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On shifted contact derived Artin stacks

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Abstract

This is a sequel of [2] on the development of derived contact geometry. In [2], we formally introduced shifted contact structures on derived stacks. We then gave a Darboux-type theorem and the notion of symplectification *only for* negatively shifted contact derived schemes.

In this paper, we extend the results of [2] from derived schemes to derived Artin stacks and provide some examples of contact derived Artin stacks. In brief, we first show that for $k < 0$, every k-shifted contact derived Artin stack admits a contact Darboux atlas. Secondly, we canonically describe the symplectification of a derived Artin stack equipped with a k-shifted contact structure, where $k < 0$. Lastly, we give several constructions of contact derived stacks using certain cotangent stacks and shifted prequantization structures.

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1. Introduction and summary

As a relatively popular mainstream in the literature, generalized versions of certain familiar geometric structures have been introduced and studied in the context of derived algebraic geometry. For example, [9, 4] focus on shifted Symplectic and Poisson geometries. Furthermore, [3, 6, 1] provide some applications and local constructions.

In [2], on the other hand, we formally described shifted contact structures on derived (Artin) stacks and investigated some interesting consequences, such as a Darboux-type theorem and the process of canonical symplectification for negatively shifted contact derived schemes. These observations essentially motivate us to make further investigations of contact structures in the context of *derived algebraic/symplectic geometry*.

Regarding the study of shifted contact structures, we note that Maglio, Tortorella and Vitagliano [8] have recently introduced and studied 0-*shifted* and +1-*shifted contact structures*

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on *differentiable stacks*, thus providing the foundations of shifted contact geometry in the stacky context.

For shifted symplectic derived schemes, in particular, it has been shown in [3, Theorem 5.18] that every k-shifted symplectic derived $\mathbb{K}$-scheme $(\mathbf{X}, \omega')$, with $k < 0$, is Zariski locally equivalent to $(\mathrm{Spec}A, \omega)$ for a pair A, ω in certain symplectic Darboux form [3, Examples 5.8, 5.9 & 5.10].

In [1], Ben-Bassat, Brav, Bussi, and Joyce extend the results of [3] from derived schemes to the case of derived Artin $\mathbb{K}$-stacks. In this regard, [1, Theorem 2.8] first proves that derived Artin $\mathbb{K}$-stacks also have nice atlases in terms of standard form cdgas. Then it is shown in [1, Theorem 2.10] that every k-shifted symplectic derived Artin $\mathbb{K}$-stack, with $k < 0$, admits the so-called *Darboux form atlas*; hence, one has a Darboux-type theorem in this case as well. These results, with some side outcomes, motivate the current work.

Main results and the outline. In this work, with the same spirit as above, our goals are to extend the results of [2] from derived schemes to derived Artin stacks and to provide several constructions of contact derived stacks.

Recall that, in [2], we introduced shifted contact structures on derived stacks and proved the following results, but *only for* (locally finitely presented) negatively shifted contact derived $\mathbb{K}$-schemes. In brief, we showed:

Theorem 1.1. *Let $\mathbf{X}$ be a (locally finitely presented) derived $\mathbb{K}$-scheme.*

a. [2, Theorem 3.13] Every k-shifted contact structure on $\mathbf{X}$, with $k < 0$, is locally equivalent to $(\mathrm{Spec}A, \alpha_0)$ for A a minimal standard form cdga and α_0 in a contact Darboux form.

b. [2, Theorem 4.7] Write $\mathcal{S}_{\mathbf{X}}$ for the total space of a certain $\mathbb{G}_m$-bundle over $\mathbf{X}$, constructed from the data of k-shifted contact structure on $\mathbf{X}$. Then $\mathcal{S}_{\mathbf{X}}$ is a derived stack equipped with a k-shifted symplectic form $\omega_{\mathbf{X}}$, which is canonically determined by the shifted contact structure of $\mathbf{X}$. We then call the pair $(\mathcal{S}_{\mathbf{X}}, \omega_{\mathbf{X}})$ the symplectification of $\mathbf{X}$.

In this paper, we extend Theorem 1.1 to the case of derived Artin $\mathbb{K}$-stacks as conjectured in [2]. In addition to that, we also provide several examples of contact derived stacks. In short, the following theorems outline the main results of this paper:

Theorem 1.2. *Theorem 1.1 also holds true for negatively shifted contact derived Artin $\mathbb{K}$-stacks locally of finite presentation (cf. Theorems 3.7 & 3.9).*

Theorem 1.3. *Let $\mathbf{X}$ be a derived Artin $\mathbb{K}$-stack locally of finite presentation. Denote by $\mathbb{G}_a, \mathbb{G}_m$ the affine additive and multiplicative group schemes, respectively. (See Sections 4.1 and 4.2.)*

1. *Let $\mathrm{T}^*[n]\mathbf{X}$ be the n-shifted cotangent stack. Then the space $\mathrm{J}^1[n]\mathbf{X} = \mathrm{T}^*[n]\mathbf{X} \times \mathbb{G}_a[n]$, called the* n-shifted 1-jet stack *of $\mathbf{X}$, carries an n-shifted contact structure.*

2. *Let $\pi_{\mathbf{X}} : \mathrm{T}^*\mathbf{X} \to \mathbf{X}$ be the natural projection. Given a prequantum 0-shifted Lagrangian fibration structure on $\pi_{\mathbf{X}}$, there is a $\mathbb{G}_m$-bundle on $\mathrm{T}^*\mathbf{X}$ with a 0-shifted contact structure.*

3. *Let $\pi_{c_1(\mathcal{G})} : \mathrm{T}^*_{c_1(\mathcal{G})}\mathbf{X} \to \mathbf{X}$ be the $c_1(\mathcal{G})$-twisted cotangent stack of $\mathbf{X}$, where $c_1(\mathcal{G}) \in \mathcal{A}^1(\mathbf{X}, 1)$ denotes the* characteristic class *of a 0-gerbe $\mathcal{G}$ - a line bundle - on $\mathbf{X}$. Given a prequantum 0-shifted Lagrangian fibration structure on $\pi_{c_1(\mathcal{G})}$, there is a $\mathbb{G}_m$-bundle on $\mathrm{T}^*_{c_1(\mathcal{G})}\mathbf{X}$ that carries a 0-shifted contact structure.*

4. *Assume that G is a simple algebraic group over $\mathbb{K}$, and C be a smooth and proper curve/$\mathbb{K}$. Then there is a $\mathbb{G}_m$-bundle on $LocSys_G(C)$ with a 0-shifted contact structure.*

Now, let us describe the content of this paper in more detail and provide an outline. In Section 2, we review derived symplectic/contact geometries and the Darboux models (for derived schemes). In Section 3, we will concentrate on the stacky generalizations. In short, Section 3.1 outlines the Darboux models for negatively shifted symplectic derived Artin stacks. With the same spirit, in Section 3.2, we consider the contact case and give the proof of Theorem 1.2 (cf. Theorems 3.7 & 3.9). Finally, Section 4 provides several examples of contact derived stacks (with some background material on prequantization) and proves Theorem 1.3. (See Sections 4.1, 4.2.2, 4.2.3, 4.2.4.) We also have two appendices presenting some relevant constructions omitted in the main text.

Conventions. Throughout the paper, $\mathbb{K}$ will be an algebraically closed field of characteristic zero. All cdgas will be graded in nonpositive degrees and over $\mathbb{K}$. We assume that all classical $\mathbb{K}$-schemes are *locally of finite type*, and that all derived $\mathbb{K}$-schemes/stacks $\mathbf{X}$ are *locally finitely presented.*

2. Recollection

2.1 Some derived algebraic geometry We provide a quick review on derived algebraic geometry. For details, we refer to [11, 12, 7].

In this paper, we essentially use the functorial approach to define (higher) spaces of interest. It is very well known that using Yoneda's embedding, spaces can be thought of as *sheaves* in addition to the standard ringed-space formulation. In brief, we have the following diagram [13]:

$$\begin{tikzcd}
CAlg_{\mathbb{K}} \arrow[r, "\text{schemes}"] \arrow[dr, "\text{stacks}"] \arrow[ddr, "\text{higher stacks}"'] \arrow[dd] & Sets \arrow[d] \\
 & Grpds \arrow[d] \\
cdga_{\mathbb{K}} \arrow[r, "\text{derived stacks}"'] & Ssets.
\end{tikzcd} \tag{1}$$

Here, $CAlg_{\mathbb{K}}$ denotes the category of commutative $\mathbb{K}$-algebras, and $cdga_{\mathbb{K}}$ is the ∞-category[1] of commutative differential graded $\mathbb{K}$-algebras in non-positive degrees. Denote by $St_{\mathbb{K}}$ the ∞-category of (higher) $\mathbb{K}$-stacks, where objects in $St_{\mathbb{K}}$ are defined via Diagram 1 above.

Recall that, in the underived setup, we have the classical "spectrum functor"

$$\mathrm{spec} : (CAlg_{\mathbb{K}})^{op} \to St_{\mathbb{K}}.$$

We then call an object X of $St_{\mathbb{K}}$ an *affine* $\mathbb{K}$-*scheme* if $X \simeq \mathrm{spec}A$ for some $A \in CAlg_{\mathbb{K}}$; and a $\mathbb{K}$-*scheme* if it has an open cover by affine $\mathbb{K}$-schemes. In DAG, there also exists an appropriate concept of a *spectrum functor*[2]

$$\mathrm{Spec} : cdga_{\mathbb{K}}^{op} \to dSt_{\mathbb{K}},$$

which leads to the following definitions.

[1] We actually mean the ∞-*category* associated to the model category $cdga_{\mathbb{K}}$, with its natural model structure for which equivalences are quasi-isomorphisms, and fibrations are epimorphisms in strictly negative degrees.

[2] In brief, it is the right adjoint to the global algebra of functions functor $\Gamma : dSt_{\mathbb{K}} \leftrightarrows cdga_{\mathbb{K}}^{op} : \mathrm{Spec}$.

Definition 2.1. Denote by $dSt_{\mathbb{K}}$ the ***∞-category of derived stacks***, where an object $\mathbf{X}$ of $dSt_{\mathbb{K}}$ is given as a certain ∞-functor[3] $\mathbf{X} : cdga_{\mathbb{K}} \to Ssets$ as in Diagram 1. More precisely, objects in $dSt_{\mathbb{K}}$ are *simplicial presheaves on the site* $(dAff)^{op} \simeq cdga_{\mathbb{K}}$ *satisfying a descent condition*. For more details, we refer to [12].

Definition 2.2. An object $\mathbf{X}$ in $dSt_{\mathbb{K}}$ is called an ***affine derived $\mathbb{K}$-scheme*** if $\mathbf{X} \simeq \mathrm{Spec}A$ for some cdga $A \in cdga_{\mathbb{K}}$. An object $\mathbf{X}$ in $dSt_{\mathbb{K}}$ is then called a ***derived $\mathbb{K}$-scheme*** if it can be covered by Zariski open affine derived $\mathbb{K}$-schemes $Y \subset X$. Denote by $dSch_{\mathbb{K}} \subset dSt_{\mathbb{K}}$ the full ∞-subcategory of derived $\mathbb{K}$-schemes, and we simply write $dAff_{\mathbb{K}} \subset dSch_{\mathbb{K}}$ for the full ∞-subcategory of affine derived $\mathbb{K}$-schemes.

Nice local models for derived $\mathbb{K}$-schemes. Let us first recall some basic concepts.

Definition 2.3. $A \in cdga_{\mathbb{K}}$ is of ***standard form*** if A^0 is a smooth finitely generated $\mathbb{K}$-algebra; the module $\Omega^1_{A^0}$ of Kähler differentials is free A^0-module of finite rank; and the graded algebra A is freely generated over A^0 by finitely many generators, all in negative degrees.

In fact, there is a systematic way of constructing such cdgas starting from a smooth $\mathbb{K}$-algebra $A^0 := A(0)$ via the use of a sequence of localizations. More precisely, for any given $n \in \mathbb{N}$, we can inductively construct a sequence of cdgas

$$A(0) \to A(1) \to \cdots \to A(i) \to \cdots A(n) =: A, \tag{2}$$

where $A^0 := A(0)$, and $A(i)$ is obtained from $A(i-1)$ by adjoining generators in degree $-i$, given by M^{-i}, for all i. Here, each M^{-i} is a free finite rank module (of degree $-i$ generators) over $A(i-1)$. Therefore, the underlying commutative graded algebra of $A = A(n)$ is freely generated over $A(0)$ by finitely many generators, all in negative degrees $-1, -2, \ldots, -n$. For more details, we refer to [3, Example 2.8].

Definition 2.4. A standard form cdga A is said to be ***minimal*** at $p \in \mathrm{spec}H^0(A)$ if $A = A(n)$ is defined by using the minimal possible numbers of graded generators in each degree ≤ 0 compared to all other cdgas locally equivalent to A near p.

Definition 2.5. Let A be a standard form cdga. $A' \in cdga_{\mathbb{K}}$ is called a ***localization*** of A if A' is obtained from A by inverting an element $f \in A^0$, by which we mean $A' = A \otimes_{A^0} A^0[f^{-1}]$.

A' is then of standard form with $A'^0 \simeq A^0[f]$. If $p \in \mathrm{spec}H^0(A)$ with $f(p) \neq 0$, we say A' is a *localization of A at p*.

With these definitions in hand, one has the following observations:

Observation 2.6. Let A be a standard form cdga. If A' is a *localization* of A, then $\mathrm{Spec}A' \subset \mathrm{Spec}A$ is a Zariski open subset. Likewise, if A' is a *localization of A at* $p \in \mathrm{spec}H^0(A) \simeq \tau(\mathrm{Spec}A)$, then $\mathrm{Spec}A' \subset \mathrm{Spec}A$ is a Zariski open neighborhood of p.

Observation 2.7. Let A be *a standard form* cdga, then there exist generators $x_1^{-i}, x_2^{-i}, \cdots, x_{m_i}^{-i}$ in A^{-i} (after localization, if necessary) with $i = 1, 2, \cdots, k$ and $m_i \in \mathbb{Z}_{\geq 0}$ such that

$$A = A(0)\big[x_j^{-i} : i = 1, 2, \ldots, k,\ j = 1, 2, \ldots, m_i\big], \tag{3}$$

[3] Using Yoneda's lemma, for a derived stack $\mathbf{X}$, we have $\mathbf{X} : A \mapsto \mathbf{X}(A) \simeq Map_{dSt_{\mathbb{K}}}(\mathrm{Spec}A, \mathbf{X})$, and hence any A-point $p \in \mathbf{X}(A)$ can be seen as a morphism $p : \mathrm{Spec}A \to \mathbf{X}$ of derived stacks.

where the subscript j in x_j^i labels the generators, and the superscript i indicates the degree of the corresponding element. So, we can consider A as a *graded polynomial algebra over* $A(0)$ *on finitely many generators, all in negative degrees.*

The following theorem outlines the central results from [3, Theorem 4.1 & 4.2] concerning the construction of useful local algebraic models for derived $\mathbb{K}$-schemes. The upshot is that given a derived $\mathbb{K}$-scheme $\mathbf{X}$ (locally of finite presentation) and a point $x \in \mathbf{X}$, one can always find a "refined" affine neighborhood $\operatorname{Spec} A$ of x, which is very useful for explicit presentations. In short, we have:

Theorem 2.8. *Every derived* $\mathbb{K}$*-scheme* X *is Zariski locally modelled on* $\operatorname{Spec} A$ *for a minimal standard form cdga* A*.*

Nice local models for cotangent complexes of derived schemes. Given$A \in cdga_{\mathbb{K}}$, d on A induces a differential on Ω^1_A, denoted again by d. This makes Ω^1_A into a dg-module (Ω^1_A, d) with the property that $\delta \circ d = d \circ \delta$, where $\delta : A \to \Omega^1_A$ is the universal derivation of degree zero. Write the decomposition of Ω^1_A into graded pieces $\Omega^1_A = \bigoplus_{k=-\infty}^{0} \left(\Omega^1_A\right)^k$ with the differential $d : \left(\Omega^1_A\right)^k \longrightarrow \left(\Omega^1_A\right)^{k+1}$. Then we define the *de Rham algebra of* A as a double complex

$$DR(A) = Sym_A(\Omega^1_A[1]) \simeq \bigoplus_{p=0}^{\infty} \bigoplus_{k=-\infty}^{0} \left(\Lambda^p \Omega^1_A\right)^k [p], \tag{4}$$

where the gradings p, k are called the *weight* and the *degree*, respectively. Also, there are two differentials on $DR(A)$, namely the *internal differential* d: $\left(\Lambda^p \Omega^1_A\right)^k [p] \longrightarrow \left(\Lambda^p \Omega^1_A\right)^{k+1} [p]$ and the *de Rham differential* $d_{dR} : \left(\Lambda^p \Omega^1_A\right)^k [p] \longrightarrow \left(\Lambda^{p+1} \Omega^1_A\right)^k [p+1]$ such that $d_{tot} = d + d_{dR}$ and

$$d^2 = d_{dR}^2 = 0, \text{ and } d \circ d_{dR} = -d_{dR} \circ d. \tag{5}$$

Here, one also has the natural multiplication on $DR(A)$:

$$\left(\Lambda^p \Omega^1_A\right)^k [p] \times \left(\Lambda^q \Omega^1_A\right)^\ell [q] \longrightarrow \left(\Lambda^{p+q} \Omega^1_A\right)^{k+\ell} [p+q]. \tag{6}$$

Note that even if both $\mathbb{L}_A$ and Ω^1_A are closely related, the identification of $\mathbb{L}_A$ with Ω^1_A is not true for an arbitrary $A \in cdga_{\mathbb{K}}$ [3]. But, when $A = A(n)$ is a standard form cdga, we have the following description for the restriction of the cotangent complex $\mathbb{L}_A$ to $\operatorname{spec} H^0(A)$. In this paper, we only give a brief version. More details and the proof can be found in [3, Prop. 2.12].

Proposition 2.9. *If* $A = A(n)$ *with* $n \in \mathbb{N}$ *is a standard form cdga constructed inductively as in (2), then the restriction of* $\mathbb{L}_A$ *to* $\operatorname{spec} H^0(A)$ *is represented by a complex of free* $H^0(A)$*-modules.*

2.2 Derived symplectic geometry and local models Pantev et al. [9] define the *simplicial sets of* p*-forms of degree* k *and closed* p*-forms of degree* k on derived stacks. Denote these simplicial sets by $\mathcal{A}^p(\mathbf{X}, k)$ and $\mathcal{A}^{p,cl}(\mathbf{X}, k)$, respectively. These definitions are in fact given first for affine derived $\mathbb{K}$-schemes. Later, both concepts are defined for general derived stacks $\mathbf{X}$ in terms of mapping stacks $\mathcal{A}^p(-, k)$ and $\mathcal{A}^{p,cl}(-, k)$, respectively.

The space $\mathcal{A}^{p,cl}(\mathbf{X}, k)$ of closed p-forms on a general derived stack $\mathbf{X}$ can be a rather complicated even when $\mathbf{X}$ is a nice derived Artin stack. However, [9, Prop. 1.14] gives the following identification for the space $\mathcal{A}^p(\mathbf{X}, k)$ of k-shifted p-forms:

$$\mathcal{A}^p(X, k) \simeq Map_{QCoh(X)}(\mathcal{O}_X, \wedge^p \mathbb{L}_X[k]).$$

Let $\mathbf{X} = \mathrm{Spec}A$ with A a standard form cdga[4], then take $\Lambda^p\mathbb{L}_A = \Lambda^p\Omega^1_A$. Therefore, elements of $\mathcal{A}^p(\mathbf{X}, k)$ form a simplicial set such that k-cohomology classes of the complex $\big(\Lambda^p\Omega^1_A, d\big)$ correspond to the connected components of this simplicial set. Likewise, the connected components of $\mathcal{A}^{p,cl}(\mathbf{X}, k)$ are identified with the k-cohomology classes of the complex $\prod_{i\geq 0}\big(\Lambda^{p+i}\Omega^1_A[-i], d_{tot}\big)$. Then we have:

Definition 2.10. Let $\mathbf{X} = \mathrm{Spec}A$ be an affine derived $\mathbb{K}$-scheme for A a minimal standard form cdga. A ***k-shifted p-form on $\mathbf{X}$*** for $p \geq 0$ and $k \leq 0$ is an element $\omega^0 \in \big(\Lambda^p\Omega^1_A\big)^k$ with $d\omega^0 = 0$.

Note that an element ω^0 defines a cohomology class $[\omega^0] \in H^k\big(\Lambda^p\Omega^1_A, d\big)$, where two p-forms ω^0_1, ω^0_2 of degrees k are *equivalent* if $\exists \alpha_{1,2} \in \big(\Lambda^p\Omega^1_A\big)^{k-1}$ s.t. $\omega^0_1 - \omega^0_2 = d\alpha_{1,2}$.

Definition 2.11. Let $\mathbf{X} = \mathrm{Spec}A$ be an affine derived $\mathbb{K}$-scheme with A a minimal standard form cdga. A ***closed k-shifted p-form on $\mathbf{X}$*** for $p \geq 0$ and $k \leq 0$ is a sequence $\omega = (\omega^0, \omega^1, \cdots)$ with $\omega^i \in \big(\Lambda^{p+i}\Omega^1_A\big)^{k-i}$ such that $d_{tot}\omega = 0$, which splits according to weights as $d\omega^0 = 0$ in $\big(\Lambda^p\Omega^1_A\big)^{k+1}$ and $d_{dR}\omega^i + d\omega^{i+1} = 0$ in $\big(\Lambda^{p+i+1}\Omega^1_A\big)^{k-i}$, $i \geq 0$.

That is, a closed k-shifted p-form consists of an actual k-shifted p-form ω^0 and the data $(\omega^i)_{i>0}$ of ω^0 being coherently d_{dR}-closed. It then follows that there also exists a natural projection morphism $\pi : \mathcal{A}^{p,cl}(\mathbf{X}, k) \longrightarrow \mathcal{A}^p(\mathbf{X}, k), \ \ \omega = (\omega^i)_{i\geq 0} \longmapsto \omega^0$.

Definition 2.12. A closed k-shifted 2-form $\omega = (\omega^i)_{i\geq 0}$ on $\mathbf{X} = \mathrm{Spec}A$ for a (minimal) standard form cdga A is called a ***k-shifted symplectic structure*** if the induced map

$$\omega^0\cdot : \mathbb{T}_A \to \Omega^1_A[k], \ Y \mapsto \iota_Y\omega^0,$$

is a quasi-isomorphism, where $\mathbb{T}_A = (\mathbb{L}_A)^\vee \simeq \mathrm{Hom}_A(\Omega^1_A, A)$[5] is the *tangent complex of A*. The requirement for the induced map $\omega^0\cdot$ is called the ***non-degeneracy condition***.

Symplectic Darboux models for derived schemes. One of the main theorems in [3] provides a k-shifted version of the classical Darboux theorem in symplectic geometry. The statement is as follows.

Theorem 2.13. *[3, Theorem 5.18] Given a derived $\mathbb{K}$-scheme $\mathbf{X}$ with a k-shifted symplectic form ω' for $k < 0$ and $x \in \mathbf{X}$, there is a local model $\big(A, f : \mathrm{Spec}A \hookrightarrow \mathbf{X}, \omega\big)$ and $p \in \mathrm{spec}(H^0(A))$ such that f is an open inclusion with $f(p) = x$, A is a standard form that is minimal at p, and ω is a k-shifted symplectic form on $\mathrm{Spec}A$ such that A, ω are in Darboux form, and $f^*(\omega') \sim \omega$ in $\mathcal{A}^{2,cl}(\boldsymbol{X}, k)$.*

In fact, Theorem 2.13 shows that such ω can be constructed explicitly depending on the integer $k < 0$. Indeed, there are three cases in total: (1) k is odd; (2) $k/2$ is even; and (3) $k/2$ is odd.

For instance, when k is odd, one can find a minimal standard form cdga A, with "coordinates" $x^{-i}_j, y^{k+i}_j \in A$, and a Zariski open inclusion $f : \mathrm{Spec}A \hookrightarrow \mathbf{X}$ so that $f^*(\omega') \sim \omega = (\omega^0, 0, 0, \ldots)$, where $\omega^0 = \sum_{i,j} d_{dR}x^{-i}_j d_{dR}y^{k+i}_j$. We will not give any further detail on the aforementioned cases in this paper. Instead, we refer to [3, Examples 5.8, 5.9, and 5.10].

[4] It should be noted that the results that are cited or to be proven in this section are all about the *local structure* of derived schemes. Thus, it is enough to consider the (refined) affine case.

[5] Thanks to the identification $\mathbb{L}_A \simeq (\Omega^1_A, d)$ for A a (minimal) standard form cdga.

We just wish to present a result that plays a significant role in constructing Darboux-type local models. The upshot is that one can always simplify the given closed 2-form $\omega = (\omega^0, \omega^1, \omega^2, \dots)$ of degree $k < 0$ on Spec$A$ so that ω^0 can be taken to be exact and $\omega^i = 0$ for all $i > 0$. More precisely, we have[6]:

Proposition 2.14. *[3, Prop. 5.7] Let $\omega = (\omega^0, \omega^1, \omega^2, \dots)$ be a closed 2-form of degree $k < 0$ on* Spec$A$ *for A a standard form cdga over $\mathbb{K}$. Then there exist $H \in A^{k+1}$ and $\phi \in (\Omega^1_A)^k$ such that $dH = 0$ in A^{k+2}, $d_{dR}H + d\phi = 0$ in $(\Omega^1_A)^{k+1}$, and $\omega \sim (d_{dR}\phi, 0, 0, \dots)$.*

2.3 Derived contact geometry and local models In this section, we review the central constructions and results from [2]. In a nutshell, a *k-shifted contact structure* on a derived Artin stack consists of a morphism $f : \mathcal{K} \to \mathbb{T}_{\mathbf{X}}$ of perfect complexes, a line bundle L such that $Cone(f) \simeq L[k]$, and a locally defined k-shifted 1-form α satisfying a non-degeneracy condition. With this structure in hand, [2] presents a Darboux-type theorem and the process of canonical symplectification for *negatively shifted contact derived schemes.*

2.3.1 Basic concepts Let us start with some terminology. Recall that the definition of a *derived Artin $\mathbb{K}$-stack* can be formalized by using an inductive construction [7, Section 5.1]. Roughly speaking, we call $\mathbf{X} \in dSt_{\mathbb{K}}$ a *derived Artin $\mathbb{K}$-stack* if it can be locally represented by an affine derived $\mathbb{K}$-scheme with respect to the "smooth topology". Thus, we require the existence of a "smooth" surjection $\varphi : U \to \mathbf{X}$ (of some relative dimension m) so that U is a disjoint union of affine derived schemes.

In order to the make sense of the smoothness of φ as above, we require that the fibers of the morphism φ are already suitable geometric objects. To this end, one should start with some subcategory S_0, called 0-*stacks*, and then define S_{n+1} to be the class of objects $\mathbf{X}$ in $dSt_{\mathbb{K}}$ having a "smooth" surjection $\varphi : U \to \mathbf{X}$ with U a disjoint union of affine derived schemes such that each fiber $U \times_{\mathbf{X}}$ Spec$A$ lies in the class S_n.

Technically speaking, an object $X \in dSt_{\mathbb{K}}$ is called a *derived Artin $\mathbb{K}$-stack* if it is m-geometric for some m, and the underlying classical stack is 1-truncated (i.e. it is just a stack, not higher stack). For details, we refer to [7, § 5.1] or [12, § 1.3.3].

The upshot is that any such object $\mathbf{X}$ of $dSt_{\mathbb{K}}$ comes with a smooth surjective morphism $\varphi : U \to \mathbf{X}$ with U a derived $\mathbb{K}$-scheme. We call such morphism an ***atlas.*** Therefore, the following definition will be sufficient for our purposes.

Definition 2.15. By a ***derived Artin $\mathbb{K}$-stack***, we mean an object $\mathbf{X}$ of $dSt_{\mathbb{K}}$ possessing an atlas (smooth of some relative dimension) near each point of $\mathbf{X}$.

Now, let us introduce *shifted contact structures* on derived Artin stacks:

Definition 2.16. Let $\mathbf{X}$ be a locally finitely presented derived (Artin) stack. A ***pre-k-shifted contact structure*** on $\mathbf{X}$ is given by a shifted line bundle $L[k]$ with a morphism $\alpha : \mathbb{T}_{\mathbf{X}} \to L[k]$. Denote such a structure by $(L[k], \alpha)$.

Note that we can consider a pre-k-shifted contact data as a perfect complex $\mathcal{K}$ and a line bundle L along with a morphism $\kappa : \mathcal{K} \to \mathbb{T}_{\mathbf{X}}$ such that $Cone(\kappa) \simeq L[k]$. Here, we have a cofiber sequence $\mathcal{K} \to \mathbb{T}_{\mathbf{X}} \to L[k]$ in $QCoh(\mathbf{X})$. Since $QCoh(\mathbf{X})$ is a stable ∞-category, the cocone of $\mathbb{T}_{\mathbf{X}} \to L[k]$ is equivalent to $\mathcal{K}$. We then may denote a pre-k-shifted contact structure on $\mathbf{X}$ by $(\mathcal{K}, \kappa, L)$.

[6] Based on the interpretation of such forms in the context of *cyclic homology theory of mixed complexes* [9, 3].

Definition 2.17. A pre-k-shifted contact structure $(\mathcal{K}, \kappa, L)$ on $\mathbf{X}$ is a ***k-shifted contact structure*** if locally on $\mathbf{X}$, where L is trivial, the induced k-shifted 1-form[7] $\alpha : \mathbb{T}_{\mathbf{X}} \to \mathcal{O}_{\mathbf{X}}[k]$ is such that the map $d_{dR}\alpha|_{\mathcal{K}} \cdot := \kappa^{\vee}[k] \circ (d_{dR}\alpha\ \cdot) \circ \kappa : \mathcal{K} \to \mathcal{K}^{\vee}[k]$ is a weak equivalence.

In that case, we say the k-shifted 2-form $d_{dR}\alpha$ is ***non-degenerate on*** $\mathcal{K}$. Also, we call such local form a ***k-contact form***.

Remark 2.18. When $k \leq 0$, the triangle $\mathcal{K} \to \mathbb{T}_{\mathbf{X}} \to L[k]$ splits locally for any affine derived scheme (so, this also holds Zariski locally for any derived scheme)[8]. In fact, the nondegeneracy condition implies that $\mathcal{K}$ has Tor-amplitude $[0, -k]$ so that $\mathcal{K}[-k]$ is connective. Then the connecting homomorphism $L[k] \to \mathcal{K}[1]$ in the exact triangle is equivalently $L \to \mathcal{K}[1-k]$. Notice that $\mathcal{K}[1-k]$ is concentrated in degrees ≤ -1, so this morphism is automatically zero on any affine derived scheme, which implies the desired splitting.

Let $\mathbf{X}$ be a locally finitely presented derived Artin stack with a k-shifted contact structure $(\mathcal{K}, \kappa, L)$. Recall from Yoneda's lemma, $\mathbf{X}(A) \simeq Map_{dPstk}(\mathrm{Spec}A, \mathbf{X})$, and hence any A-point $p \in \mathbf{X}(A)$ can be seen as a morphism $p : \mathrm{Spec}A \to \mathbf{X}$ of derived pre-stacks. Then, let us consider the pair (p, α_p), with $p \in \mathbf{X}(A)$, $\alpha_p \in p^*(\mathbb{L}_{\mathbf{X}}[k])$, such that $Cocone(\alpha_p) \simeq \mathcal{K}$. For $A \in cdga_{\mathbb{K}}$, there is a $\mathbb{G}_m(A)$-action on the pair (p, α_p) by

$$f \triangleleft (p, \alpha_p) := (p, f \cdot \alpha_p).$$

Denote by H^0 the functor sending $A \mapsto H^0(A)$. Denote the image under H^0 of an element f simply by f^0. Note that localizing A if necessary, w.l.o.g. we may assume that the image f^0 is always invertible. It follows that f^0 lies in $(A^0)^{\times}$, which is by definition $\mathbb{G}_m(A^0) = (A^0)^{\times}$.

Observation 2.19. If $\mathbf{X}, (p, \alpha_p)$, and the $\mathbb{G}_m(A)$-action are as above, then for an element $f \in \mathbb{G}_m(A)$, we can obtain $Cocone(f \cdot \alpha_p) \simeq Cocone(\alpha_p)$ by using the invertibility of f.

From Proposition 2.9, on a refined affine neighborhood, say $\mathrm{Spec}A$ with A a minimal standard form cdga, the perfect complexes $\mathbb{T}_A, \mathbb{L}_A$, when restricted to $\mathrm{spec}H^0(A)$, are both free finite complexes of $H^0(A)$-modules. In that case, Definitions 2.16 and 2.17, and Observation 2.19 will reduce to the following local descriptions, where $\mathcal{K}$ is now just equivalent to the usual $\ker\alpha$ in Mod_A; and L in the splitting corresponds to the line bundle generated by the Reeb vector field of the classical case.

More precisely, from [2, §3.2], when restricted to the (nice) local models, we equivalently have the following proposition/definition:

Proposition 2.20. (Shifted contact structures with nice affine models) *For a (minimal) standard form cdga A and $k < 0$, any k-shifted contact structure on $\boldsymbol{X} = \mathrm{Spec}A$ can be strictified in the sense that the resulting contact data consists of*

- *a submodule $\mathcal{K}$ with the natural inclusion $i : \mathcal{K} \hookrightarrow Der(A)$ such that $Cone(i) \simeq coker(i)$ is the quotient complex and of the form $L[k]$, with L a line bundle; and*
- *a k-shifted 1-form α on $\mathrm{Spec}A$ with the property that $\mathcal{K} \simeq \ker\alpha$ so that the k-shifted 2-form $d_{dR}\alpha$ is non-degenerate on $\ker\alpha$.*

Here $Der(A) = (\Omega^1_A)^{\vee} = \mathrm{Hom}_A(\Omega^1_A, A)$, where Ω^1_A is the A-module Kähler differentials such that $\Omega^1_A|_{\mathrm{spec}H^0(A)}$ is represented as a (bounded) complex of free $H^0(A)$-modules by Proposition 2.9. In

[7] We can locally identify the map α with the induced shifted one-form using the trivialization of $L^{\vee}[k]$.

[8] We thank the anonymous referee for this remark.

that case, over $p \in \mathrm{spec}H^0(A)$*, one has the splitting* $Der(A)|_{\mathrm{spec}H^0(A)} = \ker\alpha \oplus L[k]|_{\mathrm{spec}H^0(A)}$. *Adopting the classical terminology, we sometimes call the sub-module* $\ker\alpha$ *above a* k**-shifted (strict) contact structure with the defining** k**-contact form** α.

Example 2.21. *(k-contact forms, with odd* $k<0$*)* In this example, fixing $\ell \in \mathbb{N}$, we will construct a (minimal) standard form cdga carrying a k-shifted contact structure for $k=-2\ell-1$. In brief, we will coherently extend the symplectic case [3, Example 5.8]. To this end, we make use of similar notations and constructions from that example. Modifications for even shifts are outlined in Appendix A.

Step-1: Construction of a "contact" cdga.Let $A(0)$ be a smooth $\mathbb{K}$-algebra of dimension m_0. Assume that there exist degree 0 variables $x_1^0, x_2^0, \ldots, x_{m_0}^0$ in $A(0)$ defining global étale coordinates $(x_1^0, x_2^0, \ldots, x_{m_0}^0) : \mathrm{Spec}A(0) \to \mathbb{A}^{m_0}$ on $\mathrm{Spec}A(0)$ such that $d_{dR}x_1^0, \ldots, d_{dR}x_{m_0}^0$ form a $A(0)$-basis for $\Omega^1_{A(0)}$. Next, choosing non-negative integers $m_1, \ldots, m_\ell$, we define a ***commutative graded algebra*** A to be the free graded $\mathbb{K}$-algebra over $A(0)$ generated by the variables

$$
\begin{aligned}
&x_1^{-i}, x_2^{-i}, \ldots, x_{m_i}^{-i} && \text{in degree } -i \quad \text{for } i=1,\ldots,\ell, && (7)\\
&y_1^{k+i}, y_2^{k+i}, \ldots, y_{m_i}^{k+i} && \text{in degree } k+i \quad \text{for } i=1,\ldots,\ell, && (8)\\
&z^k, y_1^k, y_2^k, \ldots, y_{m_0}^k && \text{in degree } k, && (9)
\end{aligned}
$$

such that Ω^1_A is the free A-module of finite rank with basis $\{d_{dR}x_j^{-i}, d_{dR}y_j^{k+i}, d_{dR}z^k : \forall i,j\}$. Here, we call z^k the ***distinguished variable*** (of $\deg k$). Also, we choose an element $H \in A^{k+1}$ satisfying the *classical master equation*

$$\sum_{i=1}^{\ell}\sum_{j=1}^{m_i} \frac{\partial H}{\partial x_j^{-i}} \frac{\partial H}{\partial y_j^{k+i}} = 0 \text{ in } A^{k+2}. \tag{10}$$

We call such H the *Hamiltonian*. Due to degree reasons, H does not involve any of z^k, y_j^k's. Then we define the ***internal differential*** d on A by the equations

$$
\begin{aligned}
&d|_{A(0)} = 0;\ dx_j^{-i} = \frac{\partial H}{\partial y_j^{k+i}} \text{ for all } i>0, j;\quad dy_j^{k+i} = \frac{\partial H}{\partial x_j^{-i}} \text{ for all } i,j; \text{ and}\\
&-kdz^k = H + d\Big[\sum_{i,j}(-1)^i i x_j^{-i} y_j^{k+i}\Big].
\end{aligned}
\tag{11}$$

Notice that the condition on H implies $d^2=0$ on each generator [3]. Also, we get vdim $(A)=-1$.

Step-2: Pre-contact data. Next, we introduce the element $\alpha \in (\Omega^1_A)^k$ given by

$$\alpha = d_{dR}z^k + \sum_{i=0}^{\ell}\sum_{j=1}^{m_i} y_j^{k+i} d_{dR}x_j^{-i}. \tag{12}$$

Let us first show that $d\alpha = 0$. To this end, we compute $d_{dR}\alpha = \sum_{i=0}^{\ell}\sum_{j=1}^{m_i} d_{dR}x_j^{-i} d_{dR}y_j^{k+i}$. Then the element $d_{dR}\alpha$ is $(d+d_{dR})$-closed by [3, Example 5.8]. The same example also implies that if we let[9]

$$\phi := \sum_{i=0}^{\ell}\sum_{j=1}^{m_i} \left[(-i)x_j^{-i} d_{dR}y_j^{k+i} + (k+i)y_j^{k+i} d_{dR}x_j^{-i}\right]$$

[9]Alternatives like $k\sum_{i,j} y_j^{k+i} d_{dR}x_j^{-i}$ can be obtained by replacing H, ϕ by suitable $H + d[\cdots]$ and $\phi + d_{dR}[\cdots]$.

and H as above, then the pair $(\phi, H) \in (\Omega^1_A)^k \times A^{k+1}$ is a solution to the equations

$$dH = 0 \text{ in } A^{k+2},\ d_{dR}H + d\phi = 0 \text{ in } (\Omega^1_A)^{k+1}, \text{ and } d_{dR}\phi = kd_{dR}\alpha. \tag{13}$$

Observe that $\phi + d_{dR}\big[\sum_{i,j}(-1)^i ix_j^{-i}y_j^{k+i}\big] = k\sum_{i,j} y_j^{k+i}d_{dR}x_j^{-i}$, then we can write

$$k\alpha = kd_{dR}z^k + k\sum_{i=0}^{\ell}\sum_{j=1}^{m_i} y_j^{k+i}d_{dR}x_j^{-i} = kd_{dR}z^k + \phi + d_{dR}\Big[\sum_{i,j}(-1)^i ix_j^{-i}y_j^{k+i}\Big]. \tag{14}$$

As we set $-kdz^k = H + d[\cdots]$ in (11), we obtain $d(k\alpha) = d_{dR}\circ d(-kz^k) + d\phi + d\circ d_{dR}[\cdots] = d_{dR}(H + d[\cdots]) - d_{dR}H - d_{dR}\circ d[\cdots] = 0$ using (13). In that case, α is then *d-closed, and hence a 1-form of degree k*. In other words, the pair $(H, k\alpha - kd_{dR}z^k - d_{dR}[\cdots])$, with H, α as above, provides a solution to the equations (13), which implies $d\alpha = 0$.

Next, we show that there is a canonical pre-k-shifted contact structure on Spec$A$ induced by $\ker\alpha$. For $i = 0, \ldots, \ell,$ and $1 \leq j \leq m_i$, denote the vector fields annihilating α by

$$\begin{aligned} \zeta_j^i &= \partial/\partial x_j^{-i} - y_j^{k+i}\partial/\partial z^k && \text{in degree } i,\\ \eta_j^{-k-i} &= \partial/\partial y_j^{k+i} && \text{in degree } -k-i. \end{aligned} \tag{15}$$

We thus obtain $\ker\alpha = \mathrm{Span}_A\{\zeta_j^i,\ \eta_j^{-k-i}\} \hookrightarrow \mathbb{T}_A$ over Spec$A$, with the Tor-amplitude $[0, -k]$. From Prop. 2.9, the restrictions of $\mathbb{T}_A, \ker\alpha$ to Spec$H^0(A)$ are both complexes of free $H^0(A)$-modules. Likewise, the restriction of $\mathbb{T}_A\big/_{\ker\alpha}$ to Spec$H^0(A)$ is then generated by the deg $-k$ vector field $\partial/\partial z^k$ as an $H^0(A)$-module, and hence it is equivalent to the complex concentrated in degree $-k$. It follows that the quotient can also be identified with a k-shifted line bundle $L[k]$ on Spec$A$ such that we get a fiber-cofiber sequence

$$\ker\alpha \to \mathbb{T}_A \to L[k],$$

and hence a map $\mathbb{T}_A \to L[k]$. Therefore, the data of $L[k]$ and the map $\mathbb{T}_A \to L[k]$ together with the k-shifted 1-form α above define a ***pre-k-shifted contact structure*** on Spec$A$ in the sense of Definition 2.16. Call such α a ***pre-k-contact form***.

Step-3: From pre-contact to contact. So far, we have obtained a pre-k-contact form $\alpha \in (\Omega^1_A)^k$ as in (12) using the variables in (7), the differential d defined by (11), and the equations in (13). It remains to show that $d_{dR}\alpha$ is non-degenerate on $\ker\alpha$. To this end, it suffices to prove the non-degeneracy of the induced map

$$d_{dR}\alpha|_{\ker\alpha}\otimes_A \mathrm{id}_{H^0(A)} : \ker\alpha\otimes_A H^0(A) \to (\ker\alpha)^\vee[k]\otimes_A H^0(A).$$

We first observe that, at $p \in \mathrm{Spec}H^0(A)$, $d_{dR}\alpha|_p$ maps

$$\begin{aligned} \langle \partial/\partial x_1^{-i}|_p, \ldots, \partial/\partial x_{m_i}^{-i}|_p\rangle_{\mathbb{K}} &\xrightarrow{\sim} \langle d_{dR}y_1^{k+i}|_p, \ldots, d_{dR}y_{m_i}^{k+i}|_p\rangle_{\mathbb{K}},\\ \langle \partial/\partial y_1^{k+i}|_p, \ldots, \partial/\partial y_{m_i}^{k+i}|_p\rangle_{\mathbb{K}} &\xrightarrow{\sim} \langle d_{dR}x_1^{-i}|_p, \ldots, d_{dR}x_{m_i}^{-i}|_p\rangle_{\mathbb{K}} \end{aligned}$$

isomorphically for all i, such that $d_{dR}\alpha(\partial/\partial z^k|_p, -) = 0$. It follows that, at $p \in \mathrm{Spec}H^0(A)$, we have the identifications

$$\Big(\mathbb{T}_A\big/_{\ker d_{dR}\alpha}\Big)|_p \simeq \langle \partial/\partial x_j^{-i}|_p, \partial/\partial y_j^{k+i}|_p : \forall i,j\rangle_{\mathbb{K}} \simeq (\ker\alpha)|_p. \tag{16}$$

Thus, the maps $(d_{dR}\alpha|_{\ker\alpha})^i|_p : (\ker\alpha|_p)^i \to (\ker^\vee\alpha|_p)^{k+i}$ are all isomorphisms at p, hence isomorphisms in a neighborhood of p. So, localizing A at p if necessary, the induced morphism $d_{dR}\alpha|_{\ker\alpha}\otimes_A \mathrm{id}_{H^0(A)}$ is an isomorphism of complexes, and hence a quasi-isomorphism. Thus, $d_{dR}\alpha|_{\ker\alpha}$ is non-degenerate, and we get a ***k-shifted contact structure*** on Spec$A$ with a k-contact form α in the sense of Definition 2.17 as desired.

Definition 2.22. If A, d, α are as above, we then say A, α are ***in contact Darboux form***.

Note that the general expressions like "$d_{dR}z^k + \phi/k + d_{dR}[\cdots]/k$" will still be valid for the other cases (*a*) $k \equiv 0 \mod 4$, and (*b*) $k \equiv 2 \mod 4$ as well. In fact, Equations (76) – (79) in Appendix A show that the other cases involve modified versions of H, d, and ϕ with some possible extra terms. In either case, the modified A, α would also serve as the desired contact model. Following the same terminology as above, we also say A, α ***are in (contact) Darboux form*** for any k.

Observation 2.23. Sanity check: the cases $k = -1$ ($\ell = 0$) and $k = -3$ ($\ell = 1$).

- ***When*** $k = -1$, we set $A = A(0)[z^{-1}, y_1^{-1}, \ldots, y_{m_0}^{-1}]$, with $A(0)$ a $\mathbb{K}$-algebra generated by $x_1^0, \ldots, x_{m_0}^0$, such that vdim $A = m_0 - (m_0 + 1) = -1$. Choosing an arbitrary Hamiltonian $H \in A(0)$, we let $dz^{-1} = H$ and $dy_j^{-1} = \partial H/\partial x_j^0$ and $dx_j^0 = 0$ $\forall j$. Then, from (12), the element
$$\alpha = d_{dR}z^{-1} + \sum_{1 \le j \le m_0} y_j^{-1} d_{dR}x_j^0$$
defines a (-1)-contact form. In that case, $\ker \alpha$ is generated by the vector fields $\partial/\partial y_j^{-1}$ and $y_j^{-1}\partial/\partial z^{-1} - \partial/\partial x_j^0$ for $1 \le j \le m_0$.
- ***When*** $k = -3$, we let $A = A(0)[x_1^{-1}, \ldots, x_{m_1}^{-1}, y_1^{-2}, \ldots, y_{m_1}^{-2}, z^{-3}, y_1^{-3}, \ldots, y_{m_0}^{-3}]$, with $A(0)$ a $\mathbb{K}$-algebra generated by $x_1^0, \ldots, x_{m_0}^0$, such that vdim $A = -1$.
Observe that from [3, Example 5.17], a Hamiltonian $H \in A^{-2}$ is an element of the form $H = \sum_{j=1}^{m_1} y_j^{-2} s_j + \sum_{i,j=1}^{m_1} x_i^{-1} x_j^{-1} t_{ij}$, with $s_j, t_{ij} \in A(0)$, satisfying $\sum_{i,j=1}^{m_1} t_{ij}s_j = 0$ in $A(0)$ for $i = 1, \ldots, m_1$.
From (11), the differential d is defined by $dx_j^0 = 0$, $dx_j^{-1} = s_j$, $dy_j^{-2} = 2\sum_{j'=1}^{m_1} x_{j'}^{-1} t_{jj'}$, $dy_j^{-3} = \sum_{j'=1}^{m_1} y_{j'}^{-2} \frac{\partial s_{j'}}{\partial x_j^0} + \sum_{j'',j'} x_{j''}^{-1} x_{j'}^{-1} \frac{\partial t_{j''j'}}{\partial x_j^0}$, and $dz^{-3} = \frac{H}{3} - \frac{1}{3}\sum_j s_j y_j^{-1} + \frac{2}{3}\sum_{j,j'} x_j^{-1} x_{j'}^{-1} t_{jj'}$.
Then the element
$$\alpha = d_{dR}z^{-3} + \sum_{j=1}^{m_0} y_j^{-3} d_{dR}x_j^0 + \sum_{j=1}^{m_1} y_j^{-2} d_{dR}x_j^{-1}$$
is a (-3)-contact form. Here, $\ker \alpha$ is generated by the vector fields $y_j^{-3}\partial/\partial z^{-3} - \partial/\partial x_j^0$ and $\partial/\partial y_j^{-3}$ for $1 \le j \le m_0$; and by $\partial/\partial x_j^{-1} - y_j^{-2}\partial/\partial z^{-3}$ and $\partial/\partial y_j^{-2}$ for all $1 \le j \le m_1$.

Example 2.24. Let k, A, H, ϕ be as in Example 2.21. Alternatively, we can define the ***differential*** d on A as in (11), but with $-kdz^k = H$, instead. In that case, we then introduce the element $\alpha' \in \Omega_A^1[k]$ by

$$\begin{aligned}\alpha' &= d_{dR}z^k + \sum_{i,j}\Big[-\frac{i}{k}\, x_j^{-i} d_{dR} y_j^{k+i} + \frac{k+i}{k}\, y_j^{k+i} d_{dR} x_j^{-i}\Big] \\ &= d_{dR}z^k + \phi/k.\end{aligned}$$

Note that $d\alpha' = 0$ as well due to the new choice of dz^k. Modifying Example 2.21 accordingly, one can also conclude that such element α' also serves as a k-contact form, inducing a k-contact structure on Spec$A$, with the contact data constructed similarly. Details are left to the reader. Using the same terminology, we also say A, α' ***are in (contact) Darboux form***.

Notice that both versions α and α' (and the corresponding differentials) in Example 2.21 and Example 2.24 coincide for $k = -1$. These examples also suggest that suitable modifications using $H + d[\cdots]$ and $\phi + d_{dR}[\cdots]$ may lead to alternative versions of such forms.

2.3.2 Main results for negatively shifted contact derived schemes We now outline the main results of [2] for derived $\mathbb{K}$-schemes (of locally finite presentations) with negatively shifted contact structures.

A Darboux-type theorem. The first result is about Darboux-type local models, which essentially says that for $k<0$, every k-shifted contact derived $\mathbb{K}$-scheme $\mathbf{X}$ is locally equivalent to $(\mathrm{Spec}A, \alpha_0)$ for A a minimal standard form cdga and α_0 as in Example 2.21. More precisely, we have:

Theorem 2.25. *[2, Thm. 3.13.] Let $\mathbf{X}$ be a k-shifted contact derived $\mathbb{K}$-scheme for $k<0$, and $x\in\mathbf{X}$. Then there is a local contact model $\big(A,\alpha_0\big)$ and $p\in \mathrm{Spec}H^0(A)$ such that $i:\mathrm{Spec}A\hookrightarrow\mathbf{X}$ is an open inclusion with $i(p)=x$, A is a standard form that is minimal at p, and α_0 is a k-shifted contact form on $\mathrm{Spec}A$ such that A,α_0 are in Darboux form.*

Note that for $k<0$ odd, for instance, the pair (A,α_0) can be explicitly given by the graded variables as in Examples 2.21 and 2.24. For the other cases, one should use another sets of variables as in Equations (76) and (77), and modify H,ϕ,d accordingly.

Symplectification. The second main result of [2] is about the *symplectification* of a k-shifted contact derived $\mathbb{K}$-scheme. Recall from [2, Def. 4.3] that if $\mathbf{X}$ is a locally finitely presented derived $\mathbb{K}$-scheme carrying a k-shifted contact structure $(\mathcal{K},\kappa,L)$ with $k<0$, then we define its *symplectification* $\mathcal{S}_{\mathbf{X}}$ to be the total space $\widetilde{\mathbf{L}}$ of the $\mathbb{G}_m$-bundle of L over $\mathbf{X}$, provided with a canonical k-shifted symplectic structure (for which the $\mathbb{G}_m$-action is of weight 1) as defined below.

Let $(\mathbf{X};\mathcal{K},\kappa,L)$ be a k-shifted contact derived $\mathbb{K}$-scheme of locally finite presentation. Given $k<0$ and $p\in\mathbf{X}$, find an affine derived sub-scheme $\mathbf{U}:=\mathrm{Spec}A$ such that $p:\mathrm{Spec}A\to\mathbf{X}$ is Zariski open inclusion (we may further assume A is of minimal standard form). Here, we assume w.l.o.g. that L is trivial on $\mathbf{U}$. Define the functor $\mathcal{S}_{\mathbf{X}}: cdga_{\mathbb{K}}\to Spcs$ by $A\mapsto\mathcal{S}_{\mathbf{X}}(A)$, where

$$\mathcal{S}_{\mathbf{X}}(A):=\big\{(p,\alpha,v):p\in\mathbf{X}(A),\ \alpha:p^*(\mathbb{T}_{\mathbf{X}})\to\mathcal{O}[k],\ v:Cocone(\alpha)\xrightarrow{\sim}p^*(\mathcal{K})\big\},\tag{17}$$

where each v is a quasi-isomorphism respecting the natural morphisms $p^*\kappa:p^*\mathcal{K}\to p^*(\mathbb{T}_{\mathbf{X}})$ and $Cocone(\alpha)\to p^*(\mathbb{T}_{\mathbf{X}})$. Under the current assumptions, the perfect complexes $\mathbb{T}_A,\mathbb{L}_A$, when restricted to $\mathrm{Spec}H^0(A)$, are both quasi-isomorphic to free complexes of $H^0(A)$-modules. For $A\in cdga_{\mathbb{K}}$, we then define a $\mathbb{G}_m(A)$-action on $\mathcal{S}_{\mathbf{X}}(A)$ by

$$f\triangleleft(p,\alpha,v):=(p,f\cdot\alpha,v).$$

By [2, Prop. 4.4], $\mathcal{S}_{\mathbf{X}}$ is equivalent to the total space $\widetilde{\mathbf{L}}$ of the $\mathbb{G}_m$-bundle of L. Therefore, it has the structure of a derived stack together with the projection map $\pi_1:\mathcal{S}_{\mathbf{X}}\to\mathbf{X}$.

We also introduce the *canonical 1-from* λ on $\mathcal{S}_{\mathbf{X}}$. By construction, we have the projection maps $\pi_1:\mathcal{S}_{\mathbf{X}}\to\mathbf{X}$ and $\pi_2:\mathcal{S}_{\mathbf{X}}\to \mathrm{T}^*[k]\mathbf{X}$. We define the ***canonical 1-from*** λ on $\mathcal{S}_{\mathbf{X}}$ to be the pullback $\pi_2^*\lambda_{\mathbf{X}}$ of the tautological 1-form $\lambda_{\mathbf{X}}$ on $\mathrm{T}^*[k]\mathbf{X}$. We then have:

Theorem 2.26. *[2, Thm. 4.7] Let $\mathbf{X}$ be a (locally finitely presented) derived $\mathbb{K}$-scheme carrying a k-shifted contact structure $(\mathcal{K},\kappa,L)$ with $k<0$. Then the k-shifted closed 2-form $\omega:=(d_{dR}\lambda,0,0,\dots)$ is non-degenerate, and hence the derived stack $\mathcal{S}_{\mathbf{X}}$ is k-shifted symplectic.*

We then call the pair $(\mathcal{S}_{\mathbf{X}},\omega)$ the **symplectification of X**.

3. Results for shifted geometric structures on derived Artin stacks

In this section, we will explain how to extend the main results of [2], outlined in the previous section (cf. Theorems 2.25 and 2.26), from derived schemes to the more general case of derived Artin stacks. We begin with some basic definitions and results from [1]. Later, we give two theorems about the desired generalizations (cf. Theorems 3.7 & 3.9).

3.1 A Darboux-type theorem for shifted symplectic derived Artin stacks

Nice atlases for derived Artin stacks. Recall that derived schemes has nice local models. With the same spirit, derived Artin $\mathbb{K}$-stacks have *nice atlases*. In that respect, we have the following generalization of Definition 2.3:

Definition 3.1. Let $\mathbf{X}$ be a derived Artin $\mathbb{K}$-stack and $x \in \mathbf{X}$. By a ***standard form open neighborhood of*** x, we mean a pair (A, φ) and a point $p \in \mathrm{Spec}H^0(A)$ such that A is a standard form cdga in the sense of Definition 2.3, and $\varphi : \mathbf{U} = \mathrm{Spec}A \to \mathbf{X}$ is smooth of some relative dimension $n \geq 0$ with $\varphi(p) = x$.

For A, $\varphi : \mathbf{U} = \mathrm{Spec}A \to \mathbf{X}$, and a point $p \in \mathrm{spec}H^0(A)$ as above, there exists a canonical distinguished triangle

$$\varphi^*\mathbb{L}_{\mathbf{X}} \to \mathbb{L}_{\mathbf{U}} \to \mathbb{L}_{\mathbf{U}/\mathbf{X}} \to \varphi^*\mathbb{L}_{\mathbf{X}}\ [1]. \tag{18}$$

As φ is smooth of some relative dimension $n \geq 0$, $\mathbb{L}_{\mathbf{U}/\mathbf{X}}$ is locally free of rank n. Moreover, for $\varphi(p) = x$, an element of the pullback $\varphi^*\mathbb{L}_{\mathbf{X}}$ is locally of the form $(f \otimes \beta)|_p$ with $\beta \in \mathbb{L}_X|_x$ and $f \in A$, such that the map $\varphi^*\mathbb{L}_{\mathbf{X}}|_p \to \mathbb{L}_{\mathbf{U}}|_p$ sends $f \otimes \beta \mapsto f \cdot \beta$.

Observation 3.2. It follows from the sequence (18) that $H^i(\mathbb{L}_{\mathbf{X}}|_x) \simeq H^i(\mathbb{L}_{\mathbf{U}}|_p)$ for $i < 0$. Moreover, as $\mathbf{U}$ is not "stacky", it follows that $H^1(\mathbb{L}_{\mathbf{U}}|_p) = 0$. Hence, there exists an exact sequence of $\mathbb{K}$-vector spaces

$$0 \to H^0(\mathbb{L}_{\mathbf{X}}|_x) \to H^0(\mathbb{L}_{\mathbf{U}}|_p) \to H^0(\mathbb{L}_{\mathbf{U}/\mathbf{X}}|_p) \to H^1(\mathbb{L}_{\mathbf{X}}|_x) \to 0, \tag{19}$$

so that $n \geq \dim H^1(\mathbb{L}_{\mathbf{X}}|_x)$ due to the exactness.

Definition 3.3. We say that a standard form open neighborhood (A, φ, p) of x is ***minimal*** if A is minimal in the sense of Definition 2.4, and the relative dimension n attains its minimum; i.e., $n = \dim H^1(\mathbb{L}_{\mathbf{X}}|_x)$.

Observation 3.4. Given a minimal standard form open neighborhood (A, φ, p) of x, the sequence (19) implies that there are isomorphisms $H^0(\mathbb{L}_{\mathbf{U}/\mathbf{X}}|_p) \simeq H^1(\mathbb{L}_{\mathbf{X}}|_x)$ and $H^0(\mathbb{L}_{\mathbf{X}}|_x) \simeq H^0(\mathbb{L}_{\mathbf{U}}|_p)$. Therefore, we have

$$H^i(\mathbb{L}_{\mathbf{X}}|_x) \simeq H^i(\mathbb{L}_{\mathbf{U}}|_p) \text{ for } i \leq 0. \tag{20}$$

It follows that $A(0)$ is smooth of dimension $m_0 = \dim H^0(\mathbb{L}_{\mathbf{X}}|_x)$, and A is free over $A(0)$ with $m_i = \dim H^{-i}(\mathbb{L}_{\mathbf{X}}|_x)$ generators in each degree $-i$.

Ben-Bassat, Brav, Bussi and Joyce [1] proved that derived Artin $\mathbb{K}$-stacks have nice local models in terms of minimal standard form open neighborhoods. The following result summarizes key observations from [1, Theorems 2.8 & 2.9] and in fact serves as a generalization of Theorem 2.8. For more details, we refer to [1, Sections 2.4 & 2.5].

Theorem 3.5. *Let $\mathbf{X}$ be a derived Artin $\mathbb{K}$-stack and $x \in \mathbf{X}$. Then there exists a minimal standard form open neighborhood (A, φ, p) of x. Moreover, if (A, φ, p) and (A', φ', p') are two such open neighborhoods, then there exists another standard form cdga A'' which can be used to compare them in a reasonable way.*

Darboux form atlases for negatively shifted symplectic derived Artin stacks. For $k < 0$, it has been proven in [1] that given a k-shifted symplectic derived Artin $\mathbb{K}$-stack $(\mathbf{X}, \omega)$, near each $x \in \mathbf{X}$, one can find a "minimal smooth atlas" $\varphi : \mathbf{U} \to \mathbf{X}$ with $\mathbf{U} = \mathrm{Spec}A$ an affine derived scheme such that $(\mathbf{U}, \varphi^*(\omega))$ is in a standard Darboux form. More precisely, we have:

Theorem 3.6. *([1, Theorem 2.10]) Let $(\mathbf{X}, \omega)$ be a k-shifted symplectic derived Artin $\mathbb{K}$-stack for $k < 0$, and $x \in \mathbf{X}$. Then there exist a minimal standard form open neighborhood (A, φ, p) of x, with $\varphi(p) = x$, a minimal standard form cdga B with inclusion $\iota : B \hookrightarrow A$ and the diagram*

$$\mathrm{Spec}B = \mathbf{V} \xleftarrow{j:=\mathrm{Spec}(\iota)} \mathbf{U} = \mathrm{Spec}A \xrightarrow{\varphi} \mathbf{X} \tag{21}$$

such that the induced morphism $\tau(\mathbf{U}) \xrightarrow{\tau(j)} \tau(\mathbf{V})$ between truncations is an isomorphism, and there is a k-shifted symplectic structure $\omega_B = (\omega_B^0, 0, 0, \dots)$ on $\mathbf{V} = \mathrm{Spec}B$, which is in Darboux form in the sense of Theorem 2.13, with $\varphi^*(\omega) \sim j^*(\omega_B)$ in k-shifted closed 2-forms on $\mathbf{U}$.

Moreover, there exists a natural equivalence

$$\mathbb{L}_{\mathbf{U}/\mathbf{V}} \simeq \mathbb{T}_{\mathbf{U}/\mathbf{V}}[1-k]. \tag{22}$$

3.2 A Darboux-type theorem for shifted contact derived Artin stacks In this section, we will discuss how to extend Theorem 2.25 from derived schemes to derived Artin stacks. In that respect, the following result provides a Darboux-type atlas for *negatively* shifted contact derived Artin stacks. The proof will be a variation of [1, Theorem 2.10].

Theorem 3.7. *Given $k \in \mathbb{Z}_{<0}$, let $\mathbf{X}$ be a derived Artin $\mathbb{K}$-stack (locally of finite presentation) carrying a k-shifted contact structure, and $x \in \mathbf{X}$. Then we can find*

- *a minimal standard form open neighborhood $(A, \varphi : \mathbf{U} \to \mathbf{X}, p)$ of x;*
- *a dg-subalgebra B of A with inclusion $\iota : B \hookrightarrow A$ and the diagram*

$$\mathbf{V} := \mathrm{Spec}B \xleftarrow{j:=\mathrm{Spec}(\iota)} \mathbf{U} = \mathrm{Spec}A \xrightarrow{\varphi} \mathbf{X}; \quad \textit{and}$$

- *a k-contact form α_B on $\mathbf{V}$ such that for any k-contact form α on $\mathbf{X}$, we have an equivalence $\varphi^*(\alpha) \sim j^*(\alpha_B)$ in $\mathcal{A}^1(\mathbf{U}, k)$, and the pair (B, α_B) is in contact Darboux form (cf. Example 2.21 for k odd).*

Moreover, the induced morphism $\tau(j) : \tau(\mathbf{U}) \to \tau(\mathbf{V})$ between truncations is an isomorphism.

Proof. Let $(\mathcal{K} \xrightarrow{\kappa} \mathbb{T}_{\mathbf{X}}, L)$ be a k-shifted contact data on $\mathbf{X}$. Then, locally on $\mathbf{X}$, where L is trivialized, the perfect complex $\mathcal{K}$ can be given as a cocone of α, where $\alpha : \mathbb{T}_{\mathbf{X}} \to \mathcal{O}_{\mathbf{X}}[k]$ is a k-contact form, such that we have the (co)exact triangle $\mathcal{K} \to \mathbb{T}_{\mathbf{X}} \to L[k]$.

Given $k < 0$ and $x \in \mathbf{X}$, apply now Theorem 3.5 to get a minimal standard form open neighborhood $(A, \varphi : \mathbf{U} \to \mathbf{X}, p)$ of x, with $\mathbf{U} = \mathrm{Spec}A$, $x \in \mathbf{X}(A)$, $p \in \mathrm{Spec}H^0(A)$ such that $p \mapsto x$ and that the map φ is smooth of relative dimension $n = \dim H^1(\mathbb{L}_{\mathbf{X}}|_x)$. We also assume that A is a standard form cdga constructed inductively as described in (2) such that $\mathbb{L}_A$ has Tor-amplitude in $[k-1, 0]$.

When the shift is odd. As before, we focus on a particular and the simplest case: k is *odd*, say $k=-2\ell-1$ for $\ell\in\mathbb{N}$. By Definition 3.1, A can be chosen as a free algebra over $A(0)$ with m_i generators in degree $-i$ for $i=1,\dots,\ell$, and m_i generators in degree $k+i$ for $i=0,\dots,\ell$, but with additional n generators in degree $k-1$. See Example B.1.

W.l.o.g., we can assume[10] that L is trivial on the interior of the image $\varphi(\mathbf{U})$, denoted by $\widetilde{\mathbf{U}}$. Then, over $\widetilde{\mathbf{U}}$, the induced 1-form $\alpha:\mathbb{T}_{\mathbf{X}}\to\mathcal{O}_{\mathbf{X}}[k]$ is such that $\mathcal{K}$ is the cocone of α, up to quasi-isomorphism, and the 2-form $d_{dR}\alpha$ is non-degenerate on $\mathcal{K}$. In that case, we have the exact triangle $\mathcal{K}\to\mathbb{T}_{\mathbf{X}}\to L[k]$ over $\widetilde{\mathbf{U}}$. We fix this k-contact form α for the rest of the proof. Also, by abuse of notation, we simply use α for its pullback on $\mathcal{K}$ as well.

Next, we consider $d_{dR}\varphi^*(\alpha)$ as a sequence $(d_{dR}\varphi^*(\alpha),0,0,\dots)$, which defines a closed k-shifted 2-form on $\mathbf{U}=\mathrm{Spec}A$. Applying Proposition 2.14 to $d_{dR}\varphi^*(\alpha)$, we obtain elements $H\in A^{k+1}$ and $\phi\in(\Omega^1_A)^k$ such that $dH=0,\ d_{dR}H+d\phi=0$, and $kd_{dR}\varphi^*(\alpha)\sim(d_{dR}\phi,0,0,\dots)$. We denote this representative by $(\omega^0,0,0,\dots)$ or just ω^0 whenever the meaning is clear.

Note that the pullback $\varphi^*(\alpha)$ may not be a k-shifted contact form on $\mathbf{U}$, because $d_{dR}\varphi^*(\alpha)$ is *not* necessarily non-degenerate on $\varphi^*(\mathcal{K})$. But, we may ensure the non-degeneracy for some degrees. In this regard, we have the following lemma.

Lemma 3.8. *Let $\omega^0\sim d_{dR}\varphi^*(\alpha)$ be the representative of $d_{dR}\varphi^*(\alpha)$ as above. Then the induced morphism $\omega^0|_{\varphi^*(\mathcal{K})}\cdot:\varphi^*(\mathcal{K})\to\varphi^*(\mathcal{K}^\vee[k])$ is a quasi-isomorphism* only for $0\le i\le -k$ *(i.e. except in* deg $k-1$*).*

Proof of Lemma 3.8. We first note that, from Observation 3.4, $H^i(\mathbb{L}_{\mathbf{X}}|_x)\simeq H^i(\mathbb{L}_{\mathbf{U}}|_p)$ for $i\le 0$, where $\mathbb{L}_{\mathbf{U}}\simeq\mathbb{L}_A\simeq\Omega^1_A$ with A a (minimal) standard form cdga. Then the natural morphism $\mathbb{L}_\varphi[k]:\varphi^*(\mathbb{L}_{\mathbf{X}})[k]\to\mathbb{L}_A[k]$ from the triangle (18) induces an isomorphism on cohomology H^i at p for $i+k\le 0$, and it is zero if $i+k=1$. Likewise, the dual $\mathbb{L}_\varphi^\vee$ induces an isomorphism on cohomology H^i at p for $i\ge 0$, and it is zero if $i=-1$. Thus, we get the diagram[11]

$$\begin{tikzcd}
\mathbb{T}_A \arrow[r, "\mathbb{L}_\varphi^\vee"] & \varphi^*(\mathbb{T}_{\mathbf{X}}) \arrow[r, "\varphi^*(d_{dR}\alpha\ \cdot)"] \arrow[dl, dashed, "\gamma"'] & \varphi^*(\mathbb{L}_{\mathbf{X}}[k]) \arrow[r, "\mathbb{L}_\varphi[k]"] \arrow[d, "\varphi^*\kappa^\vee[k]"] & \Omega^1_A[k] \arrow[d, dashed, "\gamma^\vee[k]"] \\
\varphi^*(\mathcal{K}[-1]) \arrow[u] \arrow[r, "\simeq"] & \varphi^*(\mathcal{K}) \arrow[u, "\varphi^*\kappa"] \arrow[r, dashed, "\omega^0|_{\varphi^*(\mathcal{K})}\cdot"] & \varphi^*(\mathcal{K}^\vee[k]) \arrow[r, equal] & \varphi^*(\mathcal{K}^\vee[k]).
\end{tikzcd} \tag{23}$$

Combining the conditions for i above, we observe that for $0\le i\le -k$ *only*, Diagram 23 describes a suitable factorization of the map $\omega^0\cdot:\mathbb{T}_A\to\Omega^1_A[k]$ commuting

$$\begin{tikzcd}
\mathbb{T}_A\simeq\varphi^*(\mathbb{T}_{\mathbf{X}}) \arrow[r, "\omega^0\cdot"] & \Omega^1_A[k]\simeq\varphi^*(\mathbb{L}_{\mathbf{X}}[k]) \arrow[d] \\
\varphi^*(\mathcal{K}) \arrow[u] \arrow[r, "\omega^0|_{\varphi^*(\mathcal{K})}\cdot"] & \varphi^*(\mathcal{K}^\vee[k]),
\end{tikzcd} \tag{24}$$

where the 2nd row is the pullback of $\mathcal{K}\to\mathcal{K}^\vee[k]$, which is an equivalence as $d_{dR}\alpha$ is non-degenerate on $\mathcal{K}$. Thus, we conclude that the map $\omega^0\cdot:\mathbb{T}_A\to\Omega^1_A[k]$ representing $d_{dR}\varphi^*(\alpha)\cdot$ is non-degenerate on $\varphi^*(\mathcal{K})$ only for $0\le i\le -k$.

$\square$

[10] Otherwise, apply again Theorem 3.5 to the overlap $\varphi(\mathbf{U})\times^h_{\mathbf{X}}W$, where W is open containing x s.t. $L|_W$ is trivial, to get another minimal standard form open neighborhood $(A',\varphi':\mathbf{U}'\to\mathbf{X},p')$. Then we consider the interior of $\varphi'(\mathbf{U}')$ over which the restriction of L is still trivial. Here $\varphi(\mathbf{U})$ denotes the image $\mathrm{Im}\varphi$, with the monomorphism $\mathrm{Im}\varphi\hookrightarrow\mathbf{X}$.

[11] The diagonal map γ is the composition defined by the commuting triangle.

Variables in degrees $0,-1,\dots,k$. Now, we start with the pullback under φ of the triangle $\mathcal{K}\to\mathbb{T}_{\mathbf{X}}\to L[k]$ above, which locally splits over $\mathbf{U}$. That is, $\mathrm{cofib}(\varphi^*(\mathcal{K})\to\varphi^*(\mathbb{T}_{\mathbf{X}}))\simeq\varphi^*(L[k])$, which is concentrated in deg $-k$, and $\varphi^*(\mathbb{T}_{\mathbf{X}})=\varphi^*(\mathcal{K})\oplus\varphi^*(L[k])$. Here, we will call the 2nd summand *Rest*. Localizing A at p if necessary, first choose degree 0 variables $x_1^0,x_2^0,\dots,x_{m_0}^0$ in $A(0)$ such that $\{d_{dR}x_j^0 : j=1,\dots,m_0\}$ forms a basis for $\varphi^*(\mathcal{K}^\vee)^0$ over $A(0)$, and $(Rest^\vee)^0$ is zero.

Since the induced morphism $d_{dR}\varphi^*(\alpha)\cdot$ is an equivalence on $\varphi^*(\mathcal{K})$ *only for* $k\le -i\le 0$ due to Lemma 3.8, we can make the following choices of variables in degrees $0,-1,\dots,k$:

- When $i=0$, we find generators $y_1^k,y_2^k,\dots,y_{m_0}^k,z^k\in A^k$ such that $\{d_{dR}y_1^k,\dots,d_{dR}y_{m_0}^k\}$ is a basis for $\varphi^*(\mathcal{K}^\vee)^k$ which is dual to the basis $\{d_{dR}x_1^0,\dots,d_{dR}x_{m_0}^0\}$ for $\varphi^*(\mathcal{K}^\vee)^0$, and that the complex *Rest* is generated by the vector field $\partial/\partial z^k$ of degree $-k$.
- For $1\le i\le\ell$, again by Lemma 3.8, we can choose generators $x_1^{-i},x_2^{-i},\dots,x_{m_i}^{-i}\in A^{-i}$ and $y_1^{k+i},y_2^{k+i},\dots,y_{m_i}^{k+i}\in A^{k+i}$ (except in deg $k-1$) such that $d_{dR}y_j^{k+i}$, for $j=1,\dots,m_i$, form a basis for $\varphi^*(\mathcal{K}^\vee)^{k+i}$ which is dual to the basis $\{d_{dR}x_1^{-i},\dots,d_{dR}x_{m_i}^{-i}\}$ for $\varphi^*(\mathcal{K}^\vee)^{-i}$.

Then the map $\omega^0|_{\varphi^*(\mathcal{K})}\cdot\otimes\mathrm{id}_{H^0(A)}$ gives an equivalence[12] of $H^0(A)$-modules in each degree:

$$\langle\partial/\partial x_1^{-i},\dots,\partial/\partial x_{m_i}^{-i}\rangle\xrightarrow{\sim}\langle d_{dR}y_1^{k+i},\dots,d_{dR}y_{m_i}^{k+i}\rangle\qquad\text{for } 0\le i\le\ell,\tag{25}$$
$$\langle\partial/\partial y_1^{k+i},\dots,\partial/\partial y_{m_i}^{k+i}\rangle\xrightarrow{\sim}\langle d_{dR}x_1^{-i},\dots,d_{dR}x_{m_i}^{-i}\rangle\qquad\text{for } 0\le i\le\ell.\tag{26}$$

Note that we have $\varphi^*(\mathcal{K})=\varphi^*(Cocone(\alpha))\simeq Cocone(\varphi^*(\alpha))$ using the equivalence between two exact sequences $Cocone(\varphi^*\alpha)\to\varphi^*\mathbb{T}_{\mathbf{X}}\xrightarrow{\varphi^*\alpha}\mathcal{O}[k]$ and $\varphi^*Cocone(\alpha)\to\varphi^*\mathbb{T}_{\mathbf{X}}\xrightarrow{\varphi^*\alpha}\mathcal{O}[k]$, where the latter is the pullback of $Cocone(\alpha)\to\mathbb{T}_{\mathbf{X}}\xrightarrow{\alpha}\mathcal{O}_{\mathbf{X}}[k]$. Moreover, over $\mathrm{Spec}A$, we simply write $\ker\varphi^*(\alpha)$ instead of $Cocone(\varphi^*\alpha)$[13].

Thus, using the local coordinates, the splitting $\mathbb{T}_A=\ker\varphi^*(\alpha)\oplus Rest$ is such that $\ker\varphi^*(\alpha)$ has Tor-amplitude $[0,-k]$ and *Rest* is concentrated in deg $-k$, where

$$\begin{aligned}\ker\varphi^*(\alpha)|_{\mathrm{Spec}H^0(A)}&=\left\langle\partial/\partial x_j^{-i},\partial/\partial y_j^{k+i}:0\le i\le\ell,\ 1\le j\le m_i\right\rangle_{H^0(A)},\\ Rest|_{\mathrm{Spec}H^0(A)}&=\left\langle\partial/\partial z^k\right\rangle_{H^0(A)}.\end{aligned}\tag{27}$$

Then using the complexes in (27), the non-degeneracy condition for $d_{dR}\varphi^*(\alpha)$ on $\ker\varphi^*(\alpha)$ - *except for* deg $(k-1)$ - sending the dual basis of $d_{dR}x_a^b$ to the basis $d_{dR}y_{a'}^{b'}$ (and vice versa) as in Equations (25) - (26) implies that $d_{dR}\varphi^*(\alpha)\in\wedge^2\Omega_A^1[k]$ is given by

$$d_{dR}\varphi^*(\alpha)=\sum_{i=0}^{\ell}\sum_{j=1}^{m_i}d_{dR}x_j^{-i}d_{dR}y_j^{k+i}.\tag{28}$$

Note that the kernel of the induced map $d_{dR}\varphi^*(\alpha)\cdot$ is spanned by the degree $-k$ vector field $\partial/\partial z^k$, while the action of $d_{dR}\varphi^*(\alpha)\cdot$ on $\ker\varphi^*(\alpha)$ is given by Equations (25)-(26). Thus, we obtain the identification $\mathbb{T}_A/\ker d_{dR}\varphi^*(\alpha)\simeq\ker\varphi^*(\alpha)$.

Scaling z^k we may assume $\iota_{\partial/\partial z^k}\varphi^*(\alpha)=1$. Now, our goal is to find a unique[14] $\varphi^*(\alpha)$ satisfying Eqn. (28), the condition on the kernel in (27), and the equation $\iota_{\partial/\partial z^k}\varphi^*(\alpha)=1$.

[12] Due to the minimality at p, $d^{-i}|_p=0=(d^{-i})^\vee|_p$ for each i. So, all degree-wise maps are isomorphisms at p, and hence in a neighborhood of p. So, localizing A at p if necessary, we can assume $\omega^0|_{\varphi^*(\mathcal{K})}\cdot\otimes\mathrm{id}_{H^0(A)}$ is an isomorphism.

[13] Thanks to Proposition 2.20.

[14] The conditions uniquely determine the explicit form, up to interchange of x_j^{-i} and y_j^{k+i}. Here, the roles of x_j^{-i},y_j^{k+i} are symmetric in (28) and (27), where $d_{dR}x_j^{-i}d_{dR}y_j^{k+i}=d_{dR}y_j^{k+i}d_{dR}x_j^{-i}$ for k odd. See [2, Proof of Thm. 3.13].

Then such $\varphi^*(\alpha)$ satisfying the desired properties can be written explicitly as

$$\varphi^*(\alpha) = d_{dR} z^k + \sum_{i,j} y_j^{k+i} d_{dR} x_j^{-i}. \tag{29}$$

Variables in degree $(k-1)$ ***and the differential.*** We construct the rest by combining Example 2.21 with Example B.1. Note that Example 2.21 does *not* involve the additional finitely many generators in degree $(k-1)$. However, due to the atlas chosen at the beginning of the proof, the corresponding cdga A must admit additional n generators in degree $(k-1)$ as in Example B.1.

In our case, we identify A as a *commutative graded algebra* with the commutative graded algebra freely generated by the variables $z^k, x_j^{-i}, y_{j'}^{k+i'}$ as above, but ***with additional*** n ***generators,*** $w_1^{k-1}, \dots, w_n^{k-1}$***, in degree*** $k-1$. As discussed before, ω^0, H, ϕ above do not involve any of w_j^{k-1} for degree reasons, and hence the extra variables can be chosen arbitrarily.

Choose B with inclusion $\iota : B \hookrightarrow A$ such that $B(0)$ is the subalgebra of $A(0)$ with the same generators $x_1^0, \dots, x_{m_0}^0$ and that the sub-cdga B is the free algebra over $B(0)$ on the generators x_j^{-i}, y_j^{k+i}, z^k ***only***. That is, we identify B as a *commutative graded algebra* with the commutative graded algebra in Example 2.21.

It remains to show that H satisfies (10) and the ***differential*** d ***on*** B can be given by Equation (11). To this end, we analyze the defining equations for the pair (H, ϕ). Notice that d will not be fully determined on A, but determined only on B, which is enough for our construction. So, dw_j^{k-1} can be arbitrary.

First of all, combining the defining equation $d_{dR}\phi = k d_{dR}\varphi^*(\alpha)$ above with the equation (28), we may explicitly write[15]

$$\phi = \sum_{i=0}^{\ell} \sum_{j=1}^{m_i} [(-i) x_j^{-i} d_{dR} y_j^{k+i} + (k+i) y_j^{k+i} d_{dR} x_j^{-i}].$$

Then (the proof of) [3, Theorem 5.18] shows that expanding $d_{dR}H + d\phi = 0$ with the explicit representation of ϕ above and comparing the coefficients of d_{dR}-terms, one can get the following formulas for d:

$$d|_{B(0)} = 0; \;\; dx_j^{-i} = \frac{\partial H}{\partial y_j^{k+i}} \text{ for all } i > 0, j; \;\; \text{and} \;\; dy_j^{k+i} = \frac{\partial H}{\partial x_j^{-i}} \text{ for all } i, j. \tag{30}$$

Secondly, using these equations[16] to expand $dH = 0$, [3, Theorem 5.18] also shows that H satisfies the classical master equation (10). Before the final step, we also observe that using the explicit representation of ϕ above, $\varphi^*(\alpha)$ in (29) can also be rewritten as

$$\varphi^*(\alpha) = d_{dR} z^k + \frac{1}{k}\Big[\phi + d_{dR}\big[\sum_{i,j} (-1)^i i x_j^{-i} y_j^{k+i}\big]\Big]. \tag{31}$$

Finally, combining the defining equation $d_{dR}H + d\phi = 0$ with Eqn. (31) (and $d\varphi^*(\alpha) = 0$), we get $d_{dR}H = -d\phi = -kd\varphi^*(\alpha) + kd \circ d_{dR} z^k + d \circ d_{dR}[\cdots] = d_{dR}(-kdz^k - d[\cdots])$. So, we have

[15] See the proof of [3, Theorem 5.18]. Alternatively, one can let $\phi = k \sum_{i=0}^{\ell} \sum_{j=1}^{m_i} y_j^{k+i} d_{dR} x_j^{-i}$. Leaving $d_{dR}\phi$ unchanged, these expressions can be transformed to each other by replacing H, ϕ by suitable $H + d(\cdots)$, $\phi + d_{dR}(\cdots)$, respectively. See Examples 2.21 and 2.24.

[16] These equations will be enough as H is independent of the variables z^k, y_j^k's.

such H satisfying $-kdz^k = H + d[\cdots]$. Thus, we conclude that the differential d is given by Eqn. (11); hence, (B, d) is identified with the cdga in Example 2.21.

Contact data on Spec$B$ ***and*** Spec$A$. We let $\mathbf{V} := \mathrm{Spec}B$, along with the diagram

$$\mathrm{Spec}B = \mathbf{V} \xleftarrow{j:=\mathrm{Spec}(\iota)} \mathbf{U} = \mathrm{Spec}A \xrightarrow{\varphi} \mathbf{X}. \tag{32}$$

For degree reasons, $H \in B$, and ω^0, ϕ are all images under ι of ω^0_B, ϕ_B, respectively, where

$$\omega^0_B = \sum_{i=0}^{\ell}\sum_{j=1}^{m_i} d_{dR}x_j^{-i} d_{dR}y_j^{k+i} \text{ and } \phi_B = \sum_{i=0}^{\ell}\sum_{j=1}^{m_i}[(-i)x_j^{-i}d_{dR}y_j^{k+i} + (k+i)y_j^{k+i}d_{dR}x_j^{-i}]. \tag{33}$$

Note that from Example 2.21, Spec$B$ carries the canonical k-shifted contact structure defined by $\alpha_B = d_{dR}z^k + \sum_{i,j} y_j^{k+i} d_{dR}x_j^{-i}$. Therefore, from Eqn. (29), we conclude that

$$\varphi^*(\alpha) \sim j^*(\alpha_B) \text{ except in degree } (k-1).$$

Lastly, we can consider $j : \mathbf{U} \to \mathbf{V}$ as an embedding of $\mathbf{U}$ into $\mathbf{V}$ as a derived subscheme; and hence, the induced morphism $\tau(j) : \tau(\mathbf{U}) \to \tau(\mathbf{V})$ between truncations is an isomorphism. This completes the proof for odd k.

When the shift is not odd. For the other cases (a) $k \equiv 0 \mod 4$, and (b) $k \equiv 2 \mod 4$, one should use another sets of variables as in Equations (76) and (77), respectively, along with the additional generators w_j^{k-1} in degree $k-1$. We leave details to the reader.

□

3.3 Symplectifications of shifted contact derived Artin stacks In this section, we describe the canonical symplectification of a (negatively) shifted contact derived Artin $\mathbb{K}$-stack. More precisely, we have:

Theorem 3.9. *Let* $\mathbf{X}$ *be a* k*-shifted contact derived Artin* $\mathbb{K}$*-stack, then its* symplectization $(\mathcal{S}_{\mathbf{X}}, \omega)$ *can be canonically described as in Theorem 2.26.*

Proof. First of all, using Theorem 2.26, we define the space $\mathcal{S}_{\mathbf{X}}$ as follows. Let $(\mathbf{X}; \mathcal{K}, \kappa, L)$ be a k-shifted contact derived Artin $\mathbb{K}$-stack of locally finite presentation. Then we define the functor $\mathcal{S}_{\mathbf{X}} : cdga_{\mathbb{K}} \to \mathbb{S}$ by $A \mapsto \mathcal{S}_{\mathbf{X}}(A)$, where

$$\mathcal{S}_{\mathbf{X}}(A) := \{(x, \alpha, v) : x \in \mathbf{X}(A),\ \alpha : \varphi^*(\mathbb{T}_{\mathbf{X}}) \to \mathcal{O}[k],\ v : Cocone(\alpha) \xrightarrow{\sim} \varphi^*(\mathcal{K})\}, \tag{34}$$

where each v is a quasi-isomorphism respecting the natural morphisms $\varphi^*\kappa : \varphi^*\mathcal{K} \to \varphi^*(\mathbb{T}_{\mathbf{X}})$ and $Cocone(\alpha) \to \varphi^*(\mathbb{T}_{\mathbf{X}})$.

By [2, Prop. 4.4], $\mathcal{S}_{\mathbf{X}}$ is equivalent to the total space $\widetilde{\mathbf{L}}$ of the $\mathbb{G}_m$-bundle of L. Therefore, it has the structure of a derived stack together with the projection maps $\pi_1 : \mathcal{S}_{\mathbf{X}} \to \mathbf{X}$ and $\pi_2 : \mathcal{S}_{\mathbf{X}} \to \mathrm{T}^*[k]\mathbf{X}$. We also define the *canonical 1-from* λ on $\mathcal{S}_{\mathbf{X}}$ to be the pullback $\pi_2^*\lambda_{\mathbf{X}}$ of the tautological 1-form $\lambda_{\mathbf{X}}$ on $\mathrm{T}^*[k]\mathbf{X}$.

We set $\omega := (d_{dR}\lambda, 0, 0, \dots)$, which is a k-shifted closed 2-form on $\mathcal{S}_{\mathbf{X}}$, and hence it defines a pre-k-shifted symplectic structure on $\mathcal{S}_{\mathbf{X}}$. Now, it remains to show that ω is non-degenerate. As before, the rest of the argument is in fact local, so it is enough to prove it using suitable local models studied in the previous sections.

Let us now analyze our local data. Given $k < 0$ and $x \in \mathbf{X}$, find an affine derived scheme $\mathbf{U} := \mathrm{Spec}A$ and $p \in \mathrm{Spec}H^0(A)$ with $\varphi : \mathrm{Spec}A \to \mathbf{X}$ a smooth map of relative dimension n such that $p \mapsto x$ (we may further assume A is of minimal standard form due to Theorem 3.5). Here, we assume w.l.o.g. that L is trivial on the interior, $\tilde{\mathbf{U}}$, of $\mathrm{Im}\varphi$ as before. Under the current assumptions, the perfect complexes $\mathbb{T}_A, \mathbb{L}_A$, when restricted to $\mathrm{Spec}H^0(A)$, are both quasi-isomorphic to free complexes of $H^0(A)$-modules. For $A \in cdga_{\mathbb{K}}$, we then define a $\mathbb{G}_m(A)$-action on $\mathcal{S}_{\mathbf{X}}(A)$ as before.

By Theorem 3.7, we can construct a cdga B with inclusion $\iota : B \hookrightarrow A$ and the diagram

$$\mathrm{Spec}B = \mathbf{V} \xleftarrow{j:=\mathrm{Spec}(\iota)} \mathbf{U} = \mathrm{Spec}A \xrightarrow{\varphi} \mathbf{X}$$

such that the induced morphism $\tau(j) : \tau(\mathbf{U}) \to \tau(\mathbf{V})$ between truncations is an isomorphism. In fact, we get the pair $(\mathbf{V}, \alpha_B)$ with an equivalence $\varphi^*(\alpha) \sim j^*(\alpha_B)$ except in degree $k-1$ such that α_B is a k-shifted contact form on $\mathbf{V}$ as in Example 2.21.

Fixing the local data above, we then consider the homotopy pullback diagram

$$\begin{tikzcd} \mathcal{Z} := \mathbf{U} \times^h_{\mathbf{X}} \mathcal{S}_{\mathbf{X}} \arrow[r, "pr_2"] \arrow[d, "pr_1"'] & \mathcal{S}_{\mathbf{X}} \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & \mathrm{T}^*[k]\mathbf{X}. \\ \mathbf{U} \arrow[r, "\varphi"] & \mathbf{X} & \end{tikzcd} \tag{35}$$

Notice that, over $p \in \mathbf{U}$, we can then identify the fiber over p locally as $\mathbf{U} \times^h_{\mathbf{X}} \mathbb{G}_m$, with natural projections, because $\mathcal{S}_{\mathbf{X}}$ is identified the total space of L.

On the part of the space $\mathcal{S}_{\mathbf{X}}$ over $\tilde{\mathbf{U}}$, for the elements $\pi_1^*\alpha, \lambda \in \mathbb{L}_{\mathcal{S}_{\mathbf{X}}}[k]$, the identification of $\mathcal{S}_{\mathbf{X}}$ with the total space of L (i.e. the space of trivializations) implies that there is an element $f \in \mathbb{G}_m(A)$ such that

$$\lambda = f \cdot \pi_1^*(\alpha). \tag{36}$$

Using the homotopy $\varphi \circ pr_1 \sim \pi_1 \circ pr_2$, we get an element $\tilde{\lambda} \in pr_2^*(\mathbb{L}_{\mathcal{S}_{\mathbf{X}}}[k])$ such that

$$\tilde{\lambda} := pr_2^*(\lambda) = pr_2^*(f) \cdot (\pi_1 \circ pr_2)^*(\alpha) \sim pr_2^*(f) \cdot (\varphi \circ pr_1)^*(\alpha) = pr_2^*(f) \cdot pr_1^*(\varphi^*\alpha), \tag{37}$$

where we denote $pr_2^*(f), pr_1^*(\varphi^*\alpha)$ simply by $\tilde{f}, \widetilde{\varphi^*\alpha}$, respectively. Thus, we get a local representative of λ,

$$\tilde{\lambda} = \tilde{f} \cdot \widetilde{\varphi^*\alpha},$$

on a (minimal) standard form open neighborhood (A, φ, p) of x.

Moreover, for the map $\varphi \circ pr_1 : \mathcal{Z} \to \mathbf{X}$ we have an exact triangle

$$(\varphi \circ pr_1)^*(\mathbb{L}_{\mathbf{X}}[k]) \to pr_1^*(\mathbb{L}_{\mathbf{U}}[k]) \oplus pr_2^*(\mathbb{L}_{\mathcal{S}_{\mathbf{X}}}[k]) \to \mathbb{L}_{\mathcal{Z}}[k]. \tag{38}$$

Now, to prove $\omega := (d_{dR}\lambda, 0, 0, \dots)$ is non-degenerate, it suffices to show that $d_{dR}\tilde{\lambda}$ is non-degenerate.

Lemma 3.10. *$d_{dR}\tilde{\lambda}$ is non-degenerate.*

Proof. Recall first that there exists a natural equivalence $DR(A) \otimes_{\mathbb{K}} DR(C) \simeq DR(A \otimes_{\mathbb{K}} C)$ induced by the identification

$$\mathbb{L}_{A \otimes_{\mathbb{K}} C} \simeq (\mathbb{L}_A \otimes_{\mathbb{K}} C) \oplus (A \otimes_{\mathbb{K}} \mathbb{L}_C). \tag{39}$$

In our case, since the fiber over p is locally given as $\mathbf{U}\times^h_{\mathbf{X}}\mathbb{G}_m$, where $\mathbf{U} = \mathrm{Spec}A$ and $\mathbb{G}_m = \mathrm{Spec}C$ is the affine group scheme, with say $C := \mathbb{K}[x, x^{-1}]$, we can use the identification (39) to locally decompose $\tilde{\lambda} = \tilde{f} \cdot \widetilde{\varphi^*\alpha}$.

We then have, when restricted to $\mathrm{Spec}H^0(A)$,

$$(\mathbb{L}_{A\otimes_{\mathbb{K}}C})^\vee \simeq \big(\ker(\varphi^*\alpha)^{17} \oplus \mathit{Rest}\big) \oplus \big(H^0(A) \otimes_{\mathbb{K}} \langle \partial/\partial\tilde{f}\rangle_C\big). \tag{40}$$

Now, to prove that $d_{dR}\tilde{\lambda}$ is non-degenerate, it suffices to show that, at $p \in \mathbf{U}$, for any non-vanishing (homogeneous) vector field $\sigma \in (\mathbb{L}_{A\otimes_{\mathbb{K}}C})^\vee$, there is a vector field $\eta \in (\mathbb{L}_{A\otimes_{\mathbb{K}}C})^\vee$ such that $\iota_\eta(\iota_\sigma d_{dR}\tilde{\lambda}) \neq 0$.[18] To this end, we first compute

$$\iota_\eta(\iota_\sigma d_{dR}\tilde{\lambda}) = \mp(d_{dR}\tilde{f})(\sigma)\varphi^*\alpha(pr_{1,*}\eta) \mp (d_{dR}\tilde{f})(\eta)\varphi^*\alpha(pr_{1,*}\sigma) \mp \tilde{f} \cdot d_{dR}(\varphi^*\alpha)(pr_{1,*}\sigma, pr_{1,*}\eta).$$

From Equation (40), it is enough to consider the following cases:

1. If $\sigma \in \ker(\varphi^*\alpha)$, then $\iota_\eta(\iota_\sigma d_{dR}\tilde{\lambda}) = \mp\tilde{f}d_{dR}(\varphi^*\alpha)(pr_{1,*}\sigma, pr_{1,*}\eta)$. Since $d_{dR}(\varphi^*\alpha)|_{\ker(\varphi^*\alpha)}$ is non-degenerate by the contactness condition on $\varphi^*(\alpha) \sim j^*(\alpha_B)$ except in degree $k-1$ (and $\tilde{f} \neq 0$ as $f \in \mathbb{G}_m(A)$), it is enough to take η to be any non-zero vector in $\ker(\varphi^*\alpha)$.
2. If $\sigma \in \mathit{Rest}$, then we get $\iota_\eta(\iota_\sigma d_{dR}\tilde{\lambda}) = \mp(d_{dR}\tilde{f})(\eta)\varphi^*\alpha(\sigma)$. Observe that $\varphi^*\alpha(\sigma) \neq 0$ since $\sigma \in \mathit{Rest}$. Thus, it is enough to take η to be any non-zero vector in $H^0(A) \otimes_{\mathbb{K}} \langle\partial/\partial\tilde{f}\rangle_C$ so that $(d_{dR}\tilde{f})(\eta) \neq 0$.
3. If $\sigma \in H^0(A) \otimes_{\mathbb{K}} \langle\partial/\partial\tilde{f}\rangle_C$, then $\iota_\eta(\iota_\sigma d_{dR}\tilde{\lambda}) = \mp(d_{dR}\tilde{f})(\sigma)\varphi^*\alpha(pr_{1,*}\eta)$. Note that we have $(d_{dR}\tilde{f})(\sigma) \neq 0$, so it suffices to take η to be any non-zero vector in Rest so that $\varphi^*\alpha(pr_{1,*}\eta) \neq 0$.

□

In total, from Lemma 3.10, the k-shifted 2-form $\omega^0 := d_{dR}\lambda$ is non-degenerate (except in degree $k-1$), and hence the sequence $\omega := (\omega^0, 0, 0, \dots)$ defines a k-shifted symplectic structure on $\mathcal{S}_{\mathbf{X}}$. This completes the proof of Theorem 3.9.

□

4. Examples

In this section we present several constructions of derived Artin stacks with shifted contact structure. In brief, the first example in Section 4.1 generalizes the 1-jet bundles in the classical setup; and the second set of constructions in Section 4.2 arises from the notion of shifted geometric (pre)quantization introduced by Safronov [10].

4.1 Shifted 1-jet stacks Denote by $\mathbb{A}^1$ the *affine line*[19] as the derived stack corepresented by $\mathbb{K}[z]$. That is, $\mathbb{A}^1 = \mathrm{Spec}(\mathbb{K}[z])$ is the derived stack

$$\mathrm{Spec}(\mathbb{K}[z]) : B \in cdga_{\mathbb{K}} \mapsto Hom(\mathbb{K}[z], B). \tag{41}$$

[17] Recall that on refined local charts, we simply write $\ker(\varphi^*\alpha)$ instead of $Cocone(\varphi^*\alpha)$.

[18] This argument follows from the following observations: When restricted to $\mathrm{Spec}H^0(A)$, the induced morphism $\mathbb{T}_A \to \Omega^1_A[k]$ is just a map of finite complexes of free $H^0(A)$-modules. And, at $p \in \mathrm{Spec}H^0(A)$, both $\mathbb{T}_A|_p, \Omega^1_A|_p$ are complexes of $\mathbb{K}$-vector spaces. For non-degeneracy, we require this map to be a (degree-wise) quasi-isomorphism. Recall that localizing A at p if necessary, we may assume that the induces map is indeed an (degree-wise) isomorphism near p. Therefore, the fact we use here is just an analogous result from linear algebra.

[19] We may also call it the *affine addtive group scheme* $\mathbb{G}_a$.

Let $\mathrm{T}^*[n]\mathbf{X}$ be the n-shifted cotangent stack of the derived Artin stack $\mathbf{X}$ (locally of finite presentation). Consider the derived stack $\mathrm{T}^*[n]\mathbf{X} \times \mathbb{A}^1[n]$ given by the pullback diagram

$$\begin{tikzcd} \mathrm{J}^1[n]\mathbf{X} := \mathrm{T}^*[n]\mathbf{X} \times \mathbb{A}^1[n] \arrow[r, "pr_2"] \arrow[d, "pr_1"'] & \mathbb{A}^1[n] \arrow[d] \\ \mathrm{T}^*[n]\mathbf{X} \arrow[r] & *, \end{tikzcd} \tag{42}$$

where $\mathbb{A}^1[n]$ is the *n-shifted affine line* corepresented by the polynomial algebra on a variable in cohomological degree $-n$. Denote this resulting space by $\mathrm{J}^1[n]\mathbf{X}$ and call it ***n-shifted 1-jet stack of X,*** with the natural projection map as the composition $\mathrm{J}^1[n]\mathbf{X} \to \mathrm{T}^*[n]\mathbf{X} \to \mathbf{X}$.

Recall from [5] that there is an n-shifted 1-form λ, called the *Liouville one-form*, on the cotangent stack $\mathrm{T}^*[n]\mathbf{X}$ such that the closed 2-form $\omega := d_{dR}\lambda$ is non-degenerate. Hence, $\mathrm{T}^*[n]\mathbf{X}$ is in fact n-symplectic. Moreover, from Diagram 42, we have the identifications

$$\mathbb{L}_{\mathrm{J}^1[n]\mathbf{X}} \simeq pr_1^*\mathbb{L}_{\mathrm{T}^*[n]\mathbf{X}} \oplus pr_2^*\mathbb{L}_{\mathbb{A}^1[n]} \text{ and } \mathbb{T}_{\mathrm{J}^1[n]\mathbf{X}} \simeq pr_1^*\mathbb{T}_{\mathrm{T}^*[n]\mathbf{X}} \oplus pr_2^*\mathbb{T}_{\mathbb{A}^1[n]}. \tag{43}$$

Equivalently, we have the exact triangle

$$pr_1^*\mathbb{T}_{\mathrm{T}^*[n]\mathbf{X}} \to \mathbb{T}_{\mathrm{J}^1[n]\mathbf{X}} \to pr_2^*\mathbb{T}_{\mathbb{A}^1[n]}. \tag{44}$$

Notice that from the triangle (44), we have a natural morphism $pr_1^*\mathbb{T}_{\mathrm{T}^*[n]\mathbf{X}} \to \mathbb{T}_{\mathrm{J}^1[n]\mathbf{X}}$ with the cofiber $pr_2^*\mathbb{T}_{\mathbb{A}^1[n]}$. Since $\mathbb{T}_{\mathbb{A}^1[n]}$ is an invertible quasi-coherent sheaf, we get $pr_2^*\mathbb{T}_{\mathbb{A}^1[n]} \simeq L[n]$, where L is a line bundle. Thus, we obtain a natural ***pre-n-shifted contact data*** by using the perfect complex $\mathcal{K} := pr_1^*\mathbb{T}_{\mathrm{T}^*[n]\mathbf{X}}$ and the natural map $\kappa : pr_1^*\mathbb{T}_{\mathrm{T}^*[n]\mathbf{X}} \to \mathbb{T}_{\mathrm{J}^1[n]\mathbf{X}}$ with the cofiber $pr_2^*\mathbb{T}_{\mathbb{A}^1[n]}$ as above.

Now, it remains to extend the pre-contact data to an n-shifted contact structure. In brief, we need to represent $\mathcal{K}$ (locally) as the cocone of a shifted 1-form satisfying the non-degeneracy condition. To this end, we construct the following n-shifted 1-form on $\mathrm{J}^1[n]\mathbf{X}$.

We define (globally) an n-shifted 1-form on $\mathrm{J}^1[n]\mathbf{X}$ by[20]

$$\alpha := -d_{dR}z + \lambda \in \Gamma\big(\mathrm{J}^1[n]\mathbf{X}, \mathbb{L}_{\mathrm{J}^1[n]\mathbf{X}}[n]\big), \tag{45}$$

where we simply write z, λ instead of $pr_2^*z, pr_1^*\lambda$, respectively. Then we claim:

Lemma 4.1. *Let α, pr_1, pr_2 be as above. Then we have equivalences (at least locally)*

$$\mathbb{T}_{pr_1} \simeq pr_2^*\mathbb{T}_{\mathbb{A}^1[n]}, \tag{46}$$

$$Cocone(\alpha) \simeq pr_1^*\mathbb{T}_{\mathrm{T}^*[n]\mathbf{X}} =: \mathcal{K}, \tag{47}$$

where $\mathbb{T}_{pr_1}$ is the relative tangent complex[21] *defined by the sequence $\mathbb{T}_{pr_1} \to \mathbb{T}_{\mathrm{J}^1[n]\mathbf{X}} \to pr_1^*\mathbb{T}_{\mathrm{T}^*[n]\mathbf{X}}$. We then have the (local) splitting*

$$\mathbb{T}_{\mathrm{J}^1[n]\mathbf{X}} \simeq Cocone(\alpha) \oplus \mathbb{T}_{pr_1}.$$

[20] Thanks to the identifications (43).

[21] Its elements are called *vertical tangent vectors*.

Proof. Combining the shift of the natural fiber sequence $\mathbb{T}_{pr_1} \to \mathbb{T}_{\mathrm{J}^1[n]\mathbf{X}} \to pr_1^*\mathbb{T}_{\mathrm{T}^*[n]\mathbf{X}}$ with the exact triangle (44), we get an equivalence of triangles

$$\begin{tikzcd} pr_1^*\mathbb{T}_{\mathrm{T}^*[n]\mathbf{X}} \arrow[r] \arrow[d, "\simeq"] & \mathbb{T}_{\mathrm{J}^1[n]\mathbf{X}} \arrow[r] \arrow[d, "id"] & pr_2^*\mathbb{T}_{\mathbb{A}^1[n]} \arrow[d, dashed, "\simeq"] \\ pr_1^*\mathbb{T}_{\mathrm{T}^*[n]\mathbf{X}}[-1] \arrow[r] & \mathbb{T}_{\mathrm{J}^1[n]\mathbf{X}} \arrow[r] & \mathbb{T}_{pr_1}[1], \end{tikzcd} \tag{48}$$

which gives the identification $pr_2^*\mathbb{T}_{\mathbb{A}^1[n]} \xrightarrow{\sim} \mathbb{T}_{pr_1}$.

By the definition of α, for any vertical vector $v \in \mathbb{T}_{pr_1}$, the contraction $\iota_v\alpha$ is never nullhomotopic, and hence we write

$$Cocone(\alpha) \cap \mathbb{T}_{pr_1} = \{0\},$$

by which we mean the pullback of the diagram $Cocone(\alpha) \hookrightarrow \mathbb{T}_{\mathrm{J}^1[n]\mathbf{X}} \leftarrow \mathbb{T}_{pr_1}$ is trivial in $\mathrm{Perf}(\mathrm{J}^1[n]\mathbf{X})$. Since both $Cocone(\alpha), \mathbb{T}_{pr_1}$ are perfect, we then have the (local) splitting[22]

$$\mathbb{T}_{\mathrm{J}^1[n]\mathbf{X}} \simeq Cocone(\alpha) \oplus \mathbb{T}_{pr_1}. \tag{49}$$

The splitting then gives an exact triangle

$$Cocone(\alpha) \to \mathbb{T}_{\mathrm{J}^1[n]\mathbf{X}} \to \mathbb{T}_{pr_1}, \tag{50}$$

which induces an equivalence of triangles

$$\begin{tikzcd} pr_1^*\mathbb{T}_{\mathrm{T}^*[n]\mathbf{X}} \arrow[r] \arrow[d, dashed, "\simeq"] & \mathbb{T}_{\mathrm{J}^1[n]\mathbf{X}} \arrow[r] \arrow[d, "id"] & pr_2^*\mathbb{T}_{\mathbb{A}^1[n]} \arrow[d, "\simeq"] \\ Cocone(\alpha) \arrow[r] & \mathbb{T}_{\mathrm{J}^1[n]\mathbf{X}} \arrow[r] & \mathbb{T}_{pr_1}. \end{tikzcd} \tag{51}$$

Thus, we get the identification[23] $Cocone(\alpha) \xrightarrow{\sim} pr_1^*\mathbb{T}_{\mathrm{T}^*[n]\mathbf{X}}$ and complete the proof. □

Finally, using Lemma 4.1, we can satisfy the desired non-degeneracy condition and prove:

Theorem 4.2. *Let X be a locally finitely presented derived Artin stack. Then the n-shifted 1-jet stack $\mathrm{J}^1[n]\mathbf{X}$ has a natural n-shifted contact structure.*

Proof. Consider the *pre-n-shifted contact data* given by the perfect complex $\mathcal{K} := pr_1^*\mathbb{T}_{\mathrm{T}^*[n]\mathbf{X}}$ and the natural map $\kappa : pr_1^*\mathbb{T}_{\mathrm{T}^*[n]\mathbf{X}} \to \mathbb{T}_{\mathrm{J}^1[n]\mathbf{X}}$ with the cofiber $pr_2^*\mathbb{T}_{\mathbb{A}^1[n]}$ as above, where we have an equivalence $pr_2^*\mathbb{T}_{\mathbb{A}^1[n]} \simeq L[n]$, with L a line bundle. It remains to check the (local) contact non-degeneracy condition.

Let α, λ be as above. By Lemma 4.1, $Cocone(\alpha) \simeq pr_1^*\mathbb{T}_{\mathrm{T}^*[n]\mathbf{X}} =: \mathcal{K}$ locally. Note that $d_{dR}\lambda$ is already n-symplectic on $\mathrm{T}^*[n]\mathbf{X}$, and hence non-degenerate on $\mathbb{T}_{\mathrm{T}^*[n]\mathbf{X}}$. Due to the identification by Lemma 4.1, $d_{dR}\alpha = d_{dR}\lambda$ is then non-degenerate on $\mathcal{K} \simeq Cocone(\alpha)$. Therefore, the data $(\mathcal{K}, \kappa, L, \alpha)$ defines an n-shifted contact structure on $\mathrm{J}^1[n]\mathbf{X}$.

□

[22] Over a (minimal) standard form cdga A with an A-point $p : \mathrm{Spec}A \to \mathrm{J}^1[n]\mathbf{X}$, the complex $p^*\mathbb{T}_{\mathrm{J}^1[n]\mathbf{X}}$ is a finite complex of free $H^0(A)$-modules, so are the both complexes $Cocone(p^*\alpha), p^*\mathbb{T}_{pr_1}$. Since $Cocone(p^*\alpha) \cap p^*\mathbb{T}_{pr_1} = \{0\}$ in each degree, we get the splitting of free modules in each degree, and hence the desired (local) identification of complexes.

[23] Using the previous footnote, we actually have a local identification $Cocone(p^*\alpha) \xrightarrow{\sim} p^*(pr_1^*\mathbb{T}_{\mathrm{T}^*[n]\mathbf{X}}) = p^*\mathcal{K}$.

4.2 Contact structures via shifted prequantization

4.2.1 Background on geometric quantization

Classical geometric quantization. Let us briefly recall the notion of geometric quantization in the context of differential geometry. Given a smooth symplectic manifold (X, ω) (or a scheme over a field $\mathbb{K}$ of characteristic 0), *geometric quantization* is a 2-step procedure combining *prequantization* and *polarization*. More precisely, we have:

Definition 4.3. By a *prequantization* of (X, ω), we mean a particular line bundle with connection (L, ∇) on X such that $\mathrm{curv}(L, \nabla) = \omega$. By a *polarization*, we mean the choice of a subbundle P of TX closed under the Lie bracket which is Lagrangian with respect to ω.

Note that on (X, ω), we can define the *Hamiltonian vector field* X_f *associated to* the function f by $\imath_{X_f}\omega = df$, and hence the *Poisson bracket* $\{f, g\}_\omega := -w(X_f, X_g) = X_f(g)$ on $C^\infty(X)$.

When we have a geometric (pre)quantization[24] of (X, ω), then it allows us to define a suitable quantum Hilbert space $\mathcal{H} := \Gamma_P(X, L)$ as the *space of P-polarizaed sections of L* such that we can construct a Lie algebra representation of (a certain subalgebra A of) $(C^\infty(X), \{-,-\}_\omega)$ on $End(\mathcal{H}, [-,-])$, which acts on the functions as

$$f \mapsto \hat{f} := -i\hbar\nabla_{X_f} - f.$$

Example 4.4. Every Kähler manifold (M, ω, J) gives rise to *a holomorphic Kähler polarization* associated to (M, ω) by setting $P := T^{(0,1)}(M)$, the $(-i)$-eigenspace subbundle of the complexified tangent bundle $TM \otimes \mathbb{C}$.

Remark 4.5. *(Relation with contact geometry)* The construction above can be given in terms of the principal $U(1)$-bundle $L^\times$ associated with L and the connection 1-form α on X corresponding to ∇. We then have $\mathrm{curv}(L^\times, \alpha) = d_{dR}\alpha = \pi^*\omega$, with $\pi : L^\times \to X$. In that case, α defines a contact structure on $L^\times$. For more details, see [14].

Shifted geometric quantization. Safronov introduces in [10] the notion of shifted geometric quantization in the context of derived symplectic geometry. We now outline the main definitions and some key results of interest. We closely follow [10].

Denote by $\mathcal{A}^p(n), \mathcal{A}^{p,cl}(n)$ the *derived stacks of p-forms of degree n* and *closed p-forms of degree n* as introduced in PTVV's work [9]. Then by construction, we have equivalences

$$\mathcal{A}^1(n) \simeq \Omega\mathcal{A}^1(n+1) \text{ and } \mathcal{A}^{1,cl}(n) \simeq \Omega\mathcal{A}^{1,cl}(n+1). \tag{52}$$

By adjunction we can get a map $B\mathcal{A}^1(n) \to \mathcal{A}^1(n+1)$, which is an equivalence [10, Lemma 2.1].

Observation 4.6. Iterating these equivalences, we also get an equivalence

$$B^n\mathcal{A}^1(0) \to \mathcal{A}^1(n).$$

However, it should be noted that $B^n\mathcal{A}^{1,cl}(0) \to \mathcal{A}^{1,cl}(n)$ is not an equivalence even if the inclusion $\mathcal{A}^{n+1,cl}(0) \to \mathcal{A}^{1,cl}(n)$ is an equivalence. For details, we refer to [10, §2.1].

[24] A sufficient condition for the existence is $[\omega] \in H^2(X, \mathbb{Z})$

Note that that we have a morphism of stacks

$$d_{dR}\log : \mathbb{G}_m \to \mathcal{A}^{1,cl}(0) \tag{53}$$

which, on a cdga R, maps an invertible element $f \in R^\times$ to $\dfrac{d_{dR}f}{f} \in \mathcal{A}^{1,cl}(\mathrm{Spec}R, 0)$. Then by delooping (cf. Observation 4.6), we get the map

$$c_1 : B^n\mathbb{G}_m \to B^n\mathcal{A}^{1,cl}(0) \to \mathcal{A}^{1,cl}(n). \tag{54}$$

Now, pre-composing c_1 with the natural projection $\mathcal{A}^{1,cl}(n) \to \mathcal{A}^1(n)$, we get a morphism

$$B^n\mathbb{G}_m \to \mathcal{A}^1(n),$$

which will be denoted again by c_1 and lead to the following definition.

Definition 4.7. Let $n \geq 0$ and $\mathbf{X}$ a derived stack.

1. An ***n-gerbe*** on $\mathbf{X}$ is a map $\mathcal{G} : \mathbf{X} \to B^{n+1}\mathbb{G}_m$. We then call its image $c_1(\mathcal{G}) \in \mathcal{A}^1(n+1)$ under the map c_1 above the *characteristic class of* $\mathcal{G}$.
2. An ***n-gerbe with a connective structure*** on $\mathbf{X}$ is a pair $(\mathcal{G}, \nabla)$, where $\mathcal{G}$ is an n-gerbe on $\mathbf{X}$ and ∇ is a nullhomotopy of the characteristic class $c_1(\mathcal{G}) \in \mathcal{A}^1(n+1)$.
3. An ***n-gerbe with a flat connective structure*** on $\mathbf{X}$ is a pair $(\mathcal{G}, \nabla)$, where $\mathcal{G}$ is an n-gerbe on $\mathbf{X}$ and ∇ is a nullhomotopy of $c_1(\mathcal{G}) \in \mathcal{A}^{1,cl}(n+1)$.
4. Any gerbe $(\mathcal{G}, \nabla)$ with connective structure on $\mathbf{X}$ has a ***curvature*** $\mathrm{curv}(\mathcal{G}, \nabla) \in \mathcal{A}^{2,cl}(n)$ defined by the pullback diagram

$$\begin{tikzcd} & \mathcal{A}^{2,cl}(n) \arrow[r] \arrow[d] & 0 \arrow[d] \\ B^{n+1}\mathbb{G}_m \arrow[r, "c_1"] & \mathcal{A}^{1,cl}(n+1) \arrow[r] & \mathcal{A}^1(n+1). \end{tikzcd} \tag{55}$$

Example 4.8. Let $(\mathcal{G}, \nabla)$ be an n-gerbe with connective structure on $\mathbf{X}$. When $n = 0$, the map $\mathcal{G} : \mathbf{X} \to B\mathbb{G}_m$ corresponds to a line bundle as $B\mathbb{G}_m$ is the classifying space of the line bundles. Then the map $c_1 : B\mathbb{G}_m \to \mathcal{A}^{1,cl}(1) \simeq \mathcal{A}^{2,cl}(0)$ corresponds the first Chern class of the line bundle. Thus, the connective structure $(\mathcal{G}, \nabla)$ corresponds to the usual notion of a connection on the line bundle $\mathcal{G}$ and its curvature corresponds to the usual notion of the curvature. For more examples and details, see [10].

Now, we define the notion of prequantization for shifted symplectic derived stacks.

Definition 4.9. Let $(\mathbf{X}, \omega)$ be an n-shifted symplectic derived stack. Its ***prequantization*** is an n-gerbe with connective structure $(\mathcal{G}, \nabla)$ such that $\mathrm{curv}(\mathcal{G}, \nabla) \simeq \omega$ in $\mathcal{A}^{2,cl}(n)$.

We can also define a relative version of the prequantization notion for the maps $\pi : \mathbf{X} \to \mathbf{B}$ of derived Artin stacks carrying a shifted Lagrangian fibration structure.

Definition 4.10. Let $\pi : \mathbf{X} \to \mathbf{B}$ be a morphism of derived Artin stacks locally of finite presentation. A ***prequantum n-shifted Lagrangian fibration*** consists of the following data:

1. An n-gerbe $\mathcal{G}$ on $\mathbf{B}$.

2. An extension of the natural relative flat connection on $\pi^*\mathcal{G}$ to a connective structure ∇, such that the pair $(\pi^*\mathcal{G}, \nabla)$ defines an n-shifted Lagrangian fibration structure[25] on π with an induced n-shifted symplectic structure on $\mathbf{X}$ given by $\mathrm{curv}(\pi^*\mathcal{G}, \nabla)$.

It is also possible to define the prequantization notion for the morphisms $\pi : \mathbf{X} \to \mathbf{B}$ of derived Artin stacks carrying a shifted Lagrangian structure. For details, see [10, §2.3].

In the upcoming sections, we discuss several examples of prequantizations from [10] and provide sample constructions of the induced shifted contact structures.

4.2.2 Prequantization of the cotangent stack Let $\mathbf{X}$ be a derived Artin stack locally of finite presentation and $\pi_X : \mathrm{T}^*\mathbf{X} \to \mathbf{X}$ the natural projection. [5] shows that the cotangent stack $\mathrm{T}^*\mathbf{X}$ has a natural Liouville 1-form $\lambda \in \mathcal{A}^1(\mathrm{T}^*\mathbf{X}, 0)$ such that 0-shifted symplectic structure on $\mathrm{T}^*\mathbf{X}$ is given by $\omega := d_{dR}\lambda$ and that the map $\pi_{\mathbf{X}}$ carries a natural structure of an 0-shifted Lagrangian fibration. That is, for any $x \in \mathbf{X}$, the inclusion of the fiber $\mathbf{X}_x \to \mathrm{T}^*\mathbf{X}$ has a 0-shifted Lagrangian structure.

Now, we consider another interesting structure that $\pi_{\mathbf{X}}$ carries: It is shown in [10, Prop. 2.21] that $\pi_{\mathbf{X}}$ has a natural structure of a prequantum 0-shifted Lagrangian fibration, determined by the trivial 0-gerbe $\mathcal{G}$ on $\mathbf{X}$ together with a connective structure on $\pi^*_{\mathbf{X}}\mathcal{G}$ given by the Liouville form λ.

By definition, the trivial 0-gerbe is the map $\mathcal{G} : \mathbf{X} \to B\mathbb{G}_m$ which in fact corresponds to a trivial line bundle, say L, on $\mathbf{X}$. Moreover, in that case, the connective structure $\pi^*_{\mathbf{X}}\mathcal{G}$ corresponds to the usual connection ∇ on the trivial line bundle $\pi^*_{\mathbf{X}}L$ over $\mathrm{T}^*\mathbf{X}$, with $\pi : \pi^*_{\mathbf{X}}L \to \mathrm{T}^*\mathbf{X}$, such that the connection 1-form is given by $\nabla := \lambda \in \mathcal{A}^1(\mathrm{T}^*\mathbf{X}, 0)$.

Denote the trivial $\mathbb{G}_m$-bundle associated with $\pi^*_{\mathbf{X}}L$ by $\mathcal{L}^\times$. Here, the frame bundle $\mathcal{L}^\times$ has the trivialization as the restriction of the trivialization of the original line bundle $\pi : \pi^*_{\mathbf{X}}L \to \mathrm{T}^*\mathbf{X}$, respecting the $\mathbb{G}_m$-action. Likewise, the projection $\pi : \mathcal{L}^\times \to \mathrm{T}^*\mathbf{X}$ is the restriction of the original projection $\pi : \pi^*_{\mathbf{X}}L \to \mathrm{T}^*\mathbf{X}$.

Since $\mathcal{L}^\times$ is a trivial $\mathbb{G}_m$-bundle, we can view it as the derived stack given by the pullback

$$
\begin{tikzcd}
\mathcal{L}^\times = \mathrm{T}^*\mathbf{X} \times \mathbb{G}_m \arrow[r, "pr_2"] \arrow[d, "\pi =: pr_1"'] & \mathbb{G}_m \arrow[d] \\
\mathrm{T}^*\mathbf{X} \arrow[r] & *,
\end{tikzcd}
\tag{56}
$$

where $\mathbb{G}_m$ is the affine group scheme as the derived stack corepresented by $\mathbb{K}[t, t^{-1}]$:

$$
\mathbb{G}_m = \mathrm{Spec}\mathbb{K}[t, t^{-1}] : R \in cdga_{\mathbb{K}} \mapsto Hom(\mathbb{K}[t, t^{-1}], R), \tag{57}
$$

which also means $\mathbb{G}_m(R) = R^\times$. Moreover, from Diagram 56, we have the identifications

$$
\mathbb{L}_{\mathcal{L}^\times} \simeq pr_1^*\mathbb{L}_{\mathrm{T}^*\mathbf{X}} \oplus pr_2^*\mathbb{L}_{\mathbb{G}_m} \text{ and } \mathbb{T}_{\mathcal{L}^\times} \simeq pr_1^*\mathbb{T}_{\mathrm{T}^*\mathbf{X}} \oplus pr_2^*\mathbb{T}_{\mathbb{G}_m}. \tag{58}
$$

Recall also from the previous section that we have a morphism of stacks

$$
d_{dR}\log : \mathbb{G}_m \to \mathcal{A}^{1,cl}(0) \tag{59}
$$

[25] For the precise definition, see [5]. For our purposes, it is useful to know some key results: Once we have such structure, then (1) the source $\mathbf{X}$ is an n-symplectic stack; (2) For any $b \in \mathbf{B}$, the inclusion of the fiber $\mathbf{X}_b \hookrightarrow \mathbf{X}$ has an n-shifted Lagrangian structure.

which, on a cdga R, maps an invertible element $f \in R^\times$ to $(d_{dR}f)/f \in \mathcal{A}^{1,cl}(\mathrm{Spec}R, 0)$.

For $R = \mathbb{K}[t, t^{-1}] = \mathcal{O}(\mathbb{G}_m)$ and $f := t$, we get a global section of $\mathcal{L}^\times$ as $t \mapsto (d_{dR}t)/t$, and hence a *global 0-shifted 1-form* α on $\mathcal{L}^\times$ defined by

$$\alpha = \pi^*\lambda + pr_2^* d_{dR} \log(t). \tag{60}$$

By abuse of notation, we may omit pr_1^*, π, pr_2^* whenever the meaning is clear from the context.

Observation 4.11. For the morphism $\pi : \mathcal{L}^\times \to \mathrm{T}^*\mathbf{X}$, we have the fiber sequence

$$\mathbb{T}_\pi \to \mathbb{T}_{\mathcal{L}^\times} \to \pi^*\mathbb{T}_{\mathrm{T}^*\mathbf{X}},$$

where $\mathbb{T}_\pi$ denotes the *relative tangent complex*. An element of the relative tangent space is called a *relative or vertical tangent vector*. That is, $\mathbb{T}_\pi$ is the *space of tangent vectors along the fibers* of $\mathcal{L}^\times$. Call this space the **vertical bundle of** $\mathbb{T}_{\mathcal{L}^\times}$ and denote it by $Ver_{\mathcal{L}^\times}$. Then we have:

Lemma 4.12. *Let $\alpha, Ver_{\mathcal{L}^\times}$ be as above. Then we have the following equivalences:*

$$Ver_{\mathcal{L}^\times} \simeq \mathrm{cofib}(Cocone(\alpha) \hookrightarrow \mathbb{T}_{\mathcal{L}^\times}) \simeq pr_2^*\mathbb{T}_{\mathbb{G}_m}, \tag{61}$$

$$Cocone(\alpha) \simeq \pi^*\mathbb{T}_{\mathrm{T}^*\mathbf{X}}, \tag{62}$$

$$\mathbb{T}_{\mathcal{L}^\times} \simeq Cocone(\alpha) \oplus Ver_{\mathcal{L}^\times}. \tag{63}$$

We then call $Cocone(\alpha)$ the **horizontal bundle of** $\mathbb{T}_{\mathcal{L}^\times}$*, denoted by $Hor_{\mathcal{L}^\times}$, and write the splitting in terms of the horizontal and vertical bundles as*

$$\mathbb{T}_{\mathcal{L}^\times} \simeq Hor_{\mathcal{L}^\times} \oplus Ver_{\mathcal{L}^\times}. \tag{64}$$

Proof. Denote the natural morphism $Cocone(\alpha) \hookrightarrow \mathbb{T}_{\mathcal{L}^\times}$ by i. Then by definition, we have $Cone(i) = \mathrm{cofib}(Cocone(\alpha) \hookrightarrow \mathbb{T}_{\mathcal{L}^\times})$, the weak quotient of $\mathbb{T}_{\mathcal{L}^\times}$ by the image of $Cocone(\alpha)$.

Notice that for $v \in Ver_{\mathcal{L}^\times}$, we have $\pi_*(v) \sim 0$, and hence

$$\alpha(v) = \pi^*\lambda(v) + \frac{1}{t} d_{dR}t(v) = \lambda(\pi_* v) + \frac{1}{t}v(t) \sim \frac{1}{t}v(t).$$

Over any R-point p of $\mathcal{L}^\times$, $p : \mathrm{Spec}R \to \mathcal{L}^\times$, p^*v is a derivation on $R^\times$, and hence it maps $t_R := p^*t$ to an invertible element $p^*v(t_R)$ of R. It follows that the restriction of α to the vertical bundle $Ver_{\mathcal{L}^\times}$ will take non-zero values only. I.e., for any $v \in Ver_{\mathcal{L}^\times}$, the image $\alpha(v)$ is homotopic to a non-zero element. Therefore, we can write

$$Cocone(\alpha) \cap Ver_{\mathcal{L}^\times} = \{0\},$$

by which we mean the pullback of the diagram $Cocone(\alpha) \hookrightarrow \mathbb{T}_{\mathcal{L}^\times} \leftarrow \mathbb{T}_\pi$ is trivial in $\mathrm{Perf}(\mathcal{L}^\times)$. Since both $Cocone(\alpha), \mathbb{T}_\pi$ are perfect, we then have the (local) splitting[26]

$$\mathbb{T}_{\mathcal{L}^\times} \simeq Cocone(\alpha) \oplus \mathbb{T}_\pi. \tag{65}$$

By the splitting (65), we obtain an equivalence of exact triangles

$$\begin{tikzcd}
Cocone(\alpha) \arrow[r, "i"] \arrow[d, "id"] & \mathbb{T}_{\mathcal{L}^\times} \arrow[r] \arrow[d, "id"] & \mathbb{T}_\pi \arrow[d, dashed, "\simeq"] \\
Cocone(\alpha) \arrow[r] & \mathbb{T}_{\mathcal{L}^\times} \arrow[r] & Cone(i).
\end{tikzcd} \tag{66}$$

[26]Over an A-point p with A a minimal standard form cdga, complexes on each side are in fact finite complexes of free $H^0(A)$-modules, and we have the splitting in each degree.

Using the notation $Ver_{\mathcal{L}^\times} := \mathbb{T}_\pi$ and the rightmost vertical map on the diagram above, we conclude

$$Ver_{\mathcal{L}^\times} \simeq Cone(i) := \text{cofib}(Cocone(\alpha) \hookrightarrow \mathbb{T}_{\mathcal{L}^\times}), \tag{67}$$

which proves the first equivalence in (61).

Recall that since $\mathcal{L}^\times$ is a trivial $\mathbb{G}_m$-bundle, from the identifications (58), there is an exact triangle $\pi^*\mathbb{T}_{\mathrm{T}^*\mathbf{X}} \to \mathbb{T}_{\mathcal{L}^\times} \to pr_2^*\mathbb{T}_{\mathbb{G}_m}$. Now, using the natural triangle $\mathbb{T}_{\mathcal{L}^\times} \to \pi^*\mathbb{T}_{\mathrm{T}^*\mathbf{X}} \to \mathbb{T}_\pi[1]$, we get an equivalence of triangles

$$\begin{tikzcd}
\pi^*\mathbb{T}_{\mathrm{T}^*\mathbf{X}} \arrow[r] \arrow[d, "\simeq"] & \mathbb{T}_{\mathcal{L}^\times} \arrow[r] \arrow[d, "id"] & pr_2^*\mathbb{T}_{\mathbb{G}_m} \arrow[d, dashed, "\simeq"] \\
\pi^*\mathbb{T}_{\mathrm{T}^*\mathbf{X}}[-1] \arrow[r] & \mathbb{T}_{\mathcal{L}^\times} \arrow[r] & \mathbb{T}_\pi[1].
\end{tikzcd} \tag{68}$$

Thus, we obtain an equivalence $pr_2^*\mathbb{T}_{\mathbb{G}_m} \simeq Cone(i)$, which gives the second equivalence in (61). Using the last identification, we then obtain another equivalence of triangles

$$\begin{tikzcd}
Cocone(\alpha) \arrow[r, "i"] \arrow[d, dashed, "\simeq"] & \mathbb{T}_{\mathcal{L}^\times} \arrow[r] \arrow[d, "id"] & Cone(i) \arrow[d, "\simeq"] \\
\pi^*\mathbb{T}_{\mathrm{T}^*\mathbf{X}} \arrow[r] & \mathbb{T}_{\mathcal{L}^\times} \arrow[r] & pr_2^*\mathbb{T}_{\mathbb{G}_m},
\end{tikzcd} \tag{69}$$

which gives (62). Finally, combining (61) with (62), we then get the homotopy cofiber sequence $Cocone(\alpha) \to \mathbb{T}_{\mathcal{L}^\times} \to Ver_{\mathcal{L}^\times}$ and the desired splitting in (63).

□

Now, we are in place of proving the desired result.

Theorem 4.13. *Let $\mathcal{L}^\times$ and α be as above. Then the pair $\big(Cocone(\alpha) \hookrightarrow \mathbb{T}_{\mathcal{L}^\times}; \alpha\big)$ defines a 0-shifted contact structure on the derived stack $\mathcal{L}^\times$.*

Proof. By Lemma 4.12, we have the cofiber sequence

$$Cocone(\alpha) \hookrightarrow \mathbb{T}_{\mathcal{L}^\times} \to Ver_{\mathcal{L}^\times}, \tag{70}$$

where $Ver_{\mathcal{L}^\times} \simeq pr_2^*\mathbb{T}_{\mathbb{G}_m}$. Since $\mathbb{T}_{\mathbb{G}_m} \in QCoh(\mathbb{G}_m)$ is free of rank 1, $pr_2^*\mathbb{T}_{\mathbb{G}_m}$ corresponds to a line bundle $L[0]$, and hence cofib$(Cocone(\alpha) \hookrightarrow \mathbb{T}_{\mathcal{L}^\times})$ is a line bundle $L[0]$ up to quasi-isomorphism. Thus, the cofiber sequence (70) defines a 0-shifted pre-contact structure determined by a global 1-form α, with a cofiber $Ver_{\mathcal{L}^\times}$ as a certain line bundle.

Now, it remains to promote the pre-contact structure above to a 0-shifted contact structure. To this end, it suffices to show $d_{dR}\alpha$ is (locally) non-degenerate on $Cocone(\alpha)$. By the definition of α, we get

$$d_{dR}\alpha = \pi^* d_{dR}\lambda = \pi^*\omega, \tag{71}$$

which is non-degenerate on $\pi^*\mathbb{T}_{\mathrm{T}^*\mathbf{X}}$ as ω is a 0-shifted symplectic structure. Note that, from Lemma 4.12, we obtain $Cocone(\alpha) \simeq \pi^*\mathbb{T}_{\mathrm{T}^*\mathbf{X}}$, which gives the desired non-degeneracy condition for $d_{dR}\alpha$ on $Cocone(\alpha)$ and completes the proof. □

4.2.3 Prequantization of twisted cotangent stacks Recall that if $\mathbf{X}$ is a derived Artin stack locally of finite presentation equipped with a closed 1-form β of degree $(n+1)$, we define the ***n-shifted β-twisted cotangent stack $\mathrm{T}^*_\beta[n]\mathbf{X}$ of $\mathbf{X}$*** to be the fiber product

$$\begin{tikzcd} \mathrm{T}^*_\beta[n]\mathbf{X} \arrow[r] \arrow[d] & \mathbf{X} \arrow[d, "\Gamma_0"] \\ \mathbf{X} \arrow[r, "\Gamma_\beta"] & \mathrm{T}^*[n+1]\mathbf{X}, \end{tikzcd} \tag{72}$$

where $\Gamma_\beta, \Gamma_0 : \mathbf{X} \to \mathrm{T}^*[n+1]\mathbf{X}$ are the graphs of β and 0-section, respectively. Note that from [5, Corollary 2.4], both morphisms Γ_β, Γ_0 have $(n+1)$-shifted Lagrangian structures, and hence the resulting fiber product $\mathrm{T}^*_\beta[n]\mathbf{X}$ is an n-shifted symplectic stack. Moreover, [10, Prop. 1.21] shows that the projection $\mathrm{T}^*_\beta[n]\mathbf{X} \to \mathbf{X}$ carries a natural structure of an n-shifted Lagrangian fibration.

Regarding a possible prequantum structure on $\mathrm{T}^*_\beta[n]\mathbf{X} \to \mathbf{X}$, it has been proven in [10, Theorem 2.24] that if $\mathbf{X}$ is a derived Artin stack locally of finite presentation with an n-gerbe $\mathcal{G}$ on $\mathbf{X}$ such that $\beta := c_1(\mathcal{G}) \in \mathcal{A}^{1,cl}(X, n+1)$, the *characteristic class of* $\mathcal{G}$, then the projection

$$\pi : \mathrm{T}^*_{c_1(\mathcal{G})}[n]\mathbf{X} \to \mathbf{X}$$

has a natural structure of a prequantum n-shifted Lagrangian fibration determined by $\mathcal{G}$. In brief, the structure is given as follows:

Observation 4.14. Recall from [10, Prop. 2.21] that $\pi_{\mathbf{X}} : \mathrm{T}^*[n+1]\mathbf{X} \to \mathbf{X}$ has a natural structure of a prequantum $(n+1)$-shifted Lagrangian fibration, determined by the trivial $(n+1)$-gerbe $\mathcal{G}_{\mathbf{X}}$ on $\mathbf{X}$ together with a connective structure on $\pi^*_{\mathbf{X}}\mathcal{G}_{\mathbf{X}}$ given by the Liouville form $\lambda_{\mathbf{X}}$. In fact, $\lambda_{\mathbf{X}} \in \mathcal{A}^1(\mathrm{T}^*[n+1]\mathbf{X}, n+1)$ represents the connective structure on $\pi^*_{\mathbf{X}}\mathcal{G}_{\mathbf{X}}$, namely a null-homotopy of $c_1(\pi^*_{\mathbf{X}}\mathcal{G}_{\mathbf{X}}) \in \mathcal{A}^1(\mathrm{T}^*[n+1]\mathbf{X}, n+2)$. Note that, from [9, Theorem 2.9], there is an induced map

$$\mathcal{A}^{1,cl}(\mathrm{T}^*[n+1]\mathbf{X}, n+2) \to \mathcal{A}^{1,cl}(\mathrm{T}^*_{c_1(\mathcal{G})}[n]\mathbf{X}, n+1).$$

Now, given an n-gerbe $\mathcal{G}$ on $\mathbf{X}$, [10, Prop. 2.21] implies that the image of $c_1(\pi^*_{\mathbf{X}}\mathcal{G}_{\mathbf{X}})$ under this map is exactly $c_1(\pi^*\mathcal{G})$. Thus, the connective structure on $\pi^*_{\mathbf{X}}\mathcal{G}_{\mathbf{X}}$, given by $\lambda_{\mathbf{X}}$, determines a suitable connective structure on $\pi^*\mathcal{G}$ satisfying prequantization conditions for π.

Let us denote the ***induced connective structure*** on $\pi^*\mathcal{G}$ by $\lambda_{c_1(\mathcal{G})}$ in $\mathcal{A}^1(\mathrm{T}^*_{c_1(\mathcal{G})}[n]\mathbf{X}, n)$.

Example 4.15. *(Construction of a shifted contact structure using a twisted cotangent stack.)* Let $\mathbf{X}$ be a derived Artin stack locally of finite presentation with a trivial 0-gerbe $\mathcal{G}$ on $\mathbf{X}$ and a characteristic class $c_1(\mathcal{G}) \in \mathcal{A}^{1,cl}(\mathbf{X}, 1)$.

Consider the twisted 0-shifted cotangent stack $\pi_{c_1(\mathcal{G})} : \mathrm{T}^*_{c_1(\mathcal{G})}\mathbf{X} \to \mathbf{X}$ equipped with a prequantum 0-shifted Lagrangian fibration structure determined by $\mathcal{G}$ and a connective structure $\lambda_{c_1(\mathcal{G})} \in \mathcal{A}^1(\mathrm{T}^*_{c_1(\mathcal{G})}\mathbf{X}, 0)$ on $\pi^*\mathcal{G}$ described in the previous observation.

Since the trivial 0-gerbe $\mathcal{G}$ corresponds to a trivial line bundle, using the same approach in Section 4.2.2 with obvious modifications, we define a trivial $\mathbb{G}_m$-bundle $\pi : \mathcal{L}^\times_{c_1(\mathcal{G})} \to \mathrm{T}^*_{c_1(\mathcal{G})}\mathbf{X}$ and a global 1-form on $\mathcal{L}^\times_{c_1(\mathcal{G})}$

$$\alpha_{c_1(\mathcal{G})} := \pi^*\lambda_{c_1(\mathcal{G})} + d_{dR}\log(t). \tag{73}$$

Then we have:

Corollary 4.16. *Let $\mathcal{L}^{\times}_{c_1(\mathcal{G})}$ and $\alpha_{c_1(\mathcal{G})}$ be as above. Then the pair $\Big(Cocone(\alpha_{c_1(\mathcal{G})}) \hookrightarrow \mathbb{T}_{\mathcal{L}^{\times}_{c_1(\mathcal{G})}}; \alpha_{c_1(\mathcal{G})}\Big)$ defines a 0-shifted contact structure on the derived stack $\mathcal{L}^{\times}_{c_1(\mathcal{G})}$.*

Proof. The claim follows from Lemma 4.12 and Theorem 4.13 with obvious modifications according to Observation 4.14. □

4.2.4 Prequantization of the moduli stack of flat G-connections Let us recall some terminology and key results from [10, §4.5].

Denote by BG the ***classifying stack*** of an affine algebraic group G equipped with non-degenerate invariant symmetric bilinear pairing $\langle -,- \rangle$ on its Lie algebra. More precisely, it is defined as the quotient stack

$$BG = \operatorname{colim}\Big(* \leftleftarrows G \,\overset{\leftarrow}{\underset{\leftarrow}{\leftarrow}}\, G \times G \,\overset{\leftarrow}{\underset{\leftarrow}{\leftleftarrows}}\, \cdots \Big), \tag{74}$$

where the maps are given by the action and projection. Note that BG carries a canonical 2-shifted symplectic structure ω.

Fix the pair (BG, ω). Given a smooth and proper curve C, we let

$$LocSys_G(C) := Map(C_{dR}, BG)^{27}$$

be the ***moduli stack of flat G-connections on*** C and

$$Bun_G(C) := Map(C, BG)$$

be the ***moduli stack of G-bundles on C.*** Since BG is 2-symplectic, PTVV's results for mapping stacks in [9] imply two important consequences:

1. $LocSys_G(C)$ is 0-symplectic.
2. There is a natural closed 1-form of degree 1 on $Bun_G(C)$, which can be obtained by *integration along C*. We denote this form by $\int_C ev^*\omega$. For more details see [10, §1.5].

Note also that if $\langle -,- \rangle$ is the Killing form, it follows from the Grothendieck–Riemann–Roch theorem that the closed 1-form of degree 1 $\int_C ev^*\omega$ coincides with the first Chern class of the *determinant line bundle* $\mathcal{G}$ on $Bun_G(C)$, see [10, Example 1.26].

Regarding prequantization, we have the identification [10, Prop. 1.24]

$$LocSys_G(C) \simeq \mathrm{T}^*_{\int_C ev^*\omega} Bun_G(C), \tag{75}$$

where $\int_C ev^*\omega = c_1(\mathcal{G})$. That is, $LocSys_G(C)$ can be equivalently seen as a twisted cotangent stack of $Bun_G(C)$ with the twisting 1-form $\int_C ev^*\omega \in \mathcal{A}^{1,cl}(Bun_G(C), 1)$. From [10, Prop. 4.22], there is a natural prequantum 0-shifted Lagrangian fibration structure on

$$\pi_{c_1(\mathcal{G})} : LocSys_G(C) \to Bun_G(C)$$

determined by the determinant line bundle $\mathcal{G}$ on $Bun_G(C)$, such that $c_1(\mathcal{G}) = \int_C ev^*\omega$, with a connective structure $\lambda_{c_1(\mathcal{G})} \in \mathcal{A}^1(LocSys_G(C), 0)$ on $\pi^*_{c_1(\mathcal{G})}\mathcal{G}$ as described in Observation 4.14.

[27] Here C_{dR} denotes the ***de Rham stack*** associate with C. In general, for a derived stack $\mathbf{X}$, its de Rham stack is defined to be the functor $X_{dR} : A \mapsto X_{dR}(A) := \mathbf{X}(\pi_0(A)_{red})$, which corresponds to A-reduced points of $\mathbf{X}$ for A a cdga.

Now, we consider the $\mathbb{G}_m$-bundle $\pi : \mathcal{L}^{\times}_{\int_C ev^*\omega} \to LocSys_G(C)$ associated with $\pi^*_{c_1(\mathcal{G})}\mathcal{G}$. If, in addition, $\mathcal{L}^{\times}_{\int_C ev^*\omega}$ is trivial, we can define a global 1-form $\alpha_{\int_C ev^*\omega}$ on $\mathcal{L}^{\times}_{\int_C ev^*\omega}$ as in (73), which induces the desired contact structure on $\mathcal{L}^{\times}_{\int_C ev^*\omega}$ by Corollary 4.16.

In other words, we prove:

Corollary 4.17. *Let $LocSys_G(C), Bun_G(C)$, $\mathcal{G}$, and $\mathcal{L}^{\times}_{\int_C ev^*\omega}$ be as above. If, in addition, $\mathcal{G}$ is trivial, then the pair*

$$\Big(Cocone(\alpha_{\int_C ev^*\omega}) \hookrightarrow \mathbb{T}_{\mathcal{L}^{\times}_{\int_C ev^*\omega}};\ \alpha_{\int_C ev^*\omega}\Big)$$

defines a 0-shifted contact structure on the derived stack $\mathcal{L}^{\times}_{\int_C ev^*\omega}$, where $\int_C ev^*\omega = c_1(\mathcal{G})$.

Appendix A: Symplectic Darboux forms with even shifts

For the sake of completeness, we briefly summarize the cases when $k/2$ is even or odd integer. Here, the main difference from the case k being odd is about the existence of *middle degree variables*. In fact, when k is odd ($k/2 \notin \mathbb{Z}$), there is no such degree. But if $k/2$ is even, then 2-forms in the middle degree variables are *anti-symmetric*. On the other hand, when $k/2$ is odd, such forms are *symmetric* in the middle degree variables. We follow [3, Examples 5.9 & 5.10].

(a) When $k/2$ is *even*, say $k = -4\ell$ for $\ell \in \mathbb{N}$, the cdga A is now free over $A(0)$ generated by the new set of variables

$$\begin{aligned}
&x_1^{-i}, x_2^{-i}, \ldots, x_{m_i}^{-i} && \text{in degree } -i \text{ for } i = 1, 2, \ldots, 2\ell - 1,\\
&x_1^{-2\ell}, x_2^{-2\ell}, \ldots, x_{m_{2\ell}}^{-2\ell}, y_1^{-2\ell}, y_2^{-2\ell}, \ldots, y_{m_{2\ell}}^{-2\ell} && \text{in degree } -2\ell,\\
&y_1^{k+i}, y_2^{k+i}, \ldots, y_{m_i}^{k+i} && \text{in degree } k+i \text{ for } i = 0, 1, \ldots, 2\ell - 1.
\end{aligned} \tag{76}$$

We also define the element $\phi \in (\Omega^1_A)^k$ as before. Choose an element $H \in A^{k+1}$, the Hamiltonian, satisfying the analogue of *classical master equation*, and define d on x_j^{-i}, y_j^{k+i} (no distinguished generator z^k contrary to the contact case) as in Equation (11) using H. Then $d_{dR}\alpha_0$ defines an element $\omega^0 = \sum_{i=0}^{2\ell}\sum_{j=1}^{m_i} d_{dR}x_j^{-i} d_{dR}y_j^{k+i}$ in $(\Lambda^2\Omega^1_A)^k$, and set $\omega := (\omega^0, 0, 0, \cdots)$ as before, which satisfies the requirements by [3, Example 5.9].

(b) When $k/2$ is *odd*, say $k = -4\ell-2$ for $\ell \in \mathbb{N}$, A is freely generated over $A(0)$ by the variables

$$\begin{aligned}
&x_1^{-i}, x_2^{-i}, \ldots, x_{m_i}^{-i} && \text{in degree } -i \text{ for } i = 1, 2, \ldots, 2\ell,\\
&z_1^{-2\ell-1}, z_2^{-2\ell-1}, \ldots, z_{m_{2\ell+1}}^{-2\ell-1} && \text{in degree } -2\ell - 1,\\
&y_1^{k+i}, y_2^{k+i}, \ldots, y_{m_i}^{k+i} && \text{in degree } k+i \text{ for } i = 0, 1, \ldots, 2\ell.
\end{aligned} \tag{77}$$

Choose an element $H \in A^{k+1}$, the Hamiltonian, satisfying the analogue of *classical master equation*

$$\sum_{i=1}^{2\ell}\sum_{j=1}^{m_i} \frac{\partial H}{\partial x_j^{-i}} \frac{\partial H}{\partial y_j^{k+i}} + \frac{1}{4}\sum_{j=1}^{m_{2\ell+1}} \Big(\frac{\partial H}{\partial z_j^{-2\ell-1}}\Big)^2 = 0 \text{ in } A^{k+2}. \tag{78}$$

Define d on x_j^{-i}, y_j^{k+i} as in Equation (11) using H, and set $dz_j^{-2\ell-1} := \dfrac{1}{2}\dfrac{\partial H}{\partial z_j^{-2\ell-1}}$.

We define the element $\phi \in (\Omega^1_A)^k$ by

$$\phi := \sum_{i=0}^{2\ell}\sum_{j=1}^{m_i} \big[-ix_j^{-i} d_{dR}y_j^{k+i} + (-1)^{i+1}(k+i)y_j^{k+i} d_{dR}x_j^{i}\big] + k\sum_{j=1}^{m_{2\ell+1}} z_j^{-2\ell-1} d_{dR}z_j^{-2\ell-1}. \tag{79}$$

Then $d_{dR}\alpha_0$ defines an element $\omega^0 = \sum_{i=0}^{2\ell}\sum_{j=1}^{m_i} d_{dR}x_j^{-i}d_{dR}y_j^{k+i} + \sum_{j=1}^{m_{2\ell+1}} d_{dR}z_j^{-2\ell-1}d_{dR}z_j^{-2\ell-1}$ in $(\Lambda^2\Omega_A^1)^k$, and set $\omega := (\omega^0, 0, 0, \cdots)$ as before, which satisfies the desired properties by [3, Example 5.10].

Observation A.1. For $k \not\equiv 2 \mod 4$, the virtual dimension vdim A is always *even*. Otherwise, it can take any value in $\mathbb{Z}$. More precisely, for any $k < 0$ we have

$$\text{vdim } A = \begin{cases} 0, & \text{if } k \text{ is odd,} \\ \text{even in } \mathbb{Z}, & \text{if } k/2 \text{ is even,} \\ \text{any value in } \mathbb{Z}, & \text{if } k/2 \text{ is odd.} \end{cases}$$

Appendix B: Symplectic Darboux models for derived Artin stacks

Now, we give the prototype construction from [1, Theorem 2.10]:

Example B.1. Let $(\mathbf{X}, \omega)$ be a k-shifted symplectic derived Artin $\mathbb{K}$-stack and $x \in \mathbf{X}$. We construct a local model with an atlas for the case k *odd*, say $k = -2\ell - 1$ for $\ell \in \mathbb{N}$.

We will essentially obtain either analogous or identical equations as in the case of shifted symplectic derived schemes (cf. § 2.2), but with additional finitely many generators in degree $k-1$. It means that our model will still rely on the inductively constructed sequence of cdgas as in Equation (2) with $A = A(-k+1)$ instead of $A(-k)$.

The key idea is that the extra generators in degree $k-1$ would not play an essential role. This is because the main ingredients of the construction in § 2.2, namely ω^0, H, ϕ, do not involve any of these extra variables due to degree reasons.

Applying Theorem 3.5 to $(\mathbf{X}, \omega)$, let us start with the construction of a minimal standard form open neighborhood (A, φ, p) of x: Let $A(0)$ be a smooth $\mathbb{K}$-algebra of dim m_0, choose $x_1^0, \dots, x_{m_0}^0$ such that $d_{dR}x_1^0, \dots, d_{dR}x_{m_0}^0$ form a basis for $\Omega^1_{A(0)}$. Then we define A, as a *commutative graded algebra*, to be the free graded algebra over $A(0)$ generated by the variables

$$\begin{aligned} &x_1^{-i}, x_2^{-i}, \dots, x_{m_i}^{-i} && \text{in degree } (-i) \text{ for } i = 1, 2, \dots, \ell, \\ &y_1^{k+i}, y_2^{k+i}, \dots, y_{m_i}^{k+i} && \text{in degree } (k+i) \text{ for } i = 0, 1, \cdots \ell, \\ &w_1^{k-1}, w_2^{k-1}, \dots, w_n^{k-1} && \text{in degree } (k-1), \end{aligned} \tag{80}$$

where $m_1, \dots, m_\ell \in \mathbb{N}$ and $n = \dim H^1(\mathbb{L}_{\mathbf{X}}|_x)$, the (minimal possible) relative dimension of φ. Then we define an element $\omega^0 = \sum_{i=0}^{\ell}\sum_{j=1}^{m_i} d_{dR}x_j^{-i}d_{dR}y_j^{k+i}$ in $(\Lambda^2\Omega_A^1)^k$, and $\omega := (\omega^0, 0, 0, \dots)$ as before.

Choose an element $H \in A^{k+1}$, the Hamiltonian, satisfying the *classical master equation* in (10). Then we define the *internal differential* on A by $d = 0$ on $A(0)$ and by Equation (11) on the generators x_j^{-i}, y_j^{k+i} for each i, j. As discussed before, the condition on H above is equivalent to saying "$dH = 0$". Note that we do not specify dw_j^{k-1} for $j = 1, \dots, n$, and hence d is not completely determined on A yet. But, by [1, Theorem 2.10], $w_1^{k-1}, w_2^{k-1}, \dots, w_n^{k-1}$ do not play any significant role in the construction, and hence can be chosen arbitrarily. In fact, from the minimality argument, we have $dw_j^{k-1}|_p = 0$ for each j.

Now, choose $\phi := \sum_{i=0}^{\ell}\sum_{j=1}^{m_i}\left[-ix_j^{-i}d_{dR}y_j^{k+i} + (k+i)y_j^{k+i}d_{dR}x_j^i\right]$, then $dH = 0$ in A^{k+2}, $d_{dR}H + d\phi = 0$ in $(\Omega_A^1)^{k+1}$, and $d_{dR}\phi = k\omega^0$.

Let B be the graded sub-cdga of A over $A(0)$ generated by the variables x_j^i, y_j^i only, with inclusion $\iota : B \hookrightarrow A$. Since ω^0, H, ϕ above do not involve any of w_j^{k-1} for degree reasons, $H \in B$, and ω^0, ϕ are all images under ι of ω_B^0, ϕ_B, respectively. Then $\omega_B := (\omega_B^0, 0, 0, \dots)$ is a k-shifted symplectic structure on $\mathbf{V} = \mathrm{Spec}B$ such that the pair (B, ω_B) is in Darboux form as in Section 2.2, and B is minimal at p. By construction, we have

$$\mathrm{Spec}B = \mathbf{V} \xleftarrow{j:=\mathrm{Spec}(\iota)} \mathbf{U} = \mathrm{Spec}A \xrightarrow{\varphi} \mathbf{X}$$

such that the induced morphism $\tau(\mathbf{U}) \xrightarrow{\tau(j)} \tau(\mathbf{V})$ between truncations is an isomorphism (as $H^0(A) \simeq H^0(B)$), and $\varphi^*(\omega) \sim j^*(\omega_B)$ except in degree $k-1$.

Remark B.2. For the other cases (a) $k \equiv 0 \mod 4,$ and (b) $k \equiv 2 \mod 4$, the cdgas A are the corresponding algebras generated by the variables as in Equations (76) and (77), respectively, with modification by adding the generators $w_1^{k-1}, w_2^{k-1}, \dots, w_n^{k-1}$ in degree $k-1$.

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