical fibre functor. We now classify the fibre functors F' accessible from the canonical fibre functor F , i.e such that there exists a UPT $\alpha : F \rightarrow F'$.

We first observe another perspective on the special Frobenius monoid (44).

Definition 5.9. Let $X : r \rightarrow s$ be a 1-morphism in a dagger 2-category. We say that X is *special* if it has a right dual $[X^*, \eta_s, \epsilon_s]$ satisfying the following equation:

$$\begin{array}{c}
 \boxed{\eta_s} \\
 \swarrow \quad \searrow \\
 s \quad r \quad s \\
 \nwarrow \quad \nearrow \\
 \boxed{\eta_s}
 \end{array}
 = s
 \tag{77}$$

Lemma 5.10. *In a \mathbb{C} -linear pivotal dagger 2-category, all 1-morphisms into a simple object are special.*

Proof. Let $X : r \rightarrow s$ be a 1-morphism into a simple object, and let $[\alpha^*, \eta, \epsilon]$ be its chosen right dual. Let d_X be the nonzero scalar such that $\dim_L(X) = d_X \text{id}_s$. Now we normalise the cup and cap 2-morphisms:

$$\tilde{\eta} := \frac{1}{\sqrt{d_X}} \eta \qquad \tilde{\epsilon} := \sqrt{d_X} \epsilon$$

Clearly the snake equations will still be obeyed. \square

A 1-morphism $X : r \rightarrow s$ in a dagger 2-category with a special right dual $[X^*, \tilde{\eta}, \tilde{\epsilon}]$ induces a special Frobenius monoid on the object $X \otimes X^*$ in $\text{End}(r)$, with multiplication and unit defined as follows:

$$\begin{array}{c}
 \uparrow \quad \downarrow \\
 \boxed{\eta'} \\
 \uparrow \quad \downarrow \\
 \uparrow \quad \downarrow
 \end{array}
 \qquad
 \begin{array}{c}
 \uparrow \quad \downarrow \\
 \boxed{\epsilon'} \\
 \uparrow \quad \downarrow
 \end{array}
 \tag{78}$$

We observe that, in a pivotal dagger 2-category, when the special dual of a 1-morphism is defined as in Lemma 5.10 then (78) is precisely the the special Frobenius monoid of (44).

For our classification we use the notion of *Morita equivalence* of special Frobenius monoids.

Definition 5.11. Let A and B be special Frobenius monoids in a monoidal dagger category. An A – B -dagger bimodule is an object M together with an morphism $\rho : A \otimes M \otimes B \rightarrow M$ fulfilling the following equations:

(79)

We usually denote an A – B -dagger bimodule M by ${}_A M_B$.

Definition 5.12. A morphism of dagger bimodules ${}_A M_B \rightarrow {}_A N_B$ is a morphism $f : M \rightarrow N$ that commutes with the action of the Frobenius monoids:

(80)

Two dagger bimodules are *isomorphic*, here written ${}_A M_B \cong {}_A N_B$, if there is a unitary morphism of dagger bimodules ${}_A M_B \rightarrow {}_A N_B$.

In a monoidal dagger category in which dagger idempotents split, we can compose dagger bimodules ${}_A M_B$ and ${}_B N_C$ to obtain an A – C -dagger bimodule ${}_A M \otimes_B {}_B N_C$ as follows. First note that the following endomorphism is a dagger idempotent:

(81)

The *relative tensor product* ${}_A M \otimes_B {}_B N_C$ is defined as the image of the splitting of this idempotent. We depict the isometry $i : M \otimes_B N \rightarrow M \otimes N$ as a downwards pointing triangle:

(82)

For dagger bimodules ${}_A M_B$ and ${}_B N_C$, the relative tensor product $M \otimes_B N$ is itself an A – C -dagger bimodule with the following action $A \otimes (M \otimes_B N) \otimes C \rightarrow M \otimes_B N$:

(83)

Definition 5.13. Two special Frobenius monoids A and B are *Morita equivalent* if there are dagger bimodules ${}_A M_B$ and ${}_B N_A$ such that ${}_A M \otimes_B {}_B N_A \cong {}_A A_A$ and ${}_B N \otimes_A {}_A M_B \cong {}_B B_B$.

We make use of the following result.

Theorem 5.14 ([8, Thm. A.1]). *Let \mathcal{C} be a dagger 2-category in which all dagger idempotents split and let $X : r \rightarrow s$ and $Y : r \rightarrow t$ be special 1-morphisms. Then the special Frobenius monoids $X \otimes X^*$ and $Y \otimes Y^*$ in $\text{End}(r)$ are Morita equivalent if and only if s is dagger equivalent to t .*

Proof. The proof is identical to that of [8, Thm. A.1], which classifies objects from which there is a morphism *into* a ; one need only read the diagrams from left to right rather than from right to left. \square

We can equate dagger equivalence of objects in $\text{Fun}_u(\text{Rep}(G), \text{Hilb})$ to a more familiar notion.

Lemma 5.15. *In $\text{Fun}_u(\text{Rep}(G), \text{Hilb})$ there exists a dagger equivalence between two objects F_1, F_2 iff these functors are unitarily monoidally naturally isomorphic.*

Proof. For a pseudonatural transformation $(\alpha, H) : F_1 \rightarrow F_2$ to be a dagger equivalence in $\text{Fun}_u(\text{Rep}(G), \text{Hilb})$, there must exist a pseudonatural transformation $(\alpha^{-1}, K) : F_2 \rightarrow F_1$ and an unitary isomorphism $f : \mathbb{C} \rightarrow H \otimes K$. But then H must be 1-dimensional, and therefore unitarily isomorphic to the unit object \mathbb{C} . Conjugating (α, H) by this isomorphism, we obtain a unitary monoidal natural isomorphism $F_1 \rightarrow F_2$. In the other direction, a unitary monoidal natural isomorphism is clearly a dagger equivalence; the weak inverse is the actual inverse and the unitary 2-morphisms witnessing the equivalence are trivial. \square

Putting these results together, we obtain the following classification.

Corollary 5.16. *Let G be a compact quantum group and let $F : \text{Rep}(G) \rightarrow \text{Hilb}$ be the canonical fibre functor. There is a bijective correspondence between the following structures:*

- Unitary monoidal natural isomorphism classes of fibre functors accessible from F by a UPT.
- Morita equivalence classes of simple Frobenius monoids in $\text{End}(F) \cong \text{Rep}(A_G)$.

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